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Sequences and filters of characters characterizing subgroups of compact Abelian groups

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Abstract

Let H be a countable subgroup of the metrizable compact Abelian group G and $f : H \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$ a (not necessarily continuous) character of H. Then there exists a sequence $(\chi_n)_{n=1}^{\infty}$ of (continuous) characters of G such that $\lim_{n\to\infty} \chi_n(\alpha) = f(\alpha)$ for all $\alpha \in H$ and $(\chi_n(\alpha))_{n=1}^{\infty}$ does not converge whenever $\alpha \in G \setminus H$. If one drops the countability and metrizability requirement one can obtain similar results by using filters of characters instead of sequences. Furthermore the introduced methods allow to answer questions of Dikranjan et al.

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1. Introduction

1.1. Motivation

In [6] several techniques have been developed to prove the existence of sequences $(k_n)_{n=1}^{\infty}$ of positive integers characterizing countable subgroups H of the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ in the sense that for $\alpha \in \mathbb{T}$,

$$\alpha \in H \iff \lim_{n\to\infty} k_n\alpha = 0.$$

These methods were extended in [7] to show that if H is generated freely by finitely many elements, a characterization is possible in an even stronger sense: One can choose a characterizing sequence such that $\sum_{n=1}^{\infty} \|k_n\alpha\| < \infty$ for $\alpha \in H$, while $\limsup_{n\to\infty} \|k_n\alpha\| \geqslant 1/4$ for $\alpha \in \mathbb{T} \setminus H$. (For $x = r + \mathbb{Z} \in \mathbb{R}/\mathbb{Z},\ r \in \mathbb{R}$, the norm $\|x\|$ denotes the distance between r and the nearest integer.)

In [17] arbitrary subgroups of $\mathbb{T}$ were characterized by filters on its dual $\mathbb{Z}$. This approach was used in [3] to extend the results from [7].

A different approach to the characterization of finitely generated dense subgroups of compact Abelian groups by sums has recently been introduced in [4,5].

Dikranjan et. al. investigated related questions concerning the characterization of subgroups of more general topological Abelian groups G (cf. [1,2,8,10]). In this article we lift the techniques of [3] to this general setting and answer questions stated in [2,10].

Our Theorems 3.1 and 3.6 have contemporaneously and independently been proved by Dikranjan and Kunen, cf. [9]. They also treat questions related to descriptive set theory. More results in this directions were, for instance, also obtained by Eliaš in [12].

1.2. Content of the paper

In Section 2 we modify the filter method from [17] for our purposes. Theorem 2.1 essentially states that arbitrary subgroups of compact Abelian groups G can be characterized by filters on the (discrete) Pontryagin dual $\widehat{G}$ of G. (Such filters are intended to be the neighborhood filters of 0 w.r.t. precompact group topologies on $\widehat{G}$.)

In Section 3 this characterization is used to prove Theorem 3.1: Among the compact Abelian groups G exactly the metrizable ones have the property that every countable subgroup H is characterizable in the sense that there is a sequence of characters $(\chi_n)_{n=1}^{\infty}$ in $\widehat{G}$ such that

$$\alpha \in H \iff \lim_{n\to\infty} \chi_n(\alpha) = 0.$$

This solves Problem 5.3 from [10]. Theorem 3.6 (which solves Problem 5.1 and Question 5.2 from [10]) states that in an arbitrary compact Abelian group every countable subgroup is the intersection of subgroups characterizable in the above sense.

Section 4 is motivated by Question 5.2 from [2]: Consider the case $G = \mathbb{T}$ and $\widehat{G} = \mathbb{Z}$. It was established in [6] that for any countable $H \leqslant \mathbb{T}$ there exists a sequence of integers $k_1 < k_2 < \cdots$ characterizing H. Is it possible to choose the k_n in such a way that the quotients $\frac{k_{n+1}}{k_n}$ are bounded? We answer this question affirmatively by proving a stronger assertion, cf. Theorem 4.1. It states that, in a certain sense, characterizing sequences can be arbitrarily close to having positive density. This seems to be best possible insofar as (apart from trivial cases) characterizing sequences always have density 0. Theorem 4.2 is a counterpart to Theorem 4.1 and describes how sparse characterizing sequences of subgroups of $\mathbb{T}$ can be.

In Section 5 we introduce a refined characterization of subgroups of a compact metrizable group G by sequences: For a sequence $(\chi_n)_{n=1}^{\infty}$ in $\widehat{G}$ we consider the set H of all $\alpha \in H$ for which $(\chi_n(\alpha))_{n=1}^{\infty}$ converges (not necessarily to $0 \in \mathbb{T}$). H is easily seen to be a subgroup of G and the pointwise limit is a (not necessarily continuous) homomorphism $f : H \to \mathbb{T}$. Theorem 5.1 gives a complete description of the situation: Given any subgroup H of a metrizable compact Abelian group G and any homomorphism $f : H \to \mathbb{T}$ there is a sequence $(\chi_n)_{n=1}^{\infty}$ in $\widehat{G}$ such that $\chi_n \to f$ pointwise on H. If H is countable then one can even achieve that H is exactly the set of convergence. If G is a compact (not necessarily metrizable) group and H is an arbitrary (not necessarily countable) subgroup of G, this result is still valid when the convergence of sequences is replaced by the more general convergence of filters. By considering the trivial homomorphism $f \equiv 0$ we see that Theorem 5.1 nicely extends Theorem 3.1. Furthermore this result allows to construct counterexamples to Question 5.4 from [2]. (For the complete statement of this question cf. Section 5.)

