Derivations of the parabolic subalgebras of the general linear Lie algebra over a commutative ring

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Abstract

Let $R$ be an arbitrary commutative ring with identity, $\text{gl}(n, R)$ the general linear Lie algebra over $R$. The aim of this paper is to give an explicit description of the derivation algebras of the parabolic subalgebras of $\text{gl}(n, R)$.

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1. Introduction

Let $R$ be a commutative ring with identity, $R^*$ the group of invertible elements of $R$, $I(R)$ the set consisting of all ideals of $R$. Let $\text{gl}(n, R)$ be the general linear Lie algebra over $R$ consisting of all $n \times n$ matrices over $R$ and with the bracket operation $[x, y] = xy - yx$. For $A \in \text{gl}(n, R)$, we denote by $A'$ the transpose of $A$. $t$ (resp., $u$) denote the subset of $\text{gl}(n, R)$ consisting of all $n \times n$ upper triangular (resp., strictly upper triangular) matrices over $R$ and by $d$ the subset of $\text{gl}(n, R)$ consisting of all $n \times n$ diagonal matrices over $R$. Let

$$u_1 = u, \quad u_2 = [u, u_1], \quad u_3 = [u, u_2], \ldots$$

be the lower central series of $u$. We denote by $E$ the identity matrix in $\text{gl}(n, R)$ and by $E_{i,j}$ the matrix in $\text{gl}(n, R)$ whose sole nonzero entry 1 is in the $(i, j)$ position. Let $RE$ be the set $\{rE|r \in R\}$ of all scalar matrices, $v_k$ the set $\{A'|A \in u_k\}$.

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We denote by \( \text{Hom}_R(d, R) \) the set consisting of all homomorphisms \( \chi: d \to R \) of \( R \)-modules, which forms a new \( R \)-module. For \( 1 \leq i \leq n \), \( \chi_i: d \to R \), defined by \( \chi_i(\text{diag}(d_1, d_2, \ldots, d_n)) = d_i \), is a standard homomorphism of \( R \)-modules. It is easy to see that \( \text{Hom}_R(d, R) \) is a free \( R \)-module of rank \( n \) with a basis \( \{ \chi_i | i = 1, 2, \ldots, n \} \). By definition, any subalgebra of \( \text{gl}(n, R) \) containing \( t \) is called a parabolic subalgebra of \( \text{gl}(n, R) \).

Recently, significant work has been done in studying the automorphisms and derivations of matrix algebras and their subalgebras (see [1–7]). However, the study on the derivations or automorphisms of the parabolic subalgebras of the general linear Lie algebra has not been reported. In this paper, we will describe the parabolic subalgebras of \( \text{gl}(n, R) \) and then give an explicit description on the derivations of the parabolic subalgebras of \( \text{gl}(n, R) \), for \( R \) an arbitrary commutative ring.

2. The parabolic subalgebras of \( \text{gl}(n, R) \)

Definition 2.1. Let \( \Phi = \{ A_{j,i} \in I(R) | 1 \leq i < j \leq n \} \) be a subset of \( I(R) \) consisting of \( n(n-1)/2 \) ideals of \( R \). We call \( \Phi \) a flag of ideals of \( R \), if \( A_{j,k}A_{k,i} \subseteq A_{j,i} \subseteq A_{j,k} \cap A_{k,i} \) for any \( 1 \leq i < j \leq n \) and any \( k \) (if exists) for which \( i < k < j \).

Example 2.2. If all \( A_{j,i} \), \( 1 \leq i < j \leq n \) are taken to be 0 (resp., \( R \)), then \( \Phi = \{ A_{j,i} | 1 \leq i < j \leq n \} \) is a flag of ideals of \( R \).

Example 2.3. Let \( A_{2,1}, A_{3,2}, \ldots, A_{n,n-1} \) take any ideals of \( R \) respectively, and let \( A_{j,i} = \prod_{k=1}^{j-i} A_{i+k,i+k-1} \) for \( 1 \leq i < j \leq n \). Then \( \Phi = \{ A_{j,i} | 1 \leq i < j \leq n \} \) is a flag of ideals of \( R \).

Example 2.4. Let \( A_{2,1}, A_{3,2}, \ldots, A_{n,n-1} \) take any ideals of \( R \) respectively, and let \( A_{j,i} = \bigcap_{k=1}^{j-i} A_{i+k,i+k-1} \) for \( 1 \leq i < j \leq n \). Then \( \Phi = \{ A_{j,i} | 1 \leq i < j \leq n \} \) is a flag of ideals of \( R \).

