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The space of unipotently supported class functions on a finite reductive group

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1. Introduction

Let $\mathbf{G}$ be a connected reductive algebraic group over an algebraic closure $\mathbb{F}$ of the finite field $\mathbb{F}_q$ of q elements; assume $\mathbf{G}$ has an $\mathbb{F}_q$-structure with associated Frobenius endomorphism F and let ℓ be a prime distinct from the characteristic of $\mathbb{F}_q$. In [5, Section 7.1] and [6] we outlined a program for the determination of the irreducible $\overline{\mathbb{Q}}_\ell$-characters of the finite group $\mathbf{G}^F$, which showed that the problem may be largely reduced (by induction) to an explicit determination of the Lusztig restrictions ${}^*R^{\mathbf{G}}_{\mathbf{M}}(\chi)$ of all the irreducible characters χ of $\mathbf{G}^F$, for all rational Levi subgroups $\mathbf{M}$ of $\mathbf{G}$. Here, and throughout this paper, the word "rational" means "stable under the action of F". As shown in [6], this problem may be addressed through the determination of the Lusztig restrictions ${}^*R^{\mathbf{G}}_{\mathbf{M}}(\Gamma_u)$, where Γ_u is the generalized Gelfand–Graev character corresponding to the $\mathbf{G}^F$-conjugacy class of the rational unipotent element $u \in \mathbf{G}^F$.

Now the characters Γ_u are examples of class functions on $\mathbf{G}^F$ which vanish outside the unipotent set. Such functions form a vector space over $\overline{\mathbb{Q}}_\ell$, which we denote by $\mathcal{C}_{\mathrm{uni}}(\mathbf{G}^F)$; it is the space of unipotently supported class functions on $\mathbf{G}$. The Γ_u form a basis of this space, and our strategy in this work will be to determine the map ${}^*R^{\mathbf{G}}_{\mathbf{M}} : \mathcal{C}_{\mathrm{uni}}(\mathbf{G}^F) \to \mathcal{C}_{\mathrm{uni}}(\mathbf{M}^F)$ explicitly. We shall use Lusztig's orthogonal decomposition

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of the space $\mathcal{C}_{\mathrm{uni}}(\mathbf{G}^F)$ into summands corresponding to “rational blocks” (see below) and determine ${}^*R_{\mathbf{M}}^{\mathbf{G}}$ on each block generically, i.e. in terms of Weyl group data which is associated with the block. In particular, we obtain a simple expression for the Lusztig restriction of generalized Green functions. We then express the generalized Gelfand–Graev characters in terms of this basis to describe their Lusztig restriction. In [6] we computed ${}^*R_{\mathbf{M}}^{\mathbf{G}}$ of the generalized Gelfand–Graev character which corresponds to a regular unipotent class. In this work, we apply the general method to carry out the corresponding computation explicitly in the subregular case.

Our general result on ${}^*R_{\mathbf{M}}^{\mathbf{G}}$ of generalized Gelfand–Graev characters (Proposition 6.10) essentially reduces this computation to the two problems of finding the Poincaré polynomials $\widetilde{P}_{\iota,\kappa}$ of certain intersection cohomology complexes on closures of unipotent classes, and to the computation of induction-restriction tables for twisted characters of Weyl groups. In Section 8 we also prove a result (see Theorem 8.1) which reduces these computations in the case of SL_n to the case of $\mathrm{GL}_{n'}$, for various n'. These investigations are part of our strategy of reducing the computation of character values to the case of “high” unipotent classes in the usual partial order.

The first five sections of this paper consist largely of a recasting of the of work of Lusztig, which may be found in [11,12,14], in a form which permits practical computation. They also contain several orthogonality relations for Green functions and their generalizations, which are proved by relating the inner product in $\mathcal{C}_{\mathrm{uni}}(\mathbf{G}^F)$ to the inner product of twisted class functions on a Weyl group. In Section 6, we prove orthogonality relations for the generalized Gelfand–Graev characters in the same way, in addition to determining their Lusztig restriction. By and large we maintain the notation of [6]. We shall rely on the context to distinguish between the Frobenius endomorphism F of an $\mathbb{F}_q$-group $\mathbf{G}$ and the automorphisms, also denoted F, which are induced by F on reflection groups (such as the Weyl group) which are associated with $\mathbf{G}$. Throughout this work we shall freely use the character theory of cosets of a finite group, for which the reader is referred to [3, (0.4)] or [8]. Characters of cosets are also sometimes known as “twisted class functions”.

2. Preliminaries

Let $\iota = (C, \zeta)$ be a pair consisting of a unipotent class of $\mathbf{G}$ and an irreducible $\mathbf{G}$-equivariant $\overline{\mathbb{Q}}_\ell$-local system ζ on it; then C will be called the *support* of ι and sometimes denoted C_ι. If we fix a non-trivial additive character χ_0 of the prime field $\mathbb{F}_p$ of $\mathbb{F}_q$, as in [6, 1.6] we may define a generalized Gelfand–Graev function Γ_ι associated with ι; one of our objectives here is to express Lusztig restrictions of generalized Gelfand–Graev characters in terms of generalized Gelfand–Graev characters.

As in [6], if the pair ι is F-stable, we shall follow Lusztig [12, 24.1–24.2] in making a specific choice of an isomorphism $\sigma : F^*\zeta \xrightarrow{\sim} \zeta$, and we denote by $\mathcal{Y}_\iota$ the characteristic function of ζ which corresponds to σ, and by $\mathcal{X}_\iota$ the characteristic function of the intersection cohomology complex of ζ (for $u \in C^F$, we have $\mathcal{X}_\iota(u) = \mathcal{Y}_\iota(u)$). The set $\mathcal{P}$ of all pairs ι is partitioned into “blocks” $\mathcal{I}$, each of which has an associated cuspidal datum $(\mathbf{L}, \iota_0 = (C_0, \zeta_0))$ where $\mathbf{L}$ is a Levi subgroup of some parabolic subgroup of $\mathbf{G}$, which is unique up to $\mathbf{G}$-conjugacy. If the block concerned is rational, then as explained in [6, 1.4], both $\mathbf{L}$ and the parabolic subgroup may be assumed to be rational. The pairs in the block $\mathcal{I}$ are in bijection with the irreducible characters of the group $W_{\mathbf{G}}(\mathbf{L}) = N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$, which is a Coxeter group. If $\mathcal{I}$ is a rational block and φ_ι is the character associated in this way to $\iota = (C, \zeta) \in \mathcal{I}^F$, then an extension $\tilde{\varphi}_\iota$ of φ_ι to $W_{\mathbf{G}}(\mathbf{L}) \rtimes \langle F \rangle$ determines an isomorphism $\sigma : F^*\zeta \xrightarrow{\sim} \zeta$ as above. In this work, we shall always choose $\tilde{\varphi}_\iota$ to be the "preferred extension" described in [12, 17.2] (as Lusztig does in [12, 24.2]).

The functions $\mathcal{Y}_\iota$ form a basis of the space of unipotently supported class functions on $\mathbf{G}^F$ as ι runs over the set $\mathcal{P}^F$ of all rational pairs. For a given block $\mathcal{I}$, the functions $\mathcal{X}_\iota$ form another basis of the space spanned by $\{\mathcal{Y}_\iota\}_{\iota \in \mathcal{I}^F}$, and if we write $\mathcal{X}_\iota = \sum_\kappa P_{\kappa,\iota} \mathcal{Y}_\kappa$ then the $P_{\kappa,\iota}$ are polynomials in q with integer coefficients We have $P_{\kappa,\iota} = 0$ unless $C_\kappa \subset \overline{C}_\iota$ and if $C_\kappa = C_\iota$ then $P_{\kappa,\iota} = \delta_{\kappa,\iota}$ (see, e.g., [14, 6.5]). We will assume from now on that the pairs ι have been totally ordered in such a way that $C_\kappa \subset \overline{C}_\iota \Rightarrow \kappa \leqslant \iota$. Then the matrix $(P_{\kappa,\iota})$ is upper unitriangular.

Set $\widetilde{\mathcal{X}}_\iota = q^{c_\iota} \mathcal{X}_\iota$ and $\widetilde{\mathcal{Y}}_\iota = q^{c_\iota} \mathcal{Y}_\iota$ where $c_\iota = \frac{1}{2}(\operatorname{codim} C_\iota - \dim Z_{\mathbf{L}})$. Then we have $\widetilde{\mathcal{X}}_\iota = \sum_\kappa \widetilde{P}_{\kappa,\iota} \widetilde{\mathcal{Y}}_\kappa$, where $\widetilde{P}_{\kappa,\iota} = q^{c_\iota - c_\kappa} P_{\kappa,\iota}$.

Remark 2.1. We shall speak below of "complex conjugation" in the field $\overline{\mathbb{Q}}_\ell$, denoted by $a \mapsto \bar{a}$. This is justified by noting that $\overline{\mathbb{Q}}_\ell$ is abstractly isomorphic to $\mathbb{C}$. In practice, we shall apply this notion almost exclusively to the subfield of $\overline{\mathbb{Q}}_\ell$ which is generated by all roots of unity, on which conjugation is uniquely defined since it fixes $\mathbb{Q}$ and inverts roots of unity. We therefore speak of "real" values (meaning fixed by conjugation) and "complex conjugates" in this context. The space $\mathcal{C}_{\text{uni}}(\mathbf{G}^F)$ is then an inner product space with Hermitian form defined by

$$\langle f, g \rangle_{\mathbf{G}^F} = |\mathbf{G}|^{-1} \sum_{x \in \mathbf{G}^F} f(x) \overline{g(x)}.$$

Remark 2.2. The cuspidal datum $(\mathbf{L}, \iota_0)$ defines a unique block $\mathcal{I}_{\mathbf{M}}$ of any Levi subgroup $\mathbf{M}$ of $\mathbf{G}$ which contains a $\mathbf{G}$-conjugate of $\mathbf{L}$. Assume $\mathbf{M}$ and $\mathbf{L}$ rational, and let $\mathbf{L}' = \operatorname{Int} g(\mathbf{L})$ $(:= g\mathbf{L}g^{-1})$ be a conjugate of $\mathbf{L}$ which is rational and contained in $\mathbf{M}$; let $\mathbf{M}_0 \supset \mathbf{L}$ be the conjugate $\operatorname{Int} g^{-1}(\mathbf{M})$ of $\mathbf{M}$. Define $w \in W_{\mathbf{G}}(\mathbf{L})$ by $\dot{w} = g^{-1}F(g) \in N_{\mathbf{G}}(\mathbf{L})$. Then $(\mathbf{L}', F)$ is conjugate to $(\mathbf{L}, \dot{w}F)$ and $\mathbf{M}_0$ is $\dot{w}F$-stable; moreover we may identify (via $\operatorname{Int} g^{-1}$) $(\mathbf{M}, F)$ with $(\mathbf{M}_0, \dot{w}F)$ and hence $(W_{\mathbf{M}}(\mathbf{L}'), F)$ with $(W_{\mathbf{M}_0}(\mathbf{L}), wF)$, $\mathcal{C}_{\text{uni}}(\mathbf{M}^F)$ with $\mathcal{C}_{\text{uni}}(\mathbf{M}_0^{\dot{w}F})$ and $(\mathcal{I}_{\mathbf{M}}, F)$ with $(\mathcal{I}_{\mathbf{M}_0}, wF)$. A particular case of this occurs when $\mathbf{M}_0 = \mathbf{L}$, when we refer to the twisted version of $\mathbf{L}$ as $\mathbf{L}_w$ (for $w \in W_{\mathbf{G}}(\mathbf{L})$). The cuspidal pair ι_0 of $\mathbf{L}$ is taken by $\operatorname{Int}(g)$ to a cuspidal pair of $\mathbf{L}_w$. The corresponding characteristic function on $\mathbf{L}_w^F$ is likewise taken by g^{-1} to a function on $\mathbf{L}^{\dot{w}F}$, which we denote by $\mathcal{X}_{\iota_0,w}$.

We recall that Lusztig induction $R_{\mathbf{M}}^{\mathbf{G}}$ has an easy description in terms of the functions $\mathcal{X}_\iota$, which applies with some restrictions on p and q. The results of this paper will depend on this, and hence we shall assume, sometimes without explicit mention, for the whole of our work that (cf. [6, 3.1]) the characteristic p is good for $\mathbf{G}$ and that $q > q_0(\mathbf{G})$, a constant which depends only on the Dynkin diagram of $\mathbf{G}$.

Proposition 2.3. *Assume p good and q sufficiently large, and that* **M** *contains a rational conjugate* $\mathbf{L}_w$ *of* **L** *as in Remark* 2.2. *Assume* (*as we may, by the above discussion*) *that* $\mathbf{L}_w$ *is a split Levi subgroup of* **M**. *Then for* $\iota \in \mathcal{I}_{\mathbf{M}}^F$, *we have*:

(i) $R_{\mathbf{M}}^{\mathbf{G}}(\widetilde{\mathcal{X}}_\iota) = \sum_{\kappa \in \mathcal{I}^F} \langle \tilde{\varphi}_\iota, \operatorname{Res}_{W_{\mathbf{M}_0}(\mathbf{L}).wF}^{W_{\mathbf{G}}(\mathbf{L}).F} \tilde{\varphi}_\kappa \rangle_{W_{\mathbf{M}_0}(\mathbf{L}).wF} \widetilde{\mathcal{X}}_\kappa$, *where* $R_{\mathbf{M}}^{\mathbf{G}}$ *is the Lusztig induction functor,*

(ii) $\langle \tilde{\varphi}_\iota, \operatorname{Res}_{W_{\mathbf{M}_0}(\mathbf{L}).wF}^{W_{\mathbf{G}}(\mathbf{L}).F} \tilde{\varphi}_\kappa \rangle_{W_{\mathbf{M}_0}(\mathbf{L}).wF} = 0$ *unless* $\overline{C_\iota} \subset \overline{C_\kappa} \subset \overline{\operatorname{Ind}_{\mathbf{M}}^{\mathbf{G}} C_\iota}$.

Proof. Assertion (i) is in [6, 3.3]. Let us prove (ii). For the rightmost inclusion recall that, from the definition of the induction of perverse sheaves, only pairs κ with support smaller than that of the class induced from the support of ι can have non zero coefficient in $R_{\mathbf{M}}^{\mathbf{G}}(\widetilde{\mathcal{X}}_\iota)$. To prove the other inclusion, first notice that if $\langle \tilde{\varphi}_\iota, \operatorname{Res}_{W_{\mathbf{M}_0}(\mathbf{L}).wF}^{W_{\mathbf{G}}(\mathbf{L}).F} \tilde{\varphi}_\kappa \rangle_{W_{\mathbf{M}_0}(\mathbf{L}).wF}$ is non-zero then so is $\langle \varphi_\iota, \operatorname{Res}_{W_{\mathbf{M}_0}(\mathbf{L})}^{W_{\mathbf{G}}(\mathbf{L})} \varphi_\kappa \rangle_{W_{\mathbf{M}_0}(\mathbf{L})}$. But it follows from formula (II) in [16, 1.2] that the latter inner product is zero unless there exists a representative of C_κ in $C_\iota.\mathbf{U}$ where $\mathbf{U}$ is the unipotent radical of a parabolic subgroup admitting **M** as a Levi component. This in turn implies $C_\iota \subset \overline{C_\kappa}$ by [5, 5.8]. □

Remark 2.4. We shall often have a situation where **M** is a rational Levi subgroup of **G** which contains a rational conjugate $\mathbf{L}_w$ of **L**, as in Remark 2.2. In this situation we shall consistently assume $w \in W_{\mathbf{G}}(\mathbf{L})$ to have been chosen so that $\mathbf{L}_w$ is split in **M**, i.e., is contained in a rational parabolic subgroup of **M**. In this case $w \in W_{\mathbf{G}}(\mathbf{L})$ is determined up to F-conjugacy in $W_{\mathbf{G}}(\mathbf{L})$ and the function $R_{\mathbf{L}_w}^{\mathbf{G}}(\mathcal{X}_{\iota_0,w})$ is well defined (see [6, 3.2 and 3.3(1)]). This is implicit in the statement and proof of Proposition 2.3.

