View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

Journal of Algebra 320 (2008) 3447–3460

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Lie properties of crossed products ☆

Adalbert Bovdi [a], Alexander Grishkov [b,*]

[a] Institute of Mathematics, University of Debrecen, P.O. Box 12, H-4010 Debrecen, Hungary
[b] Departamento de Matemática (IME-USP), Rua do Matao, 1010, Cidade Universitária, CEP 05508-090, Sao Paulo, SP, Brazil

ARTICLE INFO

Article history:
Received 1 July 2007
Available online 10 September 2008
Communicated by Efim Zelmanov

Dedicated to 60th birthday of Professor I.P. Shestakov

Keywords:
Twisted group algebra
Group of units
Crossed product
Lie nilpotent

ABSTRACT

Let $F_\sigma^\lambda[G]$ be a crossed product of a group G and the field F. We study the Lie properties of $F_\sigma^\lambda[G]$ in order to obtain a characterization of those crossed products which are upper (lower) Lie nilpotent and Lie (n,m)-Engel.

© 2008 Elsevier Inc. All rights reserved.

1. Introduction

Let G be a group and let $Aut(F)$ be the group of automorphisms of a field F. Assume the mapping $\sigma : G \to Aut(F)$ and the twisting function $\lambda : G \times G \to U(F)$ satisfy the following conditions:

$$\lambda(a, bc)\lambda(b, c) = \lambda(ab, c)\lambda(a, b)^{\sigma(c)} \tag{1}$$

and

$$\alpha^{\sigma(a)\cdot\sigma(b)} = \alpha^{\sigma(ab)} \tag{2}$$

☆ Supported by OTKA No. K68383, by FAPESP (proc. 06-56203-3) and by RFFI (07-01-00392A).

* Corresponding author.
E-mail addresses: bodibela@math.klte.hu (A. Bovdi), grishkov@ime.usb.br (A. Grishkov).

0021-8693/$ – see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.jalgebra.2008.07.005

for all $a, b, c \in G$ and $\alpha(a, b) \in F$. The twisting function λ is also called a *factor system* of the group G over the field F relative to σ. It is a 2-cocycle of the group of units $U(F)$ of F with the natural G-module structure (in the cohomology group of G in $U(F)$).

Assign to every $g \in G$ a symbol $\widetilde{g}$, and let

$$F_\sigma^\lambda[G] = \left\{ \sum_{g \in G} \widetilde{g}\alpha_g \;\middle|\; \alpha_g \in F \right\}$$

be the set of all formal sums with finitely many nonzero coefficients α_g. Two elements $x = \sum_{g \in G} \widetilde{g}\alpha_g$ and $y = \sum_{g \in G} \widetilde{g}\beta_g$ from $F_\sigma^\lambda[G]$ are equal if and only if $\alpha_g = \beta_g$ for all $g \in G$. On the set $F_\sigma^\lambda[G]$ addition and multiplication are defined as follows:

(i) $\sum_{g \in G} \widetilde{g}\alpha_g + \sum_{g \in G} \widetilde{g}\beta_g = \sum_{g \in G} \widetilde{g}(\alpha_g + \beta_g)$;
(ii) $\widetilde{g}\widetilde{h} = \widetilde{gh}\lambda(g, h)$, where λ is the twisting function;
(iii) $\alpha\widetilde{g} = \widetilde{g}\alpha^{\sigma(g)}$.

The product of arbitrary elements x and y is determined by distributivity. It is easy to check that $F_\sigma^\lambda[G]$ is an associative ring which is called a *crossed product* of the group G over the field F. If σ is a trivial mapping then we shall denote this ring by $F^\lambda[G]$ and it is a *twisted group algebra*.

From (1) and (2) follows that

$$\lambda(h, 1)^{\sigma(1)} = \lambda(h, 1) = \lambda(1, 1), \qquad \lambda(1, h) = \lambda(1, 1)^{\sigma(h)}.$$

Clearly, $\widetilde{1} \cdot \lambda(1, 1)^{-1}$ is the identity element of the ring $F_\sigma^\lambda[G]$ and

$$\widetilde{g}^{-1} = \lambda\big(g^{-1}, g\big)^{-1}\lambda(1, 1)^{-1}\widetilde{g^{-1}} = \widetilde{g^{-1}}\lambda\big(g, g^{-1}\big)^{-1}\lambda(1, 1)^{-1}. \tag{3}$$

For the twisted group algebra $F^\lambda[G]$ we can always assume without loss of generality, that the twisting function is normalized, that is

$$\lambda(h, 1) = \lambda(1, 1) = \lambda(1, h) = 1,$$

so $\widetilde{1}$ is its identity element.

In the theory of ordinary group rings the Lie properties play an important role. Group algebras with the many "good" Lie properties were described during the 70s using the theory on polynomial identities. Later these results were applied to the study of the group of units. In most cases the Lie structure reflects very well the characteristics of the group of units, there is a close relationship between their properties of them. Here we describe the structure of those crossed products which are upper (lower) Lie nilpotent and Lie (n, m)-Engel. We generalize results of Passi, Passman and Sehgal [9] which were obtained for the group algebras.

2. Preliminaries results

Let $F^\lambda[G]$ be a twisted group algebra with normalized twisting function λ. Then

$$W = \big\{ w \in G \;\big|\; \lambda(g, w) = \lambda(w, g) = 1 \text{ for all } g \in G \big\}$$

is a subgroup of G. Indeed, if $w_1, w_2, w \in W$ and $g \in G$, then

$$(\widetilde{w_1 w_2})\widetilde{g} = \widetilde{w_1}\widetilde{w_2}\widetilde{g} = \widetilde{w_1}(\widetilde{w_2 g}) = \widetilde{w_1 w_2 g};$$
$$\widetilde{g}(\widetilde{w_1 w_2}) = \widetilde{g}\widetilde{w_1}\widetilde{w_2} = (\widetilde{g w_1})\widetilde{w_2} = \widetilde{g w_1 w_2}.$$

So $\lambda(w_1w_2, g) = \lambda(g, w_1w_2) = 1$ and $w_1w_2 \in W$. Furthermore, by (1) and (3), it is easy to check that

$$\begin{aligned}\widetilde{w^{-1}}\widetilde{g} = \widetilde{w}^{-1}\widetilde{g} &= \big(\widetilde{g}^{-1}\widetilde{w}\big)^{-1} = \lambda\big(g, g^{-1}\big)\big(\widetilde{g^{-1}}\widetilde{w}\big)^{-1} = \lambda\big(g, g^{-1}\big)\big(\widetilde{g^{-1}w}\big)^{-1} \\ &= \lambda\big(g, g^{-1}\big)\lambda\big(g^{-1}w, w^{-1}g\big)^{-1}\widetilde{w^{-1}g} \\ &= \lambda\big(g^{-1}, w\big)^{-1}\lambda\big(w, w^{-1}g\big)^{-1}\widetilde{w^{-1}g} = \widetilde{w^{-1}g}.\end{aligned}$$

Similarly, $\widetilde{g}\widetilde{w^{-1}} = \widetilde{gw^{-1}}$, and this shows that $w^{-1} \in W$, so W is a subgroup.

Let H be a normal subgroup of G such that $H \subseteq W$ and denote by $\mathbf{I}(H)$ the ideal of $F^\lambda[G]$ generated by the elements $\widetilde{h} - \widetilde{1}$ with $h \in H$. It is easy to see that if $\{u_i\}$ is a transversal of H in G then the elements of the form $\widetilde{u_i}(\widetilde{h} - \widetilde{1})$ with $h \neq 1$ constitute an F-basis of the ideal $\mathbf{I}(H)$.