Theorem 2.5. \( p \) is a parabolic subalgebra of \( \text{gl}(n, R) \) if and only if there exists a flag \( \Phi = \{ A_{j,i} | 1 \leq i < j \leq n \} \) of ideals of \( R \) such that \[
\begin{align*}
p &= t + \sum_{1 \leq i < j \leq n} A_{j,i} E_{j,i}.
\end{align*}
\]

Proof. Suppose that \( \Phi = \{ A_{j,i} | 1 \leq i < j \leq n \} \) is a flag of ideals of \( R \) and \( p = t + \sum_{1 \leq i < j \leq n} A_{j,i} E_{j,i} \). Let \[
\begin{align*}
x &= \sum_{1 \leq i,j \leq n} a_{i,j} E_{i,j} \in p, \quad y = \sum_{1 \leq i,j \leq n} b_{i,j} E_{i,j} \in p,
\end{align*}
\]
where all \( a_{i,j}, b_{i,j} \) lie in \( R \) and \( a_{l,k} \in A_{l,k} \), \( b_{l,k} \in A_{l,k} \) for \( 1 \leq k < l \leq n \). It is obvious that \( rx + sy \in p \) for any \( r, s \in R \). Note that
\[ [x, y] = \sum_{1 \leq i, j \leq n} c_{i,j} E_{i,j}, \quad \text{where} \quad c_{i,j} = \sum_{k=1}^{n} (a_{i,k} b_{k,j} - b_{i,k} a_{k,j}). \]

If \( 1 \leq k < l \leq n \), because \( \Phi = \{ A_{j,i} | 1 \leq i < j \leq n \} \) is a flag of ideals of \( R \), we have that \((a_{i,j} b_{j,k} - b_{i,j} a_{k,j}) \in A_{l,k}\), which says that \( c_{l,k} \in A_{l,k} \). Thus \([x, y] \in \mathfrak{p}\). Hence \( \mathfrak{p} \) is a subalgebra of \( \text{gl}(n, R) \) containing \( t \).

On the other hand, let \( \mathfrak{p} \) be a parabolic subalgebra of \( \text{gl}(n, R) \). For any \( 1 \leq i < j \leq n \), we define
\[ A_{j,i} = \{ a_{j,i} E_{j,i} \in \mathfrak{p} \}, \]
and let
\[ \Phi = \{ A_{j,i} | 1 \leq i < j \leq n \}, \]
\[ q = t + \sum_{1 \leq i < j \leq n} A_{j,i} E_{j,i}. \]

In the following, we will prove that \( \Phi \) is a flag of ideals of \( R \), and \( q = \mathfrak{p} \). It is obvious that all \( A_{j,i} \) are ideals of \( R \) for \( 1 \leq i < j \leq n \). If \( 1 \leq i < k < j \leq n \) and \( a_{j,i} \in A_{j,i} \), then by
\[ [E_{k,j}, a_{j,i} E_{j,i}] = a_{j,i} E_{k,i} \in \mathfrak{p}, \]
\[ [a_{j,i} E_{j,i}, E_{i,k}] = a_{j,i} E_{j,k} \in \mathfrak{p}. \]
we have that \( a_{j,i} \in A_{k,i} \cap A_{j,k} \), which leads to \( A_{j,i} \subseteq A_{k,i} \cap A_{j,k} \). If \( 1 \leq i < k < j \leq n \) and \( a_{j,k} \in A_{j,k}, a_{k,i} \in A_{k,i} \), then by
\[ [a_{j,k} E_{j,k}, a_{k,i} E_{k,i}] = a_{j,k} a_{k,i} E_{j,i} \in \mathfrak{p}, \]
we have that \( a_{j,k} a_{k,i} \in A_{j,i} \), which leads to \( A_{j,k} \subseteq A_{j,i} \). This implies that \( \Phi = \{ A_{j,i} | 1 \leq i < j \leq n \} \) is a flag of ideals of \( R \) and \( q = t + \sum_{1 \leq i < j \leq n} A_{j,i} E_{j,i} \) is a parabolic subalgebra of \( \text{gl}(n, R) \). It is obvious that \( q \subseteq \mathfrak{p} \). Let \( x = T + \sum_{1 \leq i < j \leq n} a_{j,i} E_{j,i} \in \mathfrak{p} \), where \( T \in t \). Then \( \sum_{1 \leq i < j \leq n} a_{j,i} E_{j,i} \in \mathfrak{p} \). For any \( 1 \leq k < l \leq n \), by
\[ [E_{l,l}, [E_{k,k}, \sum_{1 \leq i < j \leq n} a_{j,i} E_{j,i}]] = -a_{l,k} E_{l,k} \in \mathfrak{p}, \]
we have that \( a_{l,k} \in A_{l,k} \), forcing that \( x \in q \). Thus \( q \subseteq \mathfrak{p} \), which leads to \( \mathfrak{p} = q \). This completes the proof. \( \square \)

3. The standard derivations of \( \mathfrak{p} \)

Let \( \mathfrak{p} = t + \sum_{1 \leq i < j \leq n} A_{j,i} E_{j,i} \) be a fixed parabolic subalgebra of \( \text{gl}(n, R) \) with \( \Phi = \{ A_{j,i} \in I(R) | 1 \leq i < j \leq n \} \) a flag of ideals of \( R \). We denote by \( \text{Der} \mathfrak{p} \) the set consisting of all derivations of \( \mathfrak{p} \).

The standard derivations of \( \mathfrak{p} \) are as follows.

