Gröbner bases for families of affine or projective schemes

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Abstract

Let $I$ be an ideal of the polynomial ring $A[x] = A[x_1, \ldots, x_n]$ over the commutative, Noetherian ring $A$. Geometrically, $I$ defines a family of affine schemes, parameterized by $\text{Spec}(A)$: For $p \in \text{Spec}(A)$, the fibre over $p$ is the closed subscheme of the affine space over the residue field $k(p)$, which is determined by the extension of $I$ under the canonical map $\sigma_p : A[x] \to k(p)[x]$. If $I$ is homogeneous, there is an analogous projective setting, but again the ideal defining the fibre is $\langle \sigma_p(I) \rangle$. For a chosen term order, this ideal has a unique reduced Gröbner basis which is known to contain considerable geometric information about the fibre. We study the behavior of this basis for varying $p$ and prove the existence of a canonical decomposition of the base space $\text{Spec}(A)$ into finitely many, locally closed subsets over which the reduced Gröbner bases of the fibres can be parametrized in a suitable way.

Keywords: Comprehensive Gröbner basis; Gröbner cover; Canonical decomposition; Parametric polynomial system

1. Introduction

Let $A$ be a commutative, Noetherian ring with identity and $A[x] = A[x_1, \ldots, x_n]$ the polynomial ring in the variables $x_1, \ldots, x_n$ over $A$. We denote the residue field at $p \in \text{Spec}(A)$ by $k(p)$. Geometrically, an ideal $I \subset A[x]$ defines a family of affine schemes, parameterized by $\text{Spec}(A)$: The canonical map $A \to A[x]/I$ gives rise to a morphism of affine schemes

$$\varphi : \text{Spec}(A[x]/I) \to \text{Spec}(A).$$
For \( p \in \text{Spec}(A) \), the fibre \( \varphi^{-1}(p) \) is the closed subscheme of \( \mathbb{A}^n_{k(p)} = \text{Spec}(k(p)[x]) \) determined by \( \langle \sigma_p(I) \rangle \), where \( \sigma_p : A[x] \to k(p)[x] \) denotes the trivial extension of the canonical map \( A \to k(p) \).

If \( I \) is a homogeneous ideal, we analogously obtain a family of projective schemes from

\[
\varphi : \text{Proj}(A[x]/I) \to \text{Spec}(A).
\]

The fibre \( \varphi^{-1}(p) \) is the closed subscheme of \( \mathbb{P}^n_{k(p)} = \text{Proj}(k(p)[x]) \), again determined by \( \langle \sigma_p(I) \rangle \).

For a chosen term order we wish to study – simultaneously for all \( p \in \text{Spec}(A) \) – the unique reduced Gröbner basis of \( \langle \sigma_p(I) \rangle \). It is well known that such a Gröbner basis facilitates “easy access” to geometric information about the fibre \( \varphi^{-1}(p) \). It also seems reasonable to compare two fibres by “comparing” their corresponding Gröbner bases. Of course we can compare the leading terms. It is not quite clear, however, what comparing the Gröbner bases should mean. We will make this notion precise by introducing parametric sets. Rather vaguely, a parametric set with respect to \( I \) is a locally closed subset \( Y \) of \( \text{Spec}(A) \) such that the reduced Gröbner bases of the fibres can be parameterized in a suitable way over \( Y \). The main result of this article is to establish the existence and uniqueness of a canonical decomposition of the base space \( \text{Spec}(A) \) into finitely many parametric sets.

Many concrete mathematical problems can be stated in the above described framework of families of affine or projective schemes, and to know the Gröbner basis structures of the fibres may be the first step to their solution, if not the solution itself quite yet. For example, if \( A \) is a polynomial ring over some field, then we obtain the case of algebraic systems with parameters, which is important for many “real life” applications such as robotics or electrical engineering (see e.g. Cox et al. (1997), Chapter 6, and Montes and Castro (1995)). From a more theoretical point of view, parametric sets are a tool to explore the geometry of families of affine or projective schemes. Related theoretical applications range from efficient Gröbner basis computation (see e.g. Arnold (2003) and Pauer (1992)) to cohomology (see Walther (2003)). The fundamental paper (Weispfenning, 1992) also contains several applications.

The naive hope that for a Gröbner basis \( G \) of \( I \) the specialized Gröbner basis \( \sigma_p(G) \) is a Gröbner basis of the specialized ideal \( \langle \sigma_p(I) \rangle \) is in general not fulfilled. The behavior of Gröbner bases under specialization (or extension of scalars) has actually been studied by many authors, e.g. Bayer et al. (1993), Kalkbrener (1997), Aschenbrenner (2005), Gonzalez-Lopez et al. (2000), Gonzalez-Vega et al. (2005), Assi (1994), Fortuna et al. (2001) and Suzuki and Sato (2003). In Aschenbrenner (2005) the case of standard bases in the ring of formal power series is treated. Relations to flatness are explored in Assi (1994) and also in Bayer et al. (1993). The more computational articles Weispfenning (1992, 2003) and Montes (2002, 2006) also obtain more or less canonical decompositions of the parameter space, which respect the Gröbner basis structure of the fibres. However, these decompositions are established simply by giving the algorithmic constructions. In this article, we present a more systematic approach: first we define precisely which decompositions of the parameter space we want to study, and then we prove that there exists exactly one such decomposition satisfying some additional nice properties.

The outline of the article is as follows: Section 3 (Parametric sets) introduces the fundamental notion of parametric sets and their basic properties. The main theorem of Section 4 (Lucky primes and pseudo division) is a characterization of parametric sets in terms of lucky primes (see Gräbe (1993)). This theorem can also be understood as giving the geometric meaning of luckiness. Finally, in Section 5 (Gröbner covers) we achieve the main objective of the article by proving existence and uniqueness of a canonical finite covering of \( \text{Spec}(A) \) with parametric subsets.
To establish the results in their full generality (i.e. to work over an arbitrary Noetherian base ring), we need to use the language of schemes. This language may not be so familiar to people coming from the computer algebra side. But for practical computations, the base ring is usually a polynomial ring over some field, and in this situation the concepts and results of this article can be formulated in a more down-to-earth kind of way. So at the end of the article there is included an Appendix which should help to orient the reader who prefers classical varieties and concrete computations to schemes and sheaves. The last part of the Appendix also explains a key difference between our approach and previous approaches by other authors.

2. Preliminaries and notation

A parametric subset $Y$ of Spec($A$) facilitates an object which parameterizes the reduced Gröbner bases of $(\sigma_p(I))$ for $p \in Y$. To assure the uniqueness of this object, which will be called the reduced Gröbner basis of $I$ over $Y$, we have to work with reduced schemes $(Y, \mathcal{O}_Y)$. In particular, we would like to assume that our base ring $A$ is reduced. This can be done without loss of generality:

Let Nil($A$) denote the nilradical of $A$ and define $A' = A/\text{Nil}(A)$. Then there is a natural homeomorphism

$$\text{Spec}(A) \to \text{Spec}(A')$$

and $k(p) = k(p')$. Moreover if $I' \subset A'[x]$ denotes the image of $I$ under the canonical map $A[x] \to A'[x]$ then $(\sigma_p(I)) = (\sigma_p(I'))$ for all $p \in \text{Spec}(A)$.

Throughout, $A$ denotes a commutative, Noetherian, reduced ring with identity and $I$ an ideal of the polynomial ring $A[x] = A[x_1, \ldots, x_n]$. For an $A$-module $M$ the localization at $p \in \text{Spec}(A)$ is denoted by $M_p$, and $k(p) = A_p/p_p$ is the residue field at $p$. The map $\sigma_p : A[x] \to k(p)[x]$ denotes the coefficient-wise evaluation of elements in $A[x]$, according to the evaluation map $A \to k(p)$.

We will only consider reduced subschemes of Spec($A$). So by a subscheme of Spec($A$), we mean a locally closed subset $Y$ of Spec($A$) with the induced reduced subscheme structure $\mathcal{O}_Y$. (See the Appendix for an explicit description of $\mathcal{O}_Y$.) By $a$, we denote the radical ideal of $A$ such that the Zariski closure of $Y$ is equal to $V(a)$. (As usual, $V(a) \subset \text{Spec}(A)$ denotes the closed set of all prime ideals containing $a$.) In this situation, we will usually identify Spec($A/a$) and $V(a) \subset \text{Spec}(A)$.

The set of terms (i.e. powerproducts) is denoted by $T = T(x_1, \ldots, x_n)$. Throughout we fix a term order $< \circ t$ on $T$. For a nonzero polynomial $P = \sum_{t \in T} a_t t \in A[x]$, we define

- the coefficient of $P$ at $t$ by $\text{coef}(P, t) = a_t$,
- the support of $P$ by $\text{supp}(P) = \{t \in T; a_t \neq 0\}$,
- the leading term $\text{lt}(P)$ of $P$ to be the maximal element of $\text{supp}(P)$,
- the leading coefficient of $P$ by $\text{lc}(P) = \text{coef}(P, \text{lt}(P))$ and
- the leading monomial of $P$ by $\text{lm}(P) = \text{lc}(P) \text{lt}(P)$.

For $G \subset A[x]$ we set $\text{lt}(G) = \{\text{lt}(P); P \in G \setminus \{0\}\}$, and similarly $\text{lm}(G) = \{\text{lm}(P); P \in G \setminus \{0\}\}$. A finite subset $G$ of $I \subset A[x]$ is called a Gröbner basis of $I$ if $(\text{lm}(G)) = (\text{lm}(I))$. For $t \in T$, we define the ideal of leading coefficients at $t$ by

$$\text{lc}(I, t) = \{\text{lc}(P); P \in I \setminus \{0\} \text{ with } \text{lt}(P) = t\} \subset A.$$
Note that \( \text{lc}(I, t) \) can conveniently be read off from a Gröbner basis \( G \) of \( I \). In fact, \( \text{lc}(I, t) \) is generated by \( \{ \text{lc}(g); \ g \in G \text{ with } \text{lt}(g) \text{ divides } t \} \). For a general reference for Gröbner bases over rings, see Adams and Loustaunau (1994).

Before really getting started, we look at some warm-up examples:

**Example 1.** Let \( k \) be a field and \( A = k[u_1, u_2] \) the polynomial ring in the two parameters \( u_1, u_2 \). Consider the ideal

\[
I = \langle (u_1^2 - u_2)x, (u_2 - 1)y^2 + u_1x \rangle \subset A[x, y].
\]

When faced with the task of describing the Gröbner basis structure of the fibres, perhaps most mathematicians would come up with the following pictures (Figs. 1 and 2).

Figs. 1 and 2 illustrate a decomposition of the base space \( A_k^2 = \text{Spec}(A) \) into locally closed subsets. In short, the objective of this article is to find this decomposition in general.

**Example 2.** Let \( k \) be an algebraically closed field and \( A = k[u_1, u_2, u_3, u_4] \) the polynomial ring in the parameters \( u_1, u_2, u_3, u_4 \). We consider the ideal

\[
I = \langle (u_2u_3 - u_4u_1)x, u_1x^2 + u_2x, u_3x^2 + u_4x \rangle \subset A[x].
\]

(Here \( x \) denotes just one variable.) Let \( v = (v_1, v_2, v_3, v_4) \in k^4 \) and

\[
p_v = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3, u_4 - v_4 \rangle.
\]

If \( v_2v_3 - v_4v_1 \) is nonzero, then the reduced Gröbner basis of \( \langle \sigma_{p_v}(I) \rangle \) is \( x \). If \( v_1 \) and \( v_3 \) are zero and one of \( v_2, v_4 \) is nonzero, then the reduced Gröbner basis of \( \langle \sigma_{p_v}(I) \rangle \) is also \( x \). (In particular, the set of all \( v \in k^4 \) such that \( \text{lt}(\langle \sigma_{p_v}(I) \rangle) \) is generated by \( x \), is not locally closed.) If \( v \) lies in
the quasi-affine variety $Y = V(\langle u_2 u_3 - u_4 u_1 \rangle) \setminus V(\langle u_1, u_3 \rangle)$, then the reduced Gröbner basis of $\langle \sigma_{p,v}(I) \rangle$ is given by $x^2 + f(v)x$, where $f$ denotes the regular function on $Y$ defined by

$$f(v) = \begin{cases} v_2/v_1 & \text{if } v_1 \neq 0 \\ v_4/v_3 & \text{if } v_3 \neq 0. \end{cases}$$

The above example illustrates the “local nature” of the problem and suggests working with sheaves and not just with polynomials in $I$, as was the common practice in Weispfenning (2003) or Montes (2006).

Using the Buchberger algorithm, it is relatively easy to see that the equivalence relation $\sim$ defined on $\text{Spec}(A)$ by $p \sim p'$ if $\text{lt}(\langle \sigma_{p,v}(I) \rangle) = \text{lt}(\langle \sigma_{p',v}(I) \rangle)$ has only finitely many equivalence classes, and that every equivalence class is a constructible set. However, there are reasons which militate against the obvious approach to simply stratify the base space $\text{Spec}(A)$ with respect to the leading terms:

- The equivalence classes are indeed only constructible, and in general not locally closed (see Example 2).
- Even if an equivalence class $Y$ is locally closed, $\varphi$ is not necessarily flat over $Y$.
- From the constantness of the function $p \mapsto \text{lt}(\langle \sigma_{p,v}(I) \rangle)$, it does not follow that the reduced Gröbner bases of the fibres depend on $p$ “in a continuous way”.

The following simple example illustrates the two latter points.

**Example 3.** Let $k$ be a field and $A = k[u]$ the polynomial ring in one parameter $u$. Consider the ideal $I = \langle u(ux - 1), (ux - 1)x \rangle \subset A[x] = k[u,x]$. 
Geometrically, the map $\varphi : \text{Spec}(k[u, x]/I) \rightarrow \text{Spec}(A) = \mathbb{A}^1_k$ is the projection onto the $u$-axis (see Fig. 3). For every point $p \in \text{Spec}(A)$, the leading terms of $\langle \sigma_p(I) \rangle$ are generated by $x$, but $\varphi$ is not flat. Let $\mathfrak{P} \in \text{Spec}(k[u, x]/I)$ be the point corresponding to the origin in $\mathbb{A}^2_k$; then

$$\mathcal{O}_{\text{Spec}(A[x]/I)}, \mathfrak{P} = (A[x]/I)_{\mathfrak{P}} = k$$

because $ux - 1$ does not lie in $\mathfrak{P}$. For $p = \varphi(\mathfrak{P}) = \langle u \rangle$ we have $\mathcal{O}_{\text{Spec}(A), p} = k[u](u)$. The map $k[u](u) \rightarrow k$ induced by $\varphi$ is given by evaluation at the origin and is not flat. Thus $\varphi$ is not flat at $\mathfrak{P}$.

