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On Some Inequalities for Functions with Nondecreasing Increments

JOSIP E. PEČARIĆ

Faculty of Civil Engineering, Bulevar Revolucije 73, 11000 Belgrade, Yugoslavia

Submitted by K. Fan

1. INTRODUCTION

One of the fundamental inequalities of analysis is Jensen's inequality for convex functions. Essentially Jensen's inequality is a definition of a convex function of one or several variables. However, series of well-known inequalities for convex functions of one variable, such as Jensen–Steffensen's, Popoviciu's, Brunk–Olkin's and Ciesielski's, do not have their analogues for convex functions of several variables. It was shown in [1] that these inequalities have their analogues for the functions with nondecreasing increments. These functions, i.e., functions with nondecreasing increments, represent the other generalization of convex functions for functions with several variables.

In this paper we shall show that the well-known Majorization theorem holds for functions with nondecreasing increments. Using this result we shall obtain extension of some results from [1] and some similar results.

In the proof we shall use the generalization of the well-known Fan–Lorentz inequality (see [2]). We shall use the following notation:

$$\underset{x_i}{\Delta}{}_{h_i} f = \underset{x_i}{\Delta}{}_{h_i} f(x_1,\ldots, x_n) = f(x_1,\ldots, x_i + h_i,\ldots, x_n) - f(x_1,\ldots, x_i,\ldots, x_n),$$

$$\underset{x_i}{\Delta}{}_{h_i} \underset{x_j}{\Delta}{}_{h_j} f = \underset{x_i}{\Delta}{}_{h_i}\big(\underset{x_j}{\Delta}{}_{h_j} f(x_1,\ldots, x_n)\big).$$

The result from [2], in a slightly generalized form, is as follows:

THEOREM A. *Let H be a continuous real function depending on t and $u_1,\ldots, u_n$, defined for $a \leqslant t \leqslant b$, $a_i \leqslant u_i \leqslant b_i$, for $i = 1,\ldots, n$, such that*

$$\underset{t}{\Delta}{}_{p} \underset{u_i}{\Delta}{}_{h_i} H \leqslant 0 \quad (1 \leqslant i \leqslant k), \qquad \underset{t}{\Delta}{}_{p} \underset{u_i}{\Delta}{}_{h_i} H \geqslant 0 \quad (k < i \leqslant n),$$

$$\underset{u_i}{\Delta}{}_{h_i} \underset{u_j}{\Delta}{}_{h_j} H \geqslant 0 \quad (1 \leqslant i, j \leqslant k \wedge k < i, j \leqslant n),$$

$$\underset{u_i}{\Delta}{}_{h_i} \underset{u_j}{\Delta}{}_{h_j} H \leqslant 0 \quad (1 \leqslant i \leqslant k \wedge k < j \leqslant n),$$

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hold for all u_i, $h_i \geqslant 0$, t, $p \geqslant 0$ $(1 \leqslant i \leqslant n,\ t+p \in [a,b],\ u_i + h_i \in [a_i, b_i])$. *Let* $f_i, g_i\colon [a,b] \to [a_i, b_i]$ $(1 \leqslant i \leqslant n)$ *be real continuous functions, nonincreasing for* $i = 1,\dots,k$ *and nondecreasing for* $i = k+1,\dots,n$, *and let* $G\colon [a,b] \to R$ *be a function of bounded variation.*

(a) *If*

$$\begin{aligned}
&\int_a^x f_i(t)\,dG(t) \leqslant \int_a^x g_i(t)\,dG(t) && (a \leqslant x \leqslant b,\ 1 \leqslant i \leqslant k),\\
&\int_x^b f_i(t)\,dG(t) \leqslant \int_x^b g_i(t)\,dG(t) && (a \leqslant x \leqslant b,\ k < i \leqslant n),\\
&\int_a^b f_i(t)\,dG(t) = \int_a^b g_i(t)\,dG(t) && (1 \leqslant i \leqslant n),
\end{aligned} \tag{1}$$

then

$$\int_a^b H(t; f_1(t),\dots,f_n(t))\,dG(t) \leqslant \int_a^b H(t; g_1(t),\dots,g_n(t))\,dG(t). \tag{2}$$

