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Some General Families of Generating Functions for the Laguerre Polynomials

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The object of the present paper is to develop rather systematically some general families of bilinear, bilateral, or mixed multilateral generating functions for the classical Laguerre polynomials. Numerous straightforward consequences of some of the results considered here frequently appear in the literature, especially from the viewpoint of Lie groups and Lie algebras. It is also pointed out how the main generating functions can be suitably applied to derive numerous further results involving Laguerre polynomials and various other related polynomials.

1. INTRODUCTION, DEFINITIONS, AND PRELIMINARIES

In the usual notation, let

$$L_n^{(\alpha)}(x)=\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-x)^k}{k!}, \tag{1.1}$$

where $L_n^{(\alpha)}(x)$ denotes the classical Laguerre polynomial of order α and degree n in x. These polynomials are orthogonal over the interval $(0, \infty)$ with respect to the weight function $x^\alpha e^{-x}$; in fact, we have (cf., e.g., Rainville [10] and Szegő [15])

$$\int_0^\infty x^\alpha e^{-x} L_m^{(\alpha)}(x)\, L_n^{(\alpha)}(x)\, dx = \frac{\Gamma(\alpha+n+1)}{n!}\delta_{m,n}$$
$$(\mathrm{Re}(\alpha) > -1; m, n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}; \mathbb{N} = \{1, 2, 3, ...\}), \tag{1.2}$$

where $\delta_{m,n}$ denotes the Kronecker delta.

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Just as the other members of the family of classical orthogonal polynomials (e.g., Jacobi polynomials, Hermite polynomials, Gegenbauer (or ultraspherical) polynomials, Legendre (or spherical) polynomials, and the Tchebycheff polynomials of the first and second kinds), the Laguerre polynomials can be expressed as a hypergeometric function:

$$L_n^{(\alpha)}(x) = \binom{n+\alpha}{n} {}_1F_1(-n; \alpha+1; x), \tag{1.3}$$

where ${}_1F_1$ is the (Kummer's) confluent hypergeometric function which corresponds to the special case $u = v = 1$ of the generalized hypergeometric function ${}_uF_v$ (with u numerator and v denominator parameters) defined, in the notations of Leo Pochhammer (1841–1920) and Ernest William Barnes (1874–1953), by

$$\begin{aligned} &{}_uF_v(\alpha_1, ..., \alpha_u; \beta_1, ..., \beta_v; z) \\ &= {}_uF_v\left[\begin{matrix}\alpha_1, ..., \alpha_u; \\ \beta_1, ..., \beta_v;\end{matrix} \; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_u)_n}{(\beta_1)_n \cdots (\beta_v)_n} \frac{z^n}{n!} \\ &(u \leqslant v+1; u < v+1, |z| < \infty; u = v+1, z \in \mathscr{U} = \{z : |z| < 1\}; \\ &u = v+1, z \in \partial\mathscr{U} = \{z : |z| = 1\}, \operatorname{Re}(\omega) > 0), \end{aligned} \tag{1.4}$$

provided that no zeros appear in the denominator; here $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if} \quad n = 0, \\ \lambda(\lambda+1) \cdots (\lambda+n-1), & \text{if} \quad n \in \mathbb{N}, \end{cases} \tag{1.5}$$

and (for convenience)

$$\omega = \sum_{j=1}^{v} \beta_j - \sum_{j=1}^{u} \alpha_j. \tag{1.6}$$

Clearly, (1.5) yields

$$(-n)_k = 0 \qquad (k = n+1, n+2, n+3, ...; n \in \mathbb{N}_0), \tag{1.7}$$

which accounts for the fact that a hypergeometric function ${}_uF_v$ would reduce to a polynomial whenever a numerator parameter is a non-positive integer. More importantly, since

$$
{}_{u+1}F_v\left[\begin{matrix} -n, & \alpha_1, \ldots, \alpha_u; \\ & \beta_1, \ldots, \beta_v; \end{matrix}\ z\right]
$$