This example suggests that the above described problems may not appear in the projective setting. Indeed, we will see in Section 5 that for homogeneous ideals the situation is as nice as could be hoped, i.e. the sets over which $p \mapsto \text{lt}(\langle \sigma_p(I) \rangle)$ is constant are parametric.

3. Parametric sets

The idea of “parameterizing Gröbner bases” can nicely be captured by using sheaves. For every subscheme $Y$ of $\text{Spec}(A)$, we will define a quasi-coherent sheaf $\mathcal{I}_Y$ on $Y$, which intuitively might be thought of as the restriction of $I$ to $Y$. (Recall that $I \subset A[x]$ is the ideal we want to study.)

Let $Y$ be a locally closed subset of $\text{Spec}(A)$ and $a \subset A$ the radical ideal such that $Y = V(a)$, and let $\overline{T}$ denote the image of $I$ in $(A/a)[x]$. We define $\mathcal{I}_Y$ to be the restriction of the quasi-coherent sheaf associated to the $A/a$-module $\overline{T}$ on $\text{Spec}(A/a) = V(a)$ to $Y$. That is

$$\mathcal{I}_Y = \left. \overline{T} \right|_Y.$$

More explicitly, for an open subset $U$ of $Y$, the $\mathcal{O}_Y(U)$-module $\mathcal{I}_Y(U)$ consists of all functions $g$ from $U$ into the disjoint union $\bigsqcup_{p \in U} \overline{T}_p$, which are locally fractions, i.e. for every $p \in U$ there exists an open neighborhood $U'$ of $p$ in $U$ such that for all $q \in U'$, we have $g(q) = \frac{P^2}{s} \in \overline{T}_q$, where $P \in \overline{T}$ and $s \in (A/a) \smallsetminus q$ for all $q \in U'$. 

Fig. 3. Picture for Example 3.
Since $A$ is Noetherian, $\text{Spec}(A)$ is a Noetherian topological space, and thus every open subset $U$ of $Y$ is quasi-compact. This implies that we can consider $\mathcal{I}_Y(U)$ as an ideal of the polynomial ring $\mathcal{O}_Y(U)[x]$. (If $U$ were not quasi-compact, we could not be sure that an element of $\mathcal{I}_Y(U)$ has finite support.)

Note that for $p \in Y$, the stalk $\mathcal{I}_{Y,p} = \mathcal{I}_p$ is just the extension of $I$ under $A[x] \to (A/a)_p[x]$. Let $m_p$ denote the unique maximal ideal of $\mathcal{O}_{Y,p} = (A/a)_p$; then in analogy to the sequence

$$A \to \mathcal{O}_Y(Y) \to \mathcal{O}_{Y,p} \to \mathcal{O}_{Y,p}/m_p = k(p)$$

of natural maps, we obtain natural maps

$$I \to \mathcal{I}_Y(Y) \to \mathcal{I}_{Y,p} \to (\sigma_p(I)).$$

For $g \in \mathcal{I}_Y(Y)$ the image of $g$ in $(\sigma_p(I))$ is denoted by $\overline{g}^p$.

Now we are ready to give precise meaning to the intuitive idea of parameterizing Gröbner bases: we are looking for subschemes $Y$ of $\text{Spec}(A)$ with the property that there exist global sections $g_1, \ldots, g_m \in \mathcal{I}_Y(Y)$ such that for all $p \in Y$ their images $\overline{g_1}^p, \ldots, \overline{g_m}^p$ form the unique reduced Gröbner basis of $(\sigma_p(I))$. We will need the following easy lemma.

**Lemma 1.** Let $Y$ be a subscheme of $\text{Spec}(A)$ and $g, f \in \mathcal{I}_Y(Y)$. Then the set

$$\{p \in Y; \overline{g}^p = \overline{f}^p\}$$

is a closed subset of $Y$ and $\overline{g}^p = \overline{f}^p$ for all $p \in Y$ implies $g = f$.

**Proof.** It suffices to treat the case $f = 0$. We can cover $Y$ with open sets $U_i$ such that

$$g(p) = \frac{P}{s} \in \mathcal{I}_p$$

for $P \in T \subset (A/a)[x]$ and $s \in (A/a) \setminus p$ for all $p \in U_i$. We have

$$\{p \in Y; \overline{g}^p = 0\} \cap U_i = \{p \in U_i; \text{ coef}(P, t) \in p \text{ for all } t \in \text{ supp}(P)\},$$

which is a closed subset of $U_i$. Hence $\{p \in Y; \overline{g}^p = 0\}$ is closed.

If we interpret $g$ as a polynomial with coefficients $c_t$ in $\mathcal{O}_Y(Y)$, then $\overline{g}^p = 0$ is equivalent to saying that for all $t \in T$, the image of $c_t$ in the stalk $\mathcal{O}_{Y,p} = (A/a)_p$ lies in the maximal ideal $m_p$ of $\mathcal{O}_{Y,p}$. Since this holds for all $p \in Y$ and $Y$ is a reduced scheme, we obtain $c_t = 0 \in \mathcal{O}_Y(Y)$. Hence $g = 0$. □

**Theorem 1.** If $Y$ is a connected subscheme of $\text{Spec}(A)$ and there exists a finite subset $G$ of $\mathcal{I}_Y(Y)$ such that for all $p \in Y$ the set $\overline{G}^p = \{\overline{g}^p; \ g \in G\}$ is the reduced Gröbner basis of $(\sigma_p(I))$, then $G$ is uniquely determined, and for each $g \in G$ the function $p \mapsto \text{ lt}(\overline{g}^p)$ is constant on $Y$. In particular, the function $p \mapsto \text{ lt}((\sigma_p(I)))$ is constant on $Y$.

**Proof.** First we will show that for $g \in G$ and $t \in T$, the set

$$W(t) = \{p \in Y; \text{ lt}(\overline{g}^p) = t\}$$

is a closed subset of $Y$. We can cover $Y$ with open sets $U_i$ such that

$$g(p) = \frac{P}{s} \in \mathcal{I}_p \quad \text{for all } p \in U_i.$$ 

Here $P \in T \subset (A/a)[x]$ and $s \in (A/a) \setminus p$ for all $p \in U_i$. 
Let \( p \in Y \) and \( \phi : (A/\alpha)p \to (A/\alpha)p/m_p = k(p) \) the canonical map. We will need the fact that \( \phi(c/s) = 1 \) is equivalent to \( c - s \in p \) for \( c \in A/\alpha \) and \( s \in (A/\alpha) \setminus p \). Indeed, \( \phi(c/s) = 1 \) is equivalent to saying that there exists \( c' \in p \) and \( s' \in (A/\alpha) \setminus p \) such that
\[
\frac{c}{s} = 1 + \frac{c'}{s'} = \frac{s' + c'}{s'}.
\]
This implies the existence of an \( s'' \in (A/\alpha) \setminus p \) such that
\[
(cs' - s(s' + c'))s'' = 0 \in p.
\]
Hence \( cs' - ss' \in p \) and therefore \( c - s \in p \). The converse follows from \( \frac{c}{s} = 1 + \frac{c-s}{s} \).

Using the above statement and the fact that \( \overline{g}^p \) is monic, we see that for \( p \in U_i \), we have \( \text{lt}(\overline{g}^p) = t \) if and only if \( p \) contains
\[
\{\text{coef}(P, t') : t' > t\} \cup \{\text{coef}(P, t) - s\}.
\]

Therefore, \( W(t) \cap U_i \) is a closed subset of \( U_i \), and thus \( W(t) \subseteq Y \) is closed.

Since \( \text{Spec}(A) \) is a Noetherian topological space, a finite number of the \( U_i \)'s will do, and therefore the function \( p \mapsto \text{lt}(\overline{g}^p) \) takes only finitely many values on \( Y \). Consequently, \( Y \) is the disjoint union of finitely many \( W(t) \)'s. By the connectedness assumption on \( Y \), we conclude that the function \( p \mapsto \text{lt}(\overline{g}^p) \) is constant on \( Y \).

Assume that \( F \) is a finite subset of \( \mathcal{I}_Y(Y) \) such that \( \overline{F}^p \) is the reduced Gröbner basis of \( \langle \sigma_p(I) \rangle \) for every \( p \in Y \). Then for \( f \in F \) and a chosen \( p \in Y \), there exists a \( g \in G \) such that \( \overline{f}^p = \overline{g}^p \). Since the leading terms of \( \overline{f}^p \) and \( \overline{g}^p \) are independent of \( p \), this implies \( \text{lt}(\overline{f}^p) = \text{lt}(\overline{g}^p) \) for all \( p \in Y \). But as \( \overline{F}^p \) is the reduced Gröbner basis, we can conclude \( \overline{f}^p = \overline{g}^p \) for all \( p \in Y \), and therefore \( f = g \in G \) by Lemma 1.

The following example shows that both assertions of the above theorem may be false if \( Y \) is not connected.

**Example 4.** Let \( Y = \{p_1, p_2\} \), where \( p_1 \) and \( p_2 \) are two distinct closed points of \( \text{Spec}(A) \). Note that \( \mathcal{O}_Y(Y) \) is just \( k(p_1) \times k(p_2) \). For \( j = 1, 2 \) let \( G_j \) denote the reduced Gröbner bases of \( \langle \sigma_{p_j}(I) \rangle \). Then for any subset \( G \) of
\[
G_1 \times G_2 \subseteq \langle \sigma_{p_1}(I) \rangle \times \langle \sigma_{p_2}(I) \rangle = \mathcal{I}_Y(Y)
\]
with the property that the projections \( G \to G_i \) are surjective, we have that \( \overline{G}^p \) is the reduced Gröbner basis of \( \langle \sigma_p(I) \rangle \) for every \( p \in Y \).

As we wish to have a definition suitable for all (not necessarily connected) subschemes of \( \text{Spec}(A) \), we simply demand what we want.

**Definition 1.** A locally closed subset \( Y \) of \( \text{Spec}(A) \) is called *parametric for Gröbner bases with respect to \( I \) (and \( < \)) if there exists a finite subset \( G \) of \( \mathcal{I}_Y(Y) \) with the following properties:

1. \( \overline{G}^p \) is the reduced Gröbner basis of \( \langle \sigma_p(I) \rangle \) for every \( p \in Y \).
2. For each \( g \in G \) the function \( p \mapsto \text{lt}(\overline{g}^p) \) is constant on \( Y \).

Since the ideal \( I \subseteq A[x] \) is clear from the context, we usually omit the reference to \( I \) and simply talk about parametric subschemes of \( \text{Spec}(A) \).
Theorem 2. Let \(Y \subset \text{Spec}(A)\) be parametric and \(G\) a finite subset of \(\mathcal{I}_Y(Y)\) satisfying the two conditions of the above definition. Then \(G\) is uniquely determined, and the function \(p \mapsto \text{lt}(\langle \sigma_p(I) \rangle)\) is constant on \(Y\). Furthermore, every \(g \in G\) is monic with \(\text{lt}(g) = \text{lt}(g^p)\) for every \(p \in Y\).

Proof. Because of condition (2) we can repeat the uniqueness proof as given in the last paragraph of the proof of Theorem 1.

To show that every \(g \in G\) is monic with \(\text{lt}(g) = \text{lt}(g^p)\), observe that the coefficients of \(g\) are just elements of \(O_Y(Y)\). Since \((Y, O_Y)\) is a reduced scheme, every element of \(O_Y(Y)\) is uniquely determined by its images in \(k(p)\), where \(p\) ranges over all of \(Y\). \(\square\)

Definition 2. Let \(Y \subset \text{Spec}(A)\) be parametric; then the uniquely determined subset \(G_Y = G_Y(Y)\) of \(\mathcal{I}_Y(Y)\) of the above theorem is called the reduced Gröbner basis of \(I\) over \(Y\). We define the leading terms of \(Y\), denoted by \(\text{lt}(Y)\), to be the value of the constant function \(p \mapsto \text{lt}(\langle \sigma_p(I) \rangle)\).

To give the reader some idea where the journey is going, we give the following definition at this early stage — even though we will not need it before Section 5.

Definition 3. A Gröbner cover of \(\text{Spec}(A)\) with respect to \(I\) (and \(<\)) is a finite set \(G\) of pairs \((Y, G_Y)\) such that \(Y \subset \text{Spec}(A)\) is parametric, \(G_Y\) is the reduced Gröbner basis of \(I\) over \(Y\), and

\[
\bigcup_{(Y, G_Y) \in G} Y = \text{Spec}(A).
\]

Parametric sets are well behaved with respect to inclusion:

Theorem 3. Let \(Y \subset \text{Spec}(A)\) be parametric. Then every locally closed subset \(Y'\) of \(Y\) is parametric and the canonical map \(\mathcal{I}_Y(Y) \to \mathcal{I}_{Y'}(Y')\) maps the reduced Gröbner basis of \(I\) over \(Y\) to the reduced Gröbner basis of \(I\) over \(Y'\).

Proof. First of all, let us construct the canonical map of the theorem. Assume \(\overline{Y} = \text{V}(a)\) and \(\overline{Y'} = \text{V}(a')\) for radical ideals \(a\) and \(a'\) of \(A\). Let \(\overline{I} \subset (A/a)[x]\) and \(\overline{I'} \subset (A/a')[x]\) denote the corresponding extensions of \(I\). As \(\overline{Y'} \subset \overline{Y}\), we have \(a \subset a'\) and a canonical map \(A/a \to A/a'\) which extends to \(\phi : \overline{I} \to \overline{I'}\). Then for \(p \in Y' \subset Y\) we have a canonical map

\[
\phi_p : \overline{I}_p \to \overline{I'}_p.
\]

Now an element \(g \in \mathcal{I}_Y(Y)\) gives rise to a function

\[
g' : Y' \to \prod_{p \in Y'} \overline{I'}_p
\]

by \(g'(p) = \phi_p(g(p))\). One easily verifies that the map \(\mathcal{I}_Y(Y) \to \mathcal{I}_{Y'}(Y')\), \(g \mapsto g'\) is well defined. For \(p \in Y'\), the commutative diagram

\[
\begin{array}{ccc}
\overline{I}_p & \longrightarrow & \overline{I'}_p \\
& \nearrow & \\
& \langle \sigma_p(I) \rangle & \\
\end{array}
\]
gives rise to a commutative diagram

\[ \begin{array}{ccc}
\mathcal{I}_Y(Y) & \rightarrow & \mathcal{I}_Y'(Y') \\
\downarrow \quad \quad \downarrow & & \quad \quad \downarrow \quad \quad \downarrow \\
(\sigma_p(I)) & \quad \quad & \quad \quad \\
\end{array} \]

From this, the claim of the theorem follows. \( \square \)

Next we will give a characterization of parametric sets in terms of monic ideals (see Pauer (1992)).

**Definition 4.** An ideal \( I \subset A[x] \) is called monic (with respect to \( < \)) if \( \text{lct}(I, t) \in \{(0), (1)\} \) for all \( t \in T \). In other words: \( I \) is monic if for every \( t \in \text{lt}(I) \), there exists a monic polynomial \( P \in I \) with \( \text{lt}(P) = t \).

