Injective colorings of sparse graphs

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A B S T R A C T

Let $\mathrm{mad}(G)$ denote the maximum average degree (over all subgraphs) of G and let $\chi_i(G)$ denote the injective chromatic number of G. We prove that if $\mathrm{mad}(G) \leq \frac{5}{2}$, then $\chi_i(G) \leq \Delta(G) + 1$; and if $\mathrm{mad}(G) < \frac{42}{19}$, then $\chi_i(G) = \Delta(G)$. Suppose that G is a planar graph with girth $g(G)$ and $\Delta(G) \geq 4$. We prove that if $g(G) \geq 9$, then $\chi_i(G) \leq \Delta(G) + 1$; similarly, if $g(G) \geq 13$, then $\chi_i(G) = \Delta(G)$.

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1. Introduction

An *injective coloring* of a graph G is an assignment of colors to the vertices of G so that any two vertices with a common neighbor receive distinct colors. The *injective chromatic number*, $\chi_i(G)$, is the minimum number of colors needed for an injective coloring. Injective colorings have their origin in complexity theory [13], and can be used in coding theory.

Note that an injective coloring is not necessarily proper, and in fact, $\chi_i(G) = \chi(G^{(2)})$, where the *neighboring graph* $G^{(2)}$ is defined by $V(G^{(2)}) = V(G)$ and $E(G^{(2)}) = \{uv : u \text{ and } v \text{ have a common neighbor in } G\}$. It is clear that $\Delta \leq \chi_i(G) \leq \Delta^2 - \Delta + 1$, where Δ is the maximum degree of graph G. Graphs attaining the upper bound were characterized in [13], and it was also shown that for every fixed $k \geq 3$ the problem of determining if a graph is injective k-colorable is NP-complete.

As one can see, injective coloring is a close relative of the coloring of square of graphs and of $L(2, 1)$-labeling, which have both been studied extensively. Upper bounds on $\chi(G^2)$ and on the $L(2, 1)$-labeling number $\lambda(G)$ are both upper bounds on $\chi_i(G)$.

Alon and Mohar [1] showed that if G has girth at least 7, then $\chi(G^2)$ could be as large as $\frac{c\Delta^2}{\log \Delta}$ (for some constant c), but not larger. This gives an upper bound on $\chi_i(G)$ when graph G has girth at least 7. The study of $\chi(G^2)$ has been largely focused on the well-known Wenger's Conjecture [16].

Conjecture 1 (*Wenger [16]*)**.** *If G is a planar graph with maximum degree Δ, then $\chi(G^2) \leq 7$ when $\Delta = 3$, $\chi(G^2) \leq \Delta + 5$ when $4 \leq \Delta \leq 7$, and $\chi(G^2) \leq 3\Delta/2 + 1$ when $\Delta \geq 8$.*

This conjecture and its variations have been studied extensively; see [4] or [10] for a good survey.

Much effort has been spent on finding graphs with low injective chromatic numbers, namely graphs with $\chi_i(G) \leq \Delta + c$, for some small constant c. In Theorems 2 and 3, we list some of the most recent related results.

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Theorem 2. *Let G be a planar graph with maximum degree $\Delta(G) \geq D$ and girth $g(G) \geq g$. Then*

(a) [5] *if $(D, g) \in \{(3, 24), (4, 15), (5, 13), (6, 12), (7, 11), (9, 10), (15, 8), (30, 7)\}$, then $\chi_i(G) \leq \chi(G^2) = \Delta + 1$.*
(b) [4] *$\chi_i(G) \leq \chi(G^2) \leq \Delta + 2$ if $(D, g) = (36, 6)$.*
(c) [14] *$\chi_i(G) \leq \Delta + 4$ if $g = 5$; $\chi_i(G) \leq \Delta + 1$ if $g = 10$; and $\chi_i(G) = \Delta$ if $g = 19$.*
(d) [7] *$\chi_i(G) \leq \Delta + 2$ if $g = 8$; $\chi_i(G) \leq \Delta + 1$ if $g = 11$; and $\chi_i(G) = \Delta$ if $(D, g) \in \{(3, 20), (71, 7)\}$.*

Instead of studying planar graphs with high girth, some researchers consider graphs with bounded *maximum average degree*, mad(G), where the average is taken over all subgraphs of G. Note that every planar graph G with girth at least g satisfies mad(G) $< \frac{2g}{g-2}$. Below are some results in terms of mad(G).

Theorem 3. *Let G be a graph with maximum degree $\Delta(G) \geq D$ and mad(G) $< m$. Then*

(a) [9] *$\chi_i(G) \leq \Delta + 8$ if $m = 10/3$; $\chi_i(G) \leq \Delta + 4$ if $m = 3$; and $\chi_i(G) \leq \Delta + 3$ if $m = 14/5$.*
(b) [8] *$\chi_i(G) \leq \Delta + 2$ if $(D, m) = (4, 14/5)$; $\chi_i(G) \leq 5$ if $(D, m) = (3, 36/13)$ and $m = 36/13$ is sharp.*

The major tool used in the proofs of Theorems 2 and 3 is the discharging method, which relies heavily on the idea of reducible subgraphs. A *reducible subgraph H* is a subgraph such that any coloring of $G - H$ can be extended to a coloring of G. Since the coloring of $G - H$ will restrict the choice of colors on H, these arguments work well when the graph G is sparse.

In particular, if mad(G) is much smaller than $\Delta(G)$, then we are guaranteed a vertex v with degree much smaller than $\Delta(G)$. Such a vertex v is a natural candidate to be included in our reducible subgraph H, since v has at least $\Delta(G)$ allowable colors and only has a few restrictions on its color. However, if mad(G) is nearly as large as $\Delta(G)$, then we are not guaranteed the presence of such a low degree vertex v. Here it is less clear how to proceed. Thus, proving results when $\Delta(G) - $ mad(G) is small is a much harder task than proving analogous results when $\Delta(G) - $ mad(G) is larger.