Define a new function: $\mu : G/H \times G/H \to U(F)$ the following way: Choose two representatives $g_i = u_kh_1$ and $g_j = u_lh_2$ with $h_i \in H \subseteq W$ of the cosets of H in G. Then by (1) in $F^\lambda[G]$ we get

$$\begin{aligned}\lambda(g_i, g_j) &= \lambda(u_kh_1, u_lh_2) \\ &= \lambda(u_k, h_1)^{-1}\lambda(h_1, u_lh_2)\lambda(u_k, h_1u_lh_2) = \lambda(u_k, h_1u_lh_2)\end{aligned}$$

and $h_1u_lh_2 = u_lh$ for suitable $h \in H$. It is easy to check that

$$\begin{aligned}\lambda(u_k, h_1u_lh_2) &= \lambda(u_k, u_lh) \\ &= \lambda(u_l, h)^{-1}\lambda(u_k, u_l)\lambda(u_ku_l, h) = \lambda(u_k, u_l).\end{aligned}$$

We proved that the twisting function λ satisfies the condition $\lambda(g_i, g_j) = \lambda(u_k, u_l)$ and it can define the function

$$\mu(g_iH, g_jH) = \lambda(g_i, g_j), \tag{4}$$

which is a twisting function of G/H. Of course, any element of $F^\lambda[G]$ has the form $\sum_i u_ix_i$ with $x_i \in F^\lambda[H]$ and $x_i + \mathbf{I}(H) = \lambda_i + \mathbf{I}(H)$ for suitable $\lambda_i \in F$. Now it is easy to see that $F^\lambda[G]/\mathbf{I}(H)$ is an F-algebra with a basis consisting of the images $\widetilde{u_i}$ of the coset representatives u_i of H in G. We obtain the isomorphism:

$$F^\lambda[G]/\mathbf{I}(H) \cong F^\mu[G/H]. \tag{5}$$

Unlike ordinary group algebras, the twisted group algebra $F^\lambda[G]$ does not have a natural group basis. If the algebra $F^\lambda[G]$ has an F-basis $\widetilde{G} = \{\widetilde{g} \mid g \in G\}$ such that for each g of G there exists an element $d_g \in F$ such that the elements of the set $\{d_g\widetilde{g} \mid g \in G\}$ form a group basis for $F^\lambda[G]$, then $F^\lambda[G]$ is called *untwisted*. In this situation $F^\lambda[G]$ is isomorphic to FG via this diagonal change of the basis. In addition, $F^\lambda[G]$ is called *stably untwisted* if there exists an extension K of the field F such that $K^\lambda[G]$ is untwisted.

We use a criteria from [12] to verify that a twisted group algebra is untwisted or stably untwisted.

Lemma 1. *(See [12].) Let $F^\lambda[G]$ be a twisted group algebra.*

(i) *$F^\lambda[G]$ is untwisted if and only if there exists an F-algebra homomorphism $F^\lambda[G] \to F$.*
(ii) *$F^\lambda[G]$ is stably untwisted if and only if there exists an F-algebra homomorphism of $F^\lambda[G]$ into a commutative F-algebra R.*

Recall that $\{u_i\}$ is a transversal of H in G and each element from G can be written uniquely in the form $g = zu_i$ with $z \in H$.

Lemma 2. *Let $F^\lambda[G]$ be a twisted group algebra and assume that the twisting function λ of the normal subgroup H of G satisfies the condition $\lambda(h_1, h_2) = 1$ for all $h_1, h_2 \in H$. Then the algebra $F^\lambda[G]$ can be realized alternatively as a twisted group algebra $F^\tau[G]$ with the following diagonal change of the basis by*

$$\overline{g} = \begin{cases} \widetilde{g}, & \text{if } g \in H; \\ \lambda(z, u_j)\widetilde{g} & \text{if } g = zu_j \in G \setminus H, z \in H, \end{cases}$$

with twisting function τ. This twisting function has the property $\tau(h, g) = 1$ for all $h \in H$ and $g \in G$.

If $\tau(g, h) = 1$ for all $h \in H$ and $g \in G$, then

$$F^\tau[G]/\mathbf{I}(H) \cong F^\tau[G/H].$$

Proof. It is easy to see that if $g = h_1 u_j$ then

$$\begin{aligned} \tau(h, g)\overline{hg} &= \overline{h} \cdot \overline{h_1 u_j} = \widetilde{h} \cdot \widetilde{h_1 u_j}\lambda(h_1, u_j) \\ &= \widetilde{h} \cdot \widetilde{h_1} \cdot \widetilde{u_j} = \widetilde{hh_1} \cdot \widetilde{u_j} = \lambda(hh_1, u_j)\widetilde{hg} = \overline{hg}. \end{aligned}$$

This yields $\tau(h, g) = 1$.

If $\tau(g, h) = 1$ for all $h \in H$ and $g \in G$, then we can apply (5) to obtain the lemma. □

The twisted group algebras satisfying polynomial identities have been classified in [11] and this result was modified in [6] for the stably untwisted case.

Lemma 3. *(See [6].) If $F^\lambda[G]$ is a twisted group algebra of positive characteristic p with a polynomial identity of degree n then G has a subgroup A of finite index such that $F^\lambda[A]$ is stably untwisted, the commutator subgroup A' of A is a finite p-group and $|G : A||A'|$ is bounded by a fixed function of n.*

Let $F^\lambda[G]$ be a twisted group algebra. For each $h \in G$ of order k we have

$$\widetilde{h}^k = \prod_{i=1}^{k-1} \lambda\big(h^i, h\big) \cdot \widetilde{1}. \tag{6}$$

It is convenient to say that $\mu(h) = \prod_{i=1}^{k-1} \lambda(h^i, h)$ is the *twist* of $\widetilde{h}$ which plays an important role in the study of $F^\lambda[G]$. A p-element u of G is called an *untwisted p-element* if the order of u coincides with the order of $\widetilde{u}\gamma$ for some $\gamma \in F$.

Lemma 4. *Let $F^\lambda[G]$ be a twisted group algebra of positive characteristic p such that the commutator ideal $F^\lambda[G]^{[2]} = [F^\lambda[G], F^\lambda[G]]F^\lambda[G]$ is a nil ideal and let $a, b \in G$.*

(i) *If $\widetilde{a}$ is a p-element, then a is also a p-element, its order coincides with the order of $\widetilde{a}$ and its twist is $\mu(a) = 1$.*
(ii) *If a and b are untwisted p-elements, and the orders of $\widetilde{a}\gamma_1$ and $\widetilde{b}\gamma_2$ coincide with the order of a and b respectively for some $\gamma_i \in F$ then ab is also a p-element, the order p^l of ab coincides with the order of $\widetilde{ab}\gamma_1\gamma_2$ and*

$$\mu(ab) = \big(\gamma_1\gamma_2\lambda(a, b)\big)^{-p^l}.$$

(iii) *The group commutator (a,b) is an untwisted p-element for all $a,b\in G$ and if p^m is the order of (a,b), then*

$$\big(\lambda(a,b)^{-1}\lambda\big(b,b^{-1}ab\big)\lambda\big(a,(a,b)\big)\lambda\big((b,a),(a,b)\big)^{-1}\big)^{-p^m}=\mu\big((a,b)\big).$$

Proof. (i) Let $\widetilde{a}$ be an element of order p^t and let k be the order of a. Then (6) yields that $\widetilde{a}^k=\mu(a)\widetilde{1}$, so $\widetilde{a}^{kp^t}=\mu(a)^{p^t}\widetilde{1}=\widetilde{1}$. It follows that the twist $\mu(a)$ of $\widetilde{a}$ satisfies the condition $\mu(a)^{p^t}=1$. But in the field F of characteristic p this is possible only if $\mu(a)=1$. Hence the order of a coincides with the order of $\widetilde{a}$ and $\mu(a)=1$.