$$
=\frac{(\alpha_1)_n \cdots (\alpha_u)_n}{(\beta_1)_n \cdots (\beta_v)_n}(-z)^n
$$

$$
\cdot {}_{v+1}F_u\left[\begin{matrix} -n, & 1-\beta_1-n, \ldots, 1-\beta_v-n; \\ & 1-\alpha_1-n, \ldots, 1-\alpha_u-n; \end{matrix}\ \frac{(-1)^{u+v}}{z}\right] \qquad (n \in \mathbb{N}_0), \tag{1.8}
$$

which follows upon reversing the order of terms in the finite sum for either side of (1.8), the Laguerre polynomials in (1.3) can also be expressed in the form:

$$
L_n^{(\alpha)}(x) = \frac{(-x)^n}{n!}\, {}_2F_0(-n, -\alpha-n; -; -x^{-1}) \qquad (n \in \mathbb{N}_0). \tag{1.9}
$$

Some important special cases of the generalized hypergeometric function ${}_uF_v$ include (i) the confluent hypergeometric function ${}_1F_1$ studied by Ernst Eduard Kummer (1810–1893), as we pointed out above; (ii) the celebrated hypergeometric function ${}_2F_1$ introduced, in the year 1812, by Carl Friedrich Gauss (1777–1855); (iii) the function ${}_3F_2$ introduced, in the year 1828, by Thomas Clausen (1801–1885); and (iv) the function ${}_2F_0$ studied by Francesco Giacomo Tricomi (1897–1978) and Edmund Taylor Whittaker (1873–1956), among others.

For the Laguerre polynomials, the following generating functions are well-known:

$$
\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n = (1-t)^{-\alpha-1} \exp\left(-\frac{xt}{1-t}\right) \qquad (|t|<1); \tag{1.10}
$$

$$
\sum_{n=0}^{\infty} L_n^{(\alpha-n)}(x)\, t^n = (1+t)^{\alpha} e^{-xt} \qquad (|t|<1). \tag{1.11}
$$

In view of the representation (1.3), the generating functions (1.10) and (1.11) are actually contained in the hypergeometric identities

$$
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, {}_{u+1}F_v\left[\begin{matrix} -n, & \alpha_1, \ldots, \alpha_u; \\ & \beta_1, \ldots, \beta_v; \end{matrix}\ z\right] t^n
$$

$$
= (1-t)^{-\lambda}\, {}_{u+1}F_v\left[\begin{matrix} \lambda, & \alpha_1, \ldots, \alpha_u; \\ & \beta_1, \ldots, \beta_v; \end{matrix}\ -\frac{zt}{1-t}\right] \qquad (|t|<1) \tag{1.12}
$$

and

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_{u+1}F_{v+1}\left[\begin{matrix} -n, & \alpha_1, ..., \alpha_u; \\ 1-\lambda-n, & \beta_1, ..., \beta_v; \end{matrix} \; z\right] t^n$$

$$= (1-t)^{-\lambda} {}_uF_v\left[\begin{matrix} \alpha_1, ..., \alpha_u; \\ \beta_1, ..., \beta_v; \end{matrix} \; zt\right] \qquad (|t|<1), \qquad (1.13)$$

respectively. For a systematic and detailed account of these and more general hypergeometric generating functions, the reader is referred to a recent treatise on the subject by Srivastava and Manocha [14].