1.3. Conventions and notation

If not stated otherwise, G is always an infinite locally compact Abelian group. (For finite G most assertions turn out to be trivial.) Since we are only interested in Abelian groups, we use additive notation. In particular we shall do so in the group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ where characters $\chi : G \to \mathbb{T}$ take their values. Elements of G will be denoted by $\alpha, \beta, \ldots$. If H is a (not necessarily closed) subgroup of G, we write $H \leqslant G$. If $A \subseteq G$ is any subset, $\langle A \rangle$ denotes the subgroup generated by A. For finite $A = \{\alpha_1, \ldots, \alpha_n\} \subseteq G$ and $M \in \mathbb{N}$ we put $\langle A \rangle_M := \{\sum_{i=1}^{n} k_i \alpha_i\colon\ k_i \in \mathbb{Z},\ |k_i| \leqslant M\}$.

Recall that a group compactification of (any topological group) G is defined to be a pair (ι, C) where C is a compact group and $\iota : G \mapsto C$ is a continuous homomorphism with dense image. Relative topologies on G induced by group compactifications are called precompact. The so-called Bohr compactification (ι_{bG}, bG) of G is the compactification of G which is maximal in the sense that for each compactification (ι, C) of G there is a continuous homomorphism $\phi : bG \to C$ with $\phi \circ \iota_{bG} = \iota_C$.

$\widehat{G} = \{\chi\colon\ \chi$ is a continuous homomorphism from G to $\mathbb{T}\}$ denotes the dual group of G, equipped with the compact open topology. By Pontryagin's Duality Theorem (cf. for instance [11] or [15]) we know that for LCA groups $G \cong \widehat{\widehat{G}}$ in the algebraic as well as in the topological sense via the canonical mapping $\alpha \mapsto x_\alpha$, $x_\alpha : \chi \mapsto \chi(\alpha)$.

We take G_d to be G endowed with the discrete topology. Duality theory can be applied to construct the Bohr compactification of G by setting $bG := \widehat{(\widehat{G})_d}$ and $\iota_{bG} : \alpha \mapsto x_\alpha$. Accordingly, the Bohr compactification of $\widehat{G}$ is $\widehat{G_d}$. It is natural to call the precompact topology on G induced by bG the Bohr topology. On the dual group $\widehat{G}$ the Bohr topology can be described by the so-called Bohr sets which are defined by

$$B_{(\alpha_1,\dots,\alpha_t,\varepsilon)} := \bigl\{\chi \in \widehat{G}\colon\ \bigl\|\chi(\alpha_i)\bigr\| \leqslant \varepsilon \text{ for } i \in \{1,2,\dots,t\}\bigr\},$$

where $\alpha_1,\dots,\alpha_t \in G$ and $\varepsilon > 0$. These sets generate the neighborhood filter of 0 in $\widehat{G}$ endowed with the Bohr topology. Further we put $B_{(\alpha_1,\dots,\alpha_t,\varepsilon)}(E) := B_{(\alpha_1,\dots,\alpha_t,\varepsilon)} \cap E$ for $E \subseteq \widehat{G}$. For $\alpha \in G$ and $B \subseteq \widehat{G}$ we write $\|\alpha B\| := \sup\{\|\chi(\alpha)\|\colon\ \chi \in B\}$.

2. Characterizing filters

We will make use of filter limits in the following sense: Let S be any set, let $\mathcal{F}$ be a filter on S, let y be a point in a topological space X and let $f : S \to X$ be a function. Then we write

$$\mathcal{F}\text{-}\lim_{s} f(s) = y$$

iff for every neighborhood U of y, $\{s \in S\colon\ f(s) \in U\} \in \mathcal{F}$.

We remark that filter limits are more general then limits along sequences: For a sequence $(x_n)_{n=1}^{\infty}$ in X put

$$\mathcal{F}_{(x_n)_{n=1}^{\infty}} = \bigl\{A \subseteq X\colon\ \exists m \in \mathbb{N} \text{ such that } \{x_n\colon\ n \geqslant m\} \subseteq A\bigr\}.$$

Then $\mathcal{F}_{(x_n)_{n=1}^{\infty}}\text{-}\lim_s f(s)$ exists iff $\lim_{n\to\infty} f(x_n)$ exists and in this case they coincide.

Let $H \leqslant G$ be a subgroup of the compact Abelian group G. Our task is to show that H can be characterized by a filter $\mathcal{F}_H$ on $\widehat{G}$ in the sense that we have $\mathcal{F}_H\text{-}\lim_{\chi} \chi(\beta) = 0$ iff $\beta \in H$. It is clear that for all $\alpha \in H$ and all $\varepsilon > 0$ the set $B_{(\alpha,\varepsilon)}$ has to be an element of $\mathcal{F}_H$ to assure convergence for elements of H. By the filter properties of $\mathcal{F}_H$ the intersection of finitely many such sets will again be an element of $\mathcal{F}_H$. Thus it would be natural to define $\mathcal{F}_H$ to be the filter generated by the sets $B_{(\alpha_1,\dots,\alpha_t,\varepsilon)}$ where $\alpha_1,\dots,\alpha_t \in H,\ \varepsilon > 0$. This definition yields the minimal filter with the required property and corresponds to the precompact group topology on $\widehat{G}$ induced by H. Later it will be important to us that we may neglect finite sets of characters. Therefore we will also take all cofinite sets to be elements of $\mathcal{F}_H$. This leads to the following definition:

$$\mathcal{F}_H := \bigl\{F \subseteq \widehat{G}\colon\ \exists \alpha_1,\dots,\alpha_t \in H, \varepsilon > 0,\ \Gamma \subseteq \widehat{G},\ |\Gamma| < \infty \text{ such that } B_{(\alpha_1,\dots,\alpha_t,\varepsilon)}(\widehat{G} \setminus \Gamma) \subseteq F\bigr\}.$$