(A) Inner derivations

Let \( x \in \mathfrak{p} \), then \( \text{ad} x : \mathfrak{p} \to \mathfrak{p}, \ y \mapsto [x, y] \), is a derivation of \( \mathfrak{p} \), called the inner derivation of \( \mathfrak{p} \) induced by \( x \). Let \( \text{ad} \mathfrak{p} \) denote the set consisting of all \( \text{ad} x, x \in \mathfrak{p} \), which forms an ideal of \( \text{Der} \mathfrak{p} \). It is well known that \( \text{ad} \mathfrak{p} \) is isomorphic to the quotient Lie algebra of \( \mathfrak{p} \) to \( Z(\mathfrak{p}) \).
(B) Central derivations

For $1 \leq i < j \leq n$, let $B_{j,i}$ be the annihilator of $A_{j,i}$ in $R$:

$B_{j,i} = \{ r \in R \mid rA_{j,i} = 0 \}.$

For brevity, we denote $B_{i+1,i}$ by $B_i$ for $i = 1, 2, \ldots, n - 1$ and let $B_0 = B_n = R$. Let $F_i = E_{i,i} - E_{i+1,i+1}$, $i = 1, 2, \ldots, n - 1$ and let $F_n = E_{n,n}$. It is obvious that $d$ is a free $R$-module of rank $n$ with a basis $\{F_1, F_2, \ldots, F_n\}$.

**Definition 3.1.** Let $\chi : d \to R$ be a homomorphism of $R$-modules. We call $\chi$ suitable for central derivations if $\chi(F_i) \in B_i$ for all $1 \leq i \leq n$.

Using the homomorphism $\chi : d \to R$ which is suitable for central derivations, we define $\eta_\chi : p \to p$ by

$\eta_\chi(x) = \chi(D_x)E$, $x \in p$.

where $D_x$ denotes the projection of $x$ to $d$ ($D_x = \sum_{i=1}^n a_{i,i}E_{i,i}$, when $x = \sum_{1 \leq i,j \leq n} a_{i,j} E_{i,j}$).

**Lemma 3.2.** $\eta_\chi$ is a derivation of $p$, provided that $\chi$ is suitable for central derivations.

**Proof.** Suppose that $\chi(F_i) = s_i \in B_i$, $i = 1, 2, \ldots, n$. Let

$x = \sum_{1 \leq i,j \leq n} a_{i,j} E_{i,j} \in p$, $y = \sum_{1 \leq i,j \leq n} b_{i,j} E_{i,j} \in p,$

where all $a_{i,j}, b_{i,j}$ lie in $R$ and $a_{i,k}, b_{i,k} \in A_{i,k}$ for $1 \leq k < l \leq n$. It is obvious that $\eta_\chi(rx + sy) = r\eta_\chi(x) + s\eta_\chi(y)$ for any $r, s \in R$. Note that

$[x, y] = \sum_{1 \leq i,j \leq n} c_{i,j} E_{i,j}$, where $c_{i,j} = \sum_{k=1}^n (a_{i,k} b_{k,j} - b_{i,k} a_{k,j}).$

The projection of $[x, y]$ to $d$ is $\sum_{i=1}^n c_{i,i} E_{i,i} = \sum_{i=1}^n \left( \sum_{j=1}^i c_{j,j} \right) F_i$. Because $\chi$ is suitable for central derivations and $\Phi = \{A_{j,i} \in I(R) \mid 1 \leq i < j \leq n\}$ is a flag of ideals of $R$, we have that

$\eta_\chi([x, y]) = \sum_{i=1}^n \left( \sum_{j=1}^i c_{j,j} \right) s_i E = 0.$

This exactly equals $[\eta_\chi(x), y] + [x, \eta_\chi(y)]$. Hence $\eta_\chi$ is a derivation of $p$. $\square$

$\eta_\chi$ is called a central derivation of $p$, if $\chi$ is suitable for central derivations.

(C) Extreme derivations

If $n \geq 3$, we define another type of derivations of $p$ as follows.

**Definition 3.3.** Let $n \geq 3$, $\pi : A_{n,1} \to R$ be a homomorphism of $R$-modules. We call $\pi$ suitable for extreme derivations, if the following conditions are satisfied:

1. $\pi(A_{n,1}) \subseteq (\cap_{i=1}^{n-1} B_{n,i}) \cap (\cap_{i=2}^n B_{i,1});$
2. $\pi(A_{n,j} A_{j,1}) = 0$ for $j = 2, 3, \ldots, n - 1$;
3. $2\pi(A_{n,1}) = 0.$
Using the homomorphism $\pi : A_{n,1} \rightarrow R$ which is suitable for extreme derivations, we define $\rho_{\pi} : \mathfrak{p} \rightarrow \mathfrak{p}$ by

$$\rho_{\pi} \left( \sum_{1 \leq i, j \leq n} a_{i,j} E_{i,j} \right) = \pi(a_{n,1}) E_{1,n}.$$ 

Lemma 3.4. $\rho_{\pi}$ is a derivation of $\mathfrak{p}$, provided that $\pi : A_{n,1} \rightarrow R$ is suitable for extreme derivations.