(b) *If* H *is a nondecreasing function on* $u_1,\dots,u_n$, *and if, instead of* (1),

$$\begin{aligned}
&\int_a^x f_i(t)\,dG(t) \leqslant \int_a^x g_i(t)\,dG(t) && (a \leqslant x \leqslant b,\ 1 \leqslant i \leqslant k),\\
&\int_x^b f_i(t)\,dG(t) \leqslant \int_x^b g_i(t)\,dG(t) && (a \leqslant x \leqslant b,\ k < i \leqslant n)
\end{aligned} \tag{3}$$

hold, then (2) *is also valid.*

Proof. Analogously to the proof which is given in [1], we can show that function H may be approximated uniformly by polynomials which satisfy the conditions

$$\frac{\partial^2 H}{\partial t\,\partial u_i} \leqslant 0 \qquad (1 \leqslant i \leqslant k), \qquad \frac{\partial^2 H}{\partial t\,\partial u_i} \geqslant 0 \qquad (k < i \leqslant n),$$

$$\frac{\partial^2 H}{\partial u_i\,\partial u_j} \geqslant 0 \qquad (1 \leqslant i, j \leqslant k \wedge k < i, j \leqslant n),$$

$$\frac{\partial^2 H}{\partial u_i\,\partial u_j} \leqslant 0 \qquad (1 \leqslant i \leqslant k \wedge k < j \leqslant n).$$

So, there is no loss in generality in assuming that the second partial derivatives exist. Now, using the method from [2] (or the result from [2]) we can obtain Theorem A.

2. Some Results for Functions with Nondecreasing Increments

Let R^k denote the k-dimensional vector lattice of points $x=(x_1,..., x_k)$, x_i real for $i=1,..., k$ with the partial ordering $x=(x_1,..., x_k) \leqslant y=(y_1,..., y_k)$ if and only if $x_i \leqslant y_i$ for $i=1,..., k$.

A real-valued function f on an interval $I \subset R^k$ will be said to have nondecreasing increments if

$$f(a+h)-f(a) \leqslant f(b+h)-f(b)$$

whenever $a \in I$, $b+h \in I$, $0 \leqslant h \in R^k$, $a \leqslant b$.

For functions with nondecreasing increments the following results are valid (see [1]):

(i) If the partial derivatives $\partial f(x)/\partial x_i$ exist for $x \in I$ then f has nondecreasing increments if and only if each of these partial derivatives is nondecreasing in each argument.

(ii) The second partials, if they exist, are then nonnegative.

In the paper $X(t)=(X_1(t),..., X_k(t))$ denotes the map from the real interval $[a, b]$ into an interval I in k-dimensional Euclidean space R^k, such that components X_i of X are continuous and nondecreasing. Then we say that $X \in A$. By $\int_J X\,dG$ we understand the vector $(\int_J X_1\,dG,..., \int_J X_k\,dG)$.

2.1. *Majorization Theorem*

Theorem 1. *Let G: $[a, b] \rightarrow R$ be a function of bounded variation and $X, Y \in A$.*

(a) *If*

$$\begin{gathered}\int_x^b X(t)\,dG(t) \leqslant \int_x^b Y(t)\,dG(t) \qquad (\forall x \in [a, b]), \\ \int_a^b X(t)\,dG(t)=\int_a^b Y(t)\,dG(t),\end{gathered} \tag{4}$$

then for every continuous function f with nondecreasing increments on I

$$\int_a^b f(X(t))\,dG(t) \leqslant \int_a^b f(Y(t))\,dG(t). \tag{5}$$

(b) *If*

$$\int_x^b X(t)\,dG(t) \leqslant \int_x^b Y(t)\,dG(t) \qquad (\forall x \in [a, b]), \tag{6}$$

then (5) *holds for every continuous nondecreasing function f with nondecreasing increments on I.*

Proof. This is a simple consequence of Theorem A.