(ii) First note that if $F^\lambda[G]^{[2]}$ is a nil ideal, then

$$xy\equiv yx \pmod{F^\lambda[G]^{[2]}},\qquad (xy)^n\equiv x^ny^n \pmod{F^\lambda[G]^{[2]}}$$

for all $x,y\in F^\lambda[G]$ and for all n. It follows that the nilpotent elements of $F^\lambda[G]$ form an ideal N and $F^\lambda[G]/N$ is commutative.

Let a and b be untwisted p-elements, and assume that the orders of $\widetilde{a}\gamma_1$ and $\widetilde{b}\gamma_2$ coincide with the order of a and b respectively for some $\gamma_i\in F$. Since $\widetilde{a}\gamma_1-\widetilde{1}\in N$ and $\widetilde{b}\gamma_2-\widetilde{1}\in N$, we have

$$\widetilde{a}\widetilde{b}\gamma_1\gamma_2-\widetilde{1}=(\widetilde{a}\gamma_1-\widetilde{1})(\widetilde{b}\gamma_2-\widetilde{1})+(\widetilde{a}\gamma_1-\widetilde{1})+(\widetilde{b}\gamma_2-\widetilde{1})\in N,$$

because the nilpotent elements of $F^\lambda[G]$ form an ideal. It follows that

$$(\widetilde{a}\widetilde{b}\gamma_1\gamma_2-\widetilde{1})^{p^{l_1}}=0$$

for some l_1, which shows that $\widetilde{a}\widetilde{b}\gamma_1\gamma_2$ has order p^l. Now $(\widetilde{a}\widetilde{b}\lambda(a,b)\gamma_1\gamma_2)^{p^l}=\widetilde{1}$ implies that ab has order p^m and, by (6), we obtain that

$$\big(\widetilde{a}\widetilde{b}\lambda(a,b)\gamma_1\gamma_2\big)^{p^m}=\mu(ab)\cdot\big(\lambda(a,b)\gamma_1\gamma_2\big)^{p^m}\cdot\widetilde{1}$$

which yields $(\mu(ab)\cdot(\lambda(a,b)\gamma_1\gamma_2)^{p^m})^{p^{l-m}}=1$. Then $m=l$ and

$$\mu(ab)\big(\lambda(a,b)\gamma_1\gamma_2\big)^{p^m}=1.$$

Hence ab has order p^l and $\mu(ab)=(\lambda(a,b)\gamma_1\gamma_2)^{-p^m}$.

(iii) Obviously, the Lie commutator $[\widetilde{a},\widetilde{b}]$ belongs to the nil ideal $F^\lambda[G]^{[2]}$ for all $a,b\in G$ and the identity

$$[\widetilde{a},\widetilde{b}]=\widetilde{a}^{-1}\widetilde{b}^{-1}\big((\widetilde{a},\widetilde{b})-\widetilde{1}\big) \tag{7}$$

ensures that $(\widetilde{a},\widetilde{b})-\widetilde{1}$ is nilpotent. Then $(\widetilde{a},\widetilde{b})$ has order p^m and an easy computation shows that

$$(\widetilde{a},\widetilde{b})=\widetilde{(a,b)}\chi\big((a,b)\big) \tag{8}$$

for some $\chi((a,b))$ of F. The argument of the proof of (i) states that p^m is the order of (a,b) and

$$(\widetilde{a},\widetilde{b})^{p^m}=\mu((a,b))\chi((a,b))^{p^m}\widetilde{1}=\widetilde{1}. \tag{9}$$

Moreover, from (3) and (8) follows that

$$\begin{aligned}\chi((a,b)) &= \widetilde{a}^{-1}\widetilde{b}^{-1}\widetilde{a}\cdot\widetilde{a^{-1}ba}\lambda\big(b,(b,a)\big)\lambda\big((b,a),(a,b)\big)^{-1}\\ &= \widetilde{a}^{-1}\widetilde{b}^{-1}\widetilde{ba}\lambda\big(a,a^{-1}ba\big)\lambda\big(b,(b,a)\big)\lambda\big((b,a),(a,b)\big)^{-1}\\ &= \lambda(b,a)^{-1}\lambda\big(a,a^{-1}ba\big)\lambda\big(b,(b,a)\big)\lambda\big((b,a),(a,b)\big)^{-1}. \qquad \square\end{aligned}$$

Corollary 1. *If $F^{\lambda}[G]$ is a twisted group algebra of positive characteristic p such that $F^{\lambda}[G]^{[2]}$ is a nil ideal then the group commutator (a,b) is an untwisted p-element for any $a,b\in G$ and the product of untwisted p-elements of G is also an untwisted p-element.*

3. Upper and lower Lie nilpotent crossed products

Let R be an associative ring. The lower Lie central series in R is defined inductively as follows:

$$\gamma_1(R)=R,\quad \gamma_2(R)=\big[\gamma_1(R),R\big],\dots,\gamma_n(R)=\big[\gamma_{n-1}(R),R\big],\dots.$$

The two-sided ideal $R^{[n]}=\gamma_n(R)R$ of R is called the *nth lower Lie power* of R. By Gupta and Levin [4], these ideals satisfy the conditions

$$R^{[m]}R^{[n]}\subseteq R^{[n+m-2]}\quad (n,m\geqslant 2). \tag{10}$$

Let us define by induction a second set of ideals in R:

$$R^{(1)}=R,\quad R^{(2)}=\big[R^{(1)},R\big]R,\quad \dots,\quad R^{(n)}=\big[R^{(n-1)},R\big]R,\quad \dots.$$

The ideal $R^{(n)}$ is called *the nth upper Lie power* of R and these ideals have the property

$$R^{(n)}R^{(m)}\subseteq R^{(n+m-1)}\quad (n,m\geqslant 1). \tag{11}$$

Recall that R is called *upper Lie nilpotent* if $R^{(n)}=0$ for some n. Similarly, the ring R with $R^{[m]}=0$ for some m is called *lower Lie nilpotent*; these classes of rings are different.

First assume that R is lower Lie nilpotent and let $R^{[t]}=0$. For $k\geqslant 3$ we choose an l such that $l(k-1)+2\geqslant t$. Then (10) forces $(R^{[k]})^l=R^{[l(k-1)+2]}=0$, so $R^{[k]}$ $(k\geqslant 3)$ is a nilpotent ideal. Note that every element of $R^{[2]}$ has the form

$$x=[a_1,b_1]r_1+[a_2,b_2]r_2+\dots+[a_s,b_s]r_s$$

for some $a_i,b_i,r_i\in R$ and

$$\big([a_j,b_j]r_j\big)^2=r_j[a_jb_j,b_j,a_j]r_j+[a_jb_j,b_j,a_j]r_j+[a_j,b_j,a_j]b_jr_j+[a_jb_j,b_j,r_j][a_jb_j,b_j]r_j. \tag{12}$$

Let R be of characteristic p. Then, by Brauer's formula,

$$x^p=\big([a_1,b_1]r_1\big)^p+\big([a_2,b_2]r_2\big)^p+\dots+\big([a_s,b_s]r_s\big)^p+z$$

for suitable $z=[c_1,d_1]+[c_2,d_2]+\dots+[c_q,d_q]\in[R,R]$ and $c_i,d_i\in R$. Now (12) ensures that $([a_j,b_j]r_j)^p\in R^{[3]}$, because $R^{[3]}$ is an ideal. Since the elements $[c_i,d_i]$ and $[c_j,d_j]$ commute modulo $R^{[3]}$, Brauer's formula implies that

$$x^{p^2}=\big([c_1,d_1]\big)^p+\big([c_2,d_2]\big)^p+\dots+\big([c_q,d_q]\big)^p\in R^{[3]}.$$

There exists s such that $x^{p^s}=0$ for every $x\in R^{[2]}$, because $R^{[3]}$ is a nilpotent ideal, so the ideal $R^{[2]}$ is nil.