Proof. Let

$$x = \sum_{1 \leq i, j \leq n} a_{i,j} E_{i,j} \in \mathfrak{p}, \quad y = \sum_{1 \leq i, j \leq n} b_{i,j} E_{i,j} \in \mathfrak{p},$$

where all $a_{i,j}, b_{i,j}$ lie in $R$ and $a_{i,k} \in A_{i,k}, b_{l,k} \in A_{l,k}$ for $1 \leq k < l \leq n$. It is obvious that $\rho_{\pi}(rx + sy) = r\rho_{\pi}(x) + s\rho_{\pi}(y)$ for any $r, s \in R$. Note that

$$[x, y] = \sum_{1 \leq i, j \leq n} c_{i,j} E_{i,j}, \text{ where } c_{i,j} = \sum_{k=1}^{n} (a_{i,k} b_{k,j} - b_{i,k} a_{k,j}).$$

Because $\pi$ is suitable for extreme derivations, we have that

$$\rho_{\pi}([x, y]) = \pi \left( \sum_{j=1}^{n} (a_{n,j} b_{j,1} - b_{n,j} a_{j,1}) \right) E_{1,n}$$

$$= [b_{1,1}\pi(a_{n,1}) - a_{1,1}\pi(b_{n,1}) + a_{n,n}\pi(b_{n,1}) - b_{n,n}\pi(a_{n,1})] E_{1,n}.$$ 

On the other hand,

$$[\rho_{\pi}(x), y] + [x, \rho_{\pi}(y)] = \sum_{k=1}^{n} (b_{n,k}\pi(a_{n,1}) - a_{n,k}\pi(b_{n,1})) E_{1,k}$$

$$+ \sum_{k=1}^{n} (a_{k,1}\pi(b_{n,1}) - b_{k,1}\pi(a_{n,1})) E_{k,n}$$

$$= \pi(a_{n,1})(b_{n,1} - b_{1,1}) + \pi(b_{n,1})(a_{1,1} - a_{n,n}) E_{1,n}.$$ 

Thus

$$\rho_{\pi}([x, y]) = [\rho_{\pi}(x), y] + [x, \rho_{\pi}(y)].$$

Hence $\rho_{\pi}$ is a derivation of $\mathfrak{p}$. \qed

$\rho_{\pi}$ is called an extreme derivation of $\mathfrak{p}$, if $\pi$ is suitable for extreme derivations.

(D) Permutation derivations

If $n = 2$, we define another type of derivations of $\mathfrak{p}$ as follows.

Definition 3.5. Let $\sigma_1 : R \rightarrow R; \sigma_2 : R \rightarrow A_{2,1}; \sigma_3 : A_{2,1} \rightarrow A_{2,1}; \sigma_4 : A_{2,1} \rightarrow R$ be four homomorphisms of $R$-modules. If the following conditions are satisfied, we call $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ suitable for permutation derivations:

1. $2\sigma_3(A_{2,1}) = 2\sigma_4(A_{2,1}) = 2\sigma_2(R) = 0.$
2. $\sigma_1|_{A_{2,1}} = \sigma_3.
Theorem 4.1. Let a parabolic subalgebra of \( \text{gl}_7 \) be a derivation of \( p \) by
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \sigma_1(a) & \sigma_4(c) \\ \sigma_3(c) + \sigma_2(b) & \sigma_1(a) \end{pmatrix}.
\]

Lemma 3.6. \( \varphi(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \) is a derivation of \( p \), provided that \( (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \) is suitable for permutation derivations.

Proof. Let \( x = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \), \( y = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \).
It is obvious that
\[
\varphi(\sigma_1, \sigma_2, \sigma_3, \sigma_4)(rx + sy) = r\varphi(\sigma_1, \sigma_2, \sigma_3, \sigma_4)(x) + s\varphi(\sigma_1, \sigma_2, \sigma_3, \sigma_4)(y), \quad r, s \in R.
\]
Note that
\[
[x, y] = \begin{pmatrix} b_1c_2 - b_2c_1 & a_1b_2 + b_1d_2 - a_2b_1 - b_2d_1 \\ c_1a_2 + d_1c_2 - c_2a_1 - d_2c_1 & c_1b_2 - c_2b_1 \end{pmatrix},
\]
then
\[
\varphi(\sigma_1, \sigma_2, \sigma_3, \sigma_4)([x, y]) = (b_1\sigma_3(c_2) - b_2\sigma_3(c_1))E_{1,1} + (\sigma_4(c_1)(a_2 - d_2) + \sigma_4(c_2)(d_1 - a_1))E_{1,2} + ((\sigma_3(c_2) - \sigma_2(b_2))(d_1 - a_1) + (\sigma_3(c_1) - \sigma_2(b_1))(a_2 - d_2))E_{2,1} + (b_1\sigma_3(c_2) - b_2\sigma_3(c_1))E_{2,2},
\]
which exactly equals \([\varphi(\sigma_1, \sigma_2, \sigma_3, \sigma_4)(x), y] + [x, \varphi(\sigma_1, \sigma_2, \sigma_3, \sigma_4)(y)]\).
Hence \( \varphi(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \) is a derivation of \( p \). \( \square \)

4. The derivation algebra of \( p \)

If \( n > 1 \), for \( 1 \leq k \leq n - 1 \), we assume that \( n = kq + r \) with \( q \) and \( r \) two non-negative integers and \( r \leq k - 1 \). Let \( D_k = \text{diag}(E_k, 2E_k, \ldots, qE_k, (q + 1)E_r) \in d \), \( k = 1, 2, \ldots, n - 1 \), where \( E_k \) denotes the \( k \times k \) identity matrix.