There are quite a few definitions of reduced Gröbner bases over rings in the literature. We will use the one strictly paralleling the field case.

**Definition 5.** A Gröbner basis \( G = \{g_1, \ldots, g_m\} \) of \( I \) is called reduced if for \( j = 1, \ldots, m \)
- \( g_j \) is monic and
- \( \text{supp}(g_j) \cap \text{lt}(I) = \{\text{lt}(g_j)\} \).

With this definition, not every ideal has a reduced Gröbner basis, but as in the field case one can easily show that if it exists, it is unique, and that \( A[x]/I \) is a free \( A \)-module with basis \( T \setminus \text{lt}(I) \). Concerning existence, we have the following (cf. Pauer (1992) and Aschenbrenner (2005), Theorem 2.11).

**Theorem 4.** Let \( I \subset A[x] \) be an ideal; then there exists a reduced Gröbner basis of \( I \) if and only if \( I \) is monic.

**Proof.** If there exists a reduced Gröbner basis of \( I \), then clearly \( I \) is monic. Conversely, if \( I \) is monic, then we can choose monic polynomials \( g_1, \ldots, g_m \in I \) such that \( \text{lt}(g_1), \ldots, \text{lt}(g_m) \) is the unique minimal generating set of \( \text{lt}(I) \). Now if we mutually reduce the \( g_j \)'s, we end up with the desired reduced Gröbner basis of \( I \). \( \square \)

The connection to parametric subschemes is the following:

**Theorem 5.** A subscheme \( Y \) of \( \text{Spec}(A) \) is parametric if and only if \( \mathcal{I}_Y(Y) \subset \mathcal{O}_Y(Y)[x] \) is monic, and in this case the reduced Gröbner basis of \( I \) over \( Y \) is the reduced Gröbner basis of \( \mathcal{I}_Y(Y) \). In particular \( \text{lt}((\mathcal{I}_Y(Y))) = \text{lt}(Y) \).

**Proof.** Suppose that \( Y \) is parametric, and let \( G \subset \mathcal{I}_Y(Y) \) denote the reduced Gröbner basis of \( I \) over \( Y \). Then the leading term of every \( f \in \mathcal{I}_Y(Y) \) is divisible by \( \text{lt}(g) \) for some \( g \in G \). Indeed, since \( (Y, \mathcal{O}_Y) \) is a reduced scheme, there exists a \( p \in Y \) such that the image of \( \text{lct}(f) \in \mathcal{O}_Y(Y) \) in \( k(p) \) is nonzero. For this particular \( p \), we know that \( \text{lt}(f) = \text{lt}(g_p) \) is divisible by \( \text{lt}(g_p) = \text{lt}(g) \) for some \( g \in G \), as was claimed. Since the elements of \( G \) are monic, this shows in particular that \( \mathcal{I}_Y(Y) \) is monic.
Now suppose that \( \mathcal{I}_Y(Y) \) is monic and let \( G = \{g_1, \ldots, g_m\} \) denote the reduced Gröbner basis of \( \mathcal{I}_Y(Y) \). For \( f \in \mathcal{I}_Y(Y) \), the usual division (or reduction) algorithm shows that there exists a representation
\[
f = f_1 g_1 + \cdots + f_m g_m
\]
such that for \( i = 1, \ldots, m \) we have \( \text{lt}(f_i) \text{lt}(g_i) \leq \text{lt}(f) \) and
\[
\text{coef}(f_i, t) \in \{ \text{coef}(f, t'); t' \geq t \text{lt}(g_i) \} \quad \text{for all } t \in T.
\]
By the last condition, we have \( \text{lt}(f_i^p) \text{lt}(g_i^p) \leq \text{lt}(f^p) \) for every \( p \in Y \). Because \( f^p = f_1^p g_1^p + \cdots + f_m^p g_m^p \), this shows that \( \text{lt}(f^p) \) is divisible by \( \text{lt}(g_i^p) \) for some \( i \in \{1, \ldots, m\} \).

Recall that the reduced Gröbner basis \( G \) of \( I \) over \( Y \) was essentially defined by imposing the reduced Gröbner basis property for every point in \( Y \). Now Theorem 5 states that this pointwise property lifts to the global sections \( \mathcal{I}_Y(Y) \subset \mathcal{O}_Y(Y)[x] \), so that the reduced Gröbner basis of \( I \) over \( Y \) is indeed a reduced Gröbner basis. Furthermore, by Theorem 3, \( G|_U = \{g|_U; g \in G\} \) is the reduced Gröbner basis of \( \mathcal{I}_Y(U) \subset \mathcal{O}_Y(U)[x] \) for every open subset \( U \) of \( Y \).

**Corollary 1.** Spec\((A)\) is parametric with respect to \( I \) if and only if \( I \) is monic, and in this case the reduced Gröbner basis of \( I \) over Spec\((A)\) is the reduced Gröbner basis of \( I \).

**Proof.** This follows directly from the theorem, because \( \mathcal{I}_{\text{Spec}(A)}(\text{Spec}(A)) = I \) (see Hartshorne (1977), Chapter II, Proposition 5.1). \( \square \)

Next we will prove a local criterion for a locally closed subset of Spec\((A)\) to be parametric. Using this criterion, we will then show that a family of affine or projective schemes over a parametric subset of Spec\((A)\) is flat. We need two easy lemmas.

**Lemma 2.** Let \( p \in \text{Spec}(A) \) and \( f \in I_p \). Then there exists \( P \in I \) and \( s \in A \setminus p \) such that
\[
f = \frac{P}{s} \in I_p
\]
and \( \text{coef}(P, t) = 0 \) whenever \( \text{coef}(f, t) = 0 \). In particular \( \text{lt}(P) = \text{lt}(f) \).

**Proof.** By definition, there exists \( P \in I \) and \( s \in A \setminus p \) such that \( f = P/s \in I_p \). If \( \text{coef}(f, t) = \text{coef}(P, t)/s \in A_p \) is zero, there exists an \( s_i \in A \setminus p \) such that \( \text{coef}(P, t)s_i = 0 \). If we multiply \( P \) and \( s \) by the product of all \( s_i \)'s where \( t \) ranges over the support of \( P \), we obtain the desired representation of \( f \). \( \square \)

**Lemma 3.** Let \( Y \subset \text{Spec}(A) \) be locally closed and \( \mathfrak{a} \subset A \) the radical ideal such that \( \overline{Y} = \mathcal{V}(\mathfrak{a}) \). Let \( P \in \overline{I} \subset (A/\mathfrak{a})[x] \). Then the leading term of the image of \( P \) in \( \mathcal{I}_Y(Y) \) equals the leading term of \( P \).

**Proof.** It suffices to show that there exists a \( p \in Y \) which does not contain \( \text{lc}(P) \). Assume the contrary; then \( Y \) is contained in the closed set
\[
W = \{ p \in \text{Spec}(A/\mathfrak{a}); \text{lc}(P) \in p \}.
\]
But as $Y$ is dense in $V(\alpha) = \text{Spec}(A/\alpha)$, we conclude that $W = \text{Spec}(A/\alpha)$, and thus $\text{lc}(P) \in \mathfrak{p}$ for all $\mathfrak{p} \in \text{Spec}(A/\alpha)$. Because $\alpha$ is a radical ideal, this yields the contradiction $\text{lc}(P) = 0$. □

**Theorem 6.** Let $Y \subset \text{Spec}(A)$ be locally closed and $T'$ a set of terms such that $TT' = T'$. Let $\alpha \subset A$ denote the radical ideal such that $Y = V(\alpha)$ and $T$ the image of $I$ in $(A/\alpha)[x]$. Then $Y$ is parametric with $\text{lt}(Y) = T'$ if and only if $T_p$ is monic with $\text{lt}(T_p) = T'$ for every $p \in Y$.

**Proof.** To show that $T_p$ is monic with $\text{lt}(T_p) = T'$, it suffices to prove $\text{lt}(T_p) \subset T'$, because this shows that the image of the reduced Gröbner basis of $I$ over $Y$ in $T_p$ is the reduced Gröbner basis of $T_p$. Let $P \in T$ and $s \in (A/\alpha) \setminus \mathfrak{p}$. By Lemma 2, we may assume that the leading term of $P/s \in T_p$ equals the leading term of $P$. And by Lemma 3, the leading term of the image of $P$ in $\mathcal{I}_Y(Y)$ is the leading term of $P$. Thus $\text{lt}(P/s) \in \text{lt}(\mathcal{I}_Y(Y)) = \text{lt}(Y) = T'$.

For the converse direction, let $T = \{t_1, \ldots, t_m\}$ denote the minimal generating set of $T'$. For $i = 1, \ldots, m$ and $p \in Y$, let $g_i(p)$ denote the element of the reduced Gröbner basis of $T_p$ with leading term $t_i$. We want to show that $g_i$ defines an element of $\mathcal{I}_Y(Y)$. Let $p \in Y$ and $P \in T$, $s \in (A/\alpha) \setminus \mathfrak{p}$ such that $g_i(p) = P/s \in T_p$. By Lemma 2 we may assume $\text{lt}(P) = t_i$ and $\text{coeff}(P, t) = 0$ for $t \in T' \setminus \{t_i\}$. Because $g_i(p)$ is monic, there exists an $s' \in (A/\alpha) \setminus \mathfrak{p}$ such that $\text{lc}(P) - s's' = 0$. The set $U = \{q \in Y; s, s' \notin q\}$ is an open neighborhood of $p$ in $Y$, and we have $g_i(q) = P/s \in T_q$ for all $q \in U$ because $P/s \in T_q$ is monic with leading term $t_i$ and $\text{supp}(P/s) \cap T' = t_i$. Thus the $g_i$'s are elements of $\mathcal{I}_Y(Y)$.

For $f \in \mathcal{I}_Y(Y)$, there exists a $p \in Y$ such that the image of $\text{lc}(f)$ in $(A/\alpha)p$ is nonzero. This implies that the leading term of the image of $f$ in $T_p$ is the leading term of $f$, and thus we have $\text{lt}(f) \in \text{lt}(T_p) = T'$. Consequently $\text{lt}(\mathcal{I}_Y(Y)) = T'$, and because $g_i$ is monic with leading term $t_i$ for $i = 1, \ldots, m$ by Theorem 5, we see that $Y$ is parametric. □

Recall that $\varphi$ denotes the map from $\text{Spec}(A[x]/I)$ (respectively $\text{Proj}(A[x]/I)$) to $\text{Spec}(A)$.

**Corollary 2.** If $Y \subset \text{Spec}(A)$ is parametric, then $\varphi$ is flat over $Y$, i.e. the map $\varphi^{-1}(Y) \to Y$ is a flat morphism.

**Proof.** Let $\alpha \subset A$ denote the radical ideal such that $Y = V(\alpha)$, and let $T$ denote the extension of $I$ in $(A/\alpha)[x]$. The scheme structure on the set $\varphi^{-1}(Y)$ is given by identifying $\varphi^{-1}(Y)$ with $X = \text{Spec}(A[x]/I) \times_A Y$ or $X = \text{Proj}(A[x]/I) \times_A Y$ respectively. Thus, for $\mathfrak{p} \in \varphi^{-1}(Y)$, the stalk $\mathcal{O}_{X, \mathfrak{p}}$ equals $((A/\alpha)[x]/T)_{\mathfrak{p}}$ or $((A/\alpha)[x]/T)_{\mathfrak{p}}$. (Here $((A/\alpha)[x]/T)_{\mathfrak{p}}$ denotes the elements of degree zero in the localized ring $S^{-1}((A/\alpha)[x]/T)$, where $S$ is the multiplicative system of all homogeneous elements of $(A/\alpha)[x]/T$ which do not lie in $\mathfrak{p}$. ) Let $p = \varphi(\mathfrak{p}) \in Y$. We have to show that $\varphi_{\mathfrak{p}} : \mathcal{O}_{Y,p} \to \mathcal{O}_{X,\mathfrak{p}}$ is flat. In the affine case, $\varphi_{\mathfrak{p}}$ can be factored:

$$
\mathcal{O}_{Y,p} = (A/\alpha)p \to (A/\alpha)p[x]/T_p = ((A/\alpha)[x]/T)_p \to ((A/\alpha)[x]/T)_{\mathfrak{p}} = \mathcal{O}_{X,\mathfrak{p}}.
$$

By Theorem 6, the ideal $T_p \subset (A/\alpha)p[x]$ is monic, and thus $(A/\alpha)p[x]/T_p$ is a free $(A/\alpha)p$-module. In particular, $(A/\alpha)p[x]/T_p$ is a flat $(A/\alpha)p$-module. Since “localization is flat”, $((A/\alpha)[x]/T)_{\mathfrak{p}}$ is a flat $((A/\alpha)[x]/T)_{\mathfrak{p}}$-module. This completes the proof in the affine case.

In the projective case, we know that $S^{-1}((A/\alpha)[x]/T)$ is a flat $((A/\alpha)[x]/T)_{\mathfrak{p}}$-module, and therefore also a flat $(A/\alpha)p$-module. Since $((A/\alpha)[x]/T)_{\mathfrak{p}}$ is a direct summand of $S^{-1}((A/\alpha)[x]/T)$, $((A/\alpha)[x]/T)_{\mathfrak{p}}$ is a flat $(A/\alpha)p$-module as well. □
4. Lucky primes and pseudo division

Now it is time to introduce the concept of pseudo division (cf. Cox et al. (1997) and Montes (2002)). This is basically just the usual division without inverting the elements in the base ring. The idea behind pseudo division has already appeared in the proof of Theorem 5.

Definition 6. Let \( f, g_1, \ldots, g_m \in A[x] \). A representation

\[
 cf = f_1 g_1 + \cdots + f_m g_m + r
\]

is called a pseudo division of \( f \) modulo \( g_1, \ldots, g_m \) (w.r.t. \(<\)) if the following assertions are satisfied:

• \( f_1, \ldots, f_m, r \in A[x] \) and \( c \in A \) is a product of leading coefficients of the \( g_j \)'s.
• \( \text{lt}(f_j) \text{lt}(g_j) \leq \text{lt}(f) \) for \( j = 1, \ldots, m \).
• No term in \( \text{supp}(r) \) is divisible by a leading term of the \( g_j \)'s.
• \( \text{coef}(f_j, t) \in \langle \text{coef}(f, t'); t' \geq \text{lt}(g_j) t \rangle \) for all \( j \in \{1, \ldots, m\} \) and \( t \in T \).

In this situation, \( r \) is called a remainder of \( f \) after pseudo division modulo \( g_1, \ldots, g_m \). A pseudo division of \( f \) modulo \( g_1, \ldots, g_m \) can be obtained by successively applying pseudo reduction steps:

If there exists an element of the support of \( f \) which is divisible by a leading term of any of the \( g_j \)'s, then choose \( t \in \text{supp}(f) \) maximal with this property. Then \( t = t' \text{lt}(g_j) \) holds for some \( j \in \{1, \ldots, m\} \) and \( t' \in T \). Now substitute \( f \) by

\[
 \text{lc}(g_j) f - \text{coef}(f, t)t'g_j.
\]

By iterating this process and keeping track of the monomials used, we obtain the desired representation.