Some useful generalizations of the classical results (1.10) and (1.11) include the generating functions

$$\sum_{n=0}^{\infty} \binom{m+n}{n} L_{m+n}^{(\alpha)}(x)\, t^n$$

$$= (1-t)^{-\alpha-m-1} \exp\left(-\frac{xt}{1-t}\right) L_m^{(\alpha)}\left(\frac{x}{1-t}\right) \qquad (|t|<1; m \in \mathbb{N}_0) \qquad (1.14)$$

and

$$\sum_{n=0}^{\infty} \binom{m+n}{n} L_{m+n}^{(\alpha-n)}(x)\, t^n$$

$$= (1+t)^{\alpha} \exp(-xt)\, L_m^{(\alpha)}(x(1+t)) \qquad (|t|<1; m \in \mathbb{N}_0), \quad (1.15)$$

which, in the special case when $m=0$, would immediately yield (1.10) and (1.11), respectively. With a view to obtaining bilinear, bilateral, or mixed multilateral generating functions for the Laguerre polynomials $L_n^{(\alpha)}(x)$ and $L_n^{(\alpha-n)}(x)$, several workers have successfully applied the generating functions (1.14) and (1.15). The most general applications of the generating functions (1.14) and (1.15), of the type just mentioned, yield two classes of mixed multilateral generating functions for $L_n^{(\alpha)}(x)$ and $L_n^{(\alpha-n)}(x)$, which were given by Srivastava [13, p. 231, Corollaries 8, 9] and which have since been reproduced by Srivastava and Manocha [14, pp. 424–425, Corollaries 8, 9]. For the sake of ready reference, and in yet another attempt to help avoid unnecessary rederivations of obvious special cases of these results by current as well as future researchers on the subject, we choose to recall the aforementioned results of Srivastava [13] as Theorems A and B below:

THEOREM A (cf. Srivastava [13, p. 231, Corollary 8]). *Corresponding* *to a non-vanishing function* $\Omega_\mu(y_1, \ldots, y_s)$ *of* s *variables* $y_1, \ldots, y_s$ $(s \in \mathbb{N})$ *and of (complex) order* μ, *let*

$$\Lambda^{(1)}_{m,p,q}[x; y_1, \ldots, y_s; z] = \sum_{n=0}^{\infty} a_n L^{(\alpha)}_{m+qn}(x)\, \Omega_{\mu+pn}(y_1, \ldots, y_s)\, z^n$$

$$(a_n \neq 0;\ m \in \mathbb{N}_0;\ p, q \in \mathbb{N}), \tag{1.16}$$

where α *is a complex parameter. Suppose also that*

$$N^{p,\mu}_{n,m,q}(y_1, \ldots, y_s; z) = \sum_{k=0}^{[n/q]} \binom{m+n}{n-qk} a_k \Omega_{\mu+pk}(y_1, \ldots, y_s)\, z^k, \tag{1.17}$$

where, as usual, $[\lambda]$ *represents the greatest integer in* λ.

Then

$$\sum_{n=0}^{\infty} L^{(\alpha)}_{m+n}(x)\, N^{p,\mu}_{n,m,q}(y_1, \ldots, y_s; z)\, t^n$$

$$= (1-t)^{-\alpha-m-1} \exp\left(-\frac{xt}{1-t}\right)$$

$$\cdot \Lambda^{(1)}_{m,p,q}\left[\frac{x}{1-t}; y_1, \ldots, y_s; \frac{zt^q}{(1-t)^q}\right] \quad (|t| < 1), \tag{1.18}$$

provided that each member of (1.18) *exists.*

THEOREM B (cf. Srivastava [13, p. 231, Corollary 9]). *Corresponding to a non-vanishing function* $\Omega_\mu(y_1, \ldots, y_s)$ *of* s *variables* $y_1, \ldots, y_s$ $(s \in \mathbb{N})$ *and of (complex) order* μ, *let*

$$\Lambda^{(2)}_{m,p,q}[x; y_1, \ldots, y_s; z]$$

$$= \sum_{n=0}^{\infty} a_n L^{(\alpha-qn)}_{m+qn}(x)\, \Omega_{\mu+pn}(y_1, \ldots, y_s)\, z^n$$

$$(a_n \neq 0;\ m \in \mathbb{N}_0;\ p, q \in \mathbb{N}), \tag{1.19}$$

where α *is a complex parameter. Suppose also that the function*

$$N^{p,\mu}_{n,m,q}(y_1, \ldots, y_s; z)$$

is defined, as before, by Eq. (1.17).