3. Generalized Green functions and Lusztig restriction

In this section we shall interpret Lusztig induction and restriction in terms of ordinary induction and restriction of twisted class functions on cosets of parabolic subgroups of Coxeter groups. This will be done by defining a linear isomorphism between the spaces of twisted class functions on $W_{\mathbf{G}}(\mathbf{L})$ and a certain subspace of the space of unipotently supported functions. Under this map, the (normalized) characteristic functions of the F-classes of $W_{\mathbf{G}}(\mathbf{L})$ correspond to functions we define as "generalized Green functions." These are analogues of the ordinary Green functions (the latter corresponding to the "principal block," which is the unique block for which $\mathbf{L} = \mathbf{T}$, a maximal torus of **G**) which constitute a basis of the space of unipotently supported class functions. In order to compute their Lusztig restriction, we shall relate the generalized Gelfand–Graev characters to these.

For the whole of this section, we fix a rational cuspidal datum $(\mathbf{L}, \iota_0)$, where we may assume that **L** is split, i.e. is contained in a rational parabolic subgroup of **G**. Let $\mathcal{C}(W_{\mathbf{G}}(\mathbf{L}).F)$ be the space of $W_{\mathbf{G}}(\mathbf{L})$-invariant functions (i.e. class functions) on $W_{\mathbf{G}}(\mathbf{L}).F$ and recall that $\mathcal{C}_{\text{uni}}(\mathbf{G}^F)$ is the space of unipotently supported class functions on $\mathbf{G}^F$. For each $w \in W_{\mathbf{G}}(\mathbf{L})$, we fix a w-twisted rational conjugate $\mathbf{L}_w$ of **L** as in Remarks 2.2, 2.4, and $\widetilde{\mathcal{X}}_{\iota_0,w} \in \mathcal{C}_{\text{uni}}(\mathbf{L}_w^F)$ is the class function on $\mathbf{L}^F$ (see Remarks 2.2 and 2.4) associated with ι_0.

Definition 3.1. Let $\mathcal{C}_{\mathcal{I}}(\mathbf{G}^F)$ be the subspace of $\mathcal{C}_{\mathrm{uni}}(\mathbf{G}^F)$ spanned by the functions $\{\mathcal{Y}_\iota \mid \iota \in \mathcal{I}^F\}$.

(i) Define the linear isomorphism $Q^{\mathbf{G}}$ from $\mathcal{C}(W_{\mathbf{G}}(\mathbf{L}).F)$ to $\mathcal{C}_{\mathcal{I}}(\mathbf{G}^F)$ by $Q^{\mathbf{G}}(\tilde{\varphi}_\iota) = \widetilde{\mathcal{X}}_\iota$.
(ii) For $w \in W_{\mathbf{G}}(\mathbf{L})$ define $\gamma_{wF} \in \mathcal{C}(W_{\mathbf{G}}(\mathbf{L}).F)$ by

$$\gamma_{wF}(vF) = \begin{cases} 0, & \text{if } vF \text{ is not } W_{\mathbf{G}}(\mathbf{L})\text{-conjugate to } wF, \\ |C_{W_{\mathbf{G}}(\mathbf{L})}(wF)|, & \text{otherwise.} \end{cases}$$

(iii) The *generalized Green function* $Q^{\mathbf{G}}_{wF}$ is defined by $Q^{\mathbf{G}}_{wF} = Q^{\mathbf{G}}(\gamma_{wF})$.

Note that since the (distinct) γ_{wF} form a basis of $\mathcal{C}(W_{\mathbf{G}}(\mathbf{L}).F)$, the generalized Green functions $Q^{\mathbf{G}}_{wF}$ form a basis of $\mathcal{C}_{\mathcal{I}}(\mathbf{G}^F)$.

We shall omit the superscript in $Q^{\mathbf{G}}$ and $Q^{\mathbf{G}}_{wF}$ when there is no ambiguity.

Proposition 3.2. *We have* $Q_{wF} = R^{\mathbf{G}}_{\mathbf{L}_w} \widetilde{\mathcal{X}}_{\iota_0,w}$.

Proof. Since the $\tilde{\varphi}_\iota$ form an orthonormal basis of $\mathcal{C}(W_{\mathbf{G}}(\mathbf{L}).F)$, and $\langle \theta, \gamma_{wF}\rangle_{W_{\mathbf{G}}(\mathbf{L}).F} = \theta(wF)$ for any $\theta \in \mathcal{C}(W_{\mathbf{G}}(\mathbf{L}).F)$, we have

$$\gamma_{wF} = \sum_{\iota \in \mathcal{I}^F} \langle \tilde{\varphi}_\iota, \gamma_{wF}\rangle_{W_{\mathbf{G}}(\mathbf{L}).F} \tilde{\varphi}_\iota = \sum_{\iota \in \mathcal{I}^F} \tilde{\varphi}_\iota(wF)\tilde{\varphi}_\iota,$$

whence by linearity

$$Q_{wF} = \sum_{\iota \in \mathcal{I}^F} \tilde{\varphi}_\iota(wF)\widetilde{\mathcal{X}}_\iota. \tag{3.1}$$

But by [6, 3.1] we have

$$\widetilde{\mathcal{X}}_\iota = \left|W_{\mathbf{G}}(\mathbf{L})\right|^{-1} \sum_{v \in W_{\mathbf{G}}(\mathbf{L})} \tilde{\varphi}_\iota(vF) R^{\mathbf{G}}_{\mathbf{L}_v}\left(\widetilde{\mathcal{X}}_{\iota_0,v}\right). \tag{3.2}$$

Now in (3.2), the summand corresponding to $w \in W_{\mathbf{G}}(\mathbf{L})$ depends only on the $W_{\mathbf{G}}(\mathbf{L})$-class of wF. To see this, observe that the function $\widetilde{\mathcal{X}}_{\iota_0,v}$ is invariant under conjugation by $N_{\mathbf{G}}(\mathbf{L}_v)^F$, so that $R^{\mathbf{G}}_{\mathbf{L}_v}\widetilde{\mathcal{X}}_{\iota_0,v}$ depends only on the $\mathbf{G}^F$-class of $\mathbf{L}_v$, which is parametrized by the W-class of the coset $W_{\mathbf{L}}.vF$, or by the $W_{\mathbf{G}}(\mathbf{L})$-class of the element $vF \in W_{\mathbf{G}}(\mathbf{L}).F$.

Since the $\tilde{\varphi}_\iota$ take real values, the second orthogonality relation for them reads

$$\sum_\iota \tilde{\varphi}_\iota(wF)\tilde{\varphi}_\iota(vF) = \begin{cases} 0, & \text{if } vF \text{ is not } W_{\mathbf{G}}(\mathbf{L})\text{-conjugate to } wF, \\ |C_{W_{\mathbf{G}}(\mathbf{L})}(wF)|, & \text{otherwise.} \end{cases}$$

Substituting (3.2) into (3.1) and using this relation, the result follows. □

It follows from this proposition that our generalized Green functions are the same as those in [12, 8.3.1], since $q^{c_{\iota_0}}\mathcal{X}_{\iota_0}$ is the restriction to the unipotent elements of the characteristic function of the perverse sheaf denoted by $\mathrm{IC}(\overline{\Sigma},\mathcal{E})[\dim(\Sigma)]$ in [12, 8.2] and for cuspidal local systems, Lusztig's induction coincides with the induction of perverse sheaves by [13].

Both $\mathcal{C}(W_{\mathbf{G}}(\mathbf{L}).F)$ and $\mathcal{C}_{\mathcal{I}}(\mathbf{G}^F)$ have natural structures as non-degenerate inner product spaces. Although $Q^{\mathbf{G}}$ is not an isometry, its effect on scalar products can be computed.

Definition 3.3. Define the function $\mathcal{Z}_{\mathbf{L}}\in\mathcal{C}(W_{\mathbf{G}}(\mathbf{L}).F)$ by $\mathcal{Z}_{\mathbf{L}}(wF)=|Z_{\mathbf{L}}^{0wF}|=|Z_{\mathbf{L}_w}^{0F}|$.

Proposition 3.4. *We have, for any two functions $\theta,\phi\in\mathcal{C}(W_{\mathbf{G}}(\mathbf{L}).F)$,*

$$\left\langle Q^{\mathbf{G}}(\theta),Q^{\mathbf{G}}(\phi)\right\rangle_{\mathbf{G}^F}=\left\langle \mathcal{Z}_{\mathbf{L}}^{-1}\theta,\phi\right\rangle_{W_{\mathbf{G}}(\mathbf{L}).F}.$$

Proof. First note that [12, 24.3.6], suitably interpreted to take into account the distinction between our $\widetilde{\mathcal{X}}_{\iota_0}$ and Lusztig's $\mathcal{X}_{\iota_0}$, shows that

$$\left\langle\widetilde{\mathcal{X}}_\iota,\widetilde{\mathcal{X}}_\kappa\right\rangle_{\mathbf{G}^F}=\left\langle\mathcal{Z}_{\mathbf{L}}^{-1}\tilde{\varphi}_\iota,\tilde{\varphi}_\kappa\right\rangle_{W_{\mathbf{G}}(\mathbf{L}).F}. \tag{3.3}$$

Now in order to prove the proposition, it suffices to do so as θ and ϕ run over a basis of $\mathcal{C}(W_{\mathbf{G}}(\mathbf{L}).F)$ In particular, it suffices to take $\theta=\tilde{\varphi}_\iota$ and $\phi=\tilde{\varphi}_\kappa$. But then the statement is precisely Equation (3.3), whence the result. □

It follows easily from the Definition 3.1(iii) and Proposition 3.4 that the generalized Green functions form an orthogonal basis of $\mathcal{C}_{\mathcal{I}}(\mathbf{G}^F)$. More precisely, we have

Corollary 3.5.

$$\langle Q_{wF},Q_{w'F}\rangle_{\mathbf{G}^F}=\begin{cases}0, & \textit{if } wF \textit{ and } w'F \textit{ are not conjugate in } W_{\mathbf{G}}(\mathbf{L}),\\ \dfrac{|C_{W_{\mathbf{G}}(\mathbf{L})}(wF)|}{|Z_{\mathbf{L}}^{0wF}|}, & \textit{otherwise.}\end{cases} \tag{3.4}$$

The formula (3.4) superficially seems different from [12, 9.11]. However the two formulae are actually equivalent, although there is a power of q in [12] which is absent here. This is explained by the facts that in [12, 9.11] the inner product used differs from ours, in that it does not involve conjugation, and that the formula given there is for the inner product of two Green functions corresponding to contragredient local systems, with contragredient Frobenius isomorphisms. In Lusztig's notation, if the characteristic function of the sheaf $\mathcal{F}$ with Frobenius isomorphism φ_1 is f, then the characteristic function of $\mathcal{F}^\vee$ with Frobenius isomorphism $\varphi_1^\vee$ is $q^{-2c_{\iota_0}}\overline{f}$ (see the computation in the proof of [12, 9.8]); this, in conjunction with the fact that $R_{\mathbf{M}}^{\mathbf{G}}$ commutes with complex conjugation, shows the formulae are equivalent.

Remark 3.6. The preferred extension $\tilde{\varepsilon}$ of the alternating character ε of $W_{\mathbf{G}}(\mathbf{L})$ will play a prominent rôle in our work. A fact which we shall use repeatedly, and which results from the description in [12, 17.2] of the preferred extension, is that $\tilde{\varepsilon}$ is trivial on Frobenius, i.e., for $w \in W_{\mathbf{G}}(\mathbf{L})$, $\tilde{\varepsilon}(w.F) = \varepsilon(w)$. Note also that since the preferred extension is real, if $\tilde{\varphi}_\iota$ is the preferred extension corresponding to $\iota \in \mathcal{I}^F$, then there is a sign $\varepsilon_\iota = \pm 1$ such that $\tilde{\varphi}_\iota \otimes \tilde{\varepsilon} = \varepsilon_\iota \tilde{\varphi}_{\hat{\iota}}$, where $\hat{\iota}$ is defined by $\varphi_\iota \otimes \varepsilon = \varphi_{\hat{\iota}}$.

Let $\mathbf{H}$ be any linear algebraic group with a Frobenius morphism $F : \mathbf{H} \to \mathbf{H}$ which corresponds to an $\mathbb{F}_q$-structure on $\mathbf{G}$. Let $\mathbf{T}$ be a maximally split maximal torus of $\mathbf{H}$ and write $R_u(\mathbf{H})$ for the unipotent radical of $\mathbf{H}$. Then the Weyl group $W = W_{\mathbf{H}}(\mathbf{T})$ acts as a reflection group on $Y(\mathbf{T}) \otimes \mathbb{R}$, and F has an induced action as $q\phi$ on this space, where ϕ is a linear transformation of finite order (cf. [4, p. 40]). Write $\{f_1, f_2 \ldots, f_\ell\}$ for a set of basic invariants of W and let $d_i = \deg(f_i)$. It is known (cf. [17, 6.1]) that the f_i may be chosen to be eigenfunctions for ϕ, i.e. $\phi f_i = \delta_i f_i$ for each i, where $\delta_i \in \mathbb{C}$.

Lemma 3.7. *With notation as in the previous paragraph, we have*

(i) *The order of* $\mathbf{H}$ *is given by*

$$\left|\mathbf{H}^F\right| = q^{\dim R_u(\mathbf{H}) + \sum_i (d_i - 1)} \prod_i \left(q^{d_i} - \delta_i\right).$$

(ii) *If F is varied by keeping ϕ fixed and allowing q to vary, the order function in* (i) *is a polynomial in q and*

$$\left|\mathbf{H}^F\right|\left(q^{-1}\right) = q^{-\dim \mathbf{H}} \varepsilon_{\mathbf{H}} \left|\mathbf{H}^F\right|_{q'},$$

where, for any linear algebraic group $\mathbf{H}$ *we write* $\varepsilon_{\mathbf{H}} = (-1)^{\mathbb{F}_q\text{-rank of } \mathbf{H}}$ *and where we denote by* $|\mathbf{H}^F|_{q'}$ *the part prime to q of* $|\mathbf{H}^F|$.