The nice thing about pseudo divisions is that they are stable under specialization, in the sense that

\[
 \text{lt}(f_j) \text{lt}(g_j) \leq \text{lt}(\overline{f})
\]

for \( j = 1, \ldots, m \). Here \( \overline{g} \) denotes the coefficientwise reduction of \( g \in A[x] \) modulo some ideal of \( A \). (This follows directly from the last assertion of the definition.)

Observe that \( c \) may well be zero if \( A \) is not an integral domain.

Definition 7. A prime ideal of \( A \) is called lucky for \( I \) if for every \( t \in \text{lt}(I) \), it does not contain \( \text{lc}(I, t) \).

To my knowledge, the expression “lucky” was coined by mathematicians working on modular algorithms to compute Gröbner bases over \( \mathbb{Q} \) (see Arnold (2003), Pauer (1992) and Gräbe (1993)). Mod \( p \) arithmetic avoids the phenomenon of coefficient growth, but it is not a priori clear which prime numbers \( p \) can be used for lifting a Gröbner basis over \( \mathbb{Z}/\mathbb{Z}p \) to a Gröbner basis over \( \mathbb{Q} \). So mathematicians must have considered themselves lucky when they picked a prime to do the job.

Let \( T \) be the unique minimal generating set of \( \text{lt}(I) \). Because \( \text{lc}(I, t) \subset \text{lc}(I, t') \) if \( t \) divides \( t' \), a prime \( p \in \text{Spec}(A) \) is lucky for \( I \) if and only if \( p \) does not contain the ideal \( \prod_{t \in T} \text{lc}(I, t) \). In particular, the set of lucky prime ideals for a given \( I \) is an open subset of \( \text{Spec}(A) \).
Definition 8. The ideal
\[ J = J(I) = \sqrt{\prod_{t \in T} \text{lc}(I, t)} \subset A \]
is called the singular ideal of I (with respect to <).

So a prime \( p \in \text{Spec}(A) \) is unlucky (i.e. not lucky) for I if and only if it is an element of the singular variety \( V(J) \).

In Weispfenning (2003), Weispfenning introduced another discriminant ideal which, however, can only be constructed if \( A \) is an integral domain. So for the time being, assume that \( A \) is an integral domain. In this case, we can consider the reduced Gröbner basis \( G \) of \( I \) over the quotient field of \( A \). For \( g \in G \) the set
\[ J_g = \{ a \in A; \; ag \in I \} \]
clearly is an ideal of \( A \), and we can define Weispfenning’s discriminant ideal by
\[ J' = J'(I) = \sqrt{\prod_{g \in G} J_g}. \]

Clearly \( J_g \subset \text{lc}(I, \text{lt}(g)) \) always holds, but the inclusion may be strict, as illustrated by the following example.

Example 5. Let \( k \) be a field and \( A = k[u_1, u_2] \) the polynomial ring in the parameters \( u_1, u_2 \). We consider the ideal
\[ I = \langle u_1 x + u_2, \; u_1 y^2 - 1 \rangle \subset A[x, y]. \]

With respect to any term order, the reduced Gröbner basis of \( I \) over the quotient field of \( A \) is
\[ G = \left\{ x + \frac{u_2}{u_1}, \; y^2 - \frac{1}{u_1} \right\}. \]
But as \( u_2 y^2 + x = y^2(u_1 x + u_2) - x(u_1 y^2 - 1) \in I \), we have, with respect to any term order with \( y^2 \succ x \),
\[ J_{y^2 - \frac{1}{u_1}} = \langle u_1 \rangle \subset \langle u_1, u_2 \rangle \subset \text{lc}(I, y^2). \]

However, our discriminant ideal is not larger than Weispfenning’s; in fact, they are the same.

Theorem 7. Assume that, in addition to our standard assumptions, the base ring \( A \) is also an integral domain. Then the singular ideal agrees with Weispfenning’s discriminant ideal. In other words, \( J = J' \).

Proof. Let \( I' \) denote the extension of \( I \) in the polynomial ring over the quotient field of \( A \). First of all, observe that \( \text{lt}(I) = \text{lt}(I') \): as \( I \subset I' \), the inclusion \( \text{lt}(I) \subset \text{lt}(I') \) is clear. For the other inclusion it suffices to notice that every \( P \in I' \) is of the form \( P = \frac{Q}{a} \) with \( Q \in I \) and \( a \in A \).

Let \( G = \{g_1, \ldots, g_m\} \) denote the unique reduced Gröbner basis of \( I' \) over the quotient field of \( A \). Then, as \( \text{lt}(I) = \text{lt}(I') \), the unique minimal generating set \( T \) of \( \text{lt}(I) \) equals
\{\text{Lt}(g_1), \ldots, \text{Lt}(g_m)\}. With the abbreviations \(t_j = \text{Lt}(g_j)\) and \(J_j = J_{g_j}\) for \(j = 1, \ldots, m\), we may assume \(t_1 < \cdots < t_m\). We have to show

\[ V \left( \text{lc}(I, t_1) \cdots \text{lc}(I, t_m) \right) = V(J_1 \cdots J_m). \]

As \(J_j \subset \text{lc}(I, t_j)\) for \(j = 1, \ldots, m\), the inclusion “\(\subset\)” is clear. For the other inclusion it will suffice to show that for \(j \in \{1, \ldots, m\}\) and \(p \in \text{Spec}(A)\),

\[ J_j \subset p \Rightarrow \text{lc}(I, t_1) \cdots \text{lc}(I, t_j) \subset p. \]

We will prove this by contradiction. So assume \(\text{lc}(I, t_1) \cdots \text{lc}(I, t_j) \not\subset p\). Then we can find \(f_1, \ldots, f_j \in I\) with \(\text{Lt}(f_i) = t_i\) and \(\text{lc}(f_i) \not\in p\) for \(i = 1, \ldots, j\). Pseudo reduction of \(f_j\) modulo \(f_1, \ldots, f_{j-1}\) yields a polynomial \(g \in I\) with \(\text{Lt}(g) = t_j\), \(\text{lc}(g) \not\in p\) and no term in \(\text{supp}(g)\) divisible by any \(t_1, \ldots, t_{j-1}\). So no term in the support of \(g - \text{lc}(g)g_j \in I'\) is divisible by any \(t_1, \ldots, t_m\). Hence, \(\text{lc}(g)g_j = g \in I\), and we conclude \(\text{lc}(g) \in J_j \subset p\) (in contradiction to \(\text{lc}(g) \not\in p\)). \(\square\)

The above theorem asserts that the concept of (in)essential specializations, as introduced in Weispfenning (2003), is equivalent to the older concept of (un)lucky prime ideals. The advantage of the idea of luckiness is, of course, that it works for more general rings, i.e. not only for integral domains. Observe that it is quite natural to work with rings which are not integral domains, because even if you start with an integral domain (e.g. the polynomial ring over a field in some parameters), the singular ideal \(J\) will typically not be prime, and therefore \(A/J\) will not be an integral domain. The relevance of this will become clear with the next theorem, which gives a characterization of parametric subsets in terms of luckiness.

**Lemma 4.** Let \(Y \subset \text{Spec}(A)\) be parametric, and \(a \subset A\) the radical ideal such that \(\overline{Y} = V(a)\). If \(\overline{I}\) denotes the image of \(I\) in \((A/a)[x]\), then \(\text{Lt}(Y) = \text{Lt}(\overline{I})\).

**Proof.** Let \(t \in \text{Lt}(Y)\) and \(p \in Y\). From **Theorem 6**, we know that \(\overline{I}_p \subset (A/a)_p[x]\) is monic with \(\text{Lt}(\overline{I}_p) = \text{Lt}(Y)\). Thus, there exists \(P \in \overline{I}\) and \(s \in (A/a) \setminus p\) such that the leading term of \(P/s \in \overline{I}_p\) equals \(t\). By **Lemma 2**, we may assume \(t = \text{Lt}(P) \in \text{Lt}(\overline{I})\).

The inclusion \(\text{Lt}(\overline{I}) \subset \text{Lt}(Y)\) follows from **Lemma 3** and **Theorem 5**. \(\square\)

Now we are prepared to prove the main theorem of this section. It exhibits the “geometric meaning” of luckiness.

**Theorem 8.** Let \(Y\) be a locally closed subset of \(\text{Spec}(A)\) and \(a \subset A\) the radical ideal, such that \(\overline{Y} = V(a)\). Denote by \(\overline{I}\) the image of \(I\) in \((A/a)[x]\). Then \(Y\) is parametric for Gröbner bases with respect to \(I\) if and only if

\[ Y \cap V(J(\overline{I})) = \emptyset. \]

In other words, \(Y\) is parametric if and only if every \(p \in Y\) is lucky for \(\overline{I}\).

**Proof.** Assume that \(Y\) is parametric and \(\{g_1, \ldots, g_m\} \subset I_Y(Y)\) is the reduced Gröbner basis of \(I\) over \(Y\). Then by **Lemma 4**, the minimal generating set \(T\) of \(\text{Lt}(\overline{I})\) equals \(\{\text{Lt}(g_1), \ldots, \text{Lt}(g_m)\}\).

Let \(p \in Y\) and \(i \in \{1, \ldots, m\}\). By **Lemma 2**, there exists a \(P_i \in \overline{I}\) with \(\text{Lt}(P_i) = \text{Lt}(g_i(p))\) and \(s_i \in (A/a) \setminus p\) such that \(g_i(p) = P_i/s_i \in \overline{I}_p\). Because \(\text{Lt}(P_i) = \text{Lt}(g_i(p)) = \text{Lt}(g_i) = \text{Lt}(\overline{g_i}p)\), we have \(\text{lc}(P_i) \not\in p\), i.e. \(\text{lc}(\overline{I}, \text{Lt}(P_i)) \not\in p\). Hence

\[ J(\overline{I}) = \prod_{i \in \overline{I}} \text{lc}(\overline{I}, i) \not\in p. \]
For the converse direction, first fix a $p \in Y$ and let $T = \{t_1, \ldots, t_m\}$ denote the minimal generating set of $\l t(\overline{T})$. By our assumption,

$$\prod_{i=1}^{m} \lc(T, t_i) \not\subseteq p.$$  

Hence, there exist polynomials $P_1, \ldots, P_m \in \overline{T}$ with $\l t(P_i) = t_i$ and $\lc(P_i) \not\subseteq p$. For $i = 1, \ldots, m$, let $Q_i \in \overline{T}$ denote a remainder of $P_i$ after pseudo division modulo $\{P_1, \ldots, P_m\} \setminus \{P_i\}$. Note that $\l t(Q_i) = \l t(P_i) = t_i$ and $\lc(Q_i)$ is a product of leading coefficients of the $P_j$’s. Define

$$U = \{q \in Y; \lc(P_1) \cdots \lc(P_m) \not\subseteq q\}.$$  

Then $U$ is an open neighborhood of $p \in Y$ and $Q_i/\lc(Q_i)$ defines an element of $\mathcal{I}_Y(U)$, which by abuse of notation we again denote by $Q_i/\lc(Q_i)$.

We can repeat the above construction for any $p' \in Y$ to obtain $U'$ and $Q'_i$ (analogously defined). To obtain global sections $g_i \in \mathcal{I}_Y(Y)$, we have to show that

$$\frac{Q_i}{\lc(Q_i)} \bigg|_{U \cap U'} = \frac{Q'_i}{\lc(Q'_i)} \bigg|_{U \cap U'}.$$  

The leading term of

$$\lc(Q'_i)Q_i - \lc(Q_i)Q'_i \in \overline{T}$$

is strictly smaller than $t_i$, and by our construction no term in the support of $\lc(Q'_i)Q_i - \lc(Q_i)Q'_i$ is divisible by an element of $\{t_1, \ldots, t_m\} \setminus \{t_i\}$. Thus, $\lc(Q'_i)Q_i - \lc(Q_i)Q'_i = 0$, and we can glue together the sections $Q_i/\lc(Q_i) \in \mathcal{I}_Y(U)$ to obtain global sections $g_i \in \mathcal{I}_Y(Y)$.

To show that $Y$ is parametric, we prove that $G = \{g_1, \ldots, g_m\}$ satisfies the conditions of Definition 1. Clearly, $\l t(\overline{g_i}^P) = t_i$ for every $p \in Y$. So it remains to show that $\overline{G}^P$ is the reduced Gröbner basis of $\langle \sigma_p(I) \rangle$ for every $p \in Y$. Let $p \in Y$ and $P \in \overline{T}$. For a pseudo division (see Definition 6)

$$cP = P_1Q_1 + \cdots + P_mQ_m + r$$

of $P$ modulo $Q_1, \ldots, Q_m$, we have $r \in \overline{T}$, but no term in the support of $r$ is divisible by an element of $\{\l t(Q_1), \ldots, \l t(Q_m)\} = T$. Thus $r = 0$ and

$$cP = P_1Q_1 + \cdots + P_mQ_m.$$  

Let $\phi : (A/\alpha)[x] \to k(p)[x]$ denote the natural map; then

$$\phi(c)\phi(P) = \phi(P_1)\phi(Q_1) + \cdots + \phi(P_m)\phi(Q_m)$$

and $\l t(\phi(P_1)) \l t(\phi(Q_1)) \leq \l t(\phi(P))$. Since $\lc(Q_1) \not\subseteq p$ and $c$ is a product of leading coefficients of the $Q_i$’s, we know that $\phi(c)$, $\phi(\lc(Q_1))$, $\ldots$, $\phi(\lc(Q_m))$ are all nonzero. Consequently, $\l t(\phi(P))$ is divisible by $\l t(\phi(Q_1)) = t_i$ for some $i \in \{1, \ldots, m\}$. Since every element of $\langle \sigma_p(I) \rangle$ is of the form $\lambda f$ for $\lambda \in k(p)$, and $f \in \phi(T) = \sigma_p(I)$, this shows that $\l t(\langle \sigma_p(I) \rangle)$ is generated by $T$, and so indeed $\overline{G}^P$ is a Gröbner basis of $\langle \sigma_p(I) \rangle$. Clearly $\overline{g_i}^P$ is monic, and by the construction of $Q_i$ no term in the support of $\overline{g_i}^P$ is divisible by an element of $T \setminus \{t_i\}$. Thus, $\overline{G}^P$ is the reduced Gröbner basis of $\langle \sigma_p(I) \rangle$, and we are done.  \[\square\]
Definition 9. Let $Z$ be a closed subset of $\text{Spec}(A)$ and $a \subset A$ the radical ideal such that $Z = V(a)$. Furthermore, let $T$ denote the image of $I$ in $(A/a)[x]$. We define

$$Z_{\text{gen}} = Z \setminus V(J(T)).$$

Theorem 9. Let $Z \subset \text{Spec}(A)$ be closed, $a \subset A$ the radical ideal such that $Z = V(a)$, and $T$ the image of $I$ in $(A/a)[x]$. Then $Z_{\text{gen}}$ is parametric with $\text{lt}(Z_{\text{gen}}) = \text{lt}(T)$. Furthermore, if $Y$ is an open subset of $Z$ such that $Y$ is parametric with $\text{lt}(Y) = \text{lt}(T)$, then $Y \subset Z_{\text{gen}}$.