Similar results are valid for upper Lie nilpotent rings R.

Lemma 5. *Let $F^\lambda[G]$ be a twisted group algebra of positive characteristic p such that either $F^\lambda[G]^{[m]} = 0$, or $F^\lambda[G]^{(m)} = 0$. If $p^t \geqslant m$, then*

(i) *b^{p^t} is a central element of G for any $b \in G$;*
(ii) *for $q \neq p$ each p-element $a \in G$ commutes with any q-element $c \in G$ and*

$$\lambda(a, c) = \lambda(c, a).$$

Proof. For a and b of G the well-known Lie commutator formula confirms

$$\left[\widetilde{a}, \widetilde{b}, p^t\right] = [\widetilde{a}, \underbrace{\widetilde{b}, \widetilde{b}, \ldots, \widetilde{b}}_{p^t}] = \left[\widetilde{a}, \widetilde{b}^{p^t}\right] = 0.$$

Hence $\widetilde{a}\widetilde{b}^{p^t} = \widetilde{b}^{p^t}\widetilde{a}$, this yields $ab^{p^t} = b^{p^t}a$ and $\lambda(a, b^{p^t}) = \lambda(b^{p^t}, a)$ for all $a \in G$. Since any q-element c of G can be written as $c = b^{p^t}$ for some b, the desired assertion follows. □

We present two different proofs for the next result.

Lemma 6. *If $F^\lambda[G]$ is a twisted group algebra of* $\mathrm{char}(F) = p$ *such that either $F^\lambda[G]^{[p^t]} = 0$ or $F^\lambda[G]^{(p^t)} = 0$, then G' is a finite p-group.*

Proof. The first proof of the lemma uses the theory of polynomial identities. As we showed before $F^\lambda[G]$ satisfies a polynomial identity, and by Lemma 3, the group G has a normal subgroup A of finite index such that $F^\lambda[A]$ is stably untwisted and the commutator subgroup A' of A is a finite p-group.

We start with some facts which will be used freely.

1. If $g, h \in G$ and $(g, h) = 1$, then $[\widetilde{g}, \widetilde{h}] = 0$. Indeed, $[\widetilde{g}, \widetilde{h}] = \widetilde{gh}(\lambda(g, h) - \lambda(h, g))$ is nilpotent which is possible only if $\lambda(g, h) - \lambda(h, g) = 0$.

2. If h is central in G, then $\widetilde{h}$ is a central element of $F^\lambda[G]$.

3. By Lemma 4, (a, b) is a p-element for all $a, b \in G$ and

$$(\widetilde{a}, \widetilde{b}) = \widetilde{(a, b)}\chi((a, b)). \tag{13}$$

Now let $F^\lambda[G]$ be upper (lower) Lie nilpotent and assume that A is abelian. If P is the maximal p-subgroup of A, then $A = P \times Q$ for a suitable central p'-subgroup Q, because G' is p-group by Lemma 4. Moreover, P belongs to the FC-center of G and assume that $C_P(g)$ has infinite index in P for some g of G. For brevity, put $\chi((g, g_i)) = \pi_i$. Clearly, $(g, g_1) \neq 1$ for suitable $g_1 \in P$ and, using the fact that $F^\lambda[A]$ is commutative, we have

$$[\widetilde{g}, \widetilde{g_1}] = \widetilde{g}\widetilde{g_1}\left(\widetilde{1} - (\widetilde{g_1}, \widetilde{g})\right) = \widetilde{g}\widetilde{g_1}\left(\widetilde{1} - \widetilde{(g_1, g)}\pi_1\right).$$

Since the subset $\{(h, g) \mid h \in P\}$ of P is infinite, there exists $g_2 \in P$ such that

$$\left(\widetilde{1} - \widetilde{(g_2, g)}\pi_2\right)\left(\widetilde{1} - \widetilde{(g_1, g)}\pi_1\right) \neq 0.$$

Clearly, $[\widetilde{g_1}, \widetilde{g_2}] = 0$ and

$$\begin{aligned}[\widetilde{g}, \widetilde{g_1}, \widetilde{g_2}] &= \left[\widetilde{g}\widetilde{g_1}\left(\widetilde{1} - \widetilde{(g_1, g)}\pi_1\right), \widetilde{g_2}\right] = [\widetilde{g}, \widetilde{g_2}]\widetilde{g_1}\left(\widetilde{1} - \widetilde{(g_1, g)}\pi_1\right) \\ &= \widetilde{g}\widetilde{g_2}\widetilde{g_1}\left(\widetilde{1} - \widetilde{(g_2, g)}\pi_2\right)\left(\widetilde{1} - \widetilde{(g_1, g)}\pi_1\right).\end{aligned}$$

As before, it is easy to see that for each n there exist $g_1, g_2, \dots, g_n$ in P such that

$$\big(\widetilde{1} - \widetilde{(g_n, g)}\pi_n\big)\big(\widetilde{1} - \widetilde{(g_{n-1}, g)}\pi_{n-1}\big)\cdots\big(\widetilde{1} - \widetilde{(g_1, g)}\pi_1\big) \neq 0$$

and

$$[\widetilde{g}, \widetilde{g_1}, \dots, \widetilde{g_n}] = \widetilde{g}\widetilde{g_n}\widetilde{g_{n-1}}\cdots\widetilde{g_1}\big(\widetilde{1} - \widetilde{(g_n, g)}\pi_n\big)\big(\widetilde{1} - \widetilde{(g_{n-1}, g)}\pi_{n-1}\big)\cdots\big(\widetilde{1} - \widetilde{(g_1, g)}\pi_1\big).$$

Clearly $[\widetilde{g}, \widetilde{g_1}, \dots, \widetilde{g_n}] \neq 0$ for each n, contradicting to the assumption that $F^\lambda[G]$ is upper (lower) Lie nilpotent. Thus for any g of G the centralizer $C_P(g)$ has finite index in P. But this imply that $C_G(g)$ has finite index in G, because Q is a central subgroup and A has finite index. Therefore we can suppose below that G is an FC-group.

Assume that the p-group G' is infinite. Then P is infinite, $b_1 = (g_1, g_2) \neq 1$ for suitable g_1, g_2 and

$$\begin{aligned}[\widetilde{g_1}, \widetilde{g_2}] &= \widetilde{g_2}\widetilde{g_1}\big((\widetilde{g_1}, \widetilde{g_2}) - \widetilde{1}\big) = \widetilde{g_2}\widetilde{g_1}\big(\widetilde{(g_1, g_2)} - \widetilde{1}\big)\\ &= \widetilde{g_2}\widetilde{g_1}\big(\chi\big((g_1, g_2)\big)\widetilde{b_1} - \widetilde{1}\big).\end{aligned}$$

Now we claim that there exists a sequence $\{g_i\}$ with the following properties: g_{2n+1}, g_{2n+2} and $b_{2n+1} = (g_{2n+1}, g_{2n+2})$ are elements of

$$C_G\big(\{g_1, g_2, \dots, g_{2n}, b_1, b_3, \dots, b_{2n-1}\}\big)$$

and

$$(g_{2n+1}, g_{2n+2}) = b_{2n+1} \notin \langle b_1, b_3, \dots, b_{2n-1}\rangle.$$