Proof. The formula in (i) is well known (see, e.g., [10, 1.8]). Part (ii) is obtained directly from (i), taking into account the following three facts. First, it follows from [17, 6.5(i)] that the eigenvalues of ϕ on $Y(\mathbf{T}) \otimes \mathbb{R}$ are the δ_i^{-1}; secondly, if $\delta_i \neq \delta_i^{-1}$, both occur as eigenvalues of ϕ in the same degree. The latter fact follows because ϕ is real, and so its eigenvalues come in conjugate pairs. As a consequence, we have $\prod_i (q^{d_i} - \delta_i) = \prod_i (q^{d_i} - \delta_i^{-1})$, which is required for the identity (ii). Finally, one needs the fact that $\varepsilon_{\mathbf{H}} = \det_{Y(\mathbf{T}) \otimes \mathbb{R}}(-\phi)$ which holds because for any automorphism ϕ of finite order of a lattice Y, we have $\det_{Y \otimes \mathbb{R}}(\phi) = (-1)^d$ where d is the codimension of the fixed point subspace of ϕ in $Y \otimes \mathbb{R}$. □

Remark 3.8. In this work, we shall encounter several functions, whose definition generally involves the number of F-fixed points of some variety on which F acts, and which are (Laurent) polynomials in q. This means that if ϕ remains fixed but q is allowed to vary as in Lemma 3.7, they are Laurent polynomials in q. Examples of such functions include the orders of $\mathbb{F}_q$-groups (as in Lemma 3.7), $\widetilde{P}_{\iota,\kappa}$, and for a unipotent element $u \in \mathbf{G}^F$ with a fixed parametrization (e.g., in the Bala–Carter classification), $Q_{wF}(u)$, and $|C_{\mathbf{G}^F}(u)|$. In the case of functions in $\mathcal{C}_{\text{uni}}(\mathbf{G}^F)$, the term polynomial will be used when they are linear combinations of the $\mathcal{Y}_\iota$, with coefficients which are polynomials in the above sense. For any such function $f(q)$, we use the notation f^* to denote the function defined by $f^*(q) = f(q^{-1})$. The $\mathcal{Y}_\iota$ are fixed by this operation.

The next result gives some properties of the function $\mathcal{Z}_{\mathbf{L}} \in \mathcal{C}(W_{\mathbf{G}}(\mathbf{L}).F)$.

Lemma 3.9. (i) *We have* $|Z_{\mathbf{L}}^{0vF}| = |Z_{\mathbf{G}}^{0F}| \sum_{i=0}^{l} (\tilde{r})^{\wedge i}(vF) q^{l-i}(-1)^i$ *where* $l = \dim Z_{\mathbf{L}}^0 - \dim Z_{\mathbf{G}}^0$ *and where* $\tilde{r}$ *is the restriction to* $W_{\mathbf{G}}(\mathbf{L}).F$ *of the character of the representation of* $W_{\mathbf{G}}(\mathbf{L}) \rtimes \langle F\rangle$ *on* $Y(Z_{\mathbf{L}}^0/Z_{\mathbf{G}}^0) \otimes \mathbb{R}$*, which is an extension of the reflection character* r *of* $W_{\mathbf{G}}(\mathbf{L})$.

(ii) *We have* $\mathcal{Z}_{\mathbf{L}}(q^{-1}) = \varepsilon_{Z_{\mathbf{L}}} q^{-\dim Z_{\mathbf{L}}} \tilde{\varepsilon} \cdot \mathcal{Z}_{\mathbf{L}}(q)$.

Proof. We have $|Z_{\mathbf{L}}^{0vF}| = |Z_{\mathbf{G}}^{0F}| \sum_i (-1)^i \operatorname{Trace}(vF | H_c^i(Z_{\mathbf{L}}^0/Z_{\mathbf{G}}^0))$. As in [2, proof of 5.7] or [10, (1.4)], we have

$$\left|Z_{\mathbf{L}}^{0vF}\right| = \left|Z_{\mathbf{G}}^{0F}\right| \sum_i (-1)^i q^{l-i} \operatorname{Trace}\bigl(vF, \wedge^i Y\bigl(Z_{\mathbf{L}}^0/Z_{\mathbf{G}}^0\bigr)\bigr)$$

where $l = \dim Z_{\mathbf{L}}^0 - \dim Z_{\mathbf{G}}^0$. Now the space $Y(Z_{\mathbf{L}}^0/Z_{\mathbf{G}}^0) \otimes \mathbb{R}$ realizes the reflection representation of the Coxeter group $W_{\mathbf{G}}(\mathbf{L})$, as can be seen from [11, 9.2] and [7, Theorem 6], and part (i) of the lemma follows.

For (ii), let $v \in W_{\mathbf{G}}(\mathbf{L})$ and consider the torus $Z_{\mathbf{L}}^0$, with Frobenius action vF. From Lemma 3.7(ii) applied here, we have $|Z_{\mathbf{L}}^{0vF}|(q^{-1}) = \varepsilon'_{Z_{\mathbf{L}}} q^{-\dim Z_{\mathbf{L}}} |Z_{\mathbf{L}}^{0vF}|(q)$, where

$$\varepsilon'_{Z_{\mathbf{L}}} = (-1)^{\mathbb{F}_q\text{-rank of } Z_{\mathbf{L}} \text{ with Frobenius } vF}.$$

But, since $\tilde{\varepsilon}(vF) = \det_{Y(Z_{\mathbf{L}}^0)}(v)$ (recall that v acts trivially on $Z_{\mathbf{G}}$ and that $\tilde{\varepsilon}$ is the trivial extension), we have $\varepsilon'_{Z_{\mathbf{L}}} = \varepsilon_{Z_{\mathbf{L}}} \tilde{\varepsilon}(vF)$. □

When $\mathbf{G}$ is quasi-simple, $W_{\mathbf{G}}(\mathbf{L})$ is irreducible, so that r is irreducible. We then have

Lemma 3.10. *When* r *is irreducible,* $\tilde{r}$ *is the preferred extension of the reflection character.*

Proof. The lemma is a consequence of the definition of the preferred extension in [12, 17.2], and the fact (which can be checked by tracing through [11, 9.2]) that if we write $F = q\phi$ on $V = Y(Z_{\mathbf{L}}^0/Z_{\mathbf{G}}^0) \otimes \mathbb{R}$ so that $\tilde{r}$ is the extension of r in which F acts via ϕ, the automorphism ϕ stabilizes a set of positive roots of a root system for $W_{\mathbf{G}}(\mathbf{L})$ in V. We need only consider the case when ϕ is non-trivial, so that $(W_{\mathbf{G}}(\mathbf{L}), \phi)$ is of type ${}^2A_n$, ${}^2E_6$, ${}^3D_4$ or ${}^2D_n$. In the cases ${}^2A_n$, ${}^2E_6$, in the language of [12, 17.2] one has $a_r = 1$ so the preferred extension is the one where F acts by $-w_0$, which agrees with ϕ. In the case ${}^3D_4$, the preferred extension is the only rational one so again agrees with ϕ. Finally, in the case ${}^2D_n$ one checks from the description in [12, 17.2] that the preferred extension is the one which realizes the reflection representation of $B_n \simeq D_n \rtimes \langle F\rangle$, and indeed ϕ acts as a reflection, since it acts by exchanging two of the simple roots and fixing the others. □

If $\mathbf{G}$ is not quasi-simple the group $W_{\mathbf{G}}(\mathbf{L})$ is a direct product of the irreducible Coxeter groups $W_{\mathbf{G}_i}(\mathbf{L})$ where $\mathbf{G}_i$ runs over the quasi-simple components of $\mathbf{G}$. The representation of $W_{\mathbf{G}}(\mathbf{L})$ on $Y(Z^0_{\mathbf{L}}/Z^0_{\mathbf{G}})\otimes\mathbb{R}$ decomposes into the sum over i of summands isomorphic to the reflection representation r_i of the component $W_{\mathbf{G}_i}(\mathbf{L})$ on $Y(Z^0_{\mathbf{L}}/Z^0_{\mathbf{G}_i})\otimes\mathbb{R}$ tensored with the identity representations of the other components. The action of F permutes the r_i in the same way it permutes the $\mathbf{G}_i$. Since the preferred extension of the identity is the identity, it follows that if $\mathbf{G}_i$ is F-stable, the extension of r_i which appears in $Y(Z^0_{\mathbf{L}}/Z^0_{\mathbf{G}})\otimes\mathbb{R}$ is the preferred extension of r_i.

We now describe Lusztig restriction in terms of the generalized Green functions, which form a basis of the space $\mathcal{C}_{\text{uni}}(\mathbf{G}^F)$. Let $w\in W_{\mathbf{G}}(\mathbf{L})$ and suppose $\mathbf{M}$ is a rational Levi subgroup which contains a rational conjugate $\mathbf{L}_w$ of $\mathbf{L}$. Then we shall use the identifications explained in Remarks 2.2, 2.4 to consider $Q^{\mathbf{M}}$ as a linear isomorphism between $\mathcal{C}(W_{\mathbf{M}_0}(\mathbf{L}).wF)$ and $\mathcal{C}_{\mathcal{I}_{\mathbf{M}}}(\mathbf{M}^F)$.

Theorem 3.11. *Let $\mathbf{M}$ be a rational Levi subgroup of some parabolic subgroup of $\mathbf{G}$. Then ${}^*R^{\mathbf{G}}_{\mathbf{M}}\circ Q^{\mathbf{G}}=0$ unless $\mathbf{M}$ contains some rational $\mathbf{G}$-conjugate $\mathbf{L}_w$ of $\mathbf{L}$, and if this condition holds, then in the above notation, we have*

(i) ${}^*R^{\mathbf{G}}_{\mathbf{M}}\circ Q^{\mathbf{G}}=Q^{\mathbf{M}}\circ\operatorname{Res}^{W_{\mathbf{G}}(\mathbf{L}).F}_{W_{\mathbf{M}_0}(\mathbf{L}).wF}$,

(ii) $R^{\mathbf{G}}_{\mathbf{M}}\circ Q^{\mathbf{M}}=Q^{\mathbf{G}}\circ\operatorname{Ind}^{W_{\mathbf{G}}(\mathbf{L}).F}_{W_{\mathbf{M}_0}(\mathbf{L}).wF}$.

Proof. We need only verify the statements on a basis of the relevant space of functions. We start by proving (ii), for which it suffices to evaluate both sides on $\widetilde{\mathcal{X}}_\iota$ for $\iota\in\mathcal{I}^F_{\mathbf{M}}$. By Frobenius reciprocity, Proposition 2.3(i) can be written as

$$\begin{aligned}R^{\mathbf{G}}_{\mathbf{M}}(\widetilde{\mathcal{X}}_\iota)&=\sum_{\kappa\in\mathcal{I}^F}\big\langle\operatorname{Ind}^{W_{\mathbf{G}}(\mathbf{L}).F}_{W_{\mathbf{M}_0}(\mathbf{L}).wF}\tilde\varphi_\iota,\tilde\varphi_\kappa\big\rangle_{W_{\mathbf{M}_0}(\mathbf{L}).wF}Q^{\mathbf{G}}(\tilde\varphi_\kappa)\\&=Q^{\mathbf{G}}\bigg(\sum_{\kappa\in\mathcal{I}^F}\big\langle\operatorname{Ind}^{W_{\mathbf{G}}(\mathbf{L}).F}_{W_{\mathbf{M}_0}(\mathbf{L}).wF}\tilde\varphi_\iota,\tilde\varphi_\kappa\big\rangle_{W_{\mathbf{M}_0}(\mathbf{L}).wF}\tilde\varphi_\kappa\bigg)=Q^{\mathbf{G}}\big(\operatorname{Ind}^{W_{\mathbf{G}}(\mathbf{L}).F}_{W_{\mathbf{M}_0}(\mathbf{L}).wF}\tilde\varphi_\iota\big),\end{aligned}$$

whence (ii) follows.

Now take $\theta\in\mathcal{C}(W_{\mathbf{G}}(\mathbf{L}).F)$ and consider ${}^*R^{\mathbf{G}}_{\mathbf{M}}\circ Q^{\mathbf{G}}(\theta)$. The space $\mathcal{C}_{\text{uni}}(\mathbf{M}^F)$ has a basis $\bigcup_{\mathcal{I}'_{\mathbf{M}}}\{\widetilde{\mathcal{X}}_\iota\mid\iota\in{\mathcal{I}'_{\mathbf{M}}}^F\}$ where $\mathcal{I}'_{\mathbf{M}}$ runs over the F-stable blocks of $\mathbf{M}$. Now

$$\big\langle{}^*R^{\mathbf{G}}_{\mathbf{M}}\circ Q^{\mathbf{G}}(\theta),\widetilde{\mathcal{X}}_\iota\big\rangle_{\mathbf{M}^F}=\big\langle Q^{\mathbf{G}}(\theta),R^{\mathbf{G}}_{\mathbf{M}}(\widetilde{\mathcal{X}}_\iota)\big\rangle_{\mathbf{G}^F},$$

and by Proposition 2.3 the function $R^{\mathbf{G}}_{\mathbf{M}}(\widetilde{\mathcal{X}}_\iota)$ is in $\mathcal{C}_{\mathcal{I}'_{\mathbf{G}}}(\mathbf{G}^F)$, where $\mathcal{I}'_{\mathbf{G}}$ is the block of $\mathbf{G}$ corresponding to $\mathcal{I}'_{\mathbf{M}}$. Thus the scalar product is 0 if $\mathcal{I}'_{\mathbf{G}}$ is not equal to $\mathcal{I}$. Furthermore, the block $\mathcal{I}$ is of the form $\mathcal{I}'_{\mathbf{G}}$ for some (unique by [6, 1.2]) block $\mathcal{I}'_{\mathbf{M}}$ of $\mathbf{M}$ only if $\mathbf{M}$ contains a $\mathbf{G}$-conjugate $\mathbf{L}_w$ of $\mathbf{L}$, whence the first statement of the theorem.

It follows also, that to prove (i), we need only show that for any $\theta\in\mathcal{C}(W_{\mathbf{G}}(\mathbf{L}).F)$, if we apply both sides of (i) to θ, the resulting functions have the same inner product with any function in $\mathcal{C}_{\mathcal{I}_{\mathbf{M}}}(\mathbf{M}^F)$. But $\mathcal{C}_{\mathcal{I}_{\mathbf{M}}}(\mathbf{M}^F)$ is spanned by the functions $Q^{\mathbf{M}}(\psi)$ with $\psi \in \mathcal{C}(W_{\mathbf{M}}(\mathbf{L}_w).F)$, so that it suffices to consider inner products with these functions. We have

$$\begin{aligned}
\langle {}^*R_{\mathbf{M}}^{\mathbf{G}} \circ Q^{\mathbf{G}}(\theta), Q^{\mathbf{M}}(\psi)\rangle_{\mathbf{M}^F} &= \langle Q^{\mathbf{G}}(\theta), R_{\mathbf{M}}^{\mathbf{G}}(Q^{\mathbf{M}}(\psi))\rangle_{\mathbf{G}^F} \\
&= \langle Q^{\mathbf{G}}(\theta), Q^{\mathbf{G}} \circ \operatorname{Ind}_{W_{\mathbf{M}_0}(\mathbf{L}).wF}^{W_{\mathbf{G}}(\mathbf{L}).F}(\psi)\rangle_{\mathbf{G}^F} \quad \text{by (ii)} \\
&= \langle \theta \mathcal{Z}_{\mathbf{L}}^{-1}, \operatorname{Ind}_{W_{\mathbf{M}_0}(\mathbf{L}).wF}^{W_{\mathbf{G}}(\mathbf{L}).F}(\psi)\rangle_{W_{\mathbf{G}}(\mathbf{L}).F} \quad \text{by Proposition 3.4} \\
&= \langle \mathcal{Z}_{\mathbf{L}}^{-1} \operatorname{Res}_{W_{\mathbf{M}_0}(\mathbf{L}).wF}^{W_{\mathbf{G}}(\mathbf{L}).F}(\theta), \psi\rangle_{W_{\mathbf{M}_0}(\mathbf{L}).wF} \\
&= \langle Q^{\mathbf{M}} \circ \operatorname{Res}_{W_{\mathbf{M}_0}(\mathbf{L}).wF}^{W_{\mathbf{G}}(\mathbf{L}).F}(\theta), Q^{\mathbf{M}}(\psi)\rangle_{\mathbf{M}^F},
\end{aligned}$$

which completes the proof. □

Remark 3.12. Theorem 3.11 may be expressed as asserting the commutativity of the following diagrams.

$$\begin{array}{ccc}
\mathcal{C}(W_{\mathbf{G}}(\mathbf{L}).F) & \xrightarrow{Q^{\mathbf{G}}} & \mathcal{C}_{\mathcal{I}}(\mathbf{G}^F) \\
{\scriptstyle \mathrm{Ind}}\uparrow & & \uparrow{\scriptstyle R_{\mathbf{M}}^{\mathbf{G}}} \\
\mathcal{C}(W_{\mathbf{M}_0}(\mathbf{L}).wF) & \xrightarrow{Q^{\mathbf{M}_0}} & \mathcal{C}_{\mathcal{I}_{\mathbf{M}_0}}(\mathbf{M}_0^{\dot{w}F})
\end{array}$$

and

$$\begin{array}{ccc}
\mathcal{C}(W_{\mathbf{G}}(\mathbf{L}).F) & \xrightarrow{Q^{\mathbf{G}}} & \mathcal{C}_{\mathcal{I}}(\mathbf{G}^F) \\
{\scriptstyle \mathrm{Res}}\downarrow & & \downarrow{\scriptstyle {}^*R_{\mathbf{M}}^{\mathbf{G}}} \\
\mathcal{C}(W_{\mathbf{M}_0}(\mathbf{L}).wF) & \xrightarrow{Q^{\mathbf{M}_0}} & \mathcal{C}_{\mathcal{I}_{\mathbf{M}_0}}(\mathbf{M}_0^{\dot{w}F})
\end{array}$$

As an immediate corollary, we have the following explicit formula for the Lusztig restriction of the generalized Green functions.