Theorem 2.1. *Let G be an infinite compact Abelian group, let H be a subgroup of G and let the filter $\mathcal{F}_H$ be defined as above. Then for all $\beta \in G$*

$$\mathcal{F}_H\text{-}\lim_{\chi} \chi(\beta) = 0 \iff \beta \in H.$$

In the course of the proof we will employ the following lemma which will also be useful later on:

Lemma 2.2. *Let G be a compact Abelian group. Then $\widehat{G}$ is dense in $\widehat{G_d}$ w.r.t. pointwise convergence. Thus, for any countable subset H of G and any $\chi \in \widehat{G_d}$ there exists a sequence $(\chi_n)_{n=1}^{\infty}$ in $\widehat{G}$ such that $\chi_n(\alpha) \to \chi(\alpha)$ $(n \to \infty)$ for all $\alpha \in H$.*

Proof. Cf. [15, 26.16]. □

Proof of Theorem 2.1. The definition of $\mathcal{F}_H$ guarantees that $\mathcal{F}\text{-}\lim_\chi \|\chi(\beta)\| = 0$ for all $\beta \in H$. For the converse we prove that, given $\beta \notin H$, for all $\alpha_1, \ldots, \alpha_t \in H$ and every $\varepsilon > 0$ there exist infinitely many characters $\chi \in B_{(\alpha_1,\ldots,\alpha_t,\varepsilon)}$ with $\|\chi(\beta)\| \geqslant 1/4$ which implies that $\{\chi \in \widehat{G}\colon \|\chi(\beta)\| < 1/4\} \notin \mathcal{F}$. First we see that there exists at least one such character: Consider the Bohr compactification $\widehat{G_d}$ of $\widehat{G}$. $\widehat{G_d}$ separates subgroups and points of G. Hence there exists some $\phi \in \widehat{G_d}$ such that

$$\phi(\alpha) = 0 \quad \text{for all } \alpha \in \langle \alpha_1, \ldots, \alpha_t \rangle \quad \text{and} \quad c = \phi(\beta) \neq 0,$$

w.l.o.g. $\|\phi(\beta)\| \geqslant 1/3$ (otherwise take an appropriate multiple $2\phi, 3\phi, \ldots$). By Lemma 2.2 ϕ can be approximated arbitrarily well on finitely many points by a character. Thus we find some $\chi \in \widehat{G}$ such that $\|\chi(\alpha_i)\| \leqslant \varepsilon,\ 1 \leqslant i \leqslant t,\ \|\chi(\beta)\| > 1/4$.

Next we prove that for $\varepsilon > 0$ each $B_{(\alpha_1,\ldots,\alpha_t,\varepsilon)}$ contains infinitely many χ with $\|\chi(\beta)\| \geqslant 1/4$. Let $U := \{(\chi(\alpha_1), \ldots, \chi(\alpha_t), \chi(\beta))\colon \chi \in \widehat{G}\} \leqslant \mathbb{T}^{t+1}$. We distinguish two cases:

(1) U is finite, say $U = \{u_1, \ldots, u_k\}$. There is some i, say $i = 1$, with $u_1 := (0, \ldots, 0, c)$. Then the sets

$$\Upsilon_i := \bigl\{\chi \in \widehat{G}\colon \bigl(\chi(\alpha_1), \ldots, \chi(\alpha_t), \chi(\beta)\bigr) = u_i\bigr\}, \quad i = 1, \ldots, k,$$

and particularly Υ_1 are infinite, or

(2) U is an infinite subgroup of $\mathbb{T}^{t+1}$. But then each point of U is an accumulation point.

In both cases we find infinitely many χ with the required property. □

3. Characterizing countable subgroups

We solve Problem 5.3 from [10]: For which compact Abelian G can every countable subgroup H be characterized by a sequence of characters?

For $A \subseteq \widehat{G}$ we write $\lim_{\chi\in A} \chi(\beta) = 0$ iff $\{\chi \in A\colon \chi(\beta) \geqslant \varepsilon\}$ is finite for all $\varepsilon > 0$. (I.e. instead of the characterizing sequence $(\chi_n)_{n=1}^\infty$ we consider the characterizing set $A = \{\chi_n\colon n \in \mathbb{N}\}$.)

Theorem 3.1. *Let G be an infinite compact Abelian group and let $H \leqslant G$ be a countable subgroup. Then the following statements are equivalent*:

(i) *G is metrizable.*

(ii) *There exists a countable set $A \subseteq \widehat{G}$, such that*

$$\beta \in H \iff \lim_{\chi \in A} \chi(\beta) = 0.$$

Remark. The proof of Theorem 3.1 actually shows that (if G is metrizable) for every $\sigma < 1/3$ the characterizing set A can be chosen in such a way that $\beta \notin H$ implies $\limsup_{\chi\in A} \|\chi(\beta)\| \geqslant \sigma$. Using a diagonalization argument it is not difficult to achieve $\limsup_{\chi\in A} \|\chi(\beta)\| \geqslant 1/3$ and it is easy to see that this is best possible.

The proof of (i) ⇒ (ii) employs several lemmas which we formulate now and verify at the end of this section. According to our assumptions, in these lemmas G is an infinite compact Abelian metrizable group.