In this paper, we study graphs with low injective chromatic number, namely G such that $\chi_i(G) \leq \Delta + 1$. We consider sparse graphs with bounded maximum average degree, which include planar graphs with high girth. Our results below extend or generalize the corresponding results in Theorem 2.

Theorem 4. *Let G be a graph with maximum degree Δ.*

(a) *If* mad(G) $\leq \frac{5}{2}$, *then $\chi_i(G) \leq \Delta + 1$.*
(b) *If* mad(G) $< \frac{42}{19}$, *then $\chi_i(G) = \Delta$.*

Theorem 5. *Let G be a planar graph with girth $g(G)$ and maximum degree $\Delta \geq 4$.*

1. *If $g(G) \geq 9$, then $\chi_i(G) \leq \Delta + 1$.*
2. *If $g(G) \geq 13$, then $\chi_i(G) = \Delta$.*

Like many similar results, we use discharging arguments in our proofs. In most discharging arguments, the reducible subgraphs are of bounded size. Our contribution to injective coloring is using reducible configurations of arbitrary size, similar to the 2-alternating cycles introduced by Borodin [3] and generalized by Borodin, Kostochka, and Woodall [6].

Let G be a Class 2 graph, that is, suppose the edge-chromatic number of G is $\Delta + 1$. Let G' be the graph obtained from G by inserting a degree 2 vertex on each edge. Now $\chi_i(G') > \Delta$, for otherwise, we could color the edges of G by the colors of the 2-vertices in G' on the corresponding edges, which would give a Δ-edge-coloring of G. Here we list some Class 2 graphs; see [12] for more details.

Theorem 6 (*[12]*)**.**

(a) *If H is a regular graph with an even order, and G is a graph obtained from H by inserting a new vertex into one edge of H, then G is of Class 2.*
(b) *For any integers $D \geq 3$ and $g \geq 3$, there is a Class 2 graph of maximum degree D and girth g.*

By combining Theorem 6 with results on Class 2 graphs, we have the following corollary.

Corollary 7. *There are planar graphs with $g(G) = 8$ and $\chi_i(G) \geq \Delta + 1$. There are graphs with* mad(G) $= \frac{8}{3}$ *and $\chi_i(G) \geq \Delta + 1$. For any $\Delta \geq 3$ and $g \geq 3$, there are graphs with maximum degree Δ, girth 2g, and $\chi_i(G) \geq \Delta + 1$.*

There are some gaps between the bounds in the above corollary and Theorems 4 and 5(a). Our bounds on mad and girth may be further improved if some clever idea is elaborated.

When we extend a coloring of $G - H$ to a subgraph H, the colors available for vertices of H are restricted, thus we will essentially supply a list of available colors for each vertex of H. The following two classic theorems on list-coloring will be used heavily.

Theorem A (*[15]*)**.** *Let L be an assignment such that $|L(v)| \geq d(v)$ for all v in a connected graph G.*

(a) *If $|L(y)| > d(y)$ for some vertex y, then G is L-colorable.*
(b) *If G is 2-connected and the lists are not all identical, then G is L-colorable.*

We say that a graph is *degree-choosable* if it can be colored from lists when each vertex is given a list of size equal to its degree.

Theorem B (*[2,11]*)**.** *A graph G fails to be degree-choosable if and only if every block is a complete graph or an odd cycle.*

Here we introduce some notation. A k-vertex is a vertex of degree k; a k^+- and a k^--vertex have degree at least and at most k, respectively. A *thread* is a path with 2-vertices in its interior and 3^+-vertices as its endpoints. A k-thread has k interior 2-vertices. If u and v are the endpoints of a thread, then we say that u and v are *pseudo-adjacent*. If a 3^+-vertex u is the endpoint of a thread containing a 2-vertex v, then we say that v is a *nearby vertex* of u and vice versa. For other undefined notions, we refer to [17].

The paper is organized as follows. In Section 2, we prove Theorem 4(a). In Section 3, we prove Theorem 4(b). In Section 4, we prove Theorem 5.

2. Maximum average degree conditions that imply $\chi_i(G) \leq \Delta + 1$

We split the proof of Theorem 4(a) into two lemmas. We start with the case $\Delta \geq 4$, which only needs a simple discharging argument.

Lemma 8. *Let G be a graph with $\Delta \geq 4$. If* $\mathrm{mad}(G) \leq \frac{5}{2}$, *then* $\chi_i(G) \leq \Delta + 1$.

Proof. Let G be a minimal counterexample. It is easy to see that G has no 1-vertex and G has no 2-thread.

Also observe that G must not contain the following subgraph: a 3-vertex v adjacent to three 2-vertices such that one of these 2-vertices u is adjacent to a second 3-vertex. If G contains such a subgraph, let H be the set of v and its three neighbors. By minimality, we can injectively color $G - H$ with $\Delta + 1$ colors; now we greedily color H, making sure to color u last.

We now use a discharging argument, with an initial charge of $\mu(v) = d(v)$. We have two discharging rules:

(R1) Each 3-vertex divides a charge of $\frac{1}{2}$ equally among its adjacent 2-vertices.
(R2) Each 4^+-vertex sends a charge of $\frac{1}{3}$ to each adjacent 2-vertex.

Now we show that $\mu^*(v) \geq \frac{5}{2}$ for each vertex v.

$d(v) = 3$: $\mu^*(v) = 3 - \frac{1}{2} = \frac{5}{2}$.

$d(v) \geq 4$: $\mu^*(v) \geq d(v) - \frac{d(v)}{3} = \frac{2d(v)}{3} \geq \frac{8}{3} > \frac{5}{2}$.