Corollary 3.13. *With notation as in Theorem* 3.11*, we have*

$${}^*R_{\mathbf{M}}^{\mathbf{G}} Q_{vF}^{\mathbf{G}} = |W_{\mathbf{M}_0}(\mathbf{L})|^{-1} \sum_{\{x \in W_{\mathbf{G}}(\mathbf{L}) \mid x(vF)x^{-1} \in W_{\mathbf{M}_0}(\mathbf{L}).wF\}} Q_{x(vF)x^{-1}}^{\mathbf{M}}.$$

Proof. It is easy to see that

$$\operatorname{Res}^{W_{\mathbf{G}}(\mathbf{L}).F}_{W_{\mathbf{M}_0}(\mathbf{L}).wF}\gamma_{vF} = \left|W_{\mathbf{M}_0}(\mathbf{L})\right|^{-1}\sum_{\{x\in W_{\mathbf{G}}(\mathbf{L})\mid x(vF)x^{-1}\in W_{\mathbf{M}_0}(\mathbf{L}).wF\}}\gamma_{x(vF)x^{-1}}.$$

The result now follows immediately by applying Theorem 3.11(i) to the function γ_{vF}. □

The duality involution $\mathcal{D}_{\mathbf{G}}$ (restricted to $\mathcal{C}_{\mathcal{I}}(\mathbf{G}^F)$) has an elegant description in this setting.

Proposition 3.14 (Cf. [14]). *Let $\mathcal{D}_{\mathbf{G}}$ be the duality involution; then*

(i) *We have $\mathcal{D}_{\mathbf{G}}(Q_{wF}) = \eta_{\mathbf{L}}\tilde{\varepsilon}(wF)Q_{wF}$, where, for any reductive group* **G** *we write $\eta_{\mathbf{G}} = (-1)^{\text{semisimple } \mathbb{F}_q\text{-rank of } \mathbf{G}} = \varepsilon_{\mathbf{G}}\varepsilon_{Z_{\mathbf{G}}}$.*
(ii) *The duality involution $\mathcal{D}_{\mathbf{G}}:\mathcal{C}_{\mathcal{I}}(\mathbf{G}^F)\to\mathcal{C}_{\mathcal{I}}(\mathbf{G}^F)$ corresponds under $Q^{\mathbf{G}}$ to multiplication by $\eta_{\mathbf{L}}\tilde{\varepsilon}$ in $\mathcal{C}(W_{\mathbf{G}}(\mathbf{L}).F)$. In particular $\mathcal{D}_{\mathbf{G}}(\widetilde{\mathcal{X}}_\iota) = \eta_{\mathbf{L}}\varepsilon_\iota\widetilde{\mathcal{X}}_{\hat{\iota}}$, where $\hat{\iota}$ and ε_ι are defined in Remark* 3.6.

Proof. The statement (i) may be found in [14, Section 8] whose proof applies to the twisted case without change. The first statement in (ii) follows immediately since $Q^{\mathbf{G}}$ is linear, and the second statement follows from the relation $\tilde{\varepsilon}\otimes\tilde{\varphi}_\iota = \varepsilon_\iota\tilde{\varphi}_{\hat{\iota}}$ (see Remark 3.6). □

4. Unipotently supported class functions and twisted class functions on reflection groups

For $\iota\in\mathcal{I}^F$ define a function $\widetilde{Q}_\iota$ on $W_{\mathbf{G}}(\mathbf{L}).F$ by

$$\widetilde{Q}_\iota(wF) = \frac{1}{a_\iota}\sum_{a\in A(u)} q^{-c_\iota}\overline{\mathcal{Y}_\iota(u_a)}Q_{wF}(u_a) \tag{4.1}$$

where we fix $u\in C_\iota^F$ and set $A(u) = C_{\mathbf{G}}(u)/C^0_{\mathbf{G}}(u)$, $a_\iota = |A(u)|$ and take u_a to be a representative of the $\mathbf{G}^F$-orbit in C^F which corresponds to the F-class of $a\in A(u)$. The function $\widetilde{Q}_\iota$ does not actually depend on the choice of $u\in C_\iota^F$. Indeed, using the relation

$$a_\iota^{-1}\sum_{a\in A(u)}\mathcal{Y}_\iota(u_a)\overline{\mathcal{Y}_\gamma(u_a)} = \delta_{\iota,\gamma} \tag{4.2}$$

(see [6, 1.5]) and (3.1), we obtain

$$\widetilde{Q}_\iota(wF) = \sum_{\gamma\in\mathcal{I}^F}\tilde{\varphi}_\gamma(wF)\widetilde{P}_{\iota,\gamma}. \tag{4.3}$$

The relation (4.3) justifies the remark above that $\widetilde{Q}_\iota$ is independent of the choice of $u \in C_\iota^F$. Note also that the formula (4.3) makes sense even when $\iota \notin \mathcal{I}$; but then, since $\widetilde{P}_{\iota,\gamma} = 0$ when ι and γ are in different blocks, $\widetilde{Q}_\iota = 0$.

Proposition 4.1. *For any unipotent element $u \in \mathbf{G}^F$, we have*

$$Q_{wF}(u) = \sum_{\iota \in \mathcal{I}^F} \widetilde{Q}_\iota(wF)\widetilde{\mathcal{Y}}_\iota(u). \tag{4.4}$$

Proof. As remarked above, if $\iota \notin \mathcal{I}$ then (4.3) shows that the corresponding summand of the right-hand side is 0, since then $\widetilde{P}_{\iota,\gamma} = 0$ for all $\gamma \in \mathcal{I}^F$. So

$$\sum_{\iota \in \mathcal{I}^F} \widetilde{Q}_\iota(wF)\widetilde{\mathcal{Y}}_\iota(u) = \sum_{\iota \in \mathcal{P}^F} \widetilde{Q}_\iota(wF)\widetilde{\mathcal{Y}}_\iota(u).$$

We now use the second orthogonality formula for the $\mathcal{Y}_\iota(u)$:

$$\sum_{\iota \in \mathcal{P}^F} \mathcal{Y}_\iota(u)\overline{\mathcal{Y}_\iota(u')} = \begin{cases} |A(u)^F| & \text{if } u \sim_{\mathbf{G}^F} u', \\ 0 & \text{otherwise} \end{cases} \tag{4.5}$$

where $\sim_{\mathbf{G}^F}$ means $\mathbf{G}^F$-conjugacy. Thus

$$\begin{aligned} \sum_{\iota \in \mathcal{I}^F} \widetilde{Q}_\iota(wF)\widetilde{\mathcal{Y}}_\iota(u) &= \sum_{\iota \in \mathcal{P}^F, a \in A(u)} a_\iota^{-1}\overline{\mathcal{Y}_\iota(u_a)}Q_{wF}(u_a)\mathcal{Y}_\iota(u) \\ &= |A(u)|^{-1}|A(u)^F|\#\{a \mid u_a \sim_{\mathbf{G}^F} u\}Q_{wF}(u) = Q_{wF}(u). \qquad \square \end{aligned}$$

Note that Equation (4.5) will often be used when $u = u' = u''_a$ for some rational unipotent element u'' and some $a \in A(u'')$, in which case we have $|A(u)^F| = |C_{A(u'')}(aF)| = |A(u'')^{aF}|$. The functions $\widetilde{\mathcal{Y}}_\iota$ form a basis of $\mathcal{C}_\mathcal{I}(\mathbf{G}^F)$ as ι runs over $\mathcal{I}^F$. The next result relates the $\widetilde{Q}_\iota$ to expansions in terms of this basis.

Lemma 4.2. (i) *For any function $f \in \mathcal{C}_\mathcal{I}(\mathbf{G}^F)$, the coefficient of f in the basis $\widetilde{\mathcal{Y}}_\iota$ is*

$$\frac{1}{a_\iota}\sum_{a \in A(u)} q^{-c_\iota}\overline{\mathcal{Y}_\iota(u_a)}f(u_a). \tag{4.6}$$

(ii) *For any function $\theta \in \mathcal{C}(W_\mathbf{G}(\mathbf{L}).F)$, we have $Q^\mathbf{G}(\theta) = \sum_{\iota \in \mathcal{I}^F}\langle\theta, \widetilde{Q}_\iota\rangle_{W_\mathbf{G}(\mathbf{L}).F}\widetilde{\mathcal{Y}}_\iota$.*

(iii) *The functions $(Q^\mathbf{G})^{-1}(\widetilde{\mathcal{Y}}_\iota)$ form the basis of $\mathcal{C}(W_\mathbf{G}(\mathbf{L}).F)$ which is dual to the basis $\{\widetilde{Q}_\iota\}$.*

Proof. By Proposition 4.1 and the definition (4.1) of $\widetilde{Q}_\iota$, (i) holds when $f = Q_{wF}$, and since the Q_{wF} form a basis of $\mathcal{C}_\mathcal{I}(\mathbf{G}^F)$ and the formula (4.6) is linear in f, (i) holds in general. Similarly, (ii) holds when $\theta = \gamma_{wF}$, again by Proposition 4.1. By linearity, (ii) holds generally. The statement (iii) follows immediately from (ii). $\square$

5. Lusztig's algorithm and orthogonality relations for generalized Green functions

We shall require

Lemma 5.1. *Let H be a finite group, $\chi_1, \chi_2, \ldots$ the irreducible characters of H (over a field of characteristic zero) and f any class function on H which is non-zero at each element of H. Let f^{-1} be the pointwise inverse of f. Then we have the matrix equation*

$$\bigl\{\bigl\langle f^{-1}\chi_i, \chi_j\bigr\rangle_H\bigr\}_{i,j} = \bigl\{\langle f\chi_i, \chi_j\rangle_H\bigr\}_{i,j}^{-1}. \tag{5.1}$$

Proof. Since the χ_i form an orthonormal basis of the space of class functions on H, the left side of (5.1) is simply the matrix of the linear transformation induced by multiplication by f^{-1}, and the assertion is no more than the observation that multiplication by f^{-1} is the inverse of multiplication by f. □

Lemma 5.1 remains valid when H is a finite coset, the χ_i are extensions to H of the irreducible characters of the underlying group, and f is a twisted class function on H.

We now recall the algorithm outlined by Lusztig in [12, Section 24] for the computation of the polynomials $P_{\iota,\kappa}$. In the following, unless otherwise stated, we fix a block $\mathcal{I}$ and work in $\mathcal{C}_{\mathcal{I}}(\mathbf{G}^F)$. Lusztig's algorithm is based on the following matrix equation, which is an immediate consequence of the relation $\widetilde{\mathcal{X}}_\iota = \sum_\kappa \widetilde{P}_{\kappa,\iota}\widetilde{\mathcal{Y}}_\kappa$ and (3.3):

$${}^t\widetilde{P}\widetilde{\Lambda}\widetilde{P} = \bigl\{\bigl\langle\widetilde{\mathcal{X}}_\iota, \widetilde{\mathcal{X}}_\kappa\bigr\rangle_{\mathbf{G}^F}\bigr\}_{\iota,\kappa} = \bigl\{\bigl\langle\mathcal{Z}_{\mathbf{L}}^{-1}\tilde{\varphi}_\iota, \tilde{\varphi}_\kappa\bigr\rangle_{W_{\mathbf{G}}(\mathbf{L}).F}\bigr\}_{\iota,\kappa}$$

where $\widetilde{P} = \{\widetilde{P}_{\iota,\kappa}\}_{\iota,\kappa}$ and $\widetilde{\Lambda} = \{\langle\widetilde{\mathcal{Y}}_\iota, \widetilde{\mathcal{Y}}_\kappa\rangle_{\mathbf{G}^F}\}_{\iota,\kappa}$. We shall use the inverse of this equation:

$$\widetilde{P}^{-1}\widetilde{\Lambda}^{-1}\bigl({}^t\widetilde{P}^{-1}\bigr) = \widetilde{\Omega}$$

where $\widetilde{\Omega} = \{\widetilde{\omega}_{\iota,\kappa}\}_{\iota,\kappa}$ and $\widetilde{\omega}_{\iota,\kappa} = \langle\mathcal{Z}_{\mathbf{L}}\tilde{\varphi}_\iota, \tilde{\varphi}_\kappa\rangle_{W_{\mathbf{G}}(\mathbf{L}).F}$, the inverse of the matrix on the right-hand side being given by Lemma 5.1. The matrix $\widetilde{\Omega}$ may be considered known (see Definition 3.3) since it is given in terms of Weyl group data. The rows and columns of $\widetilde{\Lambda}$ and $\widetilde{P}$ may be ordered in a way compatible with the order on unipotent classes; they may further be ordered so that pairs with the same support form a connected sequence in the order. Then $\widetilde{\Lambda}$ is block-diagonal and $\widetilde{P}$ block-triangular with identity diagonal blocks, the blocks corresponding to unipotent classes. Given $\widetilde{\Omega}$, there are unique matrices $\widetilde{\Lambda}$ and $\widetilde{P}$ of this shape which satisfy the above equation.

We note for future reference that Lemma 3.9 immediately gives

$$\widetilde{\omega}_{\iota,\kappa} = \bigl|Z_{\mathbf{G}}^{0F}\bigr|\sum_{i=0}^{l} q^{l-i}(-1)^i\bigl\langle\tilde{\varphi}_\iota \otimes \tilde{\varphi}_\kappa, \tilde{r}^{\wedge i}\bigr\rangle_{W_{\mathbf{G}}(\mathbf{L}).F} \tag{5.2}$$

where $l = \dim Z_{\mathbf{L}}^0 - \dim Z_{\mathbf{G}}^0$ and where $\tilde{r}$ is the restriction to $W_{\mathbf{G}}(\mathbf{L}).F$ of the character of the representation of $W_{\mathbf{G}}(\mathbf{L}) \rtimes \langle F\rangle$ on $Y(Z_{\mathbf{L}}^0/Z_{\mathbf{G}}^0) \otimes \mathbb{R}$, which is an extension of the reflection character r of $W_{\mathbf{G}}(\mathbf{L})$.