Then

$$\sum_{n=0}^{\infty} L_{m+n}^{(\alpha-n)}(x)\, N_{n,m,q}^{p,\mu}(y_1, ..., y_s; z)\, t^n$$
$$= (1+t)^{\alpha} \exp(-xt)$$
$$\cdot \Lambda_{m,p,q}^{(2)}\left[x(1+t); y_1, ..., y_s; \frac{zt^q}{(1+t)^q}\right] \quad (|t|<1), \tag{1.20}$$

provided that each member of (1.20) *exists.*

As already observed by Srivastava [13, p. 232, Remark 9], since

$$f_n^{-\alpha}(x) = (-1)^n L_n^{(\alpha-n)}(x) = \frac{x^n}{n!} c_n(\alpha; x), \tag{1.21}$$

Theorem 3 can easily be restated in terms of the (so-called) modified Laguerre polynomials $f_n^{\alpha}(x)$ defined by [9, p. 4, Eq. (9)]

$$(1-t)^{-\alpha} \exp(xt) = \sum_{n=0}^{\infty} f_n^{\alpha}(x)\, t^n \qquad (|t|<1) \tag{1.22}$$

or of the Poisson–Charlier polynomials $c_n(x; \alpha)$ given explicitly by [15, p. 35, Eq. (2.81.2)]

$$c_n(x; \alpha) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{x}{k} k!\, \alpha^{-k} \qquad (\alpha>0; x \in \mathbb{N}_0). \tag{1.23}$$

In view of the frequent occurrences of the various obvious special cases of Theorems A and B (and of the aforementioned known consequences of Theorem B involving, for example, the modified Laguerre polynomials) in the current literature on the subject (cf., e.g., [1, 3–7, 11]), especially from the viewpoint of Lie algebras and Lie groups [14, Chap. 6], we aim here at developing some substantially more general families of mixed multilateral generating functions for the Laguerre polynomials by further applying (1.14) and (1.15). We also present an analogous (presumably new) application of the generating function (cf. [2, p. 142, Eq. (18)]; see also [14, p. 172, Problem 22(ii)]):

$$\sum_{k=0}^{\infty} L_n^{(\alpha+k)}(x) \frac{t^k}{k!} = e^t L_n^{(\alpha)}(x-t), \tag{1.24}$$

which incidentally is an immediate consequence of the Taylor expansion of

$$e^t L_n^{(\alpha)}(x-t)$$

in powers of t. (For another application of the generating function (1.24), see the recent work of Hubbell and Srivastava [8, p. 346].)

2. Generalizations of Theorems A and B

One of our main results on generating functions for the classical Laguerre polynomials is contained in

THEOREM 1. *Under the hypotheses of Theorem* A, *let*

$$\Lambda^{(3)}_{m,p,q}[x; y_1, ..., y_s; z] = \sum_{n=0}^{\infty} a_n L^{(\alpha+\rho q n)}_{m+qn}(x)\, \Omega_{\mu+pn}(y_1, ..., y_s)\, z^n$$

$$(a_n \neq 0;\ m \in \mathbb{N}_0;\ p, q \in \mathbb{N}) \tag{2.1}$$

and

$$\Theta^{\alpha,q,\rho}_{n,m,p}(x; y_1, ..., y_s; z) = \sum_{k=0}^{[n/q]} \binom{m+n}{n-qk} a_k L^{(\alpha+\rho q k)}_{m+n}(x)\, \Omega_{\mu+pk}(y_1, ..., y_s)\, z^k, \tag{2.2}$$

where ρ *is a suitable complex parameter.*

Then

$$\sum_{n=0}^{\infty} \Theta^{\alpha,q,\rho}_{n,m,p}(x; y_1, ..., y_s; z)\, t^n = (1-t)^{-\alpha-m-1} \exp\left(-\frac{xt}{1-t}\right) \cdot \Lambda^{(3)}_{m,p,q}\left[\frac{x}{1-t}; y_1, ..., y_s; \frac{zt^q}{(1-t)^{(\rho+1)q}}\right] \quad (|t|<1), \tag{2.3}$$