Remark 1. In Theorem 1 let G be a nondecreasing function. Then (5) is valid for every continuous function f with nondecreasing increments on I if and only if (4) holds; analogously (5) holds for every continuous nondecreasing function f with nondecreasing increments on I if and only if (6) holds.

Indeed, let x^+ denote the larger of the two real numbers, x and 0. The validity of (5) for $f(x) = x_j$, where $x = (x_1, ..., x_k)$ and $f(x) = -x_j$ $(j = 1, ..., k)$, implies the second condition in (4). Next, let, for fixed j $(j = 1, ..., k)$ and fixed u $(a \leqslant u \leqslant b)$, $f(x) = (x_j - Y_j(u))^+$. Then $f(x)$ is continuous on I and with nondecreasing increments, and $f(x) \geqslant 0$, $f(x) \geqslant x_j - Y_j(u)$. Since G is nondecreasing we have

$$\int_u^b X_j(t)\,dG(t) - Y_j(u) \int_u^b dG(t) \leqslant \int_a^b f(X(t))\,dG(t) \leqslant \int_a^b f(Y(t))\,dG(t)$$

$$= \int_u^b Y_j(t)\,dG(t) - Y_j(u) \int_u^b dG(t),$$

wherefrom we obtain the first condition in (4). Similarly, we can prove the analogous result for nondecreasing f. For $k = 1$ we have the known result for convex functions (see, for example, [3]).

2.2. *Jensen–Steffensen Inequality*

THEOREM 2. *Let G be a function of bounded variation on $[a, b]$ with $G(b) > G(a)$, and let $X \in A$.*

(a) *If*

$$G(a) \leqslant G(x) \leqslant G(b) \qquad (\forall x \in (a, b)), \tag{7}$$

then for all continuous function f with nondecreasing increment on I

$$f\left(\frac{\int_a^b X(t)\,dG(t)}{\int_a^b dG(t)}\right) \leqslant \frac{\int_a^b f(X(t))\,dG(t)}{\int_a^b dG(t)}. \tag{8}$$

(b) *If* $\int_a^b X(t)\,dG(t)/\int_a^b dG(t) \in I$, *and if either*

$$G(x) \leqslant G(a) \qquad (\forall x \in (a, b)) \tag{9}$$

or

$$G(x) \geqslant G(b) \qquad (\forall x \in (a, b)), \tag{10}$$

then the reverse inequality in (8) *holds.*

(c) *If for a continuous* $f: I \to R$ *inequality* (8) *holds for every corresponding* $X \in A$ *and for every function of bounded variation* G *satisfying* (7), *then* f *is a function with nondecreasing increments.*

Proof. (c) Put $a \leqslant t_1 < t_2 < t_3 \leqslant b$, $X(t_1) = A$, $X(t_2) = B$, $X(t_3) = B + H$ $(0 \leqslant H \in R^k)$, $G(t) = 0$ $(a \leqslant t \leqslant t_1,\ t_2 < t \leqslant t_3)$ and $G(t) = 1$ $(t_1 < t \leqslant t_2,\ t_3 < t \leqslant b)$, when inequality (8) reduces to $f(A+H) \leqslant f(A) - f(B) + f(B+H)$. Therefore, f is a function with nondecreasing increments.

(a), (b) Using the substitutions

$$X(t) \to \int_a^b X(t)\,dG(t) \Big/ \int_a^b dG(t), \qquad Y(t) \to X(t),$$

from Theorem 1, we have that (8) is valid if

$$\int_a^b X_j(t)\,dG(t) \int_x^b dG(t) \leqslant \int_x^b X_j(t)\,dG(t) \int_a^b dG(t) \qquad (\forall x \in (a, b),\ 1 \leqslant j \leqslant k). \tag{11}$$

Analogously, we have that if the reverse inequality in (11) holds, then the reverse inequality in (8) holds too.