The following proposition is a generalization of [9, 1.1.4].

Corollary 5.2 (Second orthogonality formula for Green functions).

$$\begin{aligned}\big\langle \mathcal{Z}_{\mathbf{L}}\widetilde{Q}_\iota, \widetilde{Q}_\gamma\big\rangle_{W_{\mathbf{G}}(\mathbf{L}).F} &= \big\{\big\langle\widetilde{\mathcal{Y}}_\iota, \widetilde{\mathcal{Y}}_\gamma\big\rangle_{\mathbf{G}^F}\big\}^{-1}_{\iota,\gamma}\\ &= \begin{cases} a_\iota^{-1}\displaystyle\sum_{a\in A(u)}\frac{|C^0_{\mathbf{G}}(u_a)^F|}{q^{2c_\iota}}\mathcal{Y}_\iota(u_a)\overline{\mathcal{Y}_\gamma(u_a)} & \text{if } C_\iota = C_\gamma,\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

where notation is as in (4.1).

Proof. Using the values given in (4.3) for $\widetilde{Q}_\iota$ and $\widetilde{Q}_\gamma$, we obtain:

$$\big\{\big\langle \mathcal{Z}_{\mathbf{L}}\widetilde{Q}_\iota, \widetilde{Q}_\gamma\big\rangle_{W_{\mathbf{G}}(\mathbf{L}).F}\big\}_{\iota,\gamma} = \widetilde{P}\big\{\langle \mathcal{Z}_{\mathbf{L}}\tilde{\varphi}_\kappa, \tilde{\varphi}_{\kappa'}\rangle_{W_{\mathbf{G}}(\mathbf{L}).F}\big\}_{\kappa,\kappa'}{}^t\widetilde{P} = \widetilde{P}\widetilde{\Omega}{}^t\widetilde{P} = \big\{\big\langle\widetilde{\mathcal{Y}}_\iota, \widetilde{\mathcal{Y}}_\gamma\big\rangle_{\mathbf{G}^F}\big\}^{-1}_{\iota,\gamma}.$$

Now $\langle\widetilde{\mathcal{Y}}_\iota, \widetilde{\mathcal{Y}}_\gamma\rangle_{\mathbf{G}^F}$ is 0 if $C_\iota \neq C_\gamma$ and otherwise is equal to

$$\sum_{a\in H^1(F,A(u))}\big|C_{\mathbf{G}^F}(u_a)\big|^{-1}\widetilde{\mathcal{Y}}_\iota(u_a)\overline{\widetilde{\mathcal{Y}}_\gamma(u_a)} = a_\iota^{-1}\sum_{a\in A(u)}\big|C^0_{\mathbf{G}}(u_a)^F\big|^{-1}\widetilde{\mathcal{Y}}_\iota(u_a)\overline{\widetilde{\mathcal{Y}}_\gamma(u_a)}. \quad (5.3)$$

To see (5.3), note that $(A(u_a), F)$ is isomorphic to $(A(u), aF)$, so that

$$\big|C_{\mathbf{G}^F}(u_a)\big| = \big|C_{A(u)}(aF)\big|\big|C^0_{\mathbf{G}}(u_a)^F\big|.$$

Finally, it follows from (4.2) and Lemma 5.1 that the matrix whose (ι,γ) entry is either side of (5.3) is the inverse of the matrix whose (ι,γ) entry is the expression in the statement. □

Remark 5.3. The matrices $\widetilde{P}$ and $\widetilde{\Omega}$ have been defined block by block, but may be extended in an obvious way to matrices for the whole of $\mathcal{C}_{\text{uni}}(\mathbf{G}^F)$, which are block-diagonal for the various blocks $\mathcal{I}$; then the computation at the start of the above proof shows in particular that $\{\langle\widetilde{\mathcal{Y}}_\iota, \widetilde{\mathcal{Y}}_\gamma\rangle_{\mathbf{G}^F}\}^{-1}_{\iota,\gamma} = 0$ if ι and γ belong to different blocks.

Corollary 5.2 in turn gives an orthogonality formula for the Q_{wF}, regarded as elements of $\mathcal{C}(W_{\mathbf{G}}(\mathbf{L}).F)$ for a fixed value of the argument:

Corollary 5.4. *For u a unipotent element of $\mathbf{G}^F$, define the function $Q_-(u) \in \mathcal{C}(W_{\mathbf{G}}(\mathbf{L}).F)$ by $Q_-(u)(wF) = Q_{wF}(u)$ (for $wF \in W_{\mathbf{G}}(\mathbf{L}).F$). Then*

$$\begin{aligned}&\big\langle Q_-(u), \mathcal{Z}_{\mathbf{L}}Q_-(u')\big\rangle_{W_{\mathbf{G}}(\mathbf{L}).F}\\ &\quad= \begin{cases} |A(u)|^{-1}\displaystyle\sum_{a\in A(u)}\big|C^0_{\mathbf{G}}(u_a)^F\big|\Big(\sum_{\iota\in\mathcal{I}^F}\overline{\mathcal{Y}_\iota(u_a)}\mathcal{Y}_\iota(u)\Big)\Big(\sum_{\iota\in\mathcal{I}^F}\mathcal{Y}_\iota(u_a)\overline{\mathcal{Y}_\iota(u')}\Big)\\ \quad \text{if } u\sim_{\mathbf{G}} u',\\ 0 \quad \text{otherwise.}\end{cases}\end{aligned} \quad (5.4)$$

Proof. Applying Proposition 4.1 and then Corollary 5.2 to the left-hand side we get

$$\begin{aligned}\big\langle Q_-(u), \mathcal{Z}_{\mathbf{L}} Q_-(u')\big\rangle_{W_{\mathbf{G}}(\mathbf{L}).F} &= \Big\langle \sum_\iota \widetilde{Q}_\iota \widetilde{\mathcal{Y}}_\iota(u), \sum_\gamma \mathcal{Z}_{\mathbf{L}} \widetilde{Q}_\gamma \widetilde{\mathcal{Y}}_\gamma(u')\Big\rangle_{W_{\mathbf{G}}(\mathbf{L}).F} \\ &= \sum_{\iota,\gamma} \widetilde{\mathcal{Y}}_\iota(u)\overline{\widetilde{\mathcal{Y}}_\gamma(u')}\big\{\langle \widetilde{\mathcal{Y}}_\iota, \widetilde{\mathcal{Y}}_\gamma\rangle_{\mathbf{G}^F}\big\}^{-1}_{\iota,\gamma}\end{aligned}$$

we then use that the matrix $\{\langle \widetilde{\mathcal{Y}}_\iota, \widetilde{\mathcal{Y}}_\gamma\rangle_{\mathbf{G}^F}\}_{\iota,\gamma}$ is real to write the complex conjugate of the expression in Corollary 5.2 and we get the result. □

If we sum formula (5.4) over all blocks, we obtain the simpler expression:

Proposition 5.5.

$$\sum_{\mathcal{I}} \big\langle Q_-^{\mathcal{I}}(u), \mathcal{Z}_{\mathbf{L}} Q_-^{\mathcal{I}}(u')\big\rangle_{W_{\mathbf{G}}(\mathbf{L}).F} = \begin{cases} |C_{\mathbf{G}^F}(u)| & \text{if } u \sim_{\mathbf{G}^F} u', \\ 0 & \text{otherwise}\end{cases}$$

where $\mathcal{I}$ runs over the rational blocks and where the superscript $\mathcal{I}$ on the Q_- indicates the block from which it comes.

Proof. If $u \not\sim_{\mathbf{G}} u'$, the left side is clearly zero. If $u \sim_{\mathbf{G}} u'$ then using Remark 5.3, the sum over all blocks of the right-hand side of (5.4) is

$$\big|A(u)\big|^{-1} \sum_{a\in A(u)} \big|C^0_{\mathbf{G}}(u_a)^F\big| \Big(\sum_{\iota\in\mathcal{P}^F} \overline{\mathcal{Y}_\iota(u_a)}\mathcal{Y}_\iota(u)\Big)\Big(\sum_{\iota\in\mathcal{P}^F} \mathcal{Y}_\iota(u_a)\overline{\mathcal{Y}_\iota(u')}\Big).$$

Applying the second orthogonality formula (4.5) for $\mathcal{Y}_\iota$, this reduces to

$$\big|A(u)\big|^{-1} \sum_{\{a\in A(u) \mid u_a\sim_{\mathbf{G}^F} u \text{ and } u_a\sim_{\mathbf{G}^F} u'\}} \big|C^0_{\mathbf{G}}(u_a)^F\big|\big|A(u)^F\big|\big|A(u')^F\big|$$

which is 0 unless $u \sim_{\mathbf{G}^F} u'$ and equal to $|C^F_{\mathbf{G}}(u)|$ otherwise. □

6. Gelfand–Graev characters and their Lusztig restriction

As in [14] and [6], for $\iota \in \mathcal{I}^F$ and $u \in C^F_\iota$, we define $\varGamma_\iota = \sum_{a\in A(u)} \mathcal{Y}_\iota(u_a)\varGamma_{u_a}$, where $\varGamma_{u_a}$ is the generalized Gelfand–Graev character attached to the class of u_a, and other notation is as in (4.3). We need here to assume that p is large enough for the generalized Gelfand–Graev characters to be defined, e.g., $p > 3(h-1)$ where h is the Coxeter number for $\mathbf{G}$.

Proposition 6.1. *We have $\varGamma_\iota = a_\iota \zeta_{\mathcal{I}}^{-1} Q^{\mathbf{G}}(\tilde{\varepsilon}\mathcal{Z}_{\mathbf{L}} \widetilde{Q}^*_\iota)$, where $\zeta_{\mathcal{I}}$ is a fourth root of unity (the one associated to $\mathcal{I}$ in* [14, 7.2] *when* **G** *is split).*

Proof. We start from the formula [14, 7.5(b)] of Lusztig, which must be modified for the case of a non-split group in a way hinted at in [14, 8.7]. We claim that for a possibly non-split group, the equation [14, 7.5(b)] should read

$$\Gamma_{\iota_0} = a_{\iota_0}\zeta_{\mathcal{I}}^{-1}\sum_{\iota,\iota_1}\left|W_{\mathbf{G}}(\mathbf{L})\right|^{-1}\sum_{w\in W_{\mathbf{G}}(\mathbf{L})}\tilde{\varphi}_{\hat{\iota}_1}(wF)\tilde{\varphi}_{\iota}(wF)\left|Z_{\mathbf{L}}^{0wF}\right|\widetilde{P}^*_{\iota_0,\iota}\varepsilon_{\iota_1}\widetilde{\mathcal{X}}_{\iota_1}. \tag{6.1}$$

The only part of the generalization which is not obvious, and which is the source of the coefficient ε_{ι_1} in the above formula, is (as indicated in [14, 8.7]) the lemma [14, 7.2] whose statement should be changed for the general situation to read $\widehat{\widetilde{\mathcal{X}}}_{\iota}\,|_{\mathbf{G}^F_{\mathrm{uni}}} = \zeta_{\mathcal{I}}q^{(\dim\mathbf{G}-\dim Z_{\mathbf{L}})/2}\varepsilon_{\iota}\widetilde{\mathcal{X}}_{\iota}$. The proof given in [14, 7.2] cannot be applied in our more general case, since $\dim V_\iota$ has to be replaced by $\mathrm{Trace}(F \mid V_\iota)$, which might vanish. Nonetheless the generalization may be proved by considering a Frobenius twisted by various $v \in W_{\mathbf{G}}(\mathbf{L})$ on the induced sheaf which Lusztig considers in that proof.

We now rewrite (6.1) as

$$\begin{aligned}
\Gamma_{\iota_0} &= a_{\iota_0}\zeta_{\mathcal{I}}^{-1}\left|W_{\mathbf{G}}(\mathbf{L})\right|^{-1}\sum_{w\in W_{\mathbf{G}}(\mathbf{L})}\left|Z_{\mathbf{L}}^{0wF}\right|\sum_{\iota}\tilde{\varphi}_{\iota}(wF)\widetilde{P}^*_{\iota_0,\iota}\sum_{\iota_1}\tilde{\varphi}_{\hat{\iota}_1}(wF)\varepsilon_{\iota_1}\widetilde{\mathcal{X}}_{\iota_1}\\
&= a_{\iota_0}\zeta_{\mathcal{I}}^{-1}\left|W_{\mathbf{G}}(\mathbf{L})\right|^{-1}\sum_{w\in W_{\mathbf{G}}(\mathbf{L})}\left|Z_{\mathbf{L}}^{0wF}\right|\sum_{\iota}\tilde{\varphi}_{\iota}(wF)\widetilde{P}^*_{\iota_0,\iota}\sum_{\iota_1}\tilde{\varepsilon}(wF)\tilde{\varphi}_{\iota_1}(wF)\widetilde{\mathcal{X}}_{\iota_1}\\
&\qquad \text{by Remark 3.6}\\
&= a_{\iota_0}\zeta_{\mathcal{I}}^{-1}\left|W_{\mathbf{G}}(\mathbf{L})\right|^{-1}\sum_{w\in W_{\mathbf{G}}(\mathbf{L})}\left|Z_{\mathbf{L}}^{0wF}\right|\sum_{\iota}\tilde{\varphi}_{\iota}(wF)\widetilde{P}^*_{\iota_0,\iota}\tilde{\varepsilon}(wF)Q_{wF} \quad \text{by (3.1)}\\
&= a_{\iota_0}\zeta_{\mathcal{I}}^{-1}\left|W_{\mathbf{G}}(\mathbf{L})\right|^{-1}\sum_{w\in W_{\mathbf{G}}(\mathbf{L})}\left|Z_{\mathbf{L}}^{0wF}\right|\tilde{\varepsilon}(wF)\widetilde{Q}^*_{\iota_0}(wF)Q_{wF} \quad \text{by (4.3).}
\end{aligned}$$

The proposition now follows by Definition 3.1(iii). □

Let us write $\tilde{\zeta}_{\mathcal{I}}$ for the root of unity denoted by $\zeta_{\mathcal{I}}$ in [6]. The point of this notation is to distinguish $\tilde{\zeta}_{\mathcal{I}}$ and $\zeta_{\mathcal{I}}$, since they turn out to be different generalizations to non-split groups of Lusztig's constant.

Proposition 6.2. *For any reductive group, let* $\sigma_{\mathbf{G}} := (-1)^{\text{semi-simple rank }(\mathbf{G})}$. *Then* $\tilde{\zeta}_{\mathcal{I}} = \eta_{\mathbf{L}}\sigma_{\mathbf{L}}\zeta_{\mathcal{I}}$.