In other words: $Z_{\text{gen}}$ is the largest open parametric subset of $Z$ with the same leading terms as $T$.

Proof. Let $Y$ be an open subset of $Z$. First we will show that $\mathcal{I}_Z|_Y$ is canonically isomorphic to $\mathcal{I}_Y$. Let $a' \subset A$ denote the radical ideal such that $\overline{Y} = V(a')$, and $\overline{T}$ is the image of $I$ in $(A/a')[x]$. Then $a \subset a'$, and the canonical map $A/a \to A/a'$ extends to $\overline{T} \to \overline{T}'$, and further to $\phi : \overline{T}_p \to \overline{T}'_p$ for $p \in Y$. It suffices to show that $\phi$ is an isomorphism.

Clearly $\phi$ is surjective. Let $P \in I$ and $s \in A \setminus p$ such that $\phi(\overline{P}/s) = \overline{T}'_p$ is zero. This means that there exists $s' \in A \setminus p$ such that $\text{coef}(s'P, t) \in a'$ for every $t \in T$. Let $a = p_1 \cap \cdots \cap p_m$ be the (unique minimal) primary decomposition of the radical ideal $a$. We may assume $p_1, \ldots, p_r \in Y$ and $p_{r+1}, \ldots, p_m \notin Y$. Note that $p_i \notin Y$ implies $V(p_i) \cap Y = \emptyset$, because $Y$ is an open subset of $Z$. So, in particular, $p_i \not\subseteq p$ for $i = r + 1, \ldots, m$. This means that there exists an $s'' \in p_{r+1} \cap \cdots \cap p_m \setminus p$. For $1 \leq i \leq r$, we have $V(p_i) \subset \overline{Y} = V(a')$, and thus $a' \subset p_i$. Combining these results, we see that every coefficient of $s''s'/P$ lies in $p_1 \cap \cdots \cap p_m = a$, and thus $\overline{P}/s$ is zero in $\overline{T}_p$. Consequently $\phi$ is injective.

An argument similar to the one above shows that for $p \in Y$, the map $(A/a)_p \to (A/a')_p$ is an isomorphism. Thus, $\phi$ also preserves leading terms.

Now to show that $Z_{\text{gen}}$ is parametric with the same leading terms as $T$, just repeat the second part of the proof of Theorem 8 (with $Z_{\text{gen}}$ instead of $Y$), and use that $\mathcal{I}_Z(Z_{\text{gen}})$ is canonically isomorphic to $\mathcal{I}_{Z_{\text{gen}}}(Z_{\text{gen}})$.

Now we additionally assume that $Y$ is a parametric subset of $Z$ with $\text{lt}(Y) = \text{lt}(T)$. Suppose $Y \not\subseteq Z_{\text{gen}}$. Then there exists a $p \in Y \setminus Z_{\text{gen}}$. Let $T$ denote the minimal generating set of $\text{lt}(T)$. Since $p \notin Z_{\text{gen}} = Z \setminus V(J(T))$, there exists a $t \in T$ such that $\text{lc}(T, t) \subset p$.

Since $Y$ is parametric with $\text{lt}(Y) = \text{lt}(T)$, we know from Theorem 6 that $\overline{T}'_p$ is monic with $\text{lt}(\overline{T}'_p) = \text{lt}(T)$. Using the isomorphism $\phi : \overline{T}_p \to \overline{T}'_p$, we see that $\overline{T}_p$ is monic, with $\text{lt}(\overline{T}_p) = \text{lt}(T)$. Thus there exists $P \in \overline{T}$ and $s \in A \setminus p$ such that $P/s \in \overline{T}_p$ is monic with leading term $t$. By Lemma 2 we may assume $\text{lt}(P) = t$. Since $P/s$ is monic, there exists $s' \in A \setminus p$ such that $(\text{lc}(P) - s)s' = 0$. Thus $\text{lc}(P) \notin p$, in contradiction to $\text{lc}(T, t) \subset p$. \qed

If we take $Z = \text{Spec}(A)$ in the above theorem, then we see that the set of all lucky primes of $A (= \text{Spec}(A) \setminus V(J(I)))$ is the largest open parametric subset of $\text{Spec}(A)$ with the same leading terms as $I$. This more or less comes down to saying that $J$ is the optimal discriminant ideal.

Caution. It is not true that $p \in \text{Spec}(A)$ is lucky for $I$ if and only if $\text{lt}(I) = \text{lt}(\langle \sigma_p(I) \rangle)$. We have seen above that the “only if” direction is correct but the “if” direction is not true in general (see Example 3). However, it is true for homogeneous ideals, as we will see in Section 5.1.

The following simple example illustrates that $Z_{\text{gen}}$ may well be the empty set.
Example 6. Assume that $A$ is not an integral domain; then there exist $a, b \in A \setminus \{0\}$ such that $ab = 0$. If we take $I$ to be the ideal of $A[x_1, x_2]$ generated by $ax_1$ and $bx_2$, then (with respect to any term order) $J(I) = \langle 0 \rangle$ and so $\text{Spec}(A)_{\text{gen}} = \emptyset$.

However, this cannot happen if $Z$ is irreducible, because then $Z = V(a)$ for some prime ideal $a$ of $A$, and since $A/a$ is an integral domain, $J(I)$ is not the zero ideal, and thus $Z_{\text{gen}}$ is nonempty. In particular, $Z_{\text{gen}}$ is dense in $Z$ and contains the generic point of $Z$.

The following examples have been included to convince the reader that the singular ideal $J$ is quite a reasonable object.

Example 7. Let $I \subset A[x]$ be the ideal generated by a square linear system

$$P_1 = b_{11}x_1 + b_{12}x_2 + \cdots + b_{1n}x_n - c_1$$
$$\vdots \quad \vdots \quad \vdots$$
$$P_n = b_{n1}x_1 + b_{n2}x_2 + \cdots + b_{nn}x_n - c_n$$

and let

$$B = (b_{ij})_{1 \leq i, j \leq n} \in A^{n \times n}$$

denote the matrix of the system. Suppose $\det = \det(B) \in A$ is not a zero divisor. Then the singular ideal $J$ of $I$ is independent of the chosen term order and $V(J)$ equals $V(\det)$. In other words, $J = \sqrt{\langle \det \rangle}$.

Proof. Let $B' \in A^{n \times n}$ denote the adjoint matrix of $B$. A classical theorem from linear algebra (see e.g. Lang (1977), Chapter 8, Section 4, Proposition 8) asserts that

$$B'B = BB' = \det \cdot \mathbb{I}, \tag{1}$$

where $\mathbb{I}$ denotes the $n \times n$ identity matrix.

First, we show that $1 \notin \text{lt}(I)$. Suppose the contrary. Let $A'$ denote the total ring of fractions of $A$, i.e. the localization at the multiplicative subset of all nonzero divisors. Then we may regard $A$ as a subring of $A'$. With the abbreviations

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \quad \text{and} \quad \xi = \frac{1}{\det} \cdot B'c$$

identity (1) shows that $\xi$ is a solution of our linear system. Now $1 \in \text{lt}(I)$ simply means that there exist an $a \in A \setminus \{0\}$ and $Q_1, \ldots, Q_n \in A[x]$ such that

$$Q_1P_1 + \cdots + Q_nP_n = a.$$ 

Evaluation at $\xi$ yields the contradiction $a = 0$.

Identity (1) also shows that $\det$ lies in $\text{lc}(I, x_i)$ for $i = 1, \ldots, m$. Therefore, $\det \in J$ and $V(J) \subset V(\det)$. Now for the converse inclusion, assume $p \in V(\det)$, i.e. $\det \in p$. From Theorem 9, we know that for every $q \in \text{Spec}(A) \setminus V(J)$ the leading terms of $\langle \sigma_q(I) \rangle$ are generated by $x_1, \ldots, x_n$. But $\det \in p$ implies that $\text{lt}(\langle \sigma_p(I) \rangle)$ is not generated by $x_1, \ldots, x_n$, and consequently $p \notin V(J)$.
Example 8. Let \( k \) be a field and \( I' \subset k[x] = k[x_1, \ldots, x_n] \) a (homogeneous) ideal. For \( 1 \leq i, j \leq n \), let \( u_{ij} \) be additional indeterminates and abbreviate
\[
ux = (u_{11}x_1 + \cdots + u_{1n}x_n, \ldots, u_{n1}x_1 + \cdots + u_{nn}x_n).
\]
Let \( A \) be the polynomial ring over \( k \) in the \( u_{ij} \)'s, and define
\[
I = \langle P(ux); \ P \in I' \rangle \subset A[x].
\]
Then the ideal of \( k[x] \) generated by \( \text{lt}(\text{Spec}(A)_{\text{gen}}) \) is the generic initial ideal of \( I' \), usually denoted by \( \text{Gin}(I') \) (see e.g. Eisenbud (1995) or Green (1998)).

Example 9. Suppose that < is a graded order and \( A \) is an integral domain, i.e. \( \text{Spec}(A) \) is irreducible. Then \( \text{Spec}(A)_{\text{gen}} \) is a nonempty, open (and thus dense) subset of \( \text{Spec}(A) \) such that the function
\[
p \mapsto \text{affine Hilbert function of } \langle \sigma_p(I) \rangle
\]
is constant on \( \text{Spec}(A)_{\text{gen}} \). This is clear because the affine Hilbert function of \( \langle \sigma_p(I) \rangle \) is determined by \( \text{lt}(\langle \sigma_p(I) \rangle) \) (see Cox et al. (1997), Chapter 9, Section 3, Proposition 4). Of course, there is also an analogous “projective” statement.

5. Gröbner covers

Now that we have (at least to some extent) explored the nature of parametric sets, it is time to see the complete picture.

Definition 10. Let \( L \) be a locally closed subset of \( \text{Spec}(A) \). A finite set \( G \) consisting of pairs \((Y, G_Y)\), with \( Y \subset \text{Spec}(A) \) parametric and \( G_Y \) the reduced Gröbner bases of \( I \) over \( Y \), is called a Gröbner cover of \( L \) with respect to \( I \) (and <) if
\[
L = \bigcup_{Y \in G} Y.
\]
A Gröbner cover \( G \) is called irreducible if every \( Y \in G \) is irreducible.

A Gröbner cover \( G \) of \( L \) is called locally maximal if, for every \( Y \in G \), the following holds: if \( Y' \subset \text{Spec}(A) \) is parametric with \( Y' \subset L \) and \( Y \subset Y' \subset \overline{Y} \), then \( Y = Y' \).

A Gröbner cover \( G \) is called small if for every \( Y \in G \), we have
\[
Y \setminus \bigcup_{Y' \in G \setminus \{Y\}} Y' = \overline{Y}.
\]
As already done in the above definition, we write \( Y \in G \) instead of unhandy \((Y, G_Y) \in G\) and refer to \( Y \) as an element of \( G \). To say that a Gröbner cover is small basically means that its elements are not unnecessarily large. Our main interest, of course, is in Gröbner covers of \( \text{Spec}(A) \), but (with a view towards applications) it seems reasonable to also treat the relative case. The basic idea behind Gröbner covers and reduced comprehensive Gröbner systems (Weispfenning, 1992) is the same, but there are crucial differences. See the last part of the Appendix for more details.

Definition 11. Let \( L \) be a locally closed subset of \( \text{Spec}(A) \) and \( G \) a finite subset of \( I \). Then \( G \) is called a comprehensive Gröbner basis of \( I \) with respect to \( L \) (and <) if \( \sigma_p(G) = \{\sigma_p(g); \ g \in G\} \) is a Gröbner basis of \( \langle \sigma_p(I) \rangle \) for every \( p \in L \).
Comprehensive Gröbner bases were introduced by Weispfenning in Weispfenning (1992) and advanced in Weispfenning (2003). There is a rather obvious connection between Gröbner covers of $L$ and comprehensive Gröbner bases of $I$ with respect to $L$, which we will now describe.

Let $\mathcal{G}$ be a Gröbner cover of $L$. Choose a $Y \in \mathcal{G}$ and let $A \subseteq A$ be the radical ideal such that $\overline{Y} = V(a)$; furthermore, let $\overline{T}$ denote the image of $I$ in $(A/a)[x]$. Since $\text{Spec}(A)$ is a Noetherian topological space, $Y$ is quasi-compact; and so for every $g \in G_Y$, we can find finitely many open subsets $U_i$ of $Y$ which cover $Y$ and have the following property: there exists a $P \in I$ and $s \in A/a$ such that

$$g(p) = \frac{P}{s} \in \overline{T}_p \quad \text{for every } p \in U_i.$$  

Here, $\overline{P}$ denotes the image of $P$ in $\overline{T} \subset (A/a)[x]$. Now taking together all such $P$’s (for all $U_i$’s, all $g \in G_Y$ and all $Y \in \mathcal{G}$), we end up with a finite subset of $I$, which clearly is a comprehensive Gröbner basis of $I$ with respect to $L$.

The following example is meant to illustrate that Gröbner covers are more predisposed to a canonical form than comprehensive Gröbner bases. See Weispfenning (2003) for a discussion of canonical comprehensive Gröbner bases.

**Example 10.** Assume that we are given a term order $<$ and $f \in A[x] = A[x_1, \ldots, x_n] \setminus \{0\}$ such that $\text{lc}(f) = 1$. Let $a$ be a proper ideal of $A$ and $I = \langle af; \ a \in a \rangle \subset A[x]$. If $p \in \text{Spec}(A)$ contains $a$, then $\langle \sigma_p(I) \rangle$ is the zero ideal. If $p$ does not contain $a$, then $\sigma_p(f)$ is the reduced Gröbner basis of $\langle \sigma_p(I) \rangle$. If $a_1, \ldots, a_m$ is a generating set of $a$, then clearly $a_1f, \ldots, a_mf$ is a comprehensive Gröbner basis of $I$. On the other hand, it does not seem very reasonable to consider a comprehensive Gröbner basis of $I$ which is not of this simple form as a candidate for the “canonical comprehensive Gröbner basis”. Thus, determining the “canonical comprehensive Gröbner basis” in this case is more or less equivalent to determining a generating set of $a$. This cannot, of course, be done in a canonical way without imposing further restrictions on the base ring $A$. For example, if $A$ is a polynomial ring over some field, we could fix a term order on $A$ and take $a_1, \ldots, a_m$ to be the reduced Gröbner basis of $a$ with respect to this term order.