Indeed, assume that the sequence $g_1, g_2, \dots, g_{2n}$ is given. The subgroup

$$C = C_G\big(\{g_1, g_2, \dots, g_{2n}, b_1, b_3, \dots, b_{2n-1}\}\big)$$

of the FC-group G has finite index. Neumann's result [8] asserts that in an FC-group with infinite commutator subgroup G' the commutator subgroup of the subgroup C is also infinite. Then C' is a group with infinite commutator subgroup and it contains elements g_{2n+1} and g_{2n+2} such that

$$b_{2n+1} = (g_{2n+1}, g_{2n+2}) \notin \langle b_1, b_3, \dots, b_{2n-1}\rangle,$$

because G' is a locally finite p-subgroup. Recall that if $(a, b) = 1$, then $[\widetilde{a}, \widetilde{b}] = 0$ and we put $\pi_{2n+1} = \chi((g_{2n+1}, g_{2n+2}))$. Now the properties of the sequence imply that

$$[\widetilde{g_1}, \widetilde{g_2}\widetilde{g_3}] = [\widetilde{g_1}, \widetilde{g_2}]\widetilde{g_3} = \widetilde{g_3}\widetilde{g_2}\widetilde{g_1}(\pi_1\widetilde{b_1} - \widetilde{1}),$$

and

$$\begin{aligned}&[\widetilde{g_1}, \widetilde{g_2}\widetilde{g_3}, \widetilde{g_4}\widetilde{g_5}, \dots, \widetilde{g_{2n}}\widetilde{g_{2n+1}}]\\ &\quad= \widetilde{g_{2n+1}}\widetilde{g_{2n}}\cdots\widetilde{g_2}\widetilde{g_1}(\pi_{2n-1}\widetilde{b_{2n-1}} - \widetilde{1})\cdots(\pi_3\widetilde{b_3} - \widetilde{1})(\pi_1\widetilde{b_1} - \widetilde{1}).\end{aligned}$$

Clearly, this is nonzero for each n, contradicting that $F^\lambda[G]$ is upper (lower) Lie nilpotent. Therefore G' is a finite p-group, as claimed.

Finally, let A' be of order p^t. Our assertion is valid for $t = 0$ and assume its truth for $t-1$. Lemma 5 says that b^{p^t} belongs to the center of G for any $b \in G$. It follows that any conjugacy class of G, which

belongs to A', has p-power order or it is central. This yields that A' has central subgroup $L = \langle c \rangle$ of order p and by Lemma 4 there exists $\gamma \in F$ such that $\widetilde{c}\gamma$ has order p. Then $F^{\lambda}[G]$ can be realized in a second way as a twisted group algebra $F^{\tau}[G]$ with the following diagonal change of the basis

$$\overline{g} = \begin{cases} \widetilde{c}^i \gamma^i, & \text{if } g = c^i; \\ \widetilde{g} & \text{if } g \in G \setminus L \end{cases}$$

with twisting function τ. Since $\overline{c^i}$ is central, Lemma 2 asserts that

$$F^{\tau}[G]/\mathbf{I}(L) \cong F^{\tau}[G/L].$$

Of course, $F^{\tau}[G/L]$ is upper (lower) Lie nilpotent, so we can apply induction to obtain that G' is a finite p-group.

The second proof for $\mathrm{char}(F) > 2$. For any $a, b, c \in G$ the Lie commutator $[\widetilde{a}, \widetilde{b}, \widetilde{c}]$ belongs to the nilpotent ideal $F^{\lambda}_{\sigma}[G]^{[3]}$ and let x be a nonzero fixed element from the annihilator of $F^{\lambda}_{\sigma}[G]^{[3]}$. Clearly,

$$\widetilde{a}\widetilde{b}\widetilde{c}x - \widetilde{b}\widetilde{a}\widetilde{c}x = \widetilde{c}\widetilde{a}\widetilde{b}x - \widetilde{c}\widetilde{b}\widetilde{a}x, \tag{14}$$

and without loss of generality we can assume that $1 \in \mathrm{Supp}(x)$. Assume that $\mathrm{Supp}(x) = \{x_1 = 1, x_2, \ldots, x_l\}$. It is convenient to distinguish the following cases:

1. $\mathrm{Supp}(\widetilde{c}\widetilde{a}\widetilde{b}x) \cap \mathrm{Supp}(\widetilde{c}\widetilde{b}\widetilde{a}x)$ is not empty. Then $cabx_i = cbax_j$ for suitable i and j, so the commutator (a, b) can be written as $x_j x_i^{-1}$.

2. $\mathrm{Supp}(\widetilde{c}\widetilde{a}\widetilde{b}x) \cap \mathrm{Supp}(\widetilde{c}\widetilde{b}\widetilde{a}x)$ is an empty set. The length of the right side of (14) is $2l$ and thus the length of the left one is also $2l$. This means that

$$\mathrm{Supp}(\widetilde{a}\widetilde{b}\widetilde{c}x) \cap \mathrm{Supp}(\widetilde{b}\widetilde{a}\widetilde{c}x) = \emptyset.$$

Now assume that $\mathrm{Supp}(\widetilde{a}\widetilde{b}\widetilde{c}x) \cap \mathrm{Supp}(\widetilde{c}\widetilde{a}\widetilde{b}x)$ and $\mathrm{Supp}(\widetilde{a}\widetilde{b}\widetilde{c}x) \cap \mathrm{Supp}(\widetilde{b}\widetilde{a}\widetilde{c}x)$ are not empty sets. There exist $x_i, x_j, x_t, x_r \in \mathrm{Supp}(x)$ such that $abcx_i = cabx_j$, $abcx_t = cbax_r$ and the commutator (a, b) coincides with an element of the form $x_{i_1}^{\pm 1} x_{i_2}^{\pm 1} x_{i_3}^{\pm 1} x_{i_4}^{\pm 1}$. Similar statement is valid if

$$\mathrm{Supp}(\widetilde{b}\widetilde{a}\widetilde{c}x) \cap \mathrm{Supp}(\widetilde{c}\widetilde{a}\widetilde{b}x) \quad \text{and} \quad \mathrm{Supp}(\widetilde{b}\widetilde{a}\widetilde{c}x) \cap \mathrm{Supp}(\widetilde{b}\widetilde{a}\widetilde{c}x)$$

are not empty sets. It remains to consider one of the following two subcases:

2.1. $\widetilde{a}\widetilde{b}\widetilde{c}x = \widetilde{c}\widetilde{a}\widetilde{b}x$ and $\widetilde{b}\widetilde{a}\widetilde{c}x = \widetilde{c}\widetilde{b}\widetilde{a}x$. Then the commutators (ab, c) and (ba, c) can be written as $x_j x_i^{-1}$.

2.2. $\widetilde{a}\widetilde{b}\widetilde{c}x = -\widetilde{c}\widetilde{b}\widetilde{a}x$ and $\widetilde{c}\widetilde{a}\widetilde{b}x = -\widetilde{b}\widetilde{a}\widetilde{c}x$. This yields that

$$abcx_i = cbax_j \quad \text{and} \quad (ab, c) = x_j x_i^{-1}.$$

Now, if $p \neq 2$, we put $c = b^{-1}$ and then $(ab, c) = (a, b)^{-1}$. The foregoing immediately implies that there are only finitely many group commutators of the form (a, b) with $a, b \in G$; each commutator (a, b) is a p-element which has only a finite number of conjugates. Then, as well known, G' is a finite p-group. □

Theorem 1. *Let $F^{\lambda}_{\sigma}[G]$ be a crossed product of a group G and the field F of characteristic 0 or p. Then*

(1) *Any upper (lower) Lie nilpotent crossed product $F^{\lambda}_{\sigma}[G]$ is a twisted group algebra.*
(2) *The twisted group algebra $F^{\lambda}[G]$ is lower (upper) Lie nilpotent if and only if one of the following condition holds:*
   (2.i) *$F^{\lambda}[G]$ is a commutative algebra (i.e. G is abelian and the twisting function is symmetric).*

(2.ii) $\mathrm{char}(F) = p$, *G is a nilpotent group with commutator subgroup of p-power order and the untwisted p-elements of G form a subgroup. Moreover, for any $a, b \in G$ the group commutator (a, b) is an untwisted p-element and*

$$\big(\lambda(a,b)^{-1}\lambda\big(b, b^{-1}ab\big)\lambda\big(a,(a,b)\big)\lambda\big((b,a),(a,b)\big)^{-1}\big)^{-p^m} = \mu((a,b)),$$

where p^m is the order of (a, b).