Proof. We have

$$\begin{aligned}
\left\langle D_{\mathbf{G}}\Gamma_{\iota}, \widetilde{\mathcal{X}}_{\kappa}\right\rangle_{\mathbf{G}^F} &= \left\langle \Gamma_{\iota}, D_{\mathbf{G}}\widetilde{\mathcal{X}}_{\kappa}\right\rangle_{\mathbf{G}^F} = \eta_{\mathbf{L}}\varepsilon_{\kappa}\left\langle \Gamma_{\iota}, \widetilde{\mathcal{X}}_{\hat{\kappa}}\right\rangle_{\mathbf{G}^F} \quad \text{by Proposition3.14(ii)}\\
&= \eta_{\mathbf{L}}\varepsilon_{\kappa}\left\langle \mathcal{Z}_{\mathbf{L}}^{-1}(Q^{\mathbf{G}})^{-1}(\Gamma_{\iota}), (Q^{\mathbf{G}})^{-1}(\widetilde{\mathcal{X}}_{\hat{\kappa}})\right\rangle_{W_{\mathbf{G}}(\mathbf{L}).F} \quad \text{by Proposition 3.4}\\
&= \eta_{\mathbf{L}}\varepsilon_{\kappa}a_{\iota}\zeta_{\mathcal{I}}^{-1}\langle\tilde{\varepsilon}\widetilde{Q}^*_{\iota}, \tilde{\varphi}_{\hat{\kappa}}\rangle_{W_{\mathbf{G}}(\mathbf{L}).F}
\end{aligned}$$

$$= \eta_{\mathbf{L}} a_\iota \zeta_{\mathcal{I}}^{-1} \langle \widetilde{Q}_\iota^*, \tilde{\varphi}_\kappa \rangle_{W_{\mathbf{G}}(\mathbf{L}).F} \quad \text{by Remark 3.6}$$
$$= \eta_{\mathbf{L}} a_\iota \zeta_{\mathcal{I}}^{-1} \widetilde{P}_{\iota,\kappa}^* \quad \text{by the } * \text{ of (4.3).}$$

The equation [6, 1.7] is transformed into this last relation if $\sigma_{\mathbf{L}} \tilde{\zeta}_{\mathcal{I}}^{-1}$ is replaced by $\eta_{\mathbf{L}} \zeta_{\mathcal{I}}^{-1}$, whence the proposition. □

It will be convenient to use the normalization $\widetilde{\Gamma}_\iota = a_\iota^{-1} \zeta_{\mathcal{I}} \Gamma_\iota$. We shall now discuss orthogonality relations among the $\widetilde{\Gamma}_\iota$ and among the Γ_u, as well as the Lusztig restriction of the $\widetilde{\Gamma}_\iota$. Note that from Proposition 6.1 it follows that if $\mathcal{I}$ is a rational block and $\iota \in \mathcal{I}^F$, then $\widetilde{\Gamma}_\iota \in \mathcal{C}_{\mathcal{I}}(\mathbf{G}^F)$.

Lemma 6.3. *For any rational block $\mathcal{I}$ define $\widetilde{\Gamma}_u^{\mathcal{I}} = \sum_{\iota \in \mathcal{I}} \overline{\mathcal{Y}_\iota^*(u)}\, \widetilde{\Gamma}_\iota$. If there is a pair $\iota \in \mathcal{I}^F$ whose support contains u, the orthogonal projection of Γ_u onto $\mathcal{C}_{\mathcal{I}}(\mathbf{G}^F)$ is $\zeta_{\mathcal{I}}^{-1} q^{c_\iota} \widetilde{\Gamma}_u^{\mathcal{I}}$; otherwise it is* 0.

Proof. Using (4.5), the defining relation for Γ_ι can be inverted to give

$$\Gamma_u = \left|A(u)\right|^{-1} \sum_{\iota \in \mathcal{P}^F} \overline{\mathcal{Y}_\iota(u)}\, \Gamma_\iota.$$

If we restrict the above sum to $\iota \in \mathcal{I}^F$ we obtain the orthogonal projection of Γ_u onto $\mathcal{C}_{\mathcal{I}}(\mathbf{G}^F)$, since the various spaces $\mathcal{C}_{\mathcal{I}}(\mathbf{G}^F)$ are mutually orthogonal. The lemma now follows in straightforward fashion from the definitions. □

Proposition 6.4. *We have $\widetilde{\Gamma}_u^{\mathcal{I}} = Q^{\mathbf{G}}(\tilde{\varepsilon} \mathcal{Z}_{\mathbf{L}} \overline{Q}_-^*(u))$.*

Proof. Apply $(Q^{\mathbf{G}})^{-1}$ to the expression in Lemma 6.3 for $\widetilde{\Gamma}_u^{\mathcal{I}}$ to get

$$\left(Q^{\mathbf{G}}\right)^{-1}\left(\widetilde{\Gamma}_u^{\mathcal{I}}\right) = \sum_{\iota \in \mathcal{I}^F} \overline{\mathcal{Y}_\iota^*(u)}\, \tilde{\varepsilon} \mathcal{Z}_{\mathbf{L}} \widetilde{Q}_\iota^*.$$

Now take the complex conjugate of the * of the relation (4.1) and substitute into this last equation. Taking into account that the functions $\widetilde{Q}_\iota$ are real valued (i.e. stable under complex conjugation), which is a consequence of (4.3) since the $\tilde{\varphi}_\iota$ are real, we obtain the proposition. □

Corollary 6.5. *We have $\langle \widetilde{\Gamma}_\iota, \mathcal{D}_{\mathbf{G}} \widetilde{\Gamma}_\kappa \rangle_{\mathbf{G}^F} = \varepsilon_{\mathbf{G}} q^{\dim Z_{\mathbf{L}}} (\{\langle \widetilde{\mathcal{Y}}_\iota, \widetilde{\mathcal{Y}}_\kappa \rangle_{\mathbf{G}^F}\}_{\iota,\kappa}^{-1})^*$, which is zero if $C_\iota \neq C_\kappa$.*

Proof. We have

$$\left\langle \widetilde{\Gamma}_\iota, \mathcal{D}_{\mathbf{G}} \widetilde{\Gamma}_\kappa \right\rangle_{\mathbf{G}^F} = \left\langle \mathcal{Z}_{\mathbf{L}}^{-1}\left(Q^{\mathbf{G}}\right)^{-1}\left(\widetilde{\Gamma}_\iota\right), \eta_{\mathbf{L}} \tilde{\varepsilon}\left(Q^{\mathbf{G}}\right)^{-1}\left(\widetilde{\Gamma}_\kappa\right)\right\rangle_{W_{\mathbf{G}}(\mathbf{L}).F}$$
by Propositions 3.4 and 3.14(ii)

$$\begin{aligned}
&= \eta_{\mathbf{L}}\big\langle \widetilde{Q}^*_\iota, \mathcal{Z}_{\mathbf{L}}\tilde{\varepsilon}\widetilde{Q}^*_\kappa\big\rangle_{W_{\mathbf{G}}(\mathbf{L}).F}\\
&= \eta_{\mathbf{L}}\varepsilon_{Z_{\mathbf{L}}}q^{\dim Z_{\mathbf{L}}}\big\langle \widetilde{Q}^*_\iota, \mathcal{Z}^*_{\mathbf{L}}\widetilde{Q}^*_\kappa\big\rangle_{W_{\mathbf{G}}(\mathbf{L}).F} \quad \text{by Lemma 3.9}\\
&= \varepsilon_{\mathbf{L}}q^{\dim Z_{\mathbf{L}}}\big\langle \widetilde{Q}^*_\iota, \mathcal{Z}^*_{\mathbf{L}}\widetilde{Q}^*_\kappa\big\rangle_{W_{\mathbf{G}}(\mathbf{L}).F} \quad \text{since } \eta_{\mathbf{L}} = \varepsilon_{\mathbf{L}}\varepsilon_{Z_{\mathbf{L}}}\\
&= \varepsilon_{\mathbf{L}}q^{\dim Z_{\mathbf{L}}}\big(\big\langle \widetilde{Q}_\iota, \mathcal{Z}_{\mathbf{L}}\widetilde{Q}_\kappa\big\rangle_{W_{\mathbf{G}}(\mathbf{L}).F}\big)^*\\
&= \varepsilon_{\mathbf{L}}q^{\dim Z_{\mathbf{L}}}\big(\big\{\big\langle \widetilde{\mathcal{Y}}_\iota, \widetilde{\mathcal{Y}}_\kappa\big\rangle_{\mathbf{G}^F}\big\}^{-1}_{\iota,\kappa}\big)^* \quad \text{by Corollary 5.2.}
\end{aligned}$$

The result now follows because $\varepsilon_{\mathbf{L}} = \varepsilon_{\mathbf{G}}$ since $\mathbf{L}$ is $\mathbf{G}$-split. □

Corollary 6.6. *Let $u, v \in \mathbf{G}^F$ be unipotent elements and $\mathcal{I}$ a rational block. Then*

$$\big\langle \widetilde{\Gamma}^{\mathcal{I}}_u, \mathcal{D}_{\mathbf{G}}\widetilde{\Gamma}^{\mathcal{I}}_v\big\rangle_{\mathbf{G}^F} = \varepsilon_{\mathbf{G}}q^{\dim Z_{\mathbf{L}}}\big(\big\langle \overline{Q}_-(u), \mathcal{Z}_{\mathbf{L}}\overline{Q}_-(v)\big\rangle_{W_{\mathbf{G}}(\mathbf{L}).F}\big)^*,$$

which is non-zero only if u and v are conjugate in $\mathbf{G}$.

Proof. We have, from Proposition 6.4, proceeding as in the proof of Corollary 6.5

$$\begin{aligned}
\big\langle \widetilde{\Gamma}^{\mathcal{I}}_u, \mathcal{D}_{\mathbf{G}}\widetilde{\Gamma}^{\mathcal{I}}_v\big\rangle_{\mathbf{G}^F} &= \big\langle \tilde{\varepsilon}\overline{Q}^*_-(u), \eta_{\mathbf{L}}\tilde{\varepsilon}^2\mathcal{Z}_{\mathbf{L}}\overline{Q}^*_-(v)\big\rangle_{W_{\mathbf{G}}(\mathbf{L}).F}\\
&= \eta_{\mathbf{L}}\big\langle \tilde{\varepsilon}\overline{Q}^*_-(u), \mathcal{Z}_{\mathbf{L}}\overline{Q}^*_-(v)\big\rangle_{W_{\mathbf{G}}(\mathbf{L}).F}\\
&= \eta_{\mathbf{L}}\varepsilon_{Z_{\mathbf{L}}}q^{\dim Z_{\mathbf{L}}}\big\langle \tilde{\varepsilon}\overline{Q}^*_-(u), \tilde{\varepsilon}\mathcal{Z}^*_{\mathbf{L}}\overline{Q}^*_-(v)\big\rangle_{W_{\mathbf{G}}(\mathbf{L}).F}\\
&= \varepsilon_{\mathbf{L}}q^{\dim Z_{\mathbf{L}}}\big\langle \overline{Q}^*_-(u), \mathcal{Z}^*_{\mathbf{L}}\overline{Q}^*_-(v)\big\rangle_{W_{\mathbf{G}}(\mathbf{L}).F}\\
&= \varepsilon_{\mathbf{L}}q^{\dim Z_{\mathbf{L}}}\big(\big\langle \overline{Q}_-(u), \mathcal{Z}_{\mathbf{L}}\overline{Q}_-(v)\big\rangle_{W_{\mathbf{G}}(\mathbf{L}).F}\big)^*,
\end{aligned}$$

and the result follows as in Corollary 6.5. The last remark is a consequence of the evaluation of the right side in (5.4). □

Corollary 6.7. *For any pair u, v of unipotent elements of $\mathbf{G}^F$, we have*

$$\langle \Gamma_u, D_{\mathbf{G}}\Gamma_v\rangle = \begin{cases} \varepsilon_{\mathbf{G}}\varepsilon_{C_{\mathbf{G}}(u)}|C_{\mathbf{G}^F}(u)|_{q'} & \textit{if } u \sim_{\mathbf{G}^F} v,\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. From Lemma 6.3, we see $\langle \Gamma_u, D_{\mathbf{G}}\Gamma_v\rangle = \sum_{\mathcal{I}}\langle \zeta_{\mathcal{I}}^{-1}q^{c_\iota}\widetilde{\Gamma}^{\mathcal{I}}_u, \zeta_{\mathcal{I}}^{-1}q^{c_\gamma}\mathcal{D}_{\mathbf{G}}\widetilde{\Gamma}^{\mathcal{I}}_v\rangle_{\mathbf{G}^F}$ where the sum is over all blocks which contain two pairs ι, γ whose support contains respectively u and v. By Corollary 6.6 this sum is 0 if u and v are not $\mathbf{G}$-conjugate; otherwise we obtain

$$\langle \Gamma_u, D_{\mathbf{G}}\Gamma_v\rangle = \varepsilon_{\mathbf{G}}q^{\operatorname{codim}(\operatorname{class}(u))}\sum_{\mathcal{I}}\big(\big\langle \overline{Q}^{\mathcal{I}}_-(u), \mathcal{Z}_{\mathbf{L}}\overline{Q}^{\mathcal{I}}_-(v)\big\rangle_{W_{\mathbf{G}}(\mathbf{L}).F}\big)^*.$$

We now apply Propositions 5.5 and 3.7(ii) to complete the proof. □

To describe the Lusztig restrictions of the $\widetilde{\Gamma}_\iota$, we shall define the notion of "sign relative to a block". Suppose **M** is a rational Levi subgroup which contains a rational **G**-conjugate of **L**, as in Remarks 2.2 and 2.4. If $\mathbf{T}_0$ is a maximally split rational maximal torus of **L** (and hence of **G**), the element $w \in W_{\mathbf{G}}(\mathbf{L}) \subseteq W_{\mathbf{G}}(\mathbf{T}_0)/W_{\mathbf{L}}(\mathbf{T}_0)$. This element is uniquely defined by **M** and the conditions on $\mathbf{L}_w$, up to F-conjugacy in $W_{\mathbf{G}}(\mathbf{L})$. The sign $\varepsilon_{\mathcal{I}}(\mathbf{M})$ of **M** relative to the block $\mathcal{I}$ is defined as -1 raised to the codimension in $Y(Z^0_{\mathbf{L}}) \otimes \mathbb{R}$ of the subspace of w-fixed points of $Y(Z^0_{\mathbf{L}}) \otimes \mathbb{R}$. It has also the following alternative definition:

Definition 6.8. With notation as in the previous paragraph, define $\varepsilon_{\mathcal{I}}(\mathbf{M}) := \varepsilon^{\mathbf{G}}(w)$ where $\varepsilon^{\mathbf{G}}$ is the sign character of $W_{\mathbf{G}}(\mathbf{L})$.

It follows from the remarks in the last paragraph that the right side depends only on (the $\mathbf{G}^F$-conjugacy class of) **M**.

Lemma 6.9. (i) *In the notation of Remarks* 2.2 *and* 2.4, *there exist Laurent polynomials* $R_{\iota,\gamma}$ *in* q $(\iota \in \mathcal{I}^F$ *and* $\gamma \in \mathcal{I}^F_{\mathbf{M}})$ *such that*

$$\operatorname{Res}^{W_{\mathbf{G}}(\mathbf{L}).F}_{W_{\mathbf{M}_0}(\mathbf{L}).wF} \widetilde{Q}_\iota = \sum_{\gamma \in \mathcal{I}^F_{\mathbf{M}}} R_{\iota,\gamma} \widetilde{Q}_\gamma.$$

We have $R_{\iota,\gamma} = 0$ *unless* $\overline{C_\gamma} \subset \overline{C_\iota} \subset \overline{\operatorname{Ind}^{\mathbf{G}}_{\mathbf{M}} C_\gamma}$.