Lemma 3.2. *Let $\tau \in \mathbb{T}$ and $n \in \mathbb{N}$. Assume that $\|i\tau\| \leqslant \sigma < 1/3$ for all $i \in \{1, 2, \ldots, n\}$. Then $\|\tau\| \leqslant \sigma/n$.*

Lemma 3.3. *Assume that $\gamma_1, \ldots, \gamma_d \in G$ freely generate $H \leqslant G$. For arbitrary nonempty open sets $I_1, \ldots, I_d$ in $\mathbb{T}$ there exists $\chi \in \widehat{G}$ such that $\chi(\gamma_i) \in I_i$ for all $i \in \{1, 2, \ldots, d\}$.*

Lemma 3.4. *Let $\alpha_1, \ldots, \alpha_t \in G$, $\varepsilon > 0$ and $\sigma < 1/3$.*

(1) *For all finite $\Gamma \subseteq \widehat{G}$ and all $\beta \in G$*

$$\left\|\beta B_{(\alpha_1,\ldots,\alpha_t,\varepsilon)}(\widehat{G} \setminus \Gamma)\right\| \leqslant \sigma \implies \beta \in \langle \alpha_1, \ldots, \alpha_t \rangle.$$

(2) *Moreover there exists $M \in \mathbb{N}$ such that for all finite $\Gamma \subseteq \widehat{G}$ and all $\beta \in G$*

$$\left\|\beta B_{(\alpha_1,\ldots,\alpha_t,\varepsilon)}(\widehat{G} \setminus \Gamma)\right\| \leqslant \sigma \implies \beta \in \langle \alpha_1, \ldots, \alpha_t \rangle_M.$$

(3) *If $V \supseteq \langle \alpha_1, \ldots, \alpha_t \rangle_M$ is an open subset of G then for all finite $\Gamma \subseteq \widehat{G}$ there exists a finite set $E \subseteq \widehat{G} \setminus \Gamma$ such that for $\beta \in G$*

$$\left\|\beta B_{(\alpha_1,\ldots,\alpha_t,\varepsilon)}(E)\right\| \leqslant \sigma \implies \beta \in V.$$

Lemma 3.5. *Let $R_1 \subseteq R_2 \subseteq \cdots$ be finite subsets of G. There exists a sequence of open sets $V_n \subseteq G$, $n \in \mathbb{N}$ such that*

(1) $V_n \supseteq R_n$.
(2) $\liminf_{n\to\infty} V_n = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} V_n = \bigcup_{n=1}^{\infty} R_n$.

Proof of Theorem 3.1. (i) ⇒ (ii): We will first construct the set $A \subseteq \widehat{G}$ and then prove that $\beta \in H$ iff $\lim_{\chi\in A} \chi(\beta) = 0$.

Let $H =: \{\alpha_t\colon t \in \mathbb{N}\}$ and pick $\varepsilon = \sigma \in (0, 1/3)$. Using Lemma 3.4(2) we can choose a sequence $(M_t)_{t=1}^{\infty}$ such that for every finite $\Gamma \subseteq \widehat{G}$ and all $\beta \in G$

$$\left\|\beta B_{(\alpha_1,\ldots,\alpha_t,\varepsilon)}(\widehat{G} \setminus \Gamma)\right\| \leqslant \varepsilon \implies \beta \in \langle \alpha_1, \ldots, \alpha_t \rangle_{M_t}.$$

Next put, for $t \in \mathbb{N}$, $R_t := \langle \alpha_1, \ldots, \alpha_t \rangle_{M_t}$ and define $V_t \supseteq R_t$ according to Lemma 3.5 such that $\liminf_{t\to\infty} V_t = H$.

Using Lemma 3.4(3) we choose a finite set $E_1 \subseteq \widehat{G}$ such that $\|\beta B_{(\alpha_1,\varepsilon)}(E_1)\| \leqslant \varepsilon$ implies $\beta \in V_1$. By employing Lemma 3.4(3) again, we find $E_2 \subseteq \widehat{G} \setminus E_1$ such that $\|\beta B_{(\alpha_1,\alpha_2,\varepsilon)}(E_2)\| \leqslant \varepsilon$ implies $\beta \in V_2$. Continuing in this fashion we arrive at a sequence $(E_t)_{t=1}^{\infty}$ of disjoint subsets of $\widehat{G}$ such that for each $t \in \mathbb{N}$

$$\left\|\beta B_{(\alpha_1,\ldots,\alpha_t,\varepsilon)}(E_t)\right\| \leqslant \varepsilon \implies \beta \in V_t.$$

Finally we put $A := \bigcup_{t=1}^{\infty} B_{(\alpha_1,\ldots,\alpha_t,\varepsilon)}(E_t)$.

Assume that $\beta \in H$. To prove $\lim_{\chi\in A} \|\chi(\beta)\| = 0$ note that, for arbitrary $n \in \mathbb{N}$, there exists $T = T(n) \in \mathbb{N}$ such that $i\beta \in \{\alpha_t\colon t \leqslant T\}$ for all $i \in \{1, 2, \ldots, n\}$. Thus whenever

$\chi \in B_{(\alpha_1,\ldots,\alpha_t,\varepsilon)}(E_t)$ for some $t \geqslant T$ we have $\|i\chi(\beta)\| \leqslant \varepsilon$ for all $1 \leqslant i \leqslant n$. By Lemma 3.2 this yields $\|\chi(\beta)\| \leqslant \varepsilon/n$. Since n was arbitrary we get $\lim_{\chi\in A} \|\chi(\beta)\| = 0$.

Conversely assume that $\limsup_{\chi\in A} \|\chi(\beta)\| < \varepsilon$ for some $\beta \in G$. Then for all but finitely many $t \in \mathbb{N}$ we have $\|\beta B_{(\alpha_1,\ldots,\alpha_t,\varepsilon)}(E_t)\| \leqslant \varepsilon$. Thus there exists $t_0 \in \mathbb{N}$ such that $\beta \in V_t$ for all $t \geqslant t_0$ which yields $\beta \in H$ by the choice of the sequence $(V_t)_{t=1}^{\infty}$.