$d(v) = 2$: If v is adjacent to a 4^+-vertex, then $\mu^*(v) \geq 2 + \frac{1}{3} + \frac{1}{6} = \frac{5}{2}$. If v has two neighboring 3-vertices and each neighbor gives v a charge of at least $\frac{1}{4}$, then $\mu^*(v) \geq 2 + 2\left(\frac{1}{4}\right) = \frac{5}{2}$. However, if v has two neighboring 3-vertices and at least one of them gives v a charge of only $\frac{1}{6}$, then v is in a copy of the forbidden subgraph described above; so $\mu^*(v) \geq \frac{5}{2}$.

Hence, each vertex v has a charge $\mu^*(v) \geq \frac{5}{2}$. Furthermore, each 4^+-vertex has a charge of at least $\frac{8}{3}$, so $\mathrm{mad}(G) > \frac{5}{2}$. $\square$

The above argument fails when $\Delta = 3$, since a 3-vertex v may be adjacent to three 2-vertices such that one of these 2-vertices u is adjacent to a second 3-vertex, which was forbidden when $\Delta \geq 4$. We will still start with a minimal counterexample, but instead of finding reducible configurations in a local neighborhood, we will identify ones from the global structure of the graphs.

Lemma 9. *Let G be a graph with $\Delta = 3$. If* $\mathrm{mad}(G) \leq \frac{5}{2}$, *then* $\chi_i(G) \leq 4$.

Proof. We prove the more general statement: If $\Delta = 3$ and $\mathrm{mad}(G) \leq \frac{5}{2}$, then G can be injectively colored from *lists* of size 4.

Let G be a minimal counterexample. It is easy to see that G has no 1-vertex and G has no 2-thread; in each case, we could delete the subgraph H, injectively color $G - H$, then greedily color H. First, we consider the case $\mathrm{mad}(G) < \frac{5}{2}$. Let G_{23} denote the subgraph induced by edges with one endpoint of degree 2 and the other of degree 3. If each component of G_{23} contains at most one cycle, then each component of G_{23} contains at least as many 3-vertices as 2-vertices, so $\mathrm{mad}(G) \geq \frac{5}{2}$; this contradicts our assumption. Hence, some component H of G contains a cycle C with a vertex u on C such that $d_H(u) = 3$. Let $J = V(C) \cup N(u)$. Because G is a minimal counterexample, we can injectively 4-color $G \setminus J$. We will now use Theorem A to extend the coloring of $G \setminus J$ to J.

Since C is an even cycle, $G^{(2)}[J]$ consists of two components; one component contains u and the other does not. We will use part (a) of Theorem A to color the component that contains u and we will use part (b) to color the other component; note that this second component is 2-connected. Let $L(v)$ be the list of colors available for each vertex v before we color $G \setminus J$ and let $L'(v)$ be the list of colors available after we color $G \setminus J$. To apply Theorem A as desired, we need to prove three facts: first, $|L'(v)| \geq d_{J^{(2)}}(v)$ for each $v \in J$; second, $|L'(u)| > d_{J^{(2)}}(u)$; and third, the lists L' for the second component of $G^{(2)}[J]$ are not all identical.

First, observe that for each vertex $v \in J$ we have the inequality $|L(v)| = 4 \geq d_{G^{(2)}}(v)$; because each colored neighbor in $G^{(2)}$ of v forbids only one color from use on v, this inequality implies that $|L'(v)| \geq d_{J^{(2)}}(v)$ for each $v \in J$. Second, note that

$|L(u)| = 4 > 3 = d_{G^{(2)}}(u)$; this inequality implies that $|L'(u)| = 3 > 2 = d_{J^{(2)}}(u)$. Third, let x and y be the two vertices on C that are adjacent to u in G; note that $|L'(x)| = 3$ and $|L'(y)| = 3$, while $|L'(v)| = 2$ for every other vertex v in the second component of $J^{(2)}$. Since vertices x and y have lists of size 3, while the other vertices have lists of size 2, it is clear that not all lists are identical.

Hence, we can color the first component of $J^{(2)}$ by Theorem A part (a), and we can color the second component of $J^{(2)}$ by Theorem A part (b).

Now consider the case $\text{mad}(G) = \frac{5}{2}$. If G_{23} does not contain any cycle C with a vertex u such that $d_{G_{23}}(u) = 3$, then each component of G_{23} is an even cycle. This means that $d_{G^{(2)}}(v) = 4$ for every vertex v. By applying Theorem B to $G^{(2)}$, we see that $\chi_i(G) \leq 4$; it is straightforward to verify that in each component of $G^{(2)}$, some block is neither an odd cycle nor a clique. $\quad\square$

This completes the proof of Theorem 4(a).

3. Maximum average degree conditions that imply $\chi_i(G) = \Delta$

We split the proof of Theorem 4(b) into two lemmas. In Lemma 10, we prove a stronger result than we need for Theorem 4(b), since our hypothesis here is $\text{mad}(G) \leq \frac{9}{4}$, rather than $\text{mad}(G) \leq \frac{42}{19}$.

Lemma 10. *Let G be a graph with $\Delta \geq 4$. If $\text{mad}(G) \leq \frac{9}{4}$, then $\chi_i(G) = \Delta$.*

Proof. We always have $\chi_i(G) \geq \Delta$, so we only need to prove $\chi_i(G) \leq \Delta$. Let G be a minimal counterexample. Clearly, G has no 1-vertex and G has no 4-thread. Note that G also has no 3-thread with a 3-vertex at one of its ends. We use a discharging argument, with initial charge: $\mu(v) = d(v)$. We have one discharging rule:

(R1) Each 3^+-vertex gives a charge of $\frac{1}{8}$ to each nearby 2-vertex.

Now we show that $\mu^*(v) \geq \frac{9}{4}$ for each vertex v.