(ii) *Maintaining the above notation, we have* $\operatorname{Res}^{W_{\mathbf{G}}(\mathbf{L}).F}_{W_{\mathbf{M}_0}(\mathbf{L}).wF} \tilde{\varepsilon}^{\mathbf{G}} = \varepsilon_{\mathcal{I}}(\mathbf{M})\tilde{\varepsilon}^{\mathbf{M}}$, *where* $\varepsilon_{\mathcal{I}}(\mathbf{M})$ *is defined in Definition* 6.8 *and where* $\tilde{\varepsilon}^{\mathbf{G}}$ (*respectively* $\tilde{\varepsilon}^{\mathbf{M}}$) *is the preferred extension of the sign character of* $W_{\mathbf{G}}(\mathbf{L})$ (*respectively* $W_{\mathbf{M}_0}(\mathbf{L})$).

Proof. Let R be the matrix with (ι,γ) coefficient $R_{\iota,\gamma}$ as in (i) of the statement. From (4.3), we obtain the matrix equation

$$\widetilde{P}^{\mathbf{G}}\bigl\{\bigl\langle\tilde{\varphi}_\gamma, \operatorname{Res}^{W_{\mathbf{G}}(\mathbf{L}).F}_{W_{\mathbf{M}_0}(\mathbf{L}).wF} \tilde{\varphi}_{\iota'}\bigr\rangle_{W_{\mathbf{M}_0}(\mathbf{L}).wF}\bigr\}_{\iota',\gamma} = R\widetilde{P}^{\mathbf{M}}.$$

The first statement in (i) is now immediate, since the entries of the unitriangular matrix $\widetilde{P}^{\mathbf{M}}$ are Laurent polynomials, whence the same is true of its inverse. The second statement in (i) follows from Proposition 2.3(ii).

For (ii), let $v.wF \in W_{\mathbf{M}_0}(\mathbf{L}).wF$. Then

$$\tilde{\varepsilon}^{\mathbf{G}}(v.wF) = \varepsilon^{\mathbf{G}}(vw) = \varepsilon^{\mathbf{G}}(v)\varepsilon^{\mathbf{G}}(w) = \tilde{\varepsilon}^{\mathbf{M}}(v.wF)\varepsilon_{\mathcal{I}}(\mathbf{M}). \qquad \square$$

Proposition 6.10. *We have* ${}^*R^{\mathbf{G}}_{\mathbf{M}}(\widetilde{\Gamma}_\iota) = 0$ *unless* **M** *contains a rational* **G**-*conjugate of* **L**; *in the latter case, we have*

$${}^*R^{\mathbf{G}}_{\mathbf{M}}(\widetilde{\Gamma}_\iota) = \varepsilon_{\mathcal{I}}(\mathbf{M}) \sum_{\gamma \in \mathcal{I}^F_{\mathbf{M}}} R^*_{\iota,\gamma} \widetilde{\Gamma}_\gamma, \tag{6.2}$$

where notation is as in Proposition 6.4 *and Lemma* 6.9.

Proof. Since $\widetilde{\Gamma}_\iota \in \mathcal{C}_{\mathcal{I}}(\mathbf{G}^F)$, it follows from Theorem 3.11 that ${}^*R^{\mathbf{G}}_{\mathbf{M}}\widetilde{\Gamma}_\iota$ is zero unless $\mathbf{M}$ contains a rational $\mathbf{G}$-conjugate of $\mathbf{L}$. We therefore take $\mathbf{M}$ as in Theorem 3.11. Now by Proposition 6.1, $\widetilde{\Gamma}_\iota = Q^{\mathbf{G}}(\tilde{\varepsilon}\mathcal{Z}_{\mathbf{L}}\widetilde{Q}^*_\iota)$, and $\mathcal{Z}_{\mathbf{L}} \in \mathcal{C}(W_{\mathbf{G}}(\mathbf{L}).F)$ is defined in Definition 3.3. By Theorem 3.11 we need only compute the restriction to $W_{\mathbf{M}_0}(\mathbf{L})$ of $\tilde{\varepsilon}\mathcal{Z}_{\mathbf{L}}\widetilde{Q}^*_\iota$, and a straightforward calculation using Lemma 6.9 yields the statement. □

Remark 6.11. As in [6], we refer to a block as regular if it contains a local system supported by the regular unipotent class. It is a consequence of [6, Section 2] that for regular blocks, $\tilde{\zeta}_{\mathcal{I}}$ is independent of the ambient group and the rational structure, i.e. depends only on the geometric data in the cuspidal system $(\mathbf{L}, \iota_0)$. This is asserted without justification in the proof of [6, 3.4] but can be seen as follows. From [6, 2.1] and [6, 2.5] one has that $\tilde{\zeta}_{\mathcal{I}}$ is equal (in the notation of [6]) to $\eta_{\mathbf{L}}\sigma_{\mathbf{L}}\sigma^{\mathbf{L}}_{\zeta}$ up to a power of q. Using the Hasse–Davenport relation, one may compare the product of Gauss sums in [6, 2.4] which applies to the case of twisted $\mathbf{L}$, to that occurring in a split group. One finds that the products also differ by a factor $\eta_{\mathbf{L}}\sigma_{\mathbf{L}}$. Thus $\tilde{\zeta}_{\mathcal{I}} = \tilde{\zeta}_{\mathcal{I}_{\mathbf{M}}}$ in this case. In particular, this applies generally to the principal block (when $\mathbf{L}$ is a maximal torus). In general, the question as to whether $\tilde{\zeta}_{\mathcal{I}} = \tilde{\zeta}_{\mathcal{I}_{\mathbf{M}}}$ in all cases amounts to the question of whether $\tilde{\zeta}_{\mathcal{I}_{\mathbf{L}}}$ is independent of the Frobenius structure on the triple $(L, C_{\iota_0}, \iota_0)$. Although this point does not affect the formulation of Proposition 6.10, it is relevant to some of the computations later in this work.

Remark 6.12. Equation (6.2) may be expressed as follows:

$$ {}^*R^{\mathbf{G}}_{\mathbf{M}}(a_\iota\Gamma_\iota) = \varepsilon_{\mathcal{I}}(\mathbf{M})\zeta_{\mathcal{I}}\zeta^{-1}_{\mathcal{I}_{\mathbf{M}}}\sum_{\gamma\in\mathcal{I}^F_{\mathbf{M}}} R^*_{\iota,\gamma}a_\gamma\Gamma_\gamma = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{M}}\tilde{\zeta}_{\mathcal{I}}\tilde{\zeta}^{-1}_{\mathcal{I}_{\mathbf{M}}}\sum_{\gamma\in\mathcal{I}^F_{\mathbf{M}}} R^*_{\iota,\gamma}a_\gamma\Gamma_\gamma, \tag{6.3}$$

and the previous remark implies that in the regular case, the factor $\tilde{\zeta}_{\mathcal{I}}\tilde{\zeta}^{-1}_{\mathcal{I}_{\mathbf{M}}}$ is equal to 1.

7. Application to the regular and subregular cases

Our objective now is to apply Proposition 6.10 to some specific cases. The general strategy will be first to compute (4.3) explicitly in $\mathbf{G}$ and in $\mathbf{M}$ by computing certain required values $\widetilde{P}_{\iota,\kappa}$, and then to use specific knowledge of restriction of characters from $W_{\mathbf{G}}(\mathbf{L}).F$ to $W_{\mathbf{M}_0}(\mathbf{L}).wF$.

As an example, consider first the case when $\iota = \rho$, where ρ is a pair in the block $\mathcal{I}$ with support the regular unipotent class (such a pair is then the unique one with regular support in the block $\mathcal{I}$, see [6, 1.10]). Then the only non-zero term in the right hand side of formula (4.3) is $\tilde{\varphi}_\rho(wF)$, as $\widetilde{P}_{\rho,\rho} = 1$ and $\widetilde{P}_{\rho,\gamma} = 0$ if $C_\rho \not\subset \overline{C}_\gamma$. Moreover, as ρ has regular support we have $\tilde{\varphi}_\rho = \mathrm{Id}$. So we get $\widetilde{Q}_{\rho^{\mathbf{G}}} = \mathrm{Id}_{W_{\mathbf{G}}(\mathbf{L})}F$, whence $\mathrm{Res}^{W_{\mathbf{G}}(\mathbf{L}).F}_{W_{\mathbf{M}_0}(\mathbf{L}).wF}\widetilde{Q}_{\rho^{\mathbf{G}}} = \widetilde{Q}_{\rho^{\mathbf{M}}}$. Applying (6.12) we get

$$ {}^*R^{\mathbf{G}}_{\mathbf{M}}\Gamma_{\rho^{\mathbf{G}}} = \frac{a_{\rho^{\mathbf{G}}}}{a_{\rho^{\mathbf{M}}}}\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{M}}\Gamma_{\rho^{\mathbf{M}}}. $$

Thus we recover Lemma 3.6 of [6].

Proposition 7.1. *Consider an F-stable pair σ with support a subregular class C_σ of* $\mathbf{G}$ *and denote by $\mathcal{I}$ the corresponding block; then one of the following holds*:

(i) *The representation φ_σ is a component of the reflection representation r of $W_{\mathbf{G}}(\mathbf{L})$. In this case, $\widetilde{Q}_\sigma = q\widetilde{\mathrm{Id}} + \tilde{\varphi}_\sigma$ and the block $\mathcal{I}$ is regular.*
(ii) *The representation φ_σ is not a component of $r^{\wedge i}$ for any i; then $\widetilde{Q}_\sigma = \tilde{\varphi}_\sigma$. In this case the block may or may not be regular.*

We shall refer to case (i) by saying that σ is *standard*. Recall that a block $\mathcal{I}$ is regular if there exists a local system in $\mathcal{I}$ with support the regular class and that in that case this local system is unique and corresponds to the identity representation of $W_{\mathbf{G}}(\mathbf{L})$ (cf. [6, 1.10]).

Proof. We prove first that one of the two properties for φ_σ and $\mathcal{I}$ holds. This is done by checking the tables of Appendix A. First we reduce to the case when $\mathbf{G}$ is quasi-simple and simply connected. If $\mathbf{G}$ is not quasi-simple, a unipotent class is a product of unipotent classes of the quasi-simple components. In particular a subregular class is the product of the regular classes of all the components but one and the subregular class in the last component. Although local systems depend on isogeny, Green functions do not, and hence it clearly suffices to treat the simply connected group in each isogeny class, in which case we may assume that the local system on such a class is the product of local systems on the components. In particular, a cuspidal datum is a product of cuspidal data for the quasi-simple components. All this shows that we can reduce the verification to the quasi-simple (simply connected) case.

It is then apparent from the tables that when φ_σ is the reflection representation, the block is regular and that otherwise φ has dimension strictly less than the reflection representation, so appears in no exterior power of the reflection representation.

We now prove the formula for $\widetilde{Q}_\sigma$ in each case. We know that $P_{\iota,\gamma}$ is zero unless $C_\iota \subsetneq \overline{C_\gamma}$ or $\iota = \gamma$. So $P_{\sigma,\iota} = 0$ unless C_ι is the regular class or $\iota = \sigma$.

Consider first the case when $\mathcal{I}$ is regular: denote by ρ the unique pair in $\mathcal{I}$ with regular support. If we take the rows and columns pertaining to σ and ρ to be the last two, the matrix equation $\widetilde{P}^{-1}\widetilde{\Lambda}^{-1}({}^t\widetilde{P}^{-1}) = \widetilde{\Omega}$ which determines $\widetilde{P}$ and $\widetilde{\Lambda}$ has the form:

$$\begin{pmatrix} \dots & \dots & \dots \\ 0 & 1 & Q \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \dots & \dots & 0 \\ 0 & \mu_\sigma & 0 \\ 0 & 0 & \mu_\rho \end{pmatrix}\begin{pmatrix} \dots & 0 & 0 \\ \dots & 1 & 0 \\ \dots & Q & 1 \end{pmatrix} = \begin{pmatrix} \dots & \dots & \dots \\ \dots & \widetilde{\omega}_{\sigma,\sigma} & \widetilde{\omega}_{\sigma,\rho} \\ \dots & \widetilde{\omega}_{\rho,\sigma} & \widetilde{\omega}_{\rho,\rho} \end{pmatrix}$$

where $Q = (\widetilde{P}^{-1})_{\sigma,\rho}$, $\mu_\sigma = (\widetilde{\Lambda}^{-1})_{\sigma,\sigma}$ and $\mu_\rho = (\widetilde{\Lambda}^{-1})_{\rho,\rho}$. We thus get: $\mu_\sigma + Q^2\mu_\rho = \widetilde{\omega}_{\sigma,\sigma}$, $Q\mu_\rho = \widetilde{\omega}_{\sigma,\rho}$ and $\mu_\rho = \widetilde{\omega}_{\rho,\rho}$.

In case (i) we apply (5.2). If $\mathbf{G}_1, \dots, \mathbf{G}_k$ are the quasi-simple components of $\mathbf{G}$, we have $r^{\wedge i} = \sum_{i_1+\dots+i_k=i} r_1^{\wedge i_1} \otimes \dots \otimes r_k^{\wedge i_k}$, where r_i is the reflection representation of the ith component of $W_{\mathbf{G}}(\mathbf{L})$. So, using the remarks following Lemma 3.10 we have

$$\langle \tilde{\varphi}_\sigma, \tilde{r}^{\wedge i} \rangle_{W_{\mathbf{G}}(\mathbf{L}).F} = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We then obtain $\widetilde{\omega}_{\sigma,\rho} = -|Z_{\mathbf{G}}^{0F}|q^{l-1}$ and $\widetilde{\omega}_{\rho,\rho} = |Z_{\mathbf{G}}^{0F}|q^{l}$ where l is as in (5.2), whence $Q = -q$, and $\widetilde{P}_{\sigma,\rho} = q$.

In case (ii), the above computation gives $\widetilde{\omega}_{\sigma,\rho} = 0$, so the only non-zero $\widetilde{P}_{\sigma,\iota}$ is $\widetilde{P}_{\sigma,\sigma} = 1$.

It remains only to consider case (ii) for a non-regular block, where dimension considerations imply that the only non-zero entry $\widetilde{P}_{\sigma,\iota}$ is $\widetilde{P}_{\sigma,\sigma}$.

In either case, the value of $\widetilde{Q}_\sigma$ by is obtained by applying (4.3). □

Proposition 7.2. *Assume that σ is an F-stable standard subregular pair in the regular block $\mathcal{I}_{\mathbf{G}}$, and that $\mathbf{G}$ is quasi-simple. Let $\mathbf{M}$ be a rational Levi subgroup of $\mathbf{G}$, and let $C_1, \dots, C_k$ be the F-stable subregular classes in $\mathbf{M}$, which are in bijection with the set of wF-stable irreducible constituents $\mathbf{M}_i$ of $\mathbf{M}_0$. Let σ_i be the pair corresponding to the reflection representation of $W_{\mathbf{M}_i}(\mathbf{L})$; then σ_i has support C_i and is a standard pair in the regular block $\mathcal{I}_{\mathbf{M}}$. Moreover we have*

$$\operatorname{Res}^{W_{\mathbf{G}}(\mathbf{L}).F}_{W_{\mathbf{M}_0}(\mathbf{L}).wF} \widetilde{Q}_\sigma = \left((1-k)q^{-1} + \tilde{\varphi}_\sigma(wF) - \sum_{i=1}^{k} \tilde{\varphi}_{\sigma_i}(wF)\right)\widetilde{Q}_{\rho^{\mathbf{M}}} + \sum_{i=1}^{k} \widetilde{Q}_{\sigma_i}$$

where $\rho^{\mathbf{M}}$ is the pair with regular support in $\mathcal{I}_{\mathbf{M}}$.