(ii) $\Rightarrow$ (i): Let $H \leqslant G$ be an arbitrary countable subgroup characterized by the countable set $A \subseteq \widehat{G}$. Define $\Lambda := \langle A \rangle$ and

$$\Lambda^0 := \{g \in G\colon\ \chi(g) = 0 \text{ for all } \chi \in \Lambda\},$$

the annihilator of Λ. Clearly $\Lambda^0 \leqslant H$, thus $|\Lambda^0| \leqslant \aleph_0$. Since $\widehat{\Lambda} \cong G/\Lambda^0$ we have $w(G/\Lambda^0) = w(\widehat{\Lambda}) = |\Lambda| = \aleph_0$, where w denotes the topological weight, i.e. the least cardinal number of an open basis (cf. [15, 24.10 and 24.14]). Hence G/Λ^0 and Λ^0 have at most countable weight and therefore also G [15, 5.38], implying that G is metrizable. $\square$

Let G be a compact Abelian group. In [10] subgroups characterized by a sequence $(\chi_n)_{n=1}^{\infty}$ in $\widehat{G}$ are denoted by

$$s_{(\chi_n)_{n=1}^{\infty}}(G) := \{\alpha \in G\colon\ \lim_{n\to\infty} \chi_n(\alpha) = 0\}.$$

Furthermore such subgroups are called basic **g**-closed subgroups. According to Theorem 3.1 every countable subgroup of G is basic **g**-closed iff G is metrizable.

A group $H \leqslant G$ is called **g**-closed if it is representable as the intersection of basic **g**-closed subgroups. The next theorem deals with **g**-closed subgroups and solves Problem 5.1 from [10].

Theorem 3.6. *Every countable subgroup H of a compact Abelian group G is* **g***-closed.*

Proof. For arbitrary $\beta \in G \setminus H$ there is a $\chi \in \widehat{G_d}$ with $\chi(\alpha) = 0$ for all $\alpha \in H$ and $\|\chi(\beta)\| \geqslant \frac{1}{3}$. Thus Lemma 2.2 immediately yields a sequence of $(\chi_n^{\beta})_{n=1}^{\infty}$ in $\widehat{G}$ characterizing a subgroup

$$H_\beta := s_{(\chi_n^{\beta})_{n=1}^{\infty}}(G) = \left\{\alpha \in G\colon\ \lim_{n\to\infty} \chi_n^{\beta}(\alpha) = 0\right\} \leqslant G$$

with $H \leqslant H_\beta$ and $\beta \notin H_\beta$. Thus $H = \bigcap_{\beta \in G\setminus H} H_\beta$. $\square$

3.1. Proofs of Lemmas 3.2–3.5

We assume the group G to be compact Abelian and metrizable. Lemma 3.2 is elementary, so we skip the proof.

Proof of Lemma 3.3. Assume that

$$A := \widehat{G}\langle\gamma_1, \ldots, \gamma_d\rangle = \{\chi(\alpha)\colon\ \chi \in \widehat{G},\ \alpha \in \langle\gamma_1, \ldots, \gamma_d\rangle\}$$

is not dense in $\mathbb{T}^d$, i.e. $\overline{A} \lneq \mathbb{T}^d$. There is a nontrivial character of $\mathbb{T}^d$ vanishing on $\overline{A}$, i.e. a nonzero vector $h = (h_1, \dots, h_d) \in \mathbb{Z}^d$ such that $\sum_{i=1}^d h_i x_i = 0$ holds for all $x = (x_1, \dots, x_d) \in \overline{A}$. Fix an arbitrary $\chi \in \widehat{G}$ and put $x_i = \chi(\gamma_i)$. Then

$$0 = \sum_{i=1}^d h_i \chi(\gamma_i) = \chi\left(\sum_{i=1}^d h_i \gamma_i\right).$$

Since this holds for all $\chi \in \widehat{G}$ we have $\sum_{i=1}^d h_i \gamma_i = 0$, contradicting the independence of the free generators γ_i, $1 \leqslant i \leqslant d$. □

Proof of Lemma 3.4. Let $B_0 := B_{(\alpha_1,\dots,\alpha_t,\varepsilon)}(\widehat{G} \setminus \Gamma)$.

(1) Let $\mathcal{F} = \mathcal{F}_{\langle \alpha_1,\dots,\alpha_t \rangle}$ be the filter of Theorem 2.1 characterizing $\langle \alpha_1, \dots, \alpha_t \rangle$ and let $\delta > 0$ be arbitrary. Under the assumption $\|\beta B_0\| \leqslant \sigma < 1/3$ we have to show that

$$F_\delta := \left\{\chi \in \widehat{G} \setminus \Gamma\colon\ \left\|\chi(\beta)\right\| \leqslant \delta\right\} \in \mathcal{F}.$$

Choose $m \in \mathbb{N}$ such that $\delta \geqslant \sigma/m$ and let $B_1 := B_{(\alpha_1,\dots,\alpha_t,\varepsilon/m)}(\widehat{G} \setminus \Gamma)$. By definition of $\mathcal{F}$ we have $B_1 \in \mathcal{F}$. For all $\chi \in B_1$, $i \in \{1, 2, \dots, m\}$, we have $i\chi \in B_0$. Thus $\|i\chi(\beta)\| \leqslant \sigma$ for all $i \in \{1, 2, \dots, m\}$ and Lemma 3.2 yields $\|\chi(\beta)\| \leqslant \sigma/m < \delta$. Thus $B_1 \subseteq F_\delta$ and hence $F_\delta \in \mathcal{F}$.