Theorem 4.1. Let \( R \) be an arbitrary commutative ring with identity,
\[
p = t + \sum_{1 \leq i < j \leq n} A_{j,i}E_{j,i}
\]
a parabolic subalgebra of \( \text{gl}(n, R) \) with \( \Phi = \{A_{j,i} \in I(R) | 1 \leq i < j \leq n\} \) a flag of ideals of \( R \).

1. Every derivation of \( p \) may be uniquely written as the sum of an inner derivation, a central derivation and an extreme derivation, if \( n \geq 3 \).
2. Every derivation of \( p \) may be uniquely written as the sum of an inner derivation and a permutation derivation, if \( n = 2 \).

Proof. If \( n = 1 \), then it is easy to determine \( \text{Der} \ p \). From now on, we assume that \( n > 1 \). Let \( \phi \) be a derivation of \( p \). Firstly, we will prove that there exists some \( P_0 \in p \) such that

\[
(\text{ad} \ P_0 + \phi)(d) \subseteq d.
\]
For any \( H \in d \), suppose that

\[
\phi(H) \equiv \left( \sum_{1 \leq i < j \leq n} a_{j,i}(H)E_{j,i} \right) \pmod{t},
\]

where \( a_{j,i}(H) \in A_{j,i} \) are relative to \( H \). By \([D_1, H] = 0\), we have that

\[
[H, \phi(D_1)] = [D_1, \phi(H)],
\]

which follows that

\[
\sum_{1 \leq i < j \leq n} (\chi_j(H) - \chi_i(H))a_{j,i}(D_1)E_{j,i} = \sum_{1 \leq i < j \leq n} (\chi_j(D_1) - \chi_i(D_1))a_{j,i}(H)E_{j,i}.
\]

This yields that

\[
(\chi_j(H) - \chi_i(H))a_{j,i}(D_1) = (\chi_j(D_1) - \chi_i(D_1))a_{j,i}(H),
\]

for any \( 1 \leq i < j \leq n - 1 \). In particular, we have that

\[
a_{i+1,i}(H) = (\chi_{i+1}(H) - \chi_i(H))a_{i+1,i}(D_1), \quad i = 1, 2, \ldots, n - 1.
\]

Let \( P_1 = \sum_{i=1}^{n-1} a_{i+1,i}(D_1)E_{i+1,i} \in p \), then \((\phi + \text{ad} P_1)(d) \subseteq t + v_2\). By replacing \( \phi \) with \( \phi + \text{ad} P_1 \), then we may assume that \( \phi(d) \subseteq t + v_2 \). If \( n = 2 \), this step is completed. If \( n > 2 \), for any \( H \in d \), we now suppose that

\[
\phi(H) \equiv \left( \sum_{1 \leq i < j \leq n-1} b_{j+1,i}(H)E_{j+1,i} \right) \pmod{t},
\]

where \( b_{j+1,i}(H) \in A_{j+1,i} \) are relative to \( H \). By \([D_2, H] = 0\), we have that

\[
[H, \phi(D_2)] = [D_2, \phi(H)],
\]

which follows that

\[
\sum_{1 \leq i < j \leq n-1} (\chi_{j+1}(H) - \chi_i(H))b_{j+1,i}(D_2)E_{j+1,i} = \sum_{1 \leq i < j \leq n-1} (\chi_{j+1}(D_2) - \chi_i(D_2))b_{j+1,i}(H)E_{j+1,i}.
\]

This yields that

\[
(\chi_{j+1}(H) - \chi_i(H))b_{j+1,i}(D_2) = (\chi_{j+1}(D_2) - \chi_i(D_2))b_{j+1,i}(H),
\]

for any \( 1 \leq i < j \leq n - 1 \). In particular, we have that

\[
b_{i+2,i}(H) = (\chi_{i+2}(H) - \chi_i(H))b_{i+2,i}(D_2), \quad i = 1, 2, \ldots, n - 2.
\]

Let \( P_2 = \sum_{i=1}^{n-2} b_{i+2,i}(D_2)E_{i+2,i} \), then \((\phi + \text{ad} P_2)(t) \subseteq t + v_3\). If \( n = 3 \), this step is completed. If \( n > 3 \), we repeat above replacement. After \( n - 2 \) steps, we may assume that \( \phi(d) \subseteq t + v_{n-1} \). For any \( H \in d \), suppose that \( \phi(H) \equiv c_{n,1}(H)E_{n,1} \pmod{t} \), where \( c_{n,1}(H) \in A_{n,1} \) is relative to \( H \). By \([D_{n-1}, H] = 0\), we have that

\[
[H, \phi(D_{n-1})] = [D_{n-1}, \phi(H)],
\]
which follows that
\[(\chi_n(H) - \chi_1(H))c_{n,1}(D_{n-1}) = (\chi_n(D_{n-1}) - \chi_1(D_{n-1}))c_{n,1}(H).\]

So we have that
\[c_{n,1}(H) = (\chi_n(H) - \chi_1(H))c_{n,1}(D_{n-1}).\]

Let \(P_n = c_{n,1}(D_{n-1})E_{n,1}\), then \((\phi + \text{ad } P_n)(d) \subseteq t\). By replacing \(\phi\) with \(\phi + \text{ad } P_n\), then we may assume that \(\phi(d) \subseteq t\).