Proof. Let $V_{\mathbf{G}} = Y(Z_{\mathbf{L}}^0/Z_{\mathbf{G}}^0) \otimes \mathbb{R}$, and $V_{\mathbf{M}_0} = Y(Z_{\mathbf{L}}^0/Z_{\mathbf{M}_0}^0) \otimes \mathbb{R}$. By Lemma 3.10, $\tilde{\varphi}_\sigma$ is the extension of the reflection representation of $W_{\mathbf{G}}(\mathbf{L})$ which occurs in $V_{\mathbf{G}}$, and by the same remarks we have $\operatorname{Trace}(vwF \mid V_{\mathbf{M}_0}) = \sum_i \tilde{\varphi}_{\sigma_i}(vwF)$ for $v \in W_{\mathbf{M}_0}(\mathbf{L})$ (only the wF-stable components occur when we take the trace of an element in the coset $W_{\mathbf{M}_0}(\mathbf{L}).wF$). Thus if V is the kernel of the natural map $V_{\mathbf{G}} \to V_{\mathbf{M}_0}$, we have $\operatorname{Res}^{W_{\mathbf{G}}(\mathbf{L}).F}_{W_{\mathbf{M}_0}(\mathbf{L}).wF} \tilde{\varphi}_\sigma = \sum_{i=1}^{i=k} \tilde{\varphi}_{\sigma_i} + \operatorname{Trace}(wF \mid V)\widetilde{\operatorname{Id}}$. Evaluating both sides at wF we get $\operatorname{Trace}(wF \mid V) = \tilde{\varphi}_\sigma(wF) - \sum_{i=1}^{i=k} \tilde{\varphi}_{\sigma_i}(wF)$.

Now by [6, 1.10] since the block $\mathcal{I}_{\mathbf{G}}$ is regular by assumption, the block $\mathcal{I}_{\mathbf{M}}$ is also regular. We know from the remark after the statement of Proposition 7.1 that the pairs which occur in the restriction of $\widetilde{Q}_\sigma$ have regular or sub-regular support. Since the regular class corresponds to $\widetilde{\operatorname{Id}}$ in any regular block, σ_i must have support C_i, and thus σ_i is standard, so that by Proposition 7.1 we have $\tilde{\varphi}_{\sigma_i} = \widetilde{Q}_{\sigma_i} - q\widetilde{Q}_{\rho^{\mathbf{M}}}$.

The formula for the restriction of $\widetilde{Q}_\sigma$ results from this and the above formula for the restriction of $\tilde{\varphi}_\sigma$. □

From Remark 6.12 and Proposition 7.2, we deduce

Proposition 7.3. *For any standard subregular pair σ, we have*

$$\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{M}}{}^*R_{\mathbf{M}}^{\mathbf{G}}\Gamma_\sigma = \frac{a_\sigma}{a_{\sigma_i}}\Gamma_{\sigma_i} + \frac{a_\sigma}{a_{\rho^{\mathbf{M}}}}\left((1-k)q + \tilde{\varphi}_\sigma(wF) - \sum_{i=1}^{k} \tilde{\varphi}_{\sigma_i}(wF)\right)\Gamma_{\rho^{\mathbf{M}}}.$$

Similar computations can be made for non-standard pairs; however the end result does not appear to have as clear a statement.

8. The case of SL_n

We now discuss the case of $\mathbf{G} = \mathrm{SL}_n$. According to [15, Section 5], cuspidal data are indexed by characters of the centre Z of SL_n. Assume that χ is a character of order d of Z where d is a divisor of n; then χ corresponds to an equivariant cuspidal local system on the regular class of a Levi subgroup of type $A_{d-1}^{n/d}$. We will denote by $\mathcal{I}_\chi$ the corresponding block of $\mathbf{G}$. The unipotent classes of $\mathbf{G}$ are indexed by partitions of n. Let C_λ be the class indexed by the partition λ of n. There is at most one local system on C_λ in $\mathcal{I}_\chi$; such a system exists when all the parts of λ are divisible by d and we will denote it by ι_λ^χ. When χ is the trivial character, ι_λ^χ is the trivial local system on C_λ, which is also the only irreducible local system on C_λ in GL_n. We will denote it simply by ι_λ in the latter case.

Theorem 8.1. *The Laurent polynomial $\widetilde{P}_{\iota_\lambda^\chi,\iota_\mu^\chi}$ for SL_n is equal to the Laurent polynomial $\widetilde{P}_{\iota_{\lambda/d},\iota_{\mu/d}}$ for $\mathrm{GL}_{n/d}$, where λ/d (respectively μ/d) denotes the partition whose parts are $1/d$ times those of λ (respectively μ).*

Proof. The proof consists of merely observing that the equations which determine $\widetilde{P}_{\iota_\lambda^\chi,\iota_\mu^\chi}$ and $\widetilde{P}_{\iota_{\lambda/d},\iota_{\mu/d}}$ coincide. In either case the equation may be written: $\widetilde{P}^{-1}\Lambda_1({}^t\widetilde{P}^{-1}) = \Omega_1$ where $\Lambda_1 = |Z_{\mathbf{G}}^{0F}|^{-1}\widetilde{\Lambda}^{-1}$ and $\Omega_1 = |Z_{\mathbf{G}}^{0F}|^{-1}\widetilde{\Omega}$. In the present case, F acts trivially on $W_{\mathbf{G}}(\mathbf{L})$. If, for $\varphi \in \mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L}))$, we denote by ι_φ the corresponding local system, we have according to (5.2):

$$(\Omega_1)_{\iota_\varphi,\iota_{\varphi'}} = \sum_{i=0}^{l} q^{l-i}(-1)^i\langle \varphi\otimes\varphi', r^{\wedge i}\rangle_{W_{\mathbf{G}}(\mathbf{L})}.$$

We have two cases to consider: firstly $\mathbf{G} = \mathrm{SL}_n$, $\mathbf{L}$ of type $A_{d-1}^{n/d}$ and secondly $\mathbf{G} = \mathrm{GL}_{n/d}$, $\mathbf{L}$ a maximal torus. In either case we have $W_{\mathbf{G}}(\mathbf{L}) \simeq \mathfrak{S}_{n/d}$ and $l = n/d - 1$. Thus the matrices Ω_1 in the two cases may be identified through the bijection which maps the local system ι_λ^χ to the local system $\iota_{\lambda/d}$ (since, according to [15, Section 5] both correspond under the generalized Springer correspondence to the character of $\mathfrak{S}_{n/d}$ indexed by the partition λ/d). To verify that the equations are the same, it remains only to check that the rows and columns of the matrix $\widetilde{P}$, both of which are indexed by the irreducible characters of $\mathfrak{S}_{n/d}$, are ordered in the same way in either case. This ordering is induced by the partial order on unipotent classes in either case, and the coincidence follows from the description of this partial order in terms of partitions: we have $C_\lambda \geqslant C_\mu$ if and only if $\lambda \geqslant \mu$ where, if $\lambda = \{\lambda_1, \lambda_2, \ldots\}$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$ (respectively $\mu = \{\mu_1, \mu_2, \ldots\}$ with $\mu_1 \geqslant \mu_2 \geqslant \cdots$) this means that for all i we have $\lambda_1 + \cdots + \lambda_i \geqslant \mu_1 + \cdots + \mu_i$. This condition is compatible with dividing all parts of λ and μ by the same integer d, whence the result. □

The significance of the previous result is that in view of Proposition 6.10, the computation of ${}^*R_{\mathbf{M}}^{\mathbf{G}}$ of the generalized Gelfand–Graev characters, hence of the $\mathcal{X}_\iota$, and through them of the $\mathcal{Y}_\iota$, and hence of the characteristic functions of the unipotent conjugacy classes for the group SL_n, is reduced to the same problem for various $\mathrm{GL}_{n'}$, which is in principle known. According to the program in [5], this is a step towards determining the character table of $\mathrm{SL}_n(q)$. The other essential step in this program is the determination of ${}^*R_{\mathbf{M}}^{\mathbf{G}}$ of the irreducible characters, for which the work of C. Bonnafé gives a solution.

Appendix A. Local systems on the subregular unipotent class in good characteristic for simply connected groups

We describe now the generalized Springer correspondence for local systems on the subregular class for simply connected quasi-simple groups. The description for arbitrary quasi-simple groups follows easily.

This appendix contains information extracted from [11,15,16]. Table A.1 contains the following information:

- The column "**G**" contains the type of **G**.
- The column "C" describes the subregular class C, in Carter's notation for exceptional groups and by giving the partition associated to the Jordan form for classical groups.
- The column "Dynkin–Richardson" contains the Dynkin–Richardson diagram of C.
- The column "$A(u)$" describes the group $A(u)$ for an element $u \in C$.
- The column "ι" describes the local system ζ considered on C; it is described by giving the name of the corresponding character of $A(u)$; this last group is when possible described as a Coxeter group so the naming scheme for characters of Coxeter groups (see below) applies. The exceptions are the cyclic group of order 3 whose characters are denoted $1, \zeta, \zeta^2$ and the cyclic group group of order 4 whose characters are denoted $1, i, -1, -i$. If $\iota = (C, \zeta)$ let $(\mathbf{L}, \iota_0)$ be the corresponding cuspidal datum, where $\iota_0 = (\zeta_0, C_0)$. In general there is only one cuspidal pair in **L** (which is in most cases a local system on the regular class) so neither C_0 nor ζ_0 is mentioned; when there is an ambiguity they are mentioned in the last column.
- When **L** is not a maximal torus **T** or equal to **G**, the column "**L**" describes the Levi by circling the nodes corresponding to simple roots of **L** on the Dynkin diagram of **G**. The simple roots of $W_{\mathbf{G}}(\mathbf{L})$ in $X(Z_{\mathbf{L}}^0/Z_{\mathbf{G}}^0) \otimes \mathbb{R}$ therefore correspond to the unmarked nodes of the same diagram.
- When $W_{\mathbf{G}}(\mathbf{L})$ is neither trivial nor equal to $W_{\mathbf{G}}$ it is described in the column "$W_{\mathbf{G}}(\mathbf{L})$" by its Dynkin diagram, which has been decorated by letters $a, b, \ldots$ which appear also on the un-circled nodes in the column "**L**" to describe the correspondence between simple reflections.
- The column "φ_ι" describes the character of $W_{\mathbf{G}}(\mathbf{L})$ corresponding to ι. The notation for characters of Coxeter groups is as follows: 1, ε and r always represent the trivial, sign and reflection representation, respectively. Other linear characters are represented by the Dynkin diagram labelled by the values of the character on the simple reflections. The notation for characters of F_4 is that from [1] (the character $\phi''_{2,4}$ factors through $W(F_4)/W(D_4) = W(A_2)$ and is trivial on the reflections corresponding to a short root; the character $\phi'_{2,4}$ is deduced from it by the diagram automorphism). The characters of $W(B_n)$ are parametrized in the usual way by pairs of partitions.

Table A.1

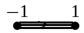
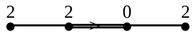
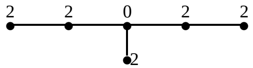
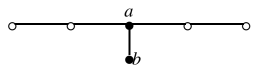
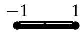
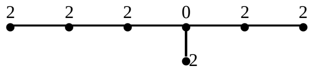
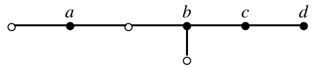
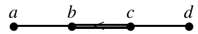
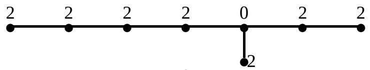
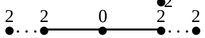
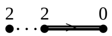
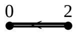
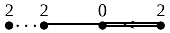
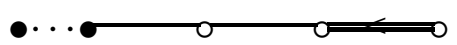
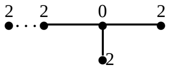
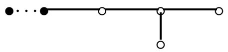
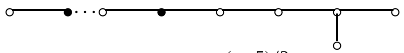
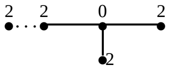
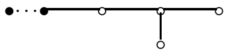
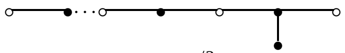
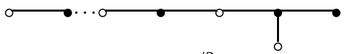
G	C	Dynkin–Richardson	$A(u)$	ι	**L**	$W_{\mathbf{G}}(\mathbf{L})$	φ_ι
G_2	$G_2(a_1)$		$W(A_2)$	1	**T**	$W(G_2)$	r
				r	**T**	$W(G_2)$	
				ε	**G**	1	1
F_4	$F_4(a_1)$		$W(A_1)$	1	**T**	$W(F_4)$	r
				ε	**T**	$W(F_4)$	$\phi'_{2,4}$
E_6	$E_6(a_1)$		$\mathbb{Z}/3\mathbb{Z}$	1	**T**	$W(E_6)$	r
				ζ			
				ζ^2	same description; the cuspidal local system is the other one on the regular class of $\mathbf{L} \simeq \mathrm{SL}_3 \times_{Z(\mathrm{SL}_3)} \mathrm{SL}_3$		
E_7	$E_7(a_1)$		$W(A_1)$	1	**T**	$W(E_7)$	r
				ε			$\phi''_{2,4}$
E_8	$E_8(a_1)$		1	1	**T**	$W(E_8)$	r
A_n n even	$(1, n-1)$		1	1	**T**	$W(A_n)$	r
A_n n odd	$(1, n-1)$		1	1	**T**	$W(A_n)$	r
B_n	$(1, 1, 2n-1)$		$W(A_1)$	1	**T**	$W(B_n)$	r
				ε	**T**	$W(B_n)$	$(1.n-1, \emptyset)$

Table A.1 (*Continued.*)

G	C	Dynkin–Richardson	$A(u)$	ι	**L**	$W_{\mathbf{G}}(\mathbf{L})$	φ_ι
C_2	$(2,2)$		$W(A_1)$	1	**T**	$W(C_2)$	r
				ε	**T**	$W(C_2)$	$(\emptyset,2)$
C_n $n>2$	$(2,2n-2)$		$W(A_1)^2$	$(1,1)$	**T**	$W(C_n)$	r
				$(\varepsilon,\varepsilon)$	**T**	$W(C_n)$	$(\emptyset,n)$
				$(\varepsilon,1)$		$W(C_{n-1})$	$(\emptyset,1.n-2)$
				$(1,\varepsilon)$		$W(C_{n-3})$	1
D_n n odd	$(3,2n-3)$		$\mathbb{Z}/4\mathbb{Z}$	1	**T**	$W(D_n)$	r
				-1		$W(B_{n-2})$	$(1.n-3,\emptyset)$
				i	 (type $D_5\times A_1^{(n-5)/2}$)	$W(B_{\frac{n-5}{2}})$	1
				$-i$	same description; the cuspidal local system is also parametrized by $-i$ on the D_5 component of **L**		
D_n n even	$(3,2n-3)$		$W(A_1)^2$	$(1,1)$	**T**	$W(D_n)$	r
				$(-1,1)$		$W(B_{n-2})$	$(1.n-3,\emptyset)$
				$(1,-1)$	 (type $A_1^{n/2}$)	$W(B_{n/2})$	$(\emptyset,n/2)$
				$(-1,-1)$	 (type $A_1^{n/2}$)	$W(B_{n/2})$	$(\emptyset,n/2)$

A more precise description of the local systems when $\mathbf{G} = \mathrm{Spin}_{2n}$ (the simply connected semi-simple group of type D_n) is as follows: $A_{\mathrm{SO}_{2n}}(u)$ is isomorphic to $W(A_1)$; when n is odd it is the unique subgroup of order 2 of $A_{\mathbf{G}}(u)$, while when n is even it is the first $W(A_1)$ in $A_{\mathbf{G}}(u)$.

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