provided that each member of (2.3) *exists.*

Proof. Denote, for convenience, the left-hand side of the assertion (2.3) by Δ. Then, upon substituting for the polynomials

$$\Theta^{\alpha,q,\rho}_{n,m,p}(x; y_1, ..., y_s; z)$$

from (2.2) into the left-hand side of (2.3), we have

$$\Delta = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{[n/q]} \binom{m+n}{n-qk} a_k L_{m+n}^{(\alpha+\rho qk)}(x)\, \Omega_{\mu+pk}(y_1, ..., y_s)\, z^k$$
$$= \sum_{k=0}^{\infty} a_k\, \Omega_{\mu+pk}(y_1, ..., y_s)(zt^q)^k$$
$$\cdot \sum_{n=0}^{\infty} \binom{m+qk+n}{n} L_{m+qk+n}^{(\alpha+\rho qk)}(x)\, t^n,$$

by inverting the order of the double summation involved.

The inner series can be summed by applying the generating function (1.14), with m and α replaced by $m+qk$ and $\alpha+\rho pk$, respectively ($q \in \mathbb{N}$; $k \in \mathbb{N}_0$), and we thus find that

$$\Delta = (1-t)^{-\alpha-m-1} \exp\left(-\frac{xt}{1-t}\right) \sum_{k=0}^{\infty} a_k L_{m+qk}^{(\alpha+\rho qk)}\left(\frac{x}{1-t}\right)$$
$$\cdot \Omega_{\mu+pk}(y_1, ..., y_s) \left\{\frac{zt^q}{(1-t)^{(\rho+1)q}}\right\}^k \qquad (|t| < 1).$$

Interpreting this last infinite series by means of the definition (2.1), we arrive immediately at the right-hand side of the assertion (2.3).

This evidently completes the proof of Theorem 1 under the assumption that the double series involved in the first two steps of our proof are absolutely convergent. Thus, in general, Theorem 1 holds true for those values of the various parameters and variables involved for which each member of the assertion (2.3) exists.

For $\rho = 0$, Theorem 1 would reduce at once to our earlier assertion (1.18) given by Theorem A. A similar generalization of Theorem B can be proven by applying the generating function (1.15) and the proof of Theorem 1 *mutatis mutandis*, and we are thus led to

THEOREM 2. *Under the hypotheses of Theorem* B, *let*

$$\Lambda_{m,p,q}^{(4)}[x; y_1, ..., y_s; z] = \sum_{n=0}^{\infty} a_n L_{m+qn}^{(\alpha+(\rho-1)qn)}(x)\, \Omega_{\mu+pn}(y_1, ..., y_s)\, z^n$$
$$(a_n \neq 0; m \in \mathbb{N}_0;\ p, q \in \mathbb{N}) \tag{2.4}$$

and

$$\Phi_{n,m,p}^{\alpha,q,\rho}(x; y_1, ..., y_s; z)$$
$$= \sum_{k=0}^{[n/q]} \binom{m+n}{n-qk} a_k L_{m+n}^{(\alpha-n+\rho qk)}(x)\, \Omega_{\mu+pk}(y_1, ..., y_s)\, z^k, \tag{2.5}$$

where ρ is a suitable complex parameter.