For Gröbner covers, the situation is straightforward: exactly one Gröbner cover stands out for being the simplest, and would therefore truly deserve to be called “canonical”. It is

$$\mathcal{G} = \{(\text{Spec}(A) \setminus V(a), \{\tilde{f}\}), (V(a), \emptyset)\}.$$  

Here, we have written $\tilde{f}$ to denote the function which assigns $f \in I_p$ to every $p \in \text{Spec}(A) \setminus V(a)$. Note that $\tilde{f}$ is an element of $\mathcal{I}_Y(Y)$ for $Y = \text{Spec}(A) \setminus V(a)$. Indeed, for every $p \in Y$, there exists an $a \in a$ such that $a \notin p$. For every $q \in U = \{q \in Y; \ a \notin q\}$, we have $\tilde{f}(q) = f = \frac{af}{a} \in I_q$. We see that the problem of choosing a generating system of $a$ has been transformed to the problem of representing $\tilde{f}$ by fractions. But all we need to know is that $\tilde{f}$ can be represented by fractions. It is irrelevant which representation we choose: they all define the same function from $Y$ to $\bigsqcup_{p \in Y} I_p$.

The main theorem of this section asserts that for every locally closed subset $L$ of $\text{Spec}(A)$, there exists a unique irreducible, small and locally maximal Gröbner cover of $L$. For the proof, we will need a few basic facts about constructible sets (cf. Hartshorne (1977)).

**Definition 12.** Let $X$ be a topological space. A constructible subset of $X$ is a subset which belongs to the smallest family $\mathcal{F}$ of subsets such that
(1) every open subset is in \( F \),
(2) a finite intersection of elements in \( F \) is in \( F \), and
(3) the complement of an element in \( F \) is in \( F \).

One easily shows that the constructible sets of a topological space are precisely those sets which can be written as finite unions of locally closed sets.

**Lemma 5.** Let \( C \) be a constructible subset of \( \text{Spec}(A) \) and
\[
\overline{C} = Z_1 \cup \cdots \cup Z_m
\]
the unique minimal decomposition of \( \overline{C} \) into irreducible and closed sets (cf. Hartshorne (1977), Chapter 1, Proposition 1.5). Then for \( j = 1, \ldots, m \), there exists a nonempty open subset of \( Z_j \) contained in \( C \).

**Proof.** A constructible set \( C \) can be written as a finite union
\[
C = L_1 \cup \cdots \cup L_{m'}
\]
of nonempty, locally closed and irreducible sets \( L_i \).
\[
Z_1 \cup \cdots \cup Z_m = \overline{C} = \overline{L_1} \cup \cdots \cup \overline{L_{m'}}.
\]
Fix a \( j \in \{1, \ldots, m\} \). As \( Z_j \) is irreducible, there exists an \( i \in \{1, \ldots, m'\} \) such that \( Z_j \subset \overline{L_i} \).
Similarly, as \( \overline{L_i} \) is irreducible, there exists a \( j' \in \{1, \ldots, m\} \) such that \( \overline{L_i} \subset Z_{j'} \). Hence
\[
Z_j \subset \overline{L_i} \subset Z_{j'}.
\]
This yields \( j = j' \) and \( Z_j = \overline{L_i} \). So \( L_i \) is a nonempty open subset of \( Z_j \) contained in \( C \). □

**Lemma 6.** Let \( L \) be a locally closed and irreducible subset of \( \text{Spec}(A) \). For a constructible subset \( C \) of \( \text{Spec}(A) \) which is contained in \( L \), we have \( \overline{C} = \overline{L} \) if and only if \( C \) contains the generic point of \( L \).

**Proof.** If \( C \) contains the generic point \( p \) of \( L \), we have \( \overline{L} = \overline{\{p\}} \subset \overline{C} \). Since also \( C \subset L \), it follows that \( \overline{L} = \overline{C} \).

Conversely, if \( \overline{C} = \overline{L} \) by Lemma 5, we know that there exists a nonempty open subset \( U \) of \( \overline{L} \) contained in \( C \). As \( U \cap L \) is a nonempty open subset of \( L \), we have
\[
p \in U \cap L \subset C. \quad \Box
\]

**Theorem 10.** Let \( L \subset \text{Spec}(A) \) be a locally closed set and \( \mathcal{G} \) an irreducible Gröbner cover of \( L \). The following are equivalent:

(1) \( \mathcal{G} \) is small.
(2) Every \( Y \in \mathcal{G} \) is the only element of \( \mathcal{G} \) containing the generic point of \( Y \).
(3) For \( Y, Y' \in \mathcal{G} \) with \( Y \neq Y' \) and \( Y \subset \overline{Y'} \), we have \( Y \cap Y' = \emptyset \).

**Proof.** The equivalence of (1) and (2) follows from Lemma 6.

For two distinct, locally closed and irreducible subsets \( Y \) and \( Y' \) of \( \text{Spec}(A) \), the generic point of \( Y \) is contained in \( Y' \) if and only if \( Y \subset \overline{Y'} \) and \( Y \cap Y' \neq \emptyset \). Therefore, (3) is equivalent to (2). □

Now we are prepared to prove the main theorem.
Theorem 11. Let $L$ be a locally closed subset of $\text{Spec}(A)$. Then there exists exactly one irreducible, small and locally maximal Gröbner cover of $L$.

Proof. First we will construct a Gröbner cover $\mathcal{G}$ of $L$ and prove that it has the desired properties. Then we will prove uniqueness. We construct $\mathcal{G}$ recursively:

Set $C_1 = L$ and $i = 1$.

(1) Let 

$$\overline{C}_i = Z_{i1} \cup \cdots \cup Z_{imi}$$

be the unique minimal decomposition of $\overline{C}_i$ into irreducible and closed sets. For $j = 1, \ldots, m_i$, define 

$$Y_{ij} = Z_{ij, \text{gen}} \cap \left( \text{union of all open subsets of } Z_{ij} \text{ contained in } L \right)$$

and

$$C_{i+1} = C_i \setminus (Y_{i1} \cup \cdots \cup Y_{imi}).$$

If $C_{i+1} \neq \emptyset$, replace $i$ by $i + 1$ and go to (1).

This yields a sequence of constructible sets $C_i$ with

$$L = C_1 \supset C_2 \supset \cdots.$$ 

To prove termination, we will show that the sequence 

$$\overline{C}_1 \supset \overline{C}_2 \supset \cdots$$

is strictly decreasing. For $i \geq 1$ and $j = 1, \ldots, m_i$, there exists a nonempty open subset of $Z_{ij}$ contained in $C_i \subset L$ by Lemma 5. Hence, $Y_{ij}$ is a nonempty open subset of $Z_{ij}$ contained in $L$.

$$C_{i+1} = C_i \setminus (Y_{i1} \cup \cdots \cup Y_{imi}) \subset Z_{i1} \cup \cdots \cup Z_{imi} \setminus Y_{i1} \cup \cdots \cup Y_{imi} \subset (Z_{i1} \setminus Y_{i1}) \cup \cdots \cup (Z_{imi} \setminus Y_{imi}) = (Z_{i1} \setminus Y_{i1}) \cup \cdots \cup (Z_{imi} \setminus Y_{imi}).$$

This shows that there exists a (minimal) $r \in \mathbb{N}$ such that $C_{r+1} = \emptyset$. Hence

$$\emptyset = C_{r+1} = C_r \setminus (Y_{r1} \cup \cdots \cup Y_{rm_r})$$

$$= C_{r-1} \setminus (Y_{r-1,1} \cup \cdots \cup Y_{r-1,m_{r-1}} \cup Y_{r1} \cup \cdots \cup Y_{rm_r}) = \cdots$$

$$= C_1 \setminus (Y_{11} \cup \cdots \cup Y_{1m} \cup \cdots \cup Y_{r1} \cup \cdots \cup Y_{rm_r}).$$

So we obtain

$$L = C_1 = Y_{11} \cup \cdots \cup Y_{1m} \cup \cdots \cup Y_{r1} \cup \cdots \cup Y_{rm_r}.$$ 

As the $Y_{ij}$'s are parametric by construction, this shows that

$$\mathcal{G} = \left\{(Y_{ij}, G_{Y_{ij}}) ; 1 \leq i \leq r, 1 \leq j \leq m_i \right\}$$

is a Gröbner cover of $L$. It is clearly irreducible.

Next, we will show that $\mathcal{G}$ is locally maximal. So let $Y \subset L$ be parametric with 

$$Y_{ij} \subset Y \subset \overline{Y_{ij}} = Z_{ij}.$$
Then $Y$ is an open parametric subset of $Z_{ij}$, and so by Theorem 9 we have $Y \subset Z_{ij,\text{gen}}$. From the definition of $Y_{ij}$, we obtain $Y \subset Y_{ij}$ and thus $Y = Y_{ij}$.

Now we will show that $\mathcal{G}$ is small. Let $Y_{ij}, Y_{ij'} \in \mathcal{G}$ with $(i, j) \neq (i', j')$.

We want to show that for $i \leq i'$, we have $Y_{ij} \not\subset \overline{Y_{ij'}}$. Assume the contrary. Then

$$\overline{Y_{ij'}} = Z_{ij'} \subset \overline{C_{ij}} \subset \overline{C_i} = Z_{i1} \cup \cdots \cup Z_{im_i}.$$  

Consequently, there exists an $l \in \{1, \ldots, m_i\}$ such that $Z_{ij'} \subset Z_{il}$. This yields

$$Z_{ij} = \overline{Y_{ij}} \subset \overline{Y_{ij'}} = Z_{ij'} \subset Z_{il}.$$  

Therefore $j = l$ and $Z_{ij} = Z_{ij'}$. For $i = i'$, this directly gives the contradiction $j = j'$. For $i < i'$, we have

$$Z_{ij} = Z_{ij'} \subset \overline{C_{ij}} \subset \overline{C_{i+1}} \subset (Z_{i1} \setminus Y_{i1}) \cup \cdots \cup (Z_{im_i} \setminus Y_{im_i}).$$  

Consequently $Z_{ij} \subset Z_{ij} \setminus Y_{ij}$, and we obtain the contradiction $Y_{ij} = \emptyset$.

To prove that $\mathcal{G}$ is small it suffices, by Theorem 10, to show that for $i > i'$ and $Y_{ij} \subset \overline{Y_{ij'}}$, we have $Y_{ij} \cap Y_{ij'} = \emptyset$. Note that $Y_{ij} \subset \overline{Y_{ij'}}$ implies that $Z_{ij} \setminus Y_{ij'}$ is a closed subset of $\text{Spec}(A)$. By our construction, we have

$$C_i = C_i \setminus \left(Y_{i1} \cup \cdots \cup Y_{i'm_i} \cup \cdots \cup Y_{i,1,1} \cup \cdots \cup Y_{i,1,m_{i,1}}\right).$$  

(2)

For subsets $B, C, D$ of an arbitrary topological space with $D \subset C$, there is the trivial identity

$$\overline{B \setminus C \setminus D} = \overline{B \setminus C}.$$  

Together with (2), this yields

$$\overline{C_i} = \overline{C_i \setminus Y_{ij'}} = Z_{i1} \cup \cdots \cup Z_{im_i} \setminus Y_{ij'} \subset Z_{i1} \cup \cdots \cup (Z_{ij} \setminus Y_{ij'}) \cup \cdots \cup Z_{im_i}$$

$$= Z_{i1} \cup \cdots \cup (Z_{ij} \setminus Y_{ij'}) \cup \cdots \cup Z_{im_i} \subset Z_{i1} \cup \cdots \cup Z_{im_i} = \overline{C_i}.$$  

Therefore

$$Z_{i1} \cup \cdots \cup Z_{im_i} = Z_{i1} \cup \cdots \cup (Z_{ij} \setminus Y_{ij'}) \cup \cdots \cup Z_{im_i}$$

and $Z_{ij} \subset Z_{ij} \setminus Y_{ij'}$. Thus $Y_{ij} \cap Y_{ij'} = \emptyset$.

So far, we have shown that $\mathcal{G}$ is an irreducible, small and locally maximal Gröbner cover of $L$. It remains to prove uniqueness. Assume $\mathcal{G}'$ is another irreducible, small and locally maximal Gröbner cover of $L$. First we will show $\mathcal{G} \subset \mathcal{G}'$. More precisely, we will show by induction on $i = 1, \ldots, r$, that $Y_{i1}, \ldots, Y_{im_i} \in \mathcal{G}'$. We denote the generic point of $Y_{ij}$ by $p_{ij}$.

First assume $i = 1$. Let $j \in \{1, \ldots, m_1\}$. As

$$\bigcup_{Y \in \mathcal{G}} Y = L = \bigcup_{Y' \in \mathcal{G}'} Y'$$

there exists a $Y'_{1j} \in \mathcal{G}'$ such that $p_{1j} \in Y'_{1j}$. We want to show $Y_{1j} = Y'_{1j}$. As $Y_{1j}$ is irreducible and $\overline{Y_{1j}} \subset \overline{L} = Z_{11} \cup \cdots \cup Z_{1m_1}$, there exists a $j' \in \{1, \ldots, m_1\}$ such that $\overline{Y_{1j}} \subset Z_{1j'}$. Together with $p_{1j} \in Y'_{1j}$, this gives

$$Z_{1j} \subset \overline{Y_{1j}} \subset Z_{1j'}.$$
Therefore \( j = j' \) and \( Y'_{ij} = Z_{ij} \). Thus, \( Y'_{ij} \) is an open subset of \( Z_{ij} \) contained in \( L \) and by Theorem 9 \( Y'_{ij} \subset Z_{1j, \text{gen}} \). So by the definition of \( Y_{1j} \), we have \( Y'_{ij} \subset Y_{1j} \). Since \( G' \) is locally maximal, we obtain \( Y_{1j} = Y'_{ij} \in G' \).

Now we do the induction step. Suppose

\[
Y_{11}, \ldots, Y_{i1m_1}, \ldots, Y_{i-1,1}, \ldots, Y_{i-1,m_{i-1}} \in G'.
\]

We have to show \( Y_{i1}, \ldots, Y_{im_i} \in G' \). For \( j \in \{1, \ldots, m_i\} \), there exists a \( Y'_{ij} \in G' \) such that \( p_{ij} \in Y'_{ij} \). Using the fact that \( G' \) is small and the induction hypothesis, we obtain

\[
\overline{Y'_{ij}} = Y'_{ij} \setminus \bigcup_{Y' \in G' \setminus \{Y'_{ij}\}} Y' \subset L \setminus \bigcup_{1 \leq i' \leq i} \bigcup_{1 \leq j' \leq m_i'} Y'_{ij'} = C_i = Z_{i1} \cup \cdots \cup Z_{im_i}.
\]

Hence, there exists a \( j' \in \{1, \ldots, m_i\} \) such that \( \overline{Y'_{ij}} \subset Z_{ij'} \). Together with \( p_{ij} \in Y'_{ij} \), this gives

\[
Z_{ij} \subset \overline{Y'_{ij}} \subset Z_{ij'}.
\]

Therefore, \( j = j' \) and \( \overline{Y'_{ij}} = Z_{ij} \). Since \( G' \) is locally maximal, a similar argument as in the case \( i = 1 \) above proves \( Y_{ij} = Y'_{ij} \in G' \). Thus, we have shown \( G \subset G' \).