By similar replacements (the process is omitted), we may further assume that \(\phi(d) \subseteq d\).

Thus there exists certain \(P_0 \in p\) such that \(\phi(P_0) \subseteq d\). Denote \(\phi(P_0)\) by \(\phi_1\), then \(\phi_1(d) \subseteq d\).

Secondly, we will prove that, for any \(1 \leq i < j \leq n\), \(A_{j,i}E_{j,i} + RE_{i,j}\) is stable under the action of \(\phi_1\).

For fixed \(i < j\) and fixed \(a \in A_{j,i}\), suppose that \(\phi_1(aE_{j,i})\) is stable under the action of \(\phi_1\). By choosing \(D\) such that \(\chi_j(D) = \chi_i(D)\), we see that \(a_{k,k} = 0\) for all \(1 \leq k \leq n\). By choosing \(D\) such that \(\chi_j(D) = \chi_i(D)\), we see that \(a_{k,k} = 0\) for all \(1 \leq k \leq n\). If \(k \neq l\) and \(\{k, l\} \neq \{i, j\}\), we may choose \(D\) such that \(\chi_k(D) = \chi_l(D)\). Therefore, \(\phi_1(A_{j,i}E_{j,i}) \subseteq A_{j,i}E_{j,i} + RE_{i,j}\). Similarly, we may prove that \(\phi_1(RE_{i,j}) \subseteq A_{i,j}E_{i,j} + RE_{i,j}\). Hence, for all \(1 \leq i < j \leq n\), \(A_{j,i}E_{j,i} + RE_{i,j}\) is stable under the action of \(\phi_1\).

For any \(1 \leq i \leq n - 1\), suppose that \(\phi_1(E_{i,i+1}) = r_i E_{i,i+1} + s_i E_{i+1,i}\), with \(r_i \in R, s_i \in A_{i+1,i+1}\). Let \(H_0 = \text{diag}(0, r_1, r_1 + r_2, \ldots, \sum_{i=1}^{n-1} r_i)\), and denote \(\phi + \text{ad } H_0\) by \(\phi_2\), then we have that \(\phi_2(E_{i,i+1}) = s_i E_{i,i+1} \in A_{i+1,i}E_{i,i+1}\).

Now, we will discuss \(\phi_2\) in two cases respectively.

**Case 1: \(n \geq 3\)**

In this case, we will show that \(\phi_2(u) = 0\) firstly.

If \(l > k\), by applying \(\phi_2\) on the two sides of
\[E_{k,l} = \{[\ldots[[E_{k,k+1}, E_{k+1,k+2}], E_{k+2,k+3}] \ldots], E_{l-1,l}\},\]
we have that \( \phi_2(E_{k,l}) = 0 \). This implies that \( \phi_2(u_2) = 0 \). If \( 1 < i < l \), \( \phi_2(E_{i,i+1}, E_{i,i+2}) = [\phi_2(E_{i,i+1}), E_{i,i+2}] = 0 \), we know that \( \phi_2(E_{l,l+1}) = 0 \). If \( i = l \), then \( i = i \), by \( \phi_2(E_{i,i+1}, E_{i-1,i+1}) = [\phi_2(E_{i,i+1}), E_{i-1,i+1}] = 0 \), we also have that \( \phi_2(E_{i,i+1}) = 0 \). Thus \( \phi_2(E_{i,i+1}) = 0 \), for all \( 1 < i < n \). Hence \( \phi_2(u) = 0 \).

By applying \( \phi_2 \) on the two sides of \( [H, E_{i,i+1}] = (\chi_i(H) - \chi_{i+1}(H))E_{i,i+1} \), \( i = 1, 2, \ldots, n - 1 \), we have that \( \phi_2(H) = r_H E \) for unique \( r_H \in R \) relative to \( H \). Moreover, for any \( 1 \leq i < j \leq n \) and \( a_{j,i} \in A_{j,i} \), by applying \( \phi_2 \) on the two sides of \( [E_{i,j}, a_{j,i}E_{j,i}] = a_{j,i}(E_{i,i} - E_{j,j}) \), we have that

\[
[E_{i,j}, \phi_2(a_{j,i}E_{j,i})] = a_{j,i}\phi_2(E_{i,i} - E_{j,j}).
\]

Since \( n \geq 3 \) and \( \phi_2(a_{j,i}E_{j,i}) \) is scalar, we know that

\[
[E_{i,j}, \phi_2(a_{j,i}E_{j,i})] = a_{j,i}\phi_2(E_{i,i} - E_{j,j}) = 0.
\]

Thus \( \phi_2(A_{j,i}E_{j,i}) \subseteq RE_{i,j} \) for all \( 1 \leq i < j \leq n \) and \( \phi_2(F_i) \in B_i E \) for \( i = 1, 2, \ldots, n \). Suppose that \( \phi_2(F_i) = b_i E \), \( i = 1, 2, \ldots, n - 1 \), where \( b_i \in B_i \). Let \( \chi : d \to R \) be a homomorphism of \( R \)-modules, defined by

\[
\chi \left( \sum_{i=1}^{n} r_i F_i \right) = \sum_{i=1}^{n} r_i b_i.
\]

Then \( \chi \) is suitable for central derivations. We use it to define a central derivation \( \eta_{\chi} \) of \( p \) as

\[
\eta_{\chi}(x) = \chi(D_x)E, \quad x \in p,
\]

where \( D_x \) is the projection of \( x \) to \( d \).