2-vertex: $\mu^*(v) \geq 2 + 2\left(\frac{1}{8}\right) = \frac{9}{4}$.
3-vertex: $\mu^*(v) \geq 3 - 6\left(\frac{1}{8}\right) = \frac{9}{4}$.
4^+-vertex: $\mu^*(v) \geq d(v) - 3d(v)\left(\frac{1}{8}\right) = \frac{5}{8}d(v) \geq \frac{5}{2} > \frac{9}{4}$.

Hence, each vertex v has a charge $\mu^*(v) \geq \frac{9}{4}$. Since each 4^+-vertex v has charge $\mu^*(v) \geq \frac{5}{2}$, we have $\text{mad}(G) > \frac{9}{4}$. $\quad\square$

When $\Delta = 3$, our proof is more involved. We consider a minimal counterexample G. As above, G has no 1-vertex and G has no 4-thread.

We form an auxiliary graph H as follows. Let $V(H)$ be the 3-vertices of G. If u and v are ends of a 3-thread in G, then add the edge uv to H. Suppose instead that u and v are ends of a 2-thread in G. If one of the other threads incident to u is a 3-thread and the third thread incident to u is either a 2-thread or 3-thread, then add edge uv to H.

Lemma 11. *If $\text{mad}(G) < \frac{42}{19}$ and $\Delta(G) = 3$, then H contains a cycle with a vertex v such that $d_H(v) = 3$.*

Proof. To prove Lemma 11, it is sufficient to show that H (or some subgraph of H) has average degree greater than 2.

Form subgraph $\widehat{H}$ from H by deleting all of the isolated vertices in H. For $i \in \{0, 1, \ldots, 9\}$, let a_i denote the number of 3-vertices of G that have exactly i nearby 2-vertices and let $\widehat{a}_i$ denote the number of 3-vertices of G that have exactly i nearby 2-vertices and also have a corresponding vertex in $\widehat{H}$. Let n and $\widehat{n}$ denote the number of vertices in H and $\widehat{H}$, respectively. Note that $\sum_{i=0}^{9} a_i = n$ and $\sum_{i=0}^{9} \widehat{a}_i = \widehat{n}$. We now consider two weighted averages of the integers $0, 1, \ldots, 9$; the first average uses the weights a_i and the second average uses the weights $\widehat{a}_i$.

Let V_2 and V_3 denote the number of 2-vertices and 3-vertices in G. Since $\text{mad}(G) < \frac{42}{19}$, by simple algebra we deduce that $2V_2/V_3 > \frac{15}{2}$. By rewriting this inequality in terms of the a_is, we get: $\frac{1}{n}\sum_{i=0}^{9} a_i i > \frac{15}{2}$. Note that 8 and 9 are the only numbers in this weighted average that are larger than the average. Thus, since $\widehat{a}_i = a_i$ for $i \in \{8, 9\}$ and $\widehat{a}_i \leq a_i$ for $0 \leq i \leq 7$, we also have $\frac{1}{\widehat{n}}\sum_{i=0}^{9} \widehat{a}_i i \geq \frac{1}{n}\sum_{i=0}^{9} a_i i > \frac{15}{2}$. We need one more inequality, which we prove in the next paragraph.

The table below lists the values of three quantities: i, the minimum degree of a vertex in $\widehat{H}$ that has i nearby 2-vertices in G, and the expression $2i/3 - 3$. Note that for all values of i, we have $d_{\widehat{H}}(v) \geq 2i/3 - 3$. We end the table at $i = 4$, since thereafter $d_{\widehat{H}}(v) = 1$ and $2i/3 - 3$ is negative.

i	9	8	7	6	5	4
$d_{\widehat{H}}(v)$	3	3	2	1	1	1
$2i/3 - 3$	3	7/3	5/3	1	1/3	−1/3

By taking the average over all vertices in $\widehat{H}$ of the inequality $d_{\widehat{H}}(v) \geq 2i/3 - 3$, we get the inequality

$$\frac{1}{\widehat{n}} \sum_{v \in V(\widehat{H})} d_{\widehat{H}}(v) \geq \frac{1}{\widehat{n}} \sum_{i=0}^{9} \widehat{a}_i \left(\frac{2}{3}i - 3\right).$$

By expanding this second sum into a difference of two sums, then substituting the values given above for these two sums, we conclude that the average degree of $\widehat{H}$ is greater than 2. This finishes the proof of Lemma 11. $\quad\square$

Now we use Lemma 11 to show that $\chi_i(G) = 3$.

Lemma 12. *Let G be a graph with $\Delta = 3$. If $\mathrm{mad}(G) < \frac{42}{19}$, then $\chi_i(G) = 3$.*

Proof. Let C be a cycle in H that contains a vertex v with $d_H(v) = 3$. Let C' be the cycle in G that corresponds to C; i.e. C' in G passes through all the vertices corresponding to vertices of C in the same order that they appear on C; furthermore, between each pair of successive 3-vertices on C' there are either two or three 2-vertices. Let J be the subgraph of G consisting of cycle C' in G, together with the neighbor w of v that is not on C'. By assumption, $G^{(2)} - J^{(2)}$ has a proper 3-coloring.

Note that each vertex u in J satisfies $d_{G^{(2)}}(u) \le 3$ unless u is a 3-vertex and is also adjacent to a 3-vertex not on J. In that case, $d_{G^{(2)}}(u) = 4$; however, then two of the vertices on C' that are adjacent to u in $G^{(2)}$ have degree 2 in $G^{(2)}$, and hence, these two vertices can be colored after all of their neighbors.

To simplify notation, we now speak of finding a proper vertex coloring of $G^{(2)}$. Let $K = J^{(2)}$. We now delete from K all vertices with degree 2 or 4 in $G^{(2)}$, since they can be colored last; call the resulting graph $\widehat{K}$. Our goal is to extend the 3-coloring of $G^{(2)} - K$ to $\widehat{K}$; we can then further extend this 3-coloring to K.