Assume this is a proper inclusion. Then there exists a \( Y' \in G' \) such that \( Y' \notin G \), and therefore

\[
\overline{Y'} = Y' \setminus \bigcup_{Y \in G \setminus \{Y'\}} Y \subset Y' \setminus \bigcup_{Y \in G} Y = \overline{Y'} \setminus L = \emptyset.
\]

This is a contradiction as, by definition, the empty set is not irreducible. \( \square \)

**Definition 13.** Let \( L \) be a locally closed subset of \( \text{Spec}(A) \). The uniquely determined irreducible, small and locally maximal Gröbner cover of \( L \) is called the **canonical irreducible Gröbner cover** of \( L \) (with respect to \( I \) and \( \prec \)).

In Weispfenning (2003), Weispfenning gave a rather ad hoc kind of construction for what he called canonical Gröbner systems. This construction bears some analogy with the existence proof of the above theorem; however, there are some differences between the concept of canonical Gröbner systems and the concept of canonical irreducible Gröbner covers. For example, the canonical Gröbner system may contain redundant elements. The persistent reader is invited to verify this with the example \( A = k[u_1, u_2] \) and \( I = \langle u_1u_2, u_1x^2 + x \rangle \). (The point is simply that if \( \text{Spec}(A) = Z_1 \cup \cdots \cup Z_{m_i} \) is the decomposition of \( \text{Spec}(A) \) into irreducible closed sets, then it may happen that the singular part of \( Z_i \) (= \( Z_i \setminus Z_{i, \text{gen}} \)) is contained in some \( Z_{j, \text{gen}} \).)

Note that Theorem 11 implies that the equivalence relation on \( \text{Spec}(A) \), given by comparing the leading terms of \( (\sigma_p(I)) \), has only finitely many equivalence classes, and that every equivalence class is a constructible set. Indeed, Examples 2 and 11 show that these equivalence classes are only constructible and not locally closed. The following example illustrates that the canonical irreducible Gröbner cover may be not of minimal cardinality among the irreducible Gröbner covers.

**Example 11.** Let \( k \) be a field and \( A = k[u_1, u_2] \) be the polynomial ring in the two parameters \( u_1, u_2 \). We consider the ideal

\[
I = \langle u_1x, (u_2^2 - 1)x^2 + x \rangle \subset A[x].
\]
(Here $x$ denotes just one variable.) Obviously, $J = J(I) = \langle u_1 \rangle$, and the affine plane without the $u_2$-axis has generic Gröbner basis $x$, i.e. $Y_1 = \mathbb{A}^2_{\text{gen}} = \text{Spec}(A) \setminus V(u_1)$ and $x \in \mathcal{I}Y_1(Y_1) = I_{u_1}$ (= localization of $I$ at $\{1, u_1, u_1^2, \ldots\}$) is the reduced Gröbner basis of $I$ over $Y_1$. By reducing mod $J = \langle u_1 \rangle$ and identifying $A/J$ with $k[u_2]$, we obtain

$$\mathcal{T} = \langle u_2^2 - 1 \rangle x^2 + x \subset k[u_2][x].$$

On the $u_2$-axis, the generic Gröbner basis is $x^2 + \frac{1}{u_2^2 - 1}x$, i.e.

$$J(\mathcal{T}) = \langle u_2^2 - 1 \rangle = \langle u_2 + 1 \rangle \cap \langle u_2 - 1 \rangle,$$}

$Y_2 = V(u_1)_{\text{gen}} = V(u_1) \setminus V(u_2^2 - 1)$ and $x^2 + \frac{1}{u_2^2 - 1}x \in \mathcal{I}Y_2(Y_2) = \mathcal{T}u_2^2 - 1$ is the reduced Gröbner basis of $I$ over $Y_2$. Finally, over the two closed points $Y_3 = \langle u_1, u_2 - 1 \rangle$ and $Y_4 = \langle u_1, u_2 + 1 \rangle$, we have the reduced Gröbner basis $x$ again. To summarize

$$\mathcal{G} = \left\{ (Y_1, \{x\}), \left( Y_2, \left\{ x^2 + \frac{1}{u_2^2 - 1}x \right\} \right), (Y_3, \{x\}), (Y_4, \{x\}) \right\}$$

is the canonical irreducible Gröbner cover of $\mathbb{A}^2 = \text{Spec}(A)$.

Let $f \in k[u_1, u_2]$ be an irreducible polynomial such that $f(0, u_2) = u_2^2 - 1$ (e.g. $f = u_1 + u_2^2 - 1$). Then there exist $h \in A = k[u_1, u_2]$ such that $f = hu_1 + u_2^2 - 1$; thus $fx^2 + x = (hx)(u_1x) + (u_2^2 - 1)x^2 + x \in I$. Therefore the extension of $I$ in $(A/\langle f \rangle)[x]$ is just $\langle x \rangle$, and $V(f)$ is parametric with a reduced Gröbner basis $x$. Consequently

$$\mathcal{G}' = \left\{ (Y_1, \{x\}), \left( Y_2, \left\{ x^2 + \frac{1}{u_2^2 - 1}x \right\} \right), (V(f), \{x\}) \right\}$$

is an irreducible Gröbner cover of $\mathbb{A}^2$ with smaller cardinality than the canonical irreducible Gröbner cover. However, choosing an irreducible Gröbner cover of $\text{Spec}(A)$ with minimal cardinality in a canonical way is as impossible as choosing a curve which meets the $u_2$-axes only in $(0, -1)$ and $(0, 1)$ in a canonical way (Fig. 4).

The above example can also be used to show that a parametric subset of $\text{Spec}(A)$ need not be contained in a maximal parametric subset.

Let $\mathcal{G}$ be the canonical irreducible Gröbner cover of a locally closed subset of $\text{Spec}(A)$ with respect to $I$. For $Y, Y' \in \mathcal{G}$, we clearly have $Y \cap Y' = \emptyset$ if $\text{lt}(Y) \neq \text{lt}(Y')$. However, if $\text{lt}(Y) = \text{lt}(Y')$, then $Y$ and $Y'$ may not be disjoint. For example, if $\text{Spec}(A)$ is parametric but not irreducible, then we can decompose $\text{Spec}(A)$ into its irreducible components

$$\text{Spec}(A) = Y_1 \cup \cdots \cup Y_m.$$

By Theorem 3, the $Y_i$'s are parametric, and using the definition of small and locally maximal, it is easy to check that

$$\mathcal{G} = \{ (Y_1, G_{Y_1}), \ldots, (Y_m, G_{Y_m}) \}$$

is the canonical irreducible Gröbner cover of $\text{Spec}(A)$. Obviously, there is no reason for the $Y_i$'s to be disjoint.
5.1. The projective case

In the projective setting, i.e. if $I$ is a homogeneous ideal, the situation is considerably nicer than in the affine setting. It actually is as nice as can be hoped: the equivalence classes of the equivalence relation $\sim$ defined on $\text{Spec}(A)$ by $p \sim p'$ if $\text{lt}((\sigma_p(I))) = \text{lt}((\sigma_p'(I)))$ are parametric. (In particular, they are locally closed.) The key to the proof is the following lemma, which is not true for arbitrary ideals (cf. Examples 3 and 11). The equivalence of (1) and (2) has already been proved for $A = \mathbb{Z}$ in Arnold (2003, Theorem 5.13).

**Lemma 7.** Let $I \subset A[x]$ be a homogeneous ideal and $p \in \text{Spec}(A)$. Then the following assertions are equivalent:

1. $p$ is lucky for $I$.
2. $\text{lt}((\sigma_p(I))) = \text{lt}(I)$.
3. $\text{lt}((\sigma_p(I))) \supset \text{lt}(I)$.

**Proof.** We have already seen that (1) implies (2) in Theorem 9. So we only have to show that (3) implies (1):

Assume that $p \in \text{Spec}(A)$ is unlucky for $I$. Then there exists $t \in \text{lt}(I)$ such that $\text{lc}(I, t) \subset p$. We may assume that $t$ is maximal in its degree, i.e. for every $t' \in \text{lt}(I)$ with $\deg(t') = \deg(t)$ and $\text{lc}(I, t') \subset p$, we have $t' \leq t$. Since $t \in \text{lt}(I) \subset \text{lt}((\sigma_p(I)))$, there exists $P \in I$ such that $\text{lt}(\sigma_p(P)) = t$. Because $I$ is homogeneous, we may assume that $P$ is homogeneous and thus $\deg(P) = \deg(t)$. We can also assume that $\text{lt}(P)$ is minimal, i.e. for $P' \in I$ with $\text{lt}((\sigma_p(P'))) = t$, we have $\text{lt}(P') \geq \text{lt}(P)$.

Because $\text{lc}(I, t) \subset p$, we have $\text{lt}(P) > t$. By the maximality of $t$, we conclude $\text{lc}(I, \text{lt}(P)) \not\subset p$. Thus, there exists $Q \in I$ with $\text{lt}(Q) = \text{lt}(P)$ and $\text{lc}(Q) \notin p$. Set

$$P' = \text{lc}(Q)P - \text{lc}(P)Q.$$
Then for \( t' > t \), we have \( \text{coef}(P', t') \in p \) because \( \text{coef}(P, t'), \text{lct}(P) \in p \). On the other hand, \( \text{coef}(P', t) \) does not lie in \( p \) because \( \text{lct}(P) \neq p \). Therefore, \( \text{lt}(\sigma(P')) = t \), but as \( \text{lt}(P') < \text{lt}(P) \), this contradicts the minimality of \( P' \). \( \square \)

Note that if \( I \subset A[x] \) is an arbitrary ideal and \( p \in \text{Spec}(A) \) is unlucky for \( I \), then we can say virtually nothing about the relation between \( \text{lt}(\sigma(p(I))) \) and \( \text{lt}(I) \). We may have \( \text{lt}(\sigma(p(I))) \subseteq \text{lt}(I) \). (This, for example, happens if \( I \) is a monomial ideal.) Or we may have \( \text{lt}(\sigma(p(I))) \equiv \text{lt}(I) \). (This, for example, happens if \( I \) is generated by a single polynomial \( P = \sum_{i=1}^{m} a_i t_i \), such that \( t_i \) divides \( t_{i+1} \) and the \( a_i \)'s generate the unit ideal in \( A \).) It may also happen that \( \text{lt}(\langle \sigma(p(I) \rangle) \) and \( \text{lt}(I) \) are incomparable, i.e. there does not hold any inclusion relation between them. Finally it may actually happen that \( \text{lt}(\langle \sigma(p(I) \rangle) \) equals \( \text{lt}(I) \) (see Example 3).

By the above lemma, at least we know that \( \text{lt}(I) \) is not contained in \( \text{lt}(\langle \sigma(p(I) \rangle) \) if \( I \) is homogeneous and \( p \) unlucky for \( I \).

**Theorem 12.** Let \( I \subset A[x] \) be a homogeneous ideal and \( L \subset \text{Spec}(A) \) be locally closed. Then the equivalence classes of the equivalence relation \( \sim \) defined on \( L \) by \( p \sim p' \) if \( \text{lt}(\langle \sigma(p(I) \rangle) = \text{lt}(\langle \sigma(p'(I) \rangle) \) are parametric with respect to \( L \).

**Proof.** By Theorem 3, every locally closed subset of a parametric subset is parametric. Thus, we may assume \( L = \text{Spec}(A) \). Let \( Y \subset \text{Spec}(A) \) be an equivalence class and \( T' \subset T \) such that \( \text{lt}(\langle \sigma(p(I)) \rangle) = T' \) for all \( p \in Y \). From Theorem 11, we already know that \( Y \) is a constructible subset of \( \text{Spec}(A) \). Let \( Z \) be the closure of \( Y \) and \( a \subset A \) the radical ideal such that \( \overline{Y} = Z = V(a) \). As usual, \( T \) denotes the image of \( I \) in \( (A/a)[x] \). To apply Lemma 7, we have to show \( \text{lt}(T) = T' \).

Let

\[
Z = Z_1 \cup \cdots \cup Z_m
\]

be the unique minimal decomposition of \( Z \) into irreducible and closed subsets. For \( i = 1, \ldots, m \), let \( a_i \subset A \) denote the radical ideal such that \( Z_i = V(a_i) \) and \( T_i \) the image of \( I \) in \( (A/a_i)[x] \). By Lemma 5, the intersection \( Z_{i, \text{gen}} \cap Y \) is nonempty. Therefore, by Theorem 9 we have \( \text{lt}(T_i) = \text{lt}(Z_{i, \text{gen}}) = T' \).

Now let \( P \in T \). If, for each \( i \in \{1, \ldots, m\} \), the leading term of the image of \( P \) in \( T_i \) is strictly smaller than the leading term of \( P \), then the leading coefficient of \( P \) must lie in the intersection of all the \( a_i \)'s, which is zero mod \( a \). Thus, there exists \( i \in \{1, \ldots, m\} \) such that \( \text{lt}(P) \in \text{lt}(T_i) = T' \). Consequently \( \text{lt}(T) \subset T' \).

For the converse direction, let \( t \in T' \). There exists a \( P \in \overline{T} \) such that the leading term of the image of \( P \) in \( \overline{T}_i \) is \( t \). This means that \( \text{coef}(P, t') \in a_1 \) for \( t' > t \) and \( \text{coef}(P, t) \notin a_1 \). The \( a_i \)'s constitute the minimal primary decomposition of \( a \), and so we can find \( c \in a_2 \cap \cdots \cap a_m \setminus a_1 \). For \( t' > t \), the coefficient of \( cP \) at \( t' \) lies in the intersection of all the \( a_i \)'s and thus equals zero. On the other hand, \( \text{coef}(cP, t) \) does not lie in \( a_1 \), and therefore \( \text{lt}(cP) = t \). Consequently \( t \in \text{lt}(T) \).

By definition, \( Y \) is the set of all primes \( p \in Z \) such that \( \text{lt}(\langle \sigma(p(I) \rangle) \) equals \( T' = \text{lt}(T) \). Thus, by Lemma 7, \( Y \) is the set of all lucky primes of \( T \), i.e. \( Y = Z_{\text{gen}} \), which is parametric by Theorem 8. \( \square \)

It is now obvious how to define the canonical Gröbner cover in the projective case:

**Definition 14.** Let \( I \) be a homogeneous ideal of \( A[x] \) and \( L \) a locally closed subset of \( \text{Spec}(A) \). The Gröbner cover corresponding to the stratification of \( L \) with respect to the leading terms of \( \langle \sigma(p(I) \rangle \) is called the canonical Gröbner cover of \( L \) with respect to \( I \) (and <).
6. Conclusion and open questions

We have introduced two concepts for studying the geometry of fibres: parametric sets and \( \text{Gröbner} \) covers. It seems possible to generalize these notions to more general (i.e. not necessarily affine) base schemes.