Let \( \phi_3 = \phi_2 - \eta_{\chi} \). It is easy to see that \( \phi_3(t) = 0 \) and \( \phi_3(A_{j,i}E_{j,i}) \subseteq RE_{i,j} \) for all \( 1 \leq i < j \leq n \). If \( 1 \leq i < j \leq n - 1 \), by applying \( \phi_3 \) on the two sides of \( [A_{j,i}E_{j,i}, E_{j,n}] = 0 \), we have that \( [\phi_3(A_{j,i}E_{j,i}), E_{j,n}] = 0 \). However, \( \phi_3(A_{j,i}E_{j,i}) \subseteq RE_{j,i} \). This implies that \( \phi_3(A_{j,i}E_{j,i}) = 0 \) for all \( 1 \leq i < j \leq n - 1 \). If \( 1 \leq j \leq n - 1 \), by applying \( \phi_3 \) on the two sides of \( [A_{n,j}E_{n,j}, E_{1,j}] = 0 \), we have that \( [\phi_3(A_{n,j}E_{n,j}), E_{1,j}] = 0 \). However, \( \phi_3(A_{n,j}E_{n,j}) \subseteq RE_{j,n} \). We have that \( \phi_3(A_{n,j}E_{n,j}) = 0 \) for all \( 1 < j \leq n - 1 \). So \( \phi_3(p) = \phi_3(A_{n,1}E_{n,1}) \subseteq R E_{1,n} \). Define \( \sigma : A_{n,1} \to R \) such that \( \phi_3(a_{n,1}E_{n,1}) = \sigma(a_{n,1})E_{1,n} \) for all \( a_{n,1} \in A_{n,1} \). Then \( \sigma \) is a homomorphism of \( R \)-modules. Let

\[
x = \sum_{1 \leq i,j \leq n} a_{i,j} E_{i,j} \in p, \quad y = \sum_{1 \leq i,j \leq n} b_{i,j} E_{i,j} \in p,
\]

where all \( a_{i,j}, b_{i,j} \) lie in \( R \) and \( a_{l,k} \in A_{l,k}, b_{l,k} \in A_{l,k} \) when \( 1 \leq k < l \leq n \). Note that

\[
[x, y] = \sum_{1 \leq i,j \leq n} c_{i,j} E_{i,j}, \quad \text{where} \quad c_{i,j} = \sum_{k=1}^{n} (a_{i,k} b_{k,j} - b_{i,k} a_{k,j}).
\]

Then

\[
\phi_3([x, y]) = \sigma \left( \sum_{k=1}^{n} (a_{n,k} b_{k,1} - b_{n,k} a_{k,1}) \right) E_{1,n}.
\]

On the other hand,

\[
[\phi_3(x), y] + [x, \phi_3(y)] = \sum_{i=1}^{n} (\sigma(a_{n,1}) b_{n,k} - \sigma(b_{n,1}) a_{n,k}) E_{1,k}
\]

\[
+ \sum_{i=1}^{n} (\sigma(b_{n,1}) a_{k,1} - \sigma(a_{n,1}) b_{k,1}) E_{k,n}.
\]
Because
\[ \phi_3([x, y]) = [\phi_3(x), y] + [x, \phi_3(y)], \]
we have that

\[
(1) \; \sigma(A_{n,1}) \subseteq (\cap_{k=1}^{n-1}B_{n,k}) \cap (\cap_{k=2}^{n}B_{k,1}), \\
(2) \; \sigma(A_{n,j}A_{j,1}) = 0, \; j = 2, 3, \ldots, n - 1, \\
(3) \; 2\sigma(A_{n,1}) = 0.
\]

Thus \( \sigma \) is a homomorphism of \( R \)-modules and is suitable for extreme derivations. Using \( \sigma \) we define an extreme derivation \( \rho_\sigma \) of \( \mathbf{p} \) by

\[
\rho_\sigma \left( \sum_{1 \leq i, j \leq n} a_{i,j}E_{i,j} \right) = \sigma(a_{n,1})E_{1,n}.
\]

It is easy to see that \( \phi_3 \) is the extreme derivation \( \rho_\sigma \). Hence \( \phi + \text{ad} \; P_0 + \text{ad} \; H_0 - \eta_\chi = \rho_\sigma \).