(2) Assume that $H := \langle \alpha_1, \dots, \alpha_t \rangle$ is infinite (otherwise the assertion follows immediately). Since H is a finitely generated Abelian group there exists a decomposition $H = T \oplus F$ where F is freely generated by $\gamma_1, \dots, \gamma_d$ and $T = \langle \nu_1, \dots, \nu_l \rangle = \bigoplus_{i=1}^l \langle \nu_i \rangle$ is the torsion subgroup of H. Hence $\langle \nu_i \rangle \cong \mathbb{Z}/e_i\mathbb{Z}$ for some $e_i \in \mathbb{N}$ and

$$\langle \alpha_1, \dots, \alpha_t \rangle = \langle \gamma_1, \dots, \gamma_d \rangle \oplus \langle \nu_1, \dots, \nu_l \rangle \cong \mathbb{Z}^d \oplus \bigoplus_{i=1}^l \mathbb{Z}/e_i\mathbb{Z}.$$

Let $\delta > 0$ be such that $\|\chi(\gamma_i)\| \leqslant \delta$ for $i \in \{1, 2, \dots, d\}$ and $\chi(\nu_j) = 0$ for $j \in \{1, 2, \dots, l\}$ implies $\|\chi(\alpha_k)\| \leqslant \varepsilon$ for $k \in \{1, 2, \dots, t\}$.

Pick now any $\beta \in G$ with $\|\beta B_0\| \leqslant \sigma < 1/3$. By (1) above we have $\beta \in H$, thus $\beta = \sum_{i=1}^d r_i \gamma_i + \sum_{j=1}^l s_j \nu_j$ for some $r_i \in \mathbb{Z}$, $s_j \in \{0, 1, \dots, e_j - 1\}$, $i \in \{1, 2, \dots, d\}$, $j \in \{1, 2, \dots, l\}$. Let $e := \prod_{j=1}^l e_j$. By Lemma 3.3 there exist infinitely many $\chi \in e\widehat{G} := \{e\chi'\colon\ \chi' \in \widehat{G}\}$ such that

$$\operatorname{sign}(r_i)\chi(\gamma_i) \in \left[\frac{1}{3\sum_{j=1}^d |r_j|}, \frac{2}{3\sum_{j=1}^d |r_j|}\right] + \mathbb{Z} \tag{1}$$

holds for $i \in \{1, 2, \dots, d\}$. Therefore we have

$$r_i\chi(\gamma_i) \in \left[\frac{|r_i|}{3\sum_{j=1}^d |r_j|}, \frac{2|r_i|}{3\sum_{j=1}^d |r_j|}\right] + \mathbb{Z}$$

for all $i \in \{1, 2, \dots, d\}$. Summing up and using that $\chi(\nu_j) = 0$ for $j \in \{1, 2, \dots, l\}$ this leads to

$$\chi(\beta) = \sum_{i=1}^d r_i\chi(\gamma_i) \in \left[\frac{1}{3}, \frac{2}{3}\right] + \mathbb{Z}.$$

Thus $\chi \notin B_{(\alpha_1,\dots,\alpha_t,\varepsilon)}$ and hence there is $j \in \{1,\dots,t\}$ with $\|\chi(\alpha_j)\| > \varepsilon$. This implies that for some $i \in \{1,2,\dots,d\}$, $\|\chi(\gamma_i)\| > \delta$. By (1) above $\frac{2}{3\sum_{j=1}^{d}|r_j|} \geqslant \|\chi(\gamma_i)\|$ and therefore $\delta < \frac{2}{3\sum_{j=1}^{d}|r_j|}$. Equivalently $\sum_{i=1}^{d}|r_i| < \frac{2}{3\delta}$. So there are only finitely many choices for β and we may put an universal bound M on the coefficients in the linear combination $\beta = \sum_{i=1}^{r} k_i\alpha_i$.

(3) Clearly, the set

$$I := \{\beta \in G\colon \|\beta B_0\| \leqslant \sigma\} = \bigcap_{\chi \in B_0} \{\gamma \in G\colon \|\chi(\gamma)\| \leqslant \sigma\}$$

is closed and by (2) we have $I \subseteq \langle\alpha_1,\dots,\alpha_t\rangle_M \subseteq V$. Thus $I \cap V^c = \emptyset$. By compactness of G there exists a finite set $E \subseteq B_0$ such that

$$\bigcap_{\chi \in E} \{\gamma \in G\colon \|\chi(\gamma)\| \leqslant \sigma\} \cap V^c = \emptyset.$$

This E is as required. □

Proof of Lemma 3.5. Let ρ be a metric on G compatible with its topology. Since the sets $R_1 \subseteq R_2 \subseteq \cdots \subseteq G$ are finite there is a sequence $(d_n)_{n=1}^{\infty}$ of positive reals decreasing to 0 such that

$$2d_n < \min\{\rho(\alpha,\alpha')\colon \alpha,\alpha' \in R_n,\ \alpha \neq \alpha'\},$$
$$d_n + d_{n+1} < \min\{\rho(\alpha,\alpha')\colon \alpha \in R_n,\ \alpha' \in R_{n+1} \setminus R_n\}.$$

Define

$$V_n := \{\beta \in G\colon \exists\alpha \in R_n \text{ with } \rho(\beta,\alpha) < d_n\}.$$