Observe that each vertex u in $\widehat{K}$ has a list size of at least 2. Furthermore, all vertices of $\widehat{K}$ have at most two uncolored neighbors in $\widehat{K}$ except for the two vertices x and y that are adjacent on C' to v; however, $d_{\widehat{K}}(x) = d_{\widehat{K}}(y) = 3$ and x and y both have three available colors. Hence, for every vertex u in $\widehat{K}$, the number of available colors is no smaller than its degree $d_{\widehat{K}}(u)$. We use this fact as follows.

If any component of $\widehat{K}$ is a path, then we can clearly color each vertex of the component from its available colors, by Theorem A part (a). In that case, each component of $\widehat{K}$ contains a vertex u with number of colors greater than $d_{\widehat{K}}(u)$; so by Theorem A part (a), we can color each vertex of $\widehat{K}$ from its available colors. If instead $\widehat{K}$ is a single component, then since the single component is neither a clique nor an odd cycle, we can color all of $\widehat{K}$, by Theorem B.

Hence, we only need to consider the case when $\widehat{K}$ has two components: one contains a cycle, where two adjacent vertices have a common neighbor not on the cycle (this is w, the off-cycle neighbor of v); the other component is a cycle containing v.

Because the first component is neither an odd cycle nor a clique, we can color it by Theorem B.

Now observe that since $d_H(v) = 3$ (and v is not adjacent to a 3-thread on C'), we know that v is adjacent in $G^{(2)}$ to a vertex z not on C' such that $d_{G^{(2)}}(z) = 2$. By uncoloring z, we make a third color available at v, so we can extend the coloring to the second component of $\widehat{K}$, using Theorem A part (a). Lastly, we recolor z. $\quad\square$

4. Planar graphs with low injective chromatic numbers

In this section, we prove Theorem 5. The condition $\Delta \ge 4$ allows us to get better girth conditions than Theorem 2, but not much.

We prove Theorem 5(a) first, and for convenience, we restate the theorem.

Theorem 5(a). *If G is planar, $\Delta(G) \ge 4$, and $g(G) \ge 9$, then $\chi_i(G) \le \Delta + 1$.*

Proof. It is easy to see that G has no 1-vertex and G has no 2-thread. We also need one more reducible configuration. Let u_1, u_2, u_3, u_4, u_5 be five consecutive vertices along a face f. Suppose that $d(u_1) = d(u_3) = d(u_5) = 2$ and $d(u_2) = d(u_4) = 3$ and the neighbor of u_2 not on f has degree at most 3. We call this subgraph H and we show that H is a reducible configuration, as follows. By assumption, $G - u_3$ has an injective coloring with $\Delta + 1$ colors; we now modify this coloring to get an injective coloring of G. First uncolor vertices u_1, u_2, u_4, and u_5. Now color the uncolored vertices in the order: u_2, u_4, u_1, u_5, u_3.

We use a discharging argument with initial charge $\mu(v) = d(v) - 4$ and $\mu(f) = d(f) - 4$. We use the following discharging rules.

(R1) Each face gives a charge of 1 to each 2-vertex and gives a charge of $1/3$ to each 3-vertex.
(R2) If face f contains the degree sequence $(4^+, 3, 2, 3, 4^+)$, then f gives a charge of $1/3$ to the face adjacent across the 2-vertex.

Now we show that $\mu^*(x) \ge 0$ for each $x \in V(G) \cup F(G)$.

$d(v) = 2$: $\mu^*(v) = -2 + 2(1) = 0$.
$d(v) = 3$: $\mu^*(v) = -1 + 3(1/3) = 0$.
$d(v) \ge 4$: $\mu^*(v) = \mu(v) \ge 0$.

To argue intuitively without handling (R2) separately, observe that wherever $(4, 3, 2, 3, 4)$ appears we could replace it with $(4, 2, 4, 2, 4)$ without creating a 2-thread; after this replacement, face f gives away $1/3$ more charge (to account for (R2), so we only consider (R1)).

For each face f, let t_2, t_3, t_4 denote the number of 2-vertices, 3-vertices, and 4^+-vertices on f, respectively. The charge of a face f is $\mu^*(f) = d(f) - 4 - t_2 - 1/3 t_3 = t_4 + 2/3 t_3 - 4$. If a face has a negative charge, then $t_4 + 2/3 t_3 < 4$. This implies $3/2 t_4 + t_3 < 6$, and hence $t_4 + t_3 \le 5$. Since G contains no 2-threads, $t_2 \le t_3 + t_4$. So, if a face has a negative charge,

$d(f) = t_2 + t_3 + t_4 \leq 2(t_3 + t_4) \leq 10$. Hence, we only need to verify that $\mu^*(f) \geq 0$ for faces of length at most 10; since girth$(G) \geq 9$, we have only two cases: $d(f) = 10$ and $d(f) = 9$.

Case 1: *face* f *of length* 10: If $\mu^*(f) < 0$, then $t_2 = 5$, $t_3 \geq 4$, and $t_4 \leq 1$. So our degree sequence around f must look like either (a) $(2, 4^+, 2, 3, 2, 3, 2, 3, 2, 3)$ or (b) $(2, 3, 2, 3, 2, 3, 2, 3, 2, 3)$.

Case 1(a) $(2, 4^+, 2, 3, 2, 3, 2, 3, 2, 3)$: Let u_1 and u_2 be the vertices not on f adjacent to the second and third vertices on f of degree 3. If G does not contain reducible configuration H, then $d(u_1) \geq 4$ and $d(u_2) \geq 4$; but then f receives a charge of $1/3$ by (R2), so $\mu^*(f) \geq 0$.