Proof. Let $F^\lambda_\sigma[G]$ be a lower (upper) Lie nilpotent crossed product of characteristic 0 or p and let $H = \ker\sigma$. The twisted group algebra $F^\lambda[H]$ is a subring of the crossed product $F^\lambda_\sigma[G]$ and, by [1], for every nonzero ideal I of $F^\lambda_\sigma[G]$ we have

$$F^\lambda[H] \cap I \neq 0. \tag{15}$$

Recall that if $\mathrm{char}(F) = p$ and $\Delta(G)$ has no element of order p, then Theorem 3.5 from [7] states that $F^\lambda_\sigma[G]$ is a semiprime ring. For $\mathrm{char}\, F = 0$ according to Corollary 6 from [13] the algebra $F^\lambda[H]$ is semiprime and (15) ensures that $F^\lambda_\sigma[G]$ is also semiprime. It follows that $F^\lambda_\sigma[G]^{[3]} = 0$ and (12) implies that $([a,b]r)^2 = 0$ for all $a, b, r \in F^\lambda_\sigma[G]$. Clearly, $[a,b]F^\lambda_\sigma[G]$ is a nilpotent ideal, but this is possible only if $F^\lambda_\sigma[G]^{[2]} = 0$ and then $F^\lambda_\sigma[G]$ is a commutative twisted group algebra, as required.

Finally assume that p divides the order of some element of $\Delta(G)$ and $F^\lambda_\sigma[G]$ is a noncommutative crossed product. Then the Lie commutator $[\widetilde{a}, \widetilde{1}\alpha] = \widetilde{a}(\alpha - \alpha^{\sigma(a)})$ belongs to the nil ideal $F^\lambda_\sigma[G]^{[2]}$ for every $\alpha \in F$ and the element $\alpha - \alpha^{\sigma(a)}$ of the field F is zero, because it is nilpotent. Thus σ is trivial and so the crossed product $F^\lambda_\sigma[G]$ is a twisted group algebra.

Let $F^\lambda[G]$ be a twisted group algebra such that $\mathrm{char}(K) = p$ and $F^\lambda[G]^{[p^t]} = 0$. By Lemma 4, the commutator subgroup G' is a p-group and Lemma 6 forces that G' is finite. Furthermore, Lemma 5 says that b^{p^t} belongs to the center of G for every $b \in G$. So the quotient G/C of G by the center C is a p-group of finite exponent with finite commutator subgroup. Clearly, the orders of those conjugacy classes of the group G/C, which are contained in the finite normal p-subgroup $(G/C)'$, are p-powers. Hence $(G/C)'$ has a nontrivial central subgroup L and thus G/C is a nilpotent group. Remark that Lemma 4 confirms the remaining statements.

The converse statement was proved in [2], so the proof is complete. □

Using the notation of untwisting, Theorem 1 can be formulated as

Corollary 2. *Let $F^\lambda[G]$ be a twisted group algebra of a group G and a field F of characteristic 0 or $p > 0$. The algebra $F^\lambda[G]$ is lower (upper) Lie nilpotent if and only if one of the following conditions holds:*

(i) *$F^\lambda[G]$ is commutative*;
(ii) $\mathrm{char}(F) = p$, *G is a nilpotent group such that G' is a finite p-group and $F^\lambda[G]$ is stably untwisted.*

Proof. Let $F^\lambda[G]$ be a noncommutative lower (upper) Lie nilpotent algebra. By Theorem 1, $\mathrm{char}(F) = p$, G is a nilpotent group and G' has p-power order. As we remarked before, the nilpotent elements of $F^\lambda[G]$ form an ideal N and $F^\lambda[G]/N$ is commutative. By Lemma 4 for any g of G' we can choose $\gamma_g \in F$ such that the order of $\widetilde{g}\gamma_g$ coincides with the order of g. Then $\widetilde{g}\gamma_g - \widetilde{1}$ is nilpotent and belongs to the radical $J(F^\lambda[G'])$ of $F^\lambda[G']$. So $J(F^\lambda[G'])$ is nilpotent of codimension 1 and from

$$J\big(F^\lambda[G']\big) \cdot F^\lambda[G] = F^\lambda[G] \cdot J\big(F^\lambda[G']\big)$$

follows that $J(F^\lambda[G']) \cdot F^\lambda[G]$ is a nilpotent ideal. For all $a, b \in G$ we have

$$[\widetilde{a}, \widetilde{b}] = \widetilde{a}^{-1}\widetilde{b}^{-1}\big((\widetilde{a}, \widetilde{b}) - \widetilde{1}\big) = \widetilde{a}^{-1}\widetilde{b}^{-1}\big(\widetilde{(a,b)}\chi((a,b)) - \widetilde{1}\big), \tag{16}$$

and if p^m is the order of the group commutator (a,b) then by Lemma 4 we obtain

$$\left(\widetilde{(a,b)}\chi((a,b))-\widetilde{1}\right)^{p^m}=\mu((a,b))\chi((a,b))^{p^m}-\widetilde{1}=0.$$

Consequently $[\widetilde{a},\widetilde{b}]\in J(F^\lambda[G'])\cdot F^\lambda[G]$, which implies that

$$F^\lambda[G]/J\left(F^\lambda[G']\right)\cdot F^\lambda[G]$$

is a commutative algebra. Now Lemma 1 states that $F^\lambda[G]$ is stably untwisted.

Conversely, if $F^\lambda[G]$ is stably untwisted, then there exists an extension K of the field F such that $K^\lambda[G]$ is isomorphic to the ordinary group algebra KG via a diagonal change of basis and the result for KG has been already known. Since $F^\lambda[G]$ is a subalgebra of $K[G]$, it is lower (upper) Lie nilpotent. □

4. (n,m)-Engel crossed products

Let R be an associative ring and let n,m be fixed positive integers. If

$$\left[a,\underbrace{b^m,b^m,\dots,b^m}_{n}\right]=0$$

for all elements $a,b\in R$, then R is called (n,m)-*Engel.*

Clearly, an (n,m)-Engel ring satisfies the polynomial identity

$$\left[x,\underbrace{y^m,y^m,\dots,y^m}_{n}\right].$$

Let p^t be the smallest positive integer such that $n\leqslant p^t$ and let $m=p^lr$ with $(p,r)=1$. If $F\langle x,y\rangle$ is a noncommutative polynomial ring with indeterminates x and y over a field F of characteristic p, then

$$\left[x,\underbrace{y^m,y^m,\dots,y^m}_{p^t}\right]=\left[x,\underbrace{y^{p^lr},y^{p^lr},\dots,y^{p^lr}}_{p^t}\right]=\left[x,y^{p^{l+t}r}\right]=\left[x,\underbrace{y^r,y^r,\dots,y^r}_{p^{l+t}}\right]. \tag{17}$$

Therefore, if R is an (n,m)-Engel ring of $\mathrm{char}(R)=p>0$, then it is (p^{l+t},r)-Engel ring, too.