By monotonicity of the sets R_n, $\beta \in \bigcup_{n=1}^{\infty} R_n$ implies $\beta \in \bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty} V_n$. Conversely, assume $\beta \in \bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty} V_n$ or, equivalently, that there exists an m with $\beta \in V_n$ for all $n \geqslant m$. According to the definition of the sets V_n there exists a unique $\alpha_n \in R_n$ such that $\rho(\alpha_n,\beta) < d_n$ for $n \geqslant m$. Moreover the choice of the d_n guarantees that $\alpha_m = \alpha_{m+1} = \cdots$ and so $\rho(\beta,\alpha_m) = \rho(\beta,\alpha_n) \leqslant d_n \to 0$. Hence $\beta = \alpha_m \in R_m \subseteq \bigcup_{n=1}^{\infty} R_n$. □

4. Thick and thin characterizing sequences

It has been shown by Erdős and Taylor (cf. [14]) that for any sequence $(k_n)_{n=1}^{\infty}$ with bounded quotients, i.e. $q_n = \frac{k_{n+1}}{k_n} \leqslant C$ for all $n \in \mathbb{N}$ and some $C \in \mathbb{R}$ the characterized group is at most countable.

Question 5.2 from [2] asks: Does every countable subgroup H of $\mathbb{T}$ admit a characterizing sequence $(k_n)_{n=1}^{\infty}$ with bounded quotients?

We answer this question affirmatively by proving a stronger result. Which type of statement can be expected? Assume that $\alpha \in H$ is irrational. Then, by uniform distribution of the sequence $(n\alpha)_{n=1}^{\infty}$, the set of all $k \in \mathbb{N}$ with $\|k\alpha\| < \varepsilon$ has density 2ε. Thus (with the exception of trivial cases) characterizing sequences have zero density. Furthermore the length of their gaps tends to infinity. In particular the thickest characterizing sequences we can expect might have a density which converges to zero very slowly in some sense. This is the content of the following result.

Theorem 4.1. *Let $H \leqslant \mathbb{T}$ be a countable subgroup and let $(\varepsilon_j)_{j=1}^{\infty}$ be a sequence with $0 \leqslant \varepsilon_j \leqslant 1$ that converges to 0 (arbitrarily slowly). Let $\mathbb{N}$ be partitioned into nonempty intervals $I_j = \{i_{j-1}+1, i_{j-1}+2, \ldots, i_j\}$, $j \in \mathbb{N}$ with $i_0 = 0$ and $\lim_{j\to\infty}(i_j - i_{j-1}) = \infty$. Then there exists an increasing sequence $(k_n)_{n=1}^{\infty}$ of nonnegative integers characterizing H such that*

$$\frac{|\{n\colon k_n \in I_j\}|}{|I_j|} \geqslant \varepsilon_j \quad \text{for all } j.$$

Proof. Let, according to Theorem 3.1 (or to [6]), $c_1 < c_2 < \cdots \in \mathbb{N}$ be any sequence characterizing H. We are going to construct a sequence $d_1 < d_2 < \cdots \in \mathbb{N}$ containing at least $\varepsilon_j |I_j|$ elements in each I_j such that $\|d_n \alpha\| \to 0$ for all $\alpha \in H$. Then $A = \{k_1 < k_2 < \cdots\} = \{d_n\colon n \in \mathbb{N}\} \cup \{c_n\colon n \in \mathbb{N}\}$ clearly has the desired properties.

Let $H =: \{\alpha_t\colon t \in \mathbb{N}\}$, put $I_j^+(0) := I_j$ and

$$I_j^+(t) := \bigl\{k \in I_j\colon \|k\alpha_i\| < 1/t \text{ for all } i \in \{1, 2, \ldots, t\}\bigr\}$$

for $t \geqslant 1$. For each $j \in \mathbb{N}$ let t_j be the maximal $t \in \{1, 2, \ldots, j\}$ such that $|I_j^+(t)| \geqslant \varepsilon_j |I_j|$ and put

$$\{d_1 < d_2 < \cdots\} = \bigcup_{j=0}^{\infty} I_j^+(t_j).$$

It suffices to show that $t_j \to \infty$ for $j \to \infty$ or, equivalently, that for each $t_0 \in \mathbb{N}$ there exists j_0 such that for all $j \geqslant j_0$

$$\Bigl|\bigl\{d \in I_j\colon \|d\alpha_i\| < 1/t_0 \text{ for all } i = 1, \ldots, t_0\bigr\}\Bigr| \geqslant \varepsilon_j |I_j|.$$

Since $\varepsilon_j \to 0$ for $j \to \infty$ this is an immediate consequence of the well distribution (cf. [16, p. 269, Corollary 4.1]) of the sequence $(ng)_{n=1}^{\infty}$ in the closed subgroup $G \leqslant \mathbb{T}^{t_0}$ generated by $g = (\alpha_1, \ldots, \alpha_{t_0}) \in \mathbb{T}^{t_0}$: The open subset $O \subseteq G$ of all $(\beta_1, \ldots, \beta_{t_0})$ with $\|\beta_i\| < 1/t_0$ has positive Haar measure $\mu(O)$ and the set of all $k \in \mathbb{Z}$ with $kg \in O$ has uniform density $\mu(O) > 0$. □

Theorem 4.1 indeed answers the question about quotients: Take, for instance, $i_j = j^2$ and choose a sequence of strictly positive ε_j. Then the quotients $q_n = \frac{k_{n+1}}{k_n}$ tend to 1. This example can be modified in many ways.