Then

$$\sum_{n=0}^{\infty} \Phi_{n,m,p}^{\alpha,q,\rho}(x; y_1, \ldots, y_s; z)\, t^n$$

$$= (1+t)^{\alpha} \exp(-xt)\, \Lambda_{m,p,q}^{(4)}[x(1+t); y_1, \ldots, y_s; zt^q(1+t)^{(\rho-1)q}]$$

$$(|t| < 1), \tag{2.6}$$

provided that each member of (2.6) *exists.*

3. An Unusual Family of Bilateral Generating Functions

In this section we consider the generating function (1.24), which obviously is unusual in the sense that the summation index k appears only in the order of the Laguerre polynomials. Nevertheless, on replacing α trivially by $\alpha + m$ $(m \in \mathbb{N}_0)$, it yields the generating function:

$$\sum_{k=0}^{\infty} L_n^{(\alpha+m+k)}(x) \frac{t^k}{k!} = e^t L_n^{(\alpha+m)}(x-t) \qquad (m \in \mathbb{N}_0), \tag{3.1}$$

which incidentally fits easily into the Singhal–Srivastava definition [12, p. 755, Eq. (1)]:

$$\sum_{k=0}^{\infty} A_{m,k} S_{m+k}(x)\, t^k = f(x,t)\{g(x,t)\}^{-m} S_m(h(x,t)) \quad (m \in \mathbb{N}_0), \tag{3.2}$$

with, of course,

$$A_{m,k} = \frac{1}{k!}, \qquad f = e^t, \qquad g = 1, \qquad h = x - t,$$

and (3.3)

$$S_k(x) = L_n^{(\alpha+k)}(x) \qquad (k \in \mathbb{N}_0).$$

Thus the entire development stemming from the Singhal–Srivastava generating function (3.1) will readily apply also to the generating function (1.24) or (3.1). Alternatively, by appealing *directly* to the generating function (1.24), we can obtain an unusual family of bilinear, bilateral, or mixed multilateral generating functions for the Laguerre polynomials, given by

THEOREM 3. *Corresponding to a non-vanishing function $\Omega_\mu(y_1, ..., y_s)$ of s variables $y_1, ..., y_s$ $(s \in \mathbb{N})$ and of (complex) order μ, let*

$$\Lambda_{p,q}^{(5)}[x; y_1, ..., y_s; z] = \sum_{k=0}^{\infty} a_k L_n^{(\alpha+(\rho+1)qk)}(x)\, \Omega_{\mu+pk}(y_1, ..., y_s) \frac{z^k}{(qk)!}$$

$$(a_k \neq 0;\ n \in \mathbb{N}_0;\ p, q \in \mathbb{N}), \tag{3.4}$$

where α and ρ are complex parameters. Suppose also that

$$\Psi_{k,p,q}^{\alpha,\mu,\rho}(x; y_1, ..., y_s; z) = \sum_{r=0}^{[k/q]} \binom{k}{qr} a_r L_n^{(\alpha+\rho qr+k)}(x)\, \Omega_{\mu+pr}(y_1, ..., y_s)\, z^r, \tag{3.5}$$

where, as before, $[\lambda]$ represents the greatest integer in λ.
Then

$$\sum_{k=0}^{\infty} \Psi_{k,p,q}^{\alpha,\mu,\rho}(x; y_1, ..., y_s; z) \frac{t^k}{k!} = e^t \Lambda_{p,q}^{(5)}[x-t; y_1, ..., y_s; zt^q], \tag{3.6}$$

provided that each member of (3.6) *exists.*

The proof of Theorem 3, using the generating function (1.24) or (3.1), is much akin to that of Theorem 1, and we choose to omit the details involved.

For each suitable choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if the multivariable function

$$\Omega_\mu(y_1, ..., y_s) \qquad (s > 1)$$

is expressed as an appropriate product of several simpler functions, each of our results (Theorems 1, 2, and 3 above) can be shown to yield various families of mixed multilateral generating functions for the Laguerre polynomials. Thus, for example, each of the numerous assertions in the recent (or current) literature on bilateral generating functions for the Laguerre (or the modified Laguerre) polynomials (cf., e.g., [1, 3–7, 11]), especially since the publication of the monograph by Srivastava and Manocha [14], is actually contained in one or the other of the results which we have presented here. The detailed demonstration of this statement may be left as an exercise for the interested reader.

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