Case 1(b) $(2, 3, 2, 3, 2, 3, 2, 3, 2, 3)$: If any neighbor not on f of a 3-vertex on f has degree at most 3, then G contains reducible configuration H. If all such neighbors have degree at least 4, then f receives a charge of $1/3$ from each adjacent face, so $\mu^*(f) = t_4 + \frac{2}{3}t_3 - 4 + \left(\frac{1}{3}\right)5 = 0 + \left(\frac{2}{3}\right)5 - 4 + \left(\frac{1}{3}\right)5 = 1 > 0$.

Case 2: *face* f *of length* 9: If $\mu^*(f) < 0$, then $t_2 = 4$, $t_3 \geq 4$, and $t_4 \leq 1$.

Our degree sequence around f, beginning and ending with vertices of degree at least 3, must look like one of the four following: (a) $(3, 2, 3, 2, 3, 2, 3, 2, 3)$, (b) $(4^+, 2, 3, 2, 3, 2, 3, 2, 3)$, (c) $(3, 2, 4^+, 2, 3, 2, 3, 2, 3)$, or (d) $(3, 2, 3, 2, 4^+, 2, 3, 2, 3)$.

Case 2(a) $(3, 2, 3, 2, 3, 2, 3, 2, 3)$: Because f contains the degree sequence $(2, 3, 2, 3, 2, 3, 2)$, either G contains the reducible configuration H or f receives a charge of $\frac{1}{3}$ from at least two faces. Hence $\mu^*(f) \geq \frac{2}{3}(5) - 4 + \frac{1}{3}(2) = 0$.

Cases 2(b) $(4^+, 2, 3, 2, 3, 2, 3, 2, 3)$ and 2(c) $(3, 2, 4^+, 2, 3, 2, 3, 2, 3)$: Again f contains the degree sequence $(2, 3, 2, 3, 2)$. So in each case, if f does not contain the reducible configuration H, then f receives a charge of $\frac{1}{3}$ from some adjacent face; hence $\mu^*(f) \geq 1 + \frac{2}{3}(4) - 4 + \frac{1}{3}(1) = 0$.

Case 2(d) $(3, 2, 3, 2, 4^+, 2, 3, 2, 3)$: Let $v_1, w_1, v_2, w_2, v_3, w_3, v_4, w_4, v_5$ denote the vertices on f, in order around the face, beginning and ending with 3-vertices. If both v_1 and v_2 are adjacent to vertices of degree at least 4, then f receives a charge of $\frac{1}{3}$ from an adjacent face, by (R2). In this case $\mu^*(f) \geq 1 + \frac{2}{3}(4) - 4 + \frac{1}{3}(1) = 0$. Conversely, we will show that if either v_1 or v_2 is not adjacent to any vertex of degree at least 4, then G contains a reducible configuration.

Let u_1 and u_2 denote the neighbors of v_1 and v_2 not on f. By minimality, we have an injective coloring of $G - \{w_1, v_2, w_2\}$ with $\Delta + 1$ colors. If $d(u_1) < 4$, then we finish as follows: uncolor v_1 and w_4, now color w_1, w_2, v_2, v_1, w_4. If instead $d(u_2) < 4$, then we finish by coloring w_1, w_2, v_2.   □

Now we prove Theorem 5(b); for convenience, we restate it.

Theorem 5(b). *If G is planar, $\Delta(G) \geq 4$, and $g(G) \geq 13$, then $\chi_i(G) = \Delta$.*

Proof. Suppose the theorem is false; let G be a minimal counterexample. Below we note six configurations that must not appear in G. In each case, we can delete the configuration H, color $G - H$ (by the minimality of G), and extend the coloring to H greedily.

(RC1) G contains no 1-vertices.
(RC2) G contains no 4-threads.
(RC3) G contains no 3-thread with one end having degree 3.
(RC4) G contains no 2-threads with both ends having degree 3.
(RC5) G contains no 3-vertex that is incident to one 1-thread and two 2-threads.
(RC6) G contains no 3-vertex that is incident to one 2-thread and two 1-threads such that the other end of the one 1-thread has degree 3.

For a specified face f, let t_2, t_3, and t_4 be the number of vertices incident to f of degrees 2, 3, and at least 4. Then by (RC2) and (RC3), for any face f, $t_2 \leq 2t_3 + 3t_4$, equality holds only if every 4^+-vertex is followed by a 3-thread and every 3-vertex is followed by a 2-thread. Thus if $t_3 > 0$, the equality does not hold. So

$$t_2 \leq 2t_3 + 3t_4; \quad \text{and if } t_3 > 0, \text{ then } t_2 < 2t_3 + 3t_4. \tag{1}$$

We use a discharging argument. Let the initial charge be $\mu(x) = d(x) - 4$ for $x \in V(G) \cup F(G)$, where $d(x)$ is the degree of vertex x or the length of face x. Then by the Euler Formula,

$$\sum_{x \in V \cup F} \mu(x) = -8. \tag{2}$$

We will distribute the charges of the vertices and faces in two phases. In Phase I, we use a simple discharging rule and show that only three types of faces have a negative charge. In Phase II, we introduce two more discharging rules and show that the final charge of every face and every vertex is nonnegative. We thus get a contradiction to Eq. (2).

Discharging Phase I

We use the following discharging rule.

(R1) Each face f gives a charge of $\frac{1}{3}$ to each incident 3-vertex and gives a charge of 1 to each incident 2-vertex.

Remark. Another way to state this discharging rule is that f gives a charge of 1 to each vertex, and every 3-vertex returns a charge of $\frac{2}{3}$ and every 4$^+$-vertex returns a charge of 1. Thus each vertex has a final charge of 0, and the final charge of each face f is

$$\mu^*(f) = \frac{2}{3}t_3 + t_4 - 4. \tag{3}$$

Claim 1. *If $\mu^*(f) < 0$, then f must have one of the following degree sequences:*

(a) $(4^+, 2, 2, 2, 4^+, 2, 2, 2, 4^+, 2, 2, 3, 2, 2)$
(b) $(4^+, 2, 2, 2, 4^+, 2, 2, 2, 4^+, 2, 3, 2, 2)$
(c) $(4^+, 2, 2, 2, 4^+, 2, 2, 4^+, 2, 2, 3, 2, 2)$.