Now, (a) and (b) are simple consequences of these results and the following lemma:

LEMMA 1. *Let* $f, G: [a, b] \to R$ *be functions of bounded variations with* $G(b) \geqslant G(a)$, *and let* f *be integrable with respect to* G. *The inequality*

$$I(f, G; c) \geqslant 0, \tag{12}$$

where

$$\begin{aligned} I(f, G; c) &= \int_a^c dG(x) \int_a^b f(x)\,dG(x) - \int_a^b dG(x) \int_a^c f(x)\,dG(x) \\ &= \int_a^b dG(x) \int_c^b f(x)\,dG(x) - \int_c^b dG(x) \int_a^b f(x)\,dG(x), \end{aligned}$$

holds for all nondecreasing functions f and for every $c \in (a, b)$, if and only if (7) *holds. The reverse inequality in* (12) *holds if and only if either* (9) *or* (10) *hold. If f is a nonincreasing function, then the reverse results are valid.*

Lemma is a simple consequence of the following identity (see [4]):

$$I(f, G; c) = (G(b) - G(c)) \int_a^c (G(t) - G(a))\, df(t)$$

$$+ (G(c) - G(a)) \int_c^b (G(b) - G(t))\, df(t).$$

Remark 2. Generalization of Theorem 2(a) is given in [1].

Remark 3. It is very interesting that Theorems 2(a) and (b) contain a well-known Čebyšev inequality for monotonic functions (functions $f(x, y) = xy$ $(x, y \in R)$ and $f(x) = x_1 \cdots x_k$ $(x_i \geqslant 0,\ i = 1,..., k)$ are with nondecreasing increments).

2.3. *Brunk–Olkin Inequality*

Using the substitution $G(t) = g(t) + s(t)$, where $s(t) = 0$ $(t \in [a, t_0))$ and $s(t) = 1 - \int_a^b dg(t)$ $(t \in [t_0, b])$, we obtain, from Theorem 2, the following result:

THEOREM 3. *Let $f: I \to R$ be a continuous function with nondecreasing increments, $X \in A$ and let $t_0 \in [a, b]$ be fixed. If $X(t_0) = 0$ $(\in R^k)$ and if $g: [a, b] \to R$ is a function of bounded variation such that*

$$0 \leqslant \int_a^t dg(t) \leqslant 1 \qquad (a \leqslant t < t_0), \qquad 0 \leqslant \int_t^b dg(t) \leqslant 1 \qquad (t_0 \leqslant t \leqslant b),$$

then

$$f\left(\int_a^b X(t)\, dg(t)\right) \leqslant \int_a^b f(X(t))\, dg(t) + \left(1 - \int_a^b dg(t)\right) f(0). \tag{13}$$

If $\int_a^b X(t)\, dg(t) \in I$, and if either

$$\int_a^t dg(t) \geqslant 1 \qquad (a \leqslant t < t_0), \qquad \int_t^b dg(t) \leqslant 0 \qquad (t_0 \leqslant t \leqslant b)$$

or

$$\int_a^t dg(t) \leqslant 0 \qquad (a \leqslant t < t_0), \qquad \int_t^b dg(t) \geqslant 1 \qquad (t_0 \leqslant t \leqslant b),$$

then the reverse inequality in (13) *holds.*

Using the substitution $g(t) = h(t)/\int_a^b |dh(t)|$, from the above result we can obtain:

THEOREM 4. *Let f and X be defined as in Theorem 3 and let h: $[a, b] \to R$ be a function of bounded variation such that*

$$\int_a^t dh(t) \geqslant 0 \qquad (a \leqslant t < t_0), \qquad \int_t^b dh(t) \geqslant 0 \qquad (t_0 \leqslant t \leqslant b), \qquad \int_a^b |dh(t)| > 0.$$

If $f(0) = 0$, then

$$f\left(\frac{\int_a^b X(t)\, dh(t)}{\int_a^b |dh(t)|}\right) \leqslant \frac{\int_a^b f(X(t))\, dh(t)}{\int_a^b |dh(t)|}.$$

Remark 4. For some similar results for convex functions see [7–10].