Theorem 2. *Let $F^\lambda_\sigma[G]$ be a crossed product of a group G and a field F of positive characteristic p.*

(1) *Any (n,m)-Engel crossed product $F^\lambda_\sigma[G]$ is a twisted group algebra.*
(2) *If $F^\lambda[G]$ is an (n,m)-Engel twisted group algebra, then either $F^\lambda[G]$ is commutative, or the following conditions hold*:
   (2.i) *G has a normal subgroup B of a finite index such that commutator subgroup B' has p-power order, the p-Sylow subgroup P/B of G/B is a normal subgroup, G/P is a finite abelian group of an exponent that divides m and P is a nilpotent subgroup.*
   (2.ii) *The untwisted p-elements of G form a subgroup and for all $a,b\in G$ the commutator (a,b) is an untwisted p-element such that*

$$\left(\lambda(a,b)^{-1}\lambda\left(b,b^{-1}ab\right)\lambda\left(a,(a,b)\right)\lambda\left((b,a),(a,b)\right)^{-1}\right)^{-p^m}=\mu((a,b)),$$

where p^m is the order of (a,b). Moreover, $F^\lambda[B]$ is stably untwisted and $|G:B||B'|$ is bounded by a fixed function of n and m.

Proof. Let $F^\lambda_\sigma[G]$ be an (n,m)-Engel crossed product of characteristic $p>0$. Then we can apply Theorem 3 of Kezlan [5] which states that $F^\lambda_\sigma[G]^{[2]}$ is a nil ideal. Hence the nilpotent elements of $F^\lambda_\sigma[G]$ form an ideal N and $F^\lambda_\sigma[G]/N$ is commutative. Clearly, $[\widetilde{a},\widetilde{1}\cdot\alpha]=\widetilde{a}(\alpha-\alpha^{\sigma(a)})$ belongs to the nil ideal $F^\lambda_\sigma[G]^{[2]}$ for every $\alpha\in F$. It follows that the element $\alpha-\alpha^\sigma(a)$ of the field F is zero, because it is nilpotent. So σ is trivial and hence $F^\lambda_\sigma[G]=F^\lambda[G]$ is a twisted group algebra.

First assume that G has no element of order p. By Corollary 2 from [12], the nil ideal $F^\lambda[G]^{[2]}$ is zero. So the twisted group algebra $F^\lambda[G]$ is commutative.

Now let G be a noncommutative group with a p-element. Without loss of generality, by (14), we can assume that $(m,p)=1$. Lemma 4 ensures that every group commutator and their products are p-elements. Therefore G' is a p-group. Choose the smallest positive integer p^t with $n\leqslant p^t$ and let $a,b\in G$. The Lie commutator formula ensures that

$$\left[\widetilde{a},\widetilde{b}^m,p^t\right]=\left[\widetilde{a},\underbrace{\widetilde{b}^m,\widetilde{b}^m,\ldots,\widetilde{b}^m}_{p^t}\right]=\left[\widetilde{a},\widetilde{b}^{mp^t}\right]=0.$$

This yields $\widetilde{a}\widetilde{b}^{mp^t}=\widetilde{b}^{mp^t}\widetilde{a}$. Hence b^{mp^t} is central for any $b\in G$. By Lemma 3, G has a normal subgroup A of finite index such that commutator subgroup A' has p-power order. Our aim is to prove that if B is the subgroup generated by A and the center of G, then $F^\lambda[B]$ is stably untwisted. Of course, B' is a finite p-group, so the ideal $I=J(F^\lambda[B'])\cdot F^\lambda[B]$ is nilpotent, because the nilpotent elements of $F^\lambda[G]$ form an ideal. Indeed, if p^r is the order of the commutator (c,d) of B, then Lemma 4 ensures that $((\widetilde{c},\widetilde{d})-1)^{p^r}=0$ and

$$(\widetilde{c},\widetilde{d})-1\in J\left(F^\lambda[B']\right)\cdot F^\lambda[B].$$

Then (16) asserts that $[\widetilde{c},\widetilde{d}]\in J(F^\lambda[B'])\cdot F^\lambda[B]$, and $F^\lambda[B]/J(F^\lambda[B'])\cdot F^\lambda[G]$ is a commutative algebra. Now, by Lemma 1, $F^\lambda_\sigma[B]$ is stably untwisted, as required.

Now b^{mp^t} is central for any $b\in G$ and it belongs to B and the p-Sylow subgroup P/B of G/B is normal, because the commutator subgroup of G is a p-group. Then, by the Schur–Zassenhaus's theorem (6.2.1 theorem of [3]), G/B is a semidirect product of the finite p-Sylow subgroup P/B and a finite abelian group M/B whose exponent divides m. Since B^{mp^t} is a central subgroup, there remains to show that P/B^{mp^t} is nilpotent. To prove this we proceed by induction on the order of the commutator subgroup L of B/B^{mp^t}. First suppose that $L'=\langle 1\rangle$. The abelian group B/B^{mp^t} is a direct product of the p-subgroup D and a subgroup H whose exponent divides m. The finite p-group P/B acts by conjugation on these subgroups. Of course for every $d\in P/B^{mp^t}$ and $h\in H$ the p'-element $d^{-1}hd$ coincides with $h(h,d)$ and the commutator (h,d) is a p-element. Therefore, the action of P/B on H is trivial and P/B acts as a finite p-group on D.

Define the subgroups $D_1=(P/B^{mp^t},D)$ and $D_j=(P/B^{mp^t},D_{j-1})$ for $j>1$. By Lemma V.4.1 [14], this sequence of subgroups is such that $D_l=\langle 1\rangle$ for some l. But the finite p-group P/B is nilpotent, so a suitable term K_s of the lower central series of the group $K=P/B^{mp^t}$ is contained in B/B^{mp^t} and

$$K_{s+1}=(K,K_s)\subseteq\left(K,B/B^{mp^t}\right)=(K,D\times H)\subseteq(K,D)=D_1,$$

because H is central in P/B^{mp^t}. Similarly, we conclude that $K_{s+i}\subseteq D_i$ and the group P/B^{mp^t} must be nilpotent, because $D_l=1$.

Finally, assume that $L'\neq\langle 1\rangle$. The orders of those conjugacy classes of the group P/B^{mp^t}, which are contained in the finite normal p-subgroup L', are p-powers. Hence L' has a nontrivial central subgroup, and by induction on the order of L', we see that P/B^{mp^t} is nilpotent. Consequence, P is nilpotent as asserted. □

Corollary 3. *The twisted group algebra $F^\lambda[G]$ of positive characteristic p is n-Engel if and only if either $F^\lambda[G]$ is commutative, or the following conditions hold*:

(i) *G is a nilpotent group with a normal subgroup B of a finite p-power index, B' is a finite p-group and $F^\lambda[B]$ is stably untwisted*;
(ii) *the untwisted p-elements of G form a subgroup, the commutator (a,b) is an untwisted p-element for all $a,b \in G$ and*

$$\big(\lambda(a,b)^{-1}\lambda\big(b,b^{-1}ab\big)\lambda\big(a,(a,b)\big)\lambda\big((b,a),(a,b)\big)^{-1}\big)^{-p^m} = \mu((a,b)), \tag{18}$$

where p^m is the order of (a,b).

Proof. There remains to prove only the sufficiency of these conditions, and as before, we can assume that F is an algebraically closed field of characteristic p.