Proof. Consider a face f with a negative charge $\mu^*(f)$. Note that $\mu^*(f) \leq -1/3$; by Eq. (3) and inequality (1), we have $-1/3 \geq \mu^*(f) = \frac{2}{3}t_3 + t_4 - 4 \geq \frac{1}{3}t_2 - 4$. We rewrite these inequalities as:

$$t_2 \leq 2t_3 + 3t_4 \leq 11. \tag{4}$$

By inequality (4), we know that $t_3 + t_4 \leq 5$. If $t_3 + t_4 \leq 3$, then $t_2 \leq 3(t_3 + t_4) \leq 9$, and hence $d(f) = t_2 + (t_3 + t_4) \leq 9 + 3 = 12$. Since girth$(G) \geq 13$, this is a contradiction. So we must have $4 \leq t_3 + t_4 \leq 5$. From inequality (4), note that $t_4 < 4$; thus $t_3 > 0$.

If $t_3 + t_4 = 5$, then inequality (4) implies that $t_4 \leq 1$, and thus $t_3 \geq 4$. By (RC2), (RC3), and (RC4), f contains at most two 2-threads, and three 1-threads; thus $d(f) \leq 2(2) + 3(1) + 5 = 12$. Again, this contradicts girth$(G) \geq 13$, so we must have $t_3 + t_4 = 4$. If $t_2 = 11$, then $t_2 = 2t_3 + 3t_4$. Now inequality (1) implies that $t_3 = 0$, which is a contradiction. So instead $t_2 \leq 10$. Combining this inequality with $t_3 + t_4 = 4$, we have $d(f) = t_2 + (t_3 + t_4) \leq 14$. We now consider two cases: $d(f) = 14$ and $d(f) = 13$.

If $d(f) = 14$, then $t_2 = d(f) - (t_3 + t_4) = 10$. Since $t_3 > 0$, inequality (1) yields $2t_3 + 3t_4 > 10$. So $t_3 = 1$ and $t_4 = 3$. By (RC2) and (RC3), the degree sequence of f must be $(4^+, 2, 2, 2, 4^+, 2, 2, 2, 4^+, 2, 2, 3, 2, 2)$.

If $d(f) = 13$, then $t_2 = d(f) - (t_3 + t_4) = 9$. Since $t_3 > 0$, inequality (1) yields $2t_3 + 3t_4 > 9$. Now $(t_3, t_4) \in \{(2, 2), (1, 3)\}$. If $t_3 = t_4 = 2$, then we have one of the following two cases. If the two 3-vertices are pseudo-adjacent, then f has at most one 1-thread, two 2-threads, and one 3-thread; so $d(f) \leq 1(1) + 2(2) + 1(3) + 4 = 12$. If the two 3-vertices are not pseudo-adjacent, then f has at most four 2-threads, so $d(f) \leq 4(2) + 4 = 12$. Both of these cases contradict girth$(G) \geq 13$, so we must have $t_4 = 3$ and $t_3 = 1$. By (RC2) and (RC3), the degree sequence of f must be $(4^+, 2, 2, 2, 4^+, 2, 2, 2, 4^+, 2, 3, 2, 2)$ or $(4^+, 2, 2, 2, 4^+, 2, 2, 4^+, 2, 2, 3, 2, 2)$. □

Discharging Phase II

Note that each type of bad face listed in Claim 1 ends Phase I with a charge of $-\frac{1}{3}$. We now introduce two new discharging rules to send an additional charge of $\frac{1}{3}$ to these bad faces.

A *type-1 3-vertex* u is a 3-vertex that is incident with one 2-thread, and two 1-threads, with the other ends of the 1-threads each having degree of at least 4. The vertex u is called a *weak vertex* in the face incident with the two 1-threads and is called a *strong vertex* in the other two faces incident to u.

A *type-2 3-vertex* u is a 3-vertex that is incident with one 0-thread, one 2-thread, and one 1$^+$-thread, with the other ends of the 2-thread and 1$^+$-thread having degree at least 4. The vertex u is called a *slim vertex* in the faces incident with the 0-thread and is called a *fat vertex* in the other face incident to u.

In our second discharging phase, we use the following two discharging rules.

(R2) Each face gives a charge of $\frac{2}{3}$ to each of its weak vertices and gives a charge of $\frac{1}{6}$ to each of its slim vertices.
(R3) Each face receives a charge of $\frac{1}{3}$ from each of its strong vertices and receives a charge of $\frac{1}{3}$ from each of its fat vertices.

Let $\mu^{**}(f)$ be the charge after the second discharging phase. Let $t_3'(f)$ be the number of non-special 3-vertices on f, i.e. 3-vertices on f that are not: slim, fat, strong, or weak. Beginning with Eq. (3) and applying rules (R2) and (R3), we write the final charge of a face f as:

$$\mu^{**}(f) = \frac{2}{3}t_3' + t_4 - 4 + 0 \cdot \#(weak) + \frac{1}{2} \cdot \#(slim) + 1 \cdot \#(strong) + 1 \cdot \#(fat).$$

Recall that each vertex had a nonnegative charge at the end of Phase I. Since rules (R2) and (R3) do not change the charge at any vertex, it is clear that every vertex has a nonnegative final charge. Now we will show that every face also has a nonnegative final charge. This will contradict Eq. (2).