Clearly, one of the main reasons for the success of \( \text{Gröbner} \) bases in the last decades has been the fact that in many cases they could actually be computed. This article was not focused on algorithms, but of course an efficient implementation of an algorithm to compute \( \text{Gröbner} \) covers is desirable. The existence proof for the canonical irreducible \( \text{Gröbner} \) cover is in principle constructive, but an algorithm for the computation of the canonical irreducible \( \text{Gröbner} \) cover would necessarily involve successive primary decompositions, and thus would be of modest practical value. The obvious solution is to skip irreducibility. For the projective case, we have the canonical \( \text{Gröbner} \) cover at hand, and makes sense to exploit this for the affine case by a process of homogenizing and dehomogenizing.

The problem of determining the \( \text{Gröbner} \) basis structure of the fibres has already been considered from an algorithmic point of view (see Montes (2006, 2002), Weispfenning (2003, 1992) and Suzuki and Sato (2006)), and there are implementations available for the case in which \( A \) is a polynomial ring over \( \mathbb{Q} \). The output of the corresponding implementations (Montes (Maple), Weispfenning (Reduce) and Sato–Suzuki (Risa/Asir)) can be interpreted as \( \text{Gröbner} \) covers, but a drawback is that it is not a priori clear which \( \text{Gröbner} \) cover the algorithm will compute. Furthermore, the result may depend on a term order on the parameters. It is a topic of current research to find the most natural and convenient way of representing the different reduced \( \text{Gröbner} \) bases for all the possible values of the parameters on a computer (see e.g. Montes and Manubens (2006)).

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The following appendix has been added to help orient the reader who is not so familiar with the language of schemes and who is mainly interested in the case in which the base ring \( A \) is a polynomial ring over some field. This important special case is most relevant for practical computations.

Appendix

If the base ring \( A \) is a polynomial ring over some algebraically closed field, then the concepts and results of this article can be formulated in a more down-to-earth kind of way. This appendix should enable the reader who prefers classical varieties to schemes to make this transition. A standard reference for algebraic geometry is Hartshorne (1977), although Mumford (1999) or Eisenbud and Harris (2000) might be found to be more easily accessible.
First, we recall briefly the classical setting: For this, we are working over a fixed algebraically closed field $k$. Affine $r$-space over $k$ is just $k^r$, and it is equipped with the Zariski topology, which is defined by taking as the closed sets all sets of the form

$$V^c(a) = \{ v \in k^r; \ P(v) = 0 \text{ for all } P \in a \},$$

where $a$ is an ideal of the polynomial ring $A = k[u] = k[u_1, \ldots, u_r]$. As in any topological space, a locally closed set is, by definition, the intersection of a closed set with an open set, or equivalently an open subset of a closed set. The geometric objects we have to deal with are the locally closed subsets of $k^r$. In the sequel, we will call them simply varieties, although there are more general types of varieties and many authors assume varieties to be irreducible (which we do not). A variety (= locally closed subset of $k^r$) is considered as a topological space with the induced topology inherited from $k^r$.

If $U^c$ is an open subset of a variety, then a function $f^c : U^c \to k$ is called regular if it is locally given as a quotient of polynomials. To be precise, we require that for every point $v \in U^c$ there exists an open neighborhood $U^{vc}$ of $v$ in $U^c$ and polynomials $P, Q \in k[u]$ such that for all $v' \in U^{vc}$ we have $Q(v') \neq 0$ and $f(v') = P(v') / Q(v')$. If $Y^c$ is a variety and $U^c$ an open subset of $Y^c$, then the regular functions on $U^c$, which we denote by $O_{Y^c}(U^c)$, obviously form a ring. In fact, $O_{Y^c}$ is a sheaf of rings on $Y^c$. One can think of $O_{Y^c}$ as an object which gives the (algebraic) structure to the topological space $Y$; indeed, regular functions are built directly into the definition of morphism of varieties.

Now assume that, as in the text, $A$ is a commutative, reduced ring with identity. As a set, $\text{Spec}(A)$ is just the set of all prime ideals of $A$. We can turn $\text{Spec}(A)$ into a topological space by taking as the closed sets all sets of the form

$$V(a) = \{ p \in \text{Spec}(A); \ p \supset a \},$$

where $a$ is an ideal of $A$. Note that a point $p \in \text{Spec}(A)$ is closed (i.e. equal to its closure) if and only if $p$ is a maximal ideal. The assignment $a \mapsto V(a)$ gives a one-to-one correspondence between radical ideals of $A$ and closed subsets of $\text{Spec}(A)$.

From a more abstract point of view, the geometric objects we are dealing with are topological spaces with a sheaf of rings (satisfying some additional properties). A subscheme of $\text{Spec}(A)$ (always reduced, as in the text) is a locally closed subset $Y$ of $\text{Spec}(A)$ together with a sheaf of rings $O_Y$, which is defined in the following way. Let $a \subset A$ denote the radical ideal such that $V(a)$ is equal to the closure of $Y$, and let $U$ be an open subset of $Y$. We define $O_Y(U)$ to be the set of all functions $f$ from $U$ into the disjoint union $\bigsqcup_{p \in U} (A/a)_p$ which are locally fractions. To be precise, we require that for every $p \in U$, there exists an open neighborhood $U'$ of $p$ in $U$, and elements $a, s \in A/a$ such that for all $p' \in U'$, we have $s \neq p'$ (identify $V(a)$ with $\text{Spec}(A/a)$) and $f(p') = a \overline{s} \in (A/a)_{p'}$.

Now if we take $A = k[u] = k[u_1, \ldots, u_r]$ ($k$ algebraically closed) then, by Hilbert’s Nullstellensatz, the closed points of $\text{Spec}(A)$ (= maximal ideals of $A$) are exactly those of the form $m_v = (u_1 - v_1, \ldots, u_r - v_r)$ for some $v = (v_1, \ldots, v_r) \in k^r$. This implies that the residue field $k(p)$ at a closed point $p \in \text{Spec}(A)$ is equal to $k$, because $k(p) = A_p/p_p = \text{quotient field of } A/p = k$. Of course the specialization map $\sigma_p : A[x] \to k(p)[x] = k[x]$, as defined in the introduction, is obtained by simply substituting the parameters $u_i$ with the concrete values $v_i$.

The assignment $v \mapsto m_v$ induces a homeomorphism between $k^r$ and the closed points of $\text{Spec}(A)$. The non-closed points (sometimes also called generic points) can safely be ignored.
because in our setting there are enough closed points, heuristically speaking. This allows us to identify classical varieties as defined above with subschemes of \( \text{Spec}(A) \). In more detail: if \( Y \) is a subscheme of \( \text{Spec}(A) \), then let \( Y^c \) denote the set of all \( v \in k^r \) such that \( m_v \) lies in \( Y \). The assignment \( Y \mapsto Y^c \) gives a one-to-one correspondence between subschemes of \( \text{Spec}(A) \) and varieties in the sense defined above. Note that the subscheme \( Y \) corresponding to a variety \( Y^c = V^c(\alpha) \setminus V^c(\alpha') \) is simply given by \( Y = V(\alpha) \setminus V(\alpha') \).

Of course the “regular functions” \( \mathcal{O}_Y \) and \( \mathcal{O}_{Y^c} \) also match: let \( U \) be an open subset of a subscheme \( Y \) of \( \text{Spec}(A) \). For \( f \in \mathcal{O}_Y(U) \), let \( f^c : Y^c \to k \) denote the function which assigns to each \( v \in U \) the image of \( f(m_v) \) under the map \( (A/a)_{m_v} \to k(p) = k \), which is obtained by factoring modulo the unique maximal ideal. Then it is not hard to see that the assignment \( f \mapsto f^c \) gives an isomorphism of \( \mathcal{O}_Y(U) \) and \( \mathcal{O}_{Y^c}(U^c) \). (The proof uses the fact that \( a \) is radical and Hilbert’s Nullstellensatz.)

With the above transformation rules, the reader should be able to convert the article into the classical language of varieties. The basic idea is to replace \( k(p) \) by \( k \) and to add a \( c \) where appropriate. For example, if \( Y^c \) is a variety and \( U^c \) an open subset of \( Y^c \), then an element \( g^c \) of \( \mathcal{I}_{Y^c}(U^c) \) is just a function from \( U^c \) to \( k[x] \) such that for every \( v \in U^c \), there exists an open neighborhood \( U^c \) of \( v \) in \( U^c \) and \( P \in I \), (Here \( I \subseteq A[x] = k[u, x] = k[u_1, \ldots, u_r, x_1, \ldots, x_n] \) is the ideal we want to discuss.) \( Q \in k[u] \) such that for all \( v' \in U^c \) we have \( Q(v') \neq 0 \) and \( g^c(v') = \frac{P(u', x)}{Q(v')} \in k[x] \). Notice that varieties are quasi-compact, and thus a finite number of \( U^c \)'s as above will be enough. So it is in principle possible to represent elements of \( \mathcal{I}_{Y^c}(U^c) \) on a computer.

The problem of determining a decomposition of the parameter space which respects the Gröbner basis structure of the fibres has already been considered by other authors (e.g. Weispfenning, Montes, Sato–Suzuki). In this last part of the appendix, I would like to explain a key difference to previous approaches, which enables us to obtain a “more canonical” decomposition.

In the very first articles on this subject, there was a tendency to decompose the parameter space \( k^r \) into locally closed sets \( Y^c \) of a somewhat restricted kind of form; namely

\[
Y^c = \{ v \in k^r; \ P_1(v) = 0, \ldots, P_m(v) = 0, \ Q_1(v) \neq 0, \ldots, Q_d(v) \neq 0 \} \\
= V^c((P_1, \ldots, P_m)) \setminus V^c((Q_1 \cdots Q_d)),
\]

where the \( P_i \)'s and \( Q_i \)'s are some polynomials in \( k[u] \). This remark does not apply to the more recent articles. I believe that – from a systematic point of view – there is no reason for restricting our considerations to this special kind of locally closed set, which is geometrically obtained by cutting out a hypersurface from a closed subset of \( k^r \). In fact, restricting to this special kind of locally closed set poses a severe obstruction to obtaining a canonical decomposition: consider the simple example \( I = (u_1 x, u_2 x) \subset k[u_1, u_2, x] \). (Here \( x \) denotes just one variable.) I hope the reader will agree that there really is just one decomposition of \( k^2 \) which deserves to be called canonical: it consists of the plane \( k^2 \) without the origin and the origin. However, the plane without the origin cannot be obtained by cutting out a hypersurface. Also note that there does not exist a polynomial in \( I \) which specializes to a multiple of \( x \) for all \( v \in Y^c = k^2 \setminus \{(0, 0)\} \). But the function which assigns \( x \in k[x] \) to every \( v \in Y^c \) clearly is an element of \( \mathcal{I}_{Y^c}(Y^c) \).

An advantage of locally closed sets of the form (3) is that the (global) regular functions on them are rather simple to describe: if \( Y^c = V^c(\alpha) \setminus V^c(\langle Q \rangle) \) where \( \alpha \subset k[u] \) is an ideal and \( Q \in k[u] \) a polynomial, then for every regular function \( f^c \) on \( Y^c \) there exists a polynomial \( P \in k[u] \) and an integer \( m \geq 0 \) such that \( f^c(v) = \frac{P(v)}{Q(v)^m} \) for all \( v \in Y^c \). Similarly, for every...
element \( g^c \in \mathcal{I}_Y(Y^c) \), there exists a polynomial \( P \in I \subset k[u, x] \) and an integer \( m \geq 0 \) such that \( g^c(v) = \frac{P(v, x)}{Q(v)} \) for all \( v \in Y^c \). (See Hartshorne (1977), Chapter II, Proposition 5.1. (c)). In other words, we can take \( U^c = Y^c \) in the above definitions. For arbitrary locally closed subsets of \( k^r \) this is no longer true, and so, since we have settled to allow arbitrary locally closed subsets of \( k^r \) in our decomposition, we have to put up with the fact that the elements of \( \mathcal{O}_Y(Y^c) \) (respectively \( \mathcal{I}_Y(Y^c) \)) can no longer be described by a single fraction. Instead, an element of \( \mathcal{I}_Y(Y^c) \) is given by a finite number of fractions \( \frac{P_1}{Q_1}, \ldots, \frac{P_m}{Q_m} \) where \( P_1, \ldots, P_m \in I \), \( Q_1, \ldots, Q_m \in k[u] \) and \( \frac{P_i(v, x)}{Q_i(v)} = \frac{P_j(v, x)}{Q_j(v)} \) for all \( v \in Y^c \) with \( Q_i(v)Q_j(v) \neq 0 \). This patching together of “local data” is somehow the nucleus of sheaf theory, and to some extent explains the (omni)presence of sheaves in this article.

A variety \( Y^c \) is said to be parametric w.r.t. \( I \) (and \( < \)) if there exist elements \( g_1^c, \ldots, g_m^c \in \mathcal{I}_Y(Y^c) \) such that

- the function \( v \mapsto \text{lt}(g_i^c(v)) \) is constant on \( Y^c \) for \( i = 1, \ldots, m \) and
- \( g_1^c(v), \ldots, g_m^c(v) \) is the reduced Gröbner basis (w.r.t. \( < \)) of the specialized ideal \( \langle \sigma_m(I) \rangle \subset k[x] \) for every \( v \in Y^c \).

At first glance (at least from an algebraic point of view), it might seem fairly natural to call a variety \( Y^c \) parametric w.r.t. \( I \) if there exist polynomials \( P_1, \ldots, P_m \in I \) such that

- the function \( v \mapsto \text{lt}(P_i(v, x)) \) is constant on \( Y^c \) for \( i = 1, \ldots, m \) and
- \( P_1(v, x), \ldots, P_m(v, x) \) is, up to normalization, the reduced Gröbner basis of the specialized ideal \( \langle \sigma_m(I) \rangle \) for every \( v \in Y^c \). (Most authors prefer to say “up to normalization” to working with fractions, but this does not make an essential difference.)

But from a more geometric point of view, it is much more natural to use elements of \( \mathcal{I}_Y(Y^c) \) rather than just polynomials in \( I \) to define parametric sets. Anyway, parametric sets defined by using \( \mathcal{I}_Y(Y^c) \) have much nicer properties. For example, the union of two open parametric subsets of a closed subset of \( k^r \) with the same leading terms is obviously parametric. (This is crucial to construct decompositions of the base space \( k^r \) such that the corresponding segments are large but few in number.) Also note that, of course, parametric, in the sense of the second definition implies parametric in the proper sense.

References