Let G be a nilpotent group with a normal abelian subgroup B of a finite p-power index. Then for any a,b of B the twist of $\widetilde{(a,b)}$ is equal to 1. So (18) asserts that $F^\lambda[B]$ is a commutative algebra. Now we try to adapt the method of the proof of Theorem V.6.1 from [14]. For this we need additional information about the nilpotent groups G with a normal abelian subgroup B of index p^s and about the twisted group algebras. Certainly, $(B, G^{p^s}) = 1$ and Lemma V.6.2 from [14] implies that $(B,G)^{p^m} = 1$ for some $m \geqslant s$. It follows that

$$\big(G^{p^{s+m}}, G\big) \subseteq \big(B^{p^m}, G\big) = \langle 1\rangle.$$

Hence, if $t = s+m$, then g^{p^t} is central in G and (18) confirms that $[\widetilde{g^{p^t}}, \widetilde{h}] = 0$ for any $h \in G$. So $\widetilde{g}^{p^t}$ is central. Now for every $y = \sum_{g\in G} c_g\widetilde{g}$ of $F^\lambda[G]$ we have that $\sum_{g\in G} c_g^{p^t}\widetilde{g}^{p^t}$ is central in $F^\lambda[G]$ and

$$y^{p^t} = \sum_{g\in G} c_g^{p^t}\widetilde{g}^{p^t} + y_1,$$

for suitable $y_1 \in [F^\lambda[G], F^\lambda[G]]$. For all $a,b \in G$ we have

$$[\widetilde{a},\widetilde{b}] = \widetilde{a}^{-1}\widetilde{b}^{-1}\big(\widetilde{(a,b)} - \widetilde{1}\big) = \widetilde{a}^{-1}\widetilde{b}^{-1}\big(\widetilde{(a,b)}\chi((a,b)) - \widetilde{1}\big) \tag{19}$$

and if p^m is the order of (a,b) then by (18),

$$\big(\widetilde{(a,b)}\chi((a,b)) - \widetilde{1}\big)^{p^m} = \mu((a,b))\chi((a,b))^{p^m} - \widetilde{1} = 0.$$

Consequently $[F^\lambda[G], F^\lambda[G]] \subseteq F^\lambda[G]\cdot J(F^\lambda[G'])$. Since the subgroup $D = (G,B)$ is normal in G and B/D is central in G/D of index $|G/D : B/D| = p^s$, it follows that G'/D is a finite group of order p^l. By Theorem 1.6 [10], we have

$$\big(F^\lambda[G]\cdot J\big(F^\lambda[G']\big)\big)^{p^l} \subseteq J\big(F^\lambda[D]\big)\cdot F^\lambda[G].$$

It follows that $y_1^{p^l} = \sum_{i=1}^{p^s} z_i\widetilde{t_i}$, where $t_1 = 1, t_2, \ldots, t_{p^s}$ is a transversal of B in G. Furthermore,

$$z_i \in F^\lambda[B] \cap J\big(F^\lambda[D]\big) \quad \big(i = 1,2,\ldots,p^s\big)$$

are nilpotent elements with nilpotency index at most p^m and each of these elements commute, because B is abelian. The inner automorphisms of $\widetilde{G}$ induces a finite group of automorphisms T on $\widetilde{B}$. The action of T on $z_1, z_2, \ldots, z_{p^s}$ produces only finitely many images and denote by L the subring

of the commutative algebra $F^\lambda[B]$ generated by these images. Of course L is nilpotent, its nilpotency index is at most $p^r = p^{m+s+1}|T|$ which does not depend on L. Clearly, $y_1^{p^{l+r}} = 0$, and by the foregoing,

$$y^{p^{t+l+r}} = \sum_{g\in G} c_g^{p^{t+l+r}} \widetilde{g}^{p^{t+l+r}}$$

is central in FG. By the Lie commutator identity we obtain that

$$\left[x, y, p^{t+l+r}\right] = \left[x, y^{p^{t+l+r}}\right] = 0$$

and FG is Lie p^{t+l+r}-Engel.

Finally, let B' be of order p^t. Our assertion is valid for $t = 0$, assume its truth for $t-1$. The normal subgroup B' of a nilpotent group contains a central cyclic subgroup $L = \langle c \mid c^p = 1\rangle$. Now we take $F^\lambda G$ and make the following change of basis:

$$\bar{g} = \begin{cases} \widetilde{g}, & \text{if } g \in G \setminus L; \\ \widetilde{c^i}\sqrt{\mu(c)^{-i}} & \text{if } g = c^i \in L. \end{cases}$$

Then $\rho(c^i, c^j) = 1$ and $\bar{c}^p = \widetilde{1}$. By Lemma 2, $F^\lambda[G]$ can be realized in a second way as a twisted group ring with new basis $\{\bar{\bar{g}}\}$ and a twisting function $\tau(g,h)$ which satisfies $\tau(c^k, g) = 1$. Clearly, $\tau(c^k, g) = \tau(g, c^k) = 1$. By (5), we have that

$$F^\lambda G/\mathbf{I}(L) \cong F^\tau[G/L]$$

and by (4), $\tau(g_iH, g_jH) = \lambda(g_i, g_j)$. Hence the twisting function $\tau(a,b)$ satisfies the required conditions, and by the inductive hypothesis, $F^\mu[G/L]$ is p^m-Engel for some m. It follows that $[x, y, p^m] = [x, y^{p^m}] \in \mathbf{I}(L)$ for all $x, y \in F^\lambda G$, so $[x, y^{p^m}] = (\bar{c} - 1)z$ for some $z \in F^\lambda G$. Since $\bar{c}$ is central, we have

$$\left[x, y^{p^m}, p\right] = \left[x, \underbrace{y^{p^m}, y^{p^m}, \ldots, y^{p^m}}_{p}\right] \in (\bar{c} - 1)^p F^\lambda G = 0.$$

This implies that $[x, y, p^{m+1}] = 0$ and the proof is complete. □

References

[1] A.A. Bovdi, Cross products of semigroups and rings, Sibirsk. Mat. Zh. 4 (1963) 481–499.
[2] A.A. Bovdi, K.K. Kolikov, Lie nilpotency and ideals of crossed products, Serdica 15 (4) (1990) 275–286, 1989.
[3] D. Gorenstein, Finite Groups, second ed., Chelsea Publishing Co., New York, 1980.
[4] N. Gupta, F. Levin, On the Lie ideals of a ring, J. Algebra 81 (1) (1983) 225–231.
[5] T.P. Kezlan, Some rigs with nil commutator ideals, Michigan Math. J. 12 (1965) 105–111.
[6] C.-H. Liu, On units of twisted group algebras, J. Algebra 250 (1) (2002) 271–282.
[7] S. Montgomery, D.S. Passman, Crossed products over prime rings, Israel J. Math. 31 (3–4) (1978) 224–256.
[8] B.H. Neumann, Groups with finite classes of conjugate elements, Proc. London Math. Soc. (3) 1 (1951) 178–187.
[9] I.B.S. Passi, D.S. Passman, S.K. Sehgal, Lie solvable group rings, Canad. J. Math. 25 (1973) 748–757.
[10] D.S. Passman, Radicals of twisted group rings, Proc. London Math. Soc. (3) 20 (1970) 409–437.
[11] D.S. Passman, Group rings satisfying a polynomial identity. II, Pacific J. Math. 39 (1971) 425–438.
[12] D.S. Passman, Trace methods in twisted group algebras, Proc. Amer. Math. Soc. 129 (4) (2001) 943–946.
[13] D.S. Passman, Twisted group algebras satisfying a generalized polynomial identity, in: Special Issue Dedicated to Alexei Ivanovich Kostrikin, Comm. Algebra 29 (9) (2001) 3683–3710.
[14] S.K. Sehgal, Topics in Group Rings, Monogr. Textbooks Pure Appl. Math., vol. 50, Marcel Dekker Inc., New York, 1978.