For a face f with a negative charge after Phase I, by Claim 1, it contains a 3-vertex v incident to a 2-thread and a 1$^+$-thread with the other ends having degree at least 4. By (RC3) and (RC5), the third thread incident to v is either a 0-thread, or a 1-thread. If it is a 0-thread, then v is a fat vertex; if it is a 1-thread, then v is a strong vertex. In each case, either rule (R2) or (R3) sends an additional charge of $\frac{1}{3}$ to f. Since $\mu^*(f) = -\frac{1}{3}$, we conclude that $\mu^{**}(f) = 0$.

If a face has a nonnegative charge after Phase I and contains no weak or slim vertices, then it does not give away charge, and therefore remains nonnegative. So we only consider the faces containing weak and slim vertices. Before proceeding, we have the following claims. Note that even after applying (R2) and (R3), the net charge given from each 3-vertex to each face is nonnegative; we use this fact implicitly when we prove Claims 2 and 4.

Claim 2. *If a face f contains a path $P = u \cdots v$ with $d(u), d(v) \geq 3$, and if $|V(P) - \{u, v\}| \geq 4$, then f receives a total charge of at least 1 from the vertices in $V(P) - \{u, v\}$.*

Proof. Let $P_0 = P - \{u, v\}$ and assume $|V(P_0)| \geq 4$. By (RC2), path P_0 contains at least one 3^+-vertex. If P_0 contains a 4^+-vertex, we are done. Thus we may assume that P_0 contains no 4^+-vertices. Recall that if a 3-vertex v is weak on face f, then each pseudo-neighbor of v that is on f must be a 4^+-vertex. Hence, if P_0 contains more than one 3-vertex, then none of these 3-vertices can be weak, since each one has a pseudo-neighbor on f that is a 3-vertex; thus f gains at least $\frac{2}{3}$ from each 3-vertex, and hence gains more than 1 from P_0. So we assume that P_0 contains exactly one 3-vertex; call it x. By (RC3), vertex x splits P_0 into a 2-thread and a 1^+-thread. By (RC5) the third thread incident to x must be a 1-thread or a 0-thread. Therefore x is either a fat vertex or a strong vertex; hence, f receives a total charge of 1 from x. $\square$

To prove Claims 2–4 we will be interested in the total charge that f receives from a slim vertex and its pseudo-neighbors and the total charge that f receives from a weak vertex and its pseudo-neighbors.

Claim 3. *A face with a negative charge after Phase II must satisfy the following properties.*

(C1) *It contains at most one weak vertex.*
(C2) *It contains at most one slim vertex.*
(C3) *It does not contain both weak and slim vertices.*

Proof. Call weak vertices and slim vertices *bad* vertices. Suppose that f has at least 2 bad vertices v_1 and v_2. Note that each bad vertex and its pseudo-neighbors give at least 2 to f, so if the set of v_1 and its pseudo-neighbors is disjoint from the set of v_2 and its pseudo-neighbors, then f receives a total of at least $2(2) = 4$. Hence, we must assume these sets are not disjoint; however, in this case v_1, v_2, and their pseudo-neighbors give f a total of at least three (this case analysis is straightforward). By Claim 2, the contribution from the remaining vertices is at least 1 (we need to verify that these vertices contain a path of at least four vertices, but this is again straightforward). $\square$

Similar to Claim 2, we have the following claim.

Claim 4. *If a face f contains a path $P = u \cdots v$ with $d(u), d(v) \geq 3$ and $|V(P) \setminus \{u, v\}| \geq 8$, then either P contains a slim vertex or else f receives a total charge of at least 2 from the vertices of $P \setminus \{u, v\}$.*

Proof. Let $P_0 = P \setminus \{u, v\}$ and assume $|P_0| \geq 8$. If P_0 contains at least two 4^+-vertices, then f receives a total charge of 1 from each of these 4^+-vertices, and hence receives a total charge at least 2 from the vertices of P_0. If P_0 contains a single 4^+-vertex, then call the 4^+-vertex y; note that P_0 contains a path P_1 with at least 5 vertices, such that one endpoint is y and the other endpoint is adjacent to either u or v. Clearly f receives a charge of 1 from y, and by Claim 2, f receives a charge of at least 1 from the vertices of $P_1 - y$; hence, f receives a total charge of 2.

Assume instead that P_0 contains no 4^+-vertices. Note that by (RC2), path P_0 must contain at least two 3-vertices. If P_0 contains at least three 3-vertices, then none of them can be weak, since each is pseudo-adjacent to a 3-vertex. If also none is slim, then f receives at least $3\left(\frac{2}{3}\right)$ from P_0; if P_0 contains a slim vertex, then the claim holds. Finally, if P_0 contains at most two 3-vertices, then by (RC3) and (RC4), it contains at most two 2-threads and one 1-thread; thus $|V(P_0)| \leq 2(2) + 1(1) + 2 = 7$, which is a contradiction. $\square$

Claim 5. *A face with a negative charge after Phase II must contain no slim vertex and no weak vertex.*

Proof. Assume that a face f contains exactly one vertex that is weak or slim; call this vertex y. Let the pseudo-neighbors of y on f be v_1 and v_2. There is a path P in f from v_1 to v_2 such that $y \notin P$, and $|V(P) \setminus \{v_1, v_2\}| \geq 8$. Note that $d(v_1), d(v_2) \geq 3$. Furthermore, since y is weak or slim, $V(P) \setminus \{v_1, v_2\}$ cannot contain a slim vertex; hence, by Claim 4, f receives a total charge of at least two from $P \setminus \{v_1, v_2\}$.

If y is weak, then f receives $1 + 0 + 1$ from v_1, y, v_2; similarly, if y is slim, then f receives at least $1 + \frac{1}{2} + \frac{1}{2}$ from v_1, y, v_2. In each case, we see that $\mu^*(f) \geq -4 + 2 + 2 = 0$; this is a contradiction. $\square$

This completes the proof of Theorem 5(b). $\square$

Acknowledgements

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