2.4. *Favard Inequality*

THEOREM 5. *If $x \in A$ with*

$$\bar{x} = \frac{1}{b-a} \int_a^b X_i(t)\, dt, \qquad i = 1,..., k, \tag{14}$$

then inequality

$$\frac{1}{2\bar{x}} \int_0^{2\bar{x}} f(y,..., y)\, dy \geqslant \frac{1}{b-a} \int_a^b f(X(t))\, dt \tag{15}$$

holds for every continuous function f: $I \to R$ with nondecreasing increments if and only if

$$\frac{1}{(x-a)^2} \int_a^x X(t)\, dt \geqslant \frac{1}{(b-a)^2} \int_a^b X(t)\, dt \qquad (\forall x \in (a, b)). \tag{16}$$

If the reverse inequality in (16) *holds, then the reverse inequality in* (15) *holds too.*

Proof. Since

$$\frac{1}{2\bar{x}} \int_0^{2\bar{x}} f(y,..., y)\, dy = \frac{1}{b-a} \int_a^b f(Y'(t))\, dt,$$

where $Y'_j(t) = 2\bar{x}(t-a)/(b-a)$ $(j = 1,..., k)$, so $Y' \in A$, and (15) becomes $\int_a^b f(Y'(t)) \geqslant \int_a^b f(X(t))\, dt$. Therefore, using Theorem 1 (i.e., Remark 1) we obtain Theorem 5.

THEOREM 6. *Let $X_i(t)$ $(1 \leqslant i \leqslant k)$ be nonnegative concave functions such that* (14) *are valid, and let $f: I \to R$ be a continuous function with nondecreasing increments. Then* (15) *is valid.*

If $X_i(t)$ $(1 \leqslant i \leqslant k)$ are nonnegative convex functions such that (14) *and $X_i(a) = 0$ $(1 \leqslant i \leqslant k)$ are valid, then the reverse inequality in* (15) *is also valid.*

Proof. Since the function $x \mapsto f(x)/(x-a)$ is nonincreasing on $(a, b]$ for every concave function, using Lemma 1 we have that the condition (16) is satisfied. So, inequality (15) holds if f is a nondecreasing concave function. On the other hand, the monotonic rearrangement of positive concave functions is also a concave function (see, for example, [11]), so using the result from [12] we have that (15) holds for arbitrary positive concave functions too.

If X_i $(1 \leqslant i \leqslant k)$ is a convex function with $X_i(a) = 0$, then X_i is also nondecreasing, so from Theorem 5 we obtain that Theorem 6 holds in this case too.

Remark 5. The above results are generalizations of some results from [13] and [14].

3. APPLICATIONS

Denote by

$$H_n(X; G) = \int_a^b \left(\prod_{j=1}^k X_j(t)\right) dG(t) \bigg/ \left(\prod_{j=1}^k \left(\int_a^b X_j(t)^{\alpha_j}\, dG(t)\right)^{1/\alpha_j}\right).$$

Using the substitutions $f(x) = x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ $(0 < \alpha_j \leqslant 1, j = 1, ..., k)$, and

$$Y_j(t) \to X_j(t)^{\alpha_j} \bigg/ \int_a^b X_j(t)^{\alpha_j}\, dG(t), \qquad X_j(t) \to Y_j(t)^{\alpha_j} \bigg/ \int_a^b Y_j(t)^{\alpha_j}\, dG(t),$$

from Theorem 1 we obtain the following result:

COROLLARY 1. *Let $G: [a, b] \to R$ be a nondecreasing function and let $X, Y \in A$ have nonnegative components. If $0 < \alpha_j \leqslant 1$ $(j = 1, ..., k)$ and if*

$$\int_a^b X_j(t)^{\alpha_j}\, dG(t) \int_x^b Y_j(t)^{\alpha_j}\, dG(t) \leqslant \int_a^b Y_j(t)^{\alpha_j}\, dG(t) \int_x^b X_j(t)^{\alpha_j}\, dG(t) \quad (17)$$

$(\forall x \in (a, b))$, *then*

$$H_n(X; G) \geqslant H_n(Y; G). \quad (18)$$

COROLLARY 2. *Let* $G: [a, b] \to R$ *be a nondecreasing function and let* $X, Y \in A$ *have positive components. If* $0 < \alpha_j \leqslant 1$ $(1 \leqslant j \leqslant k)$ *and if* $x \mapsto Y_j(t)/X_j(t)$ $(1 \leqslant j \leqslant k)$ *are nonincreasing functions, then* (18) *is valid.*

Proof. Using Lemma 1, we can easily show that (17) is satisfied. So, from Corollary 1 follows Corollary 2.

Using Theorem 1 for $k = 1$, the Fuchs generalization of Majorization theorem, we can obtain the generalization of results from [21, 22].

COROLLARY 3. *Let* $G: [a, b] \to R$ *be a nondecreasing function and let* $f, g: [a, b] \to R$ *be two continuous positive functions. If* g *and* g/f *are monotonous in the opposite sense, function* F *defined by*

$$F(r) = \left(\frac{\int_a^b f(x)^r \, dG(x)}{\int_a^b g(x)^r \, dG(x)}\right)^{1/r} \qquad (r \neq 0; |r| < +\infty),$$

$$F(0) = \exp\left(\int_a^b \log(f(x)/g(x)) \, dG(x) \Big/ \int_a^b dG(x)\right)$$

is nondecreasing.

Proof. If g and g/f are monotonic in the opposite sense, then g and f are monotonic in the same sense. We shall suppose that they are nondecreasing functions. By substitutions $f(x) = x^{r/s}$ $(r \geqslant s > 0)$, $X_1(t) = g(t)^s / \int_a^b g(x)^s \, dG(x)$, $Y_1(t) = f(t)^s / \int_a^b f(x)^s \, dG(x)$, from Theorem 1 with $k = 1$ we get that

$$F(r) \geqslant F(s) \tag{19}$$

holds if

$$\int_x^b g(t)^s \, dG(t) \int_a^b f(t)^s \, dG(t)$$
$$\leqslant \int_x^b f(t)^s \, dG(t) \int_a^b g(t)^s \, dG(t) \qquad (\forall x \in (a, b)). \tag{20}$$

Now, using Lemma 1 and the fact that g/f is a nondecreasing function, we have that Corollary 3 is valid for $r \geqslant s > 0$. In the case when $s \to 0$ we obtain that (19) is also valid. Analogously, we can prove that (19) is valid in the cases $r \geqslant 0 \geqslant s$ and $0 \geqslant r \geqslant s$ too.

From the above proof, we have that the following result is also valid:

COROLLARY 4. *Let* $G: [a, b] \to R$ *be a nondecreasing function and let* $f, g: [a, b] \to R$ *be two nondecreasing, nonnegative and continuous functions*

($g \not\equiv 0$). *If* $r \geqslant s > 0$ *and if* (20) *holds, then* (19) *is also valid. If the reverse inequality in* (20) *is valid, then the reverse inequality in* (19) *is also valid.*

Remark 6. The result similar to Corollary 4 can be obtained in the cases when $r > 0 > s$ and $0 > r \geqslant s$ too.

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