In the end of this case, we consider the uniqueness. Firstly, we show that if \( \phi = \text{ad} \; P = \eta_\chi + \rho_\pi \), with \( \chi \) suitable for central derivations and \( \pi \) suitable for extreme derivations, then \( \text{ad} \; P = \eta_\chi = \rho_\pi = 0 \). For any \( a_{n,1} \in A_{n,1} \), by \( \phi(a_{n,1}E_{n,1}) = \pi(a_{n,1})E_{1,n} = [P, a_{n,1}E_{n,1}] \), we have that \( \pi(a_{n,1}) = 0 \). This forces that \( \pi = 0 \). Then \( \phi = \text{ad} \; P = \eta_\chi \). Because \( \eta_\chi(u) = 0 \), we see that \( \text{ad} \; P(u) = 0 \). This forces that \( P \subseteq C_\mathbf{p}(u) = Z(\mathbf{p}) \) (where \( C_\mathbf{p}(u) \) means the centralizer of \( u \) in \( \mathbf{p} \)). Therefore, \( \text{ad} \; P = \eta_\chi = 0 \). To show uniqueness, suppose

\[
\text{ad} \; P + \eta_\chi + \rho_\pi = \text{ad} \; P_0 + \eta_\chi_0 + \rho_\pi_0,
\]

where \( P, P_0 \in \mathbf{p}, \chi, \chi_0 \) are suitable for central derivations and \( \pi, \pi_0 \) are suitable for extreme derivations. Then

\[
\text{ad}(P - P_0) = \eta_\chi_0 - \chi + \rho_{\pi_0 - \pi}.
\]

It follows that

\[
\text{ad}(P - P_0) = \eta_\chi_0 - \chi = \rho_{\pi_0 - \pi} = 0.
\]

Thus \( \text{ad} \; P = \text{ad} \; P_0, \eta_\chi = \eta_\chi_0 \) and \( \rho_\pi = \rho_\pi_0 \).

We are done in this case.

**Case 2: \( n = 2 \)**

As before, \( \phi_2(A_{2,1}E_{2,1}) \subseteq A_{2,1}E_{2,1} + RE_{1,2} \), and \( \phi_2(RE_{1,2}) \subseteq A_{2,1}E_{2,1} \). Thus we may define three homomorphisms of \( R \)-modules:

\[
\sigma_2 : R \to A_{2,1}, \\
\sigma_3 : A_{2,1} \to A_{2,1}, \\
\sigma_4 : A_{2,1} \to R,
\]

such that

\[
(1) \; \phi_2(a_{1,2}E_{1,2}) = \sigma_2(a_{1,2})E_{2,1}, \; a_{1,2} \in A_{1,2}, \\
(2) \; \phi_2(a_{2,1}E_{2,1}) = \sigma_3(a_{2,1})E_{2,1} + \sigma_4(a_{2,1})E_{1,2}, \; a_{2,1} \in A_{2,1}.
\]

For any \( D \in \mathbf{d} \), by applying \( \phi_2 \) on the two sides of \( [D, E_{1,2}] = (\chi_1(D) - \chi_2(D))E_{1,2} \), We have that
This implies that $\phi_2(D)$ is scalar and $2\sigma_2(R) = 0$.

For any $a_{2,1} \in A_{2,1}$, by applying $\phi_2$ on the two sides of $[E_{1,2}, a_{2,1} E_{2,1}] = a_{2,1} (E_{1,1} - E_{2,2})$, We know that

$$[E_{1,2}, \sigma_3(a_{2,1}) E_{2,1} + \sigma_4(a_{2,1}) E_{1,2}] = \sigma_3(a_{2,1})(E_{1,1} - E_{2,2}) = \text{a scalar matrix.}$$

It follows that $2\sigma_3(A_{2,1}) = 0$ and $\phi_2(a_{2,1} F_1) = \sigma_3(a_{2,1}) E$ for any $a_{2,1} \in A_{2,1}$.

For any $a_{2,1} \in A_{2,1}$, $H \in \mathfrak{d}$, by applying $\phi_2$ on the two sides of $[H, a_{2,1} E_{2,1}] = (\chi_2(H) - \chi_1(H))a_{2,1} E_{2,1}$, we have that

$$(\chi_1(H) - \chi_2(H))\sigma_4(a_{2,1}) E_{1,2} = (\chi_2(H) - \chi_1(H))\sigma_4(a_{2,1}) E_{1,2}.$$
if \( 2 = 0 \), and \( \sigma_1, \sigma_2, \sigma_3 \) are all endomorphisms of \( R \), then \( \varphi(\sigma_1, \sigma_2, \sigma_3) : \text{gl}(2, R) \to \text{gl}(2, R) \), defined by

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \mapsto \begin{pmatrix}
\sigma_1(a) & \sigma_3(c) \\
\sigma_1(c) + \sigma_2(b) & \sigma_1(a)
\end{pmatrix}
\]

is a permutation derivation of \( \text{gl}(2, R) \). If fact, any permutation derivations of \( \text{gl}(2, R) \) may be written in this form.

By applying Theorem 4.1 to \( \text{gl}(n, R) \), we obtain the results on \( \text{gl}(n, R) \) as follows.

**Corollary 5.1.** Let \( R \) be an arbitrary commutative ring with identity.

1. If \( n \geq 3 \), or \( n = 2 \) and \( 2 \neq 0 \), then every derivation of \( \text{gl}(n, R) \) may be uniquely written as the sum of an inner derivation and a central derivation.
2. If \( n = 2 \) and \( 2 = 0 \), then every derivation of \( \text{gl}(n, R) \) may be uniquely written as the sum of an inner derivation, a central derivation and a permutation derivation.

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**References**