It has been proved in [13] that $q_n \to \infty$ implies that the corresponding characterized group H is uncountable. Thus, for a given countable H, characterizing sequences cannot be arbitrarily sparse in this sense. Nevertheless we have:

Theorem 4.2. *Let H be a countable subgroup of $\mathbb{T}$ and let $m_1 < m_2 < \cdots$, be an (arbitrarily fast) increasing sequence of positive integers. Then there is a characterizing sequence $k_1 < k_2 < \cdots$ for H with $m_n < k_n$ for all $n \in \mathbb{N}$.*

Proof. Let $(c_n)_{n=1}^{\infty}$ be any characterizing sequence of H. Put $k_{2n} := c_{j_n}$ and $k_{2n+1} := c_{j_n} + c_n$ where j_n is large enough in the sense that $k_{2n} > m_{2n}$ and $k_{2n+1} > m_{2n+1}$. Clearly $\alpha \in H$ implies $k_n\alpha \to 0$. On the other hand, if $\beta \in \mathbb{T}$ and $k_n\beta \to 0$ then also $(k_{2n+1} - k_{2n})\beta = c_n\beta \to 0$. $(c_n)_{n=1}^{\infty}$ characterizes H, therefore $\beta \in H$. □

Theorem 4.2 implies that for any countable $H \leqslant \mathbb{T}$ there are sequences $(k_n)_{n=1}^{\infty}$ characterizing H with $\limsup_{n\to\infty} \frac{k_{n+1}}{k_n} = \infty$: In Theorem 4.2 put $m_n = n^n$ and let $k_1 < k_2 < \cdots$, be a characterizing sequence of H such that $m_n \leqslant k_n$ for all $n \in \mathbb{N}$. Then

$$\sup_{n\in\mathbb{N}} \frac{k_{n+1}}{k_n} \geqslant \sup_{n\in\mathbb{N}} \sqrt[n]{\prod_{i=1}^{n} \frac{k_{i+1}}{k_i}} \geqslant \sup_{n\in\mathbb{N}} \sqrt[n]{\frac{k_n}{k_1}} \geqslant \sup_{n\in\mathbb{N}} \sqrt[n]{\frac{n^n}{k_1}} = \infty.$$

Note that this is also contained in [2, Remark 3.5]. In [6] this fact is shown in the special case that H is a cyclic group. The proof is essentially the same as the one of Theorem 4.2.

For more sophisticated methods to generate sparse characterizing sequences we refer to [3,7]: E.g. for a countable subgroup $H \leqslant \mathbb{T}$ one can construct a characterizing sequence $(k_n)_{n=1}^{\infty}$ such that for all $r > 0$ and $\alpha \in H$, $\sum_{n=1}^{\infty} \|k_n\alpha\|^r < \infty$.

The idea of the proof of Theorem 4.2 has further remarkable extensions. We will analyze them more detailed in the next section.

5. Groups as sets of convergence

The following theorem presents a generalized approach to the characterization of subgroups.

Theorem 5.1. *Let G be a compact Abelian group.*

(1) *Let $\mathcal{F}$ be a filter on $\widehat{G}$. Then the set H of all $\alpha \in G$ for which $\mathcal{F}\text{-}\lim_\chi \chi(\alpha)$ exists is a subgroup of G. The mapping $f: H \mapsto \mathbb{T}$, $\alpha \mapsto \mathcal{F}\text{-}\lim_\chi \chi(\alpha)$ is a group homomorphism.*
In particular, if $(\chi_n)_{n=1}^{\infty}$ is a sequence in $\widehat{G}$, the set H of all $\alpha \in G$ for which $\lim_{n\to\infty} \chi_n(\alpha)$ exists is a subgroup and the mapping $f: H \mapsto \mathbb{T}$, $\alpha \mapsto \lim_{n\to\infty} \chi_n(\alpha)$ is a group homomorphism.

Let, conversely, H be a subgroup of G and let $f: H \mapsto \mathbb{T}$ be a homomorphism.

(2) *There exists a filter $\mathcal{F}$ on $\widehat{G}$ such that $\mathcal{F}\text{-}\lim_\chi \chi(\alpha) = f(\alpha)$ for all $\alpha \in H$ and $\mathcal{F}\text{-}\lim_\chi \chi(\beta)$ does not exist whenever $\beta \notin H$.*
(3) *If furthermore H is countable then there exists a sequence $(\chi_n)_{n=1}^{\infty}$ in $\widehat{G}$ such that*
$$\chi_n(\alpha) \to f(\alpha) \quad \text{for all } \alpha \in H.$$
(4) *If G is metrizable and H is countable then there exists a sequence $(\chi_n')_{n=1}^{\infty}$ in $\widehat{G}$ such that*
$$\chi_n'(\alpha) \to f(\alpha) \quad \text{for all } \alpha \in H$$
and $(\chi_n'(\beta))_{n=1}^{\infty}$ does not converge if $\beta \notin H$.

Proof. (1) Clear.

(2) For $\alpha_1, \ldots, \alpha_t \in H$ and $\varepsilon > 0$ put

$$F(\alpha_1, \ldots, \alpha_t, \varepsilon) := \{\chi \in \widehat{G}: \|\chi(\alpha_i) - f(\alpha_i)\| \leqslant \varepsilon \text{ for } i = \{1, 2, \ldots, t\}\}$$

and

$$\mathcal{F} = \mathcal{F}(H, f) := \{F \subseteq \widehat{G}: \exists \alpha_1, \ldots, \alpha_t \in H,\ \exists \varepsilon > 0 \text{ such that } F(\alpha_1, \ldots, \alpha_t, \varepsilon) \subseteq F\}.$$

We have to show that

(a) $\mathcal{F}$ is a filter.
(b) For all $\alpha \in H$: $\mathcal{F}\text{-}\lim_\chi \chi(\alpha) = f(\alpha)$.
(c) For all $\beta \notin H$: $\mathcal{F}\text{-}\lim_\chi \chi(\beta)$ does not exist.

ad (a): Since the set $F(\alpha_1, \ldots, \alpha_t, \beta_1, \ldots, \beta_t, \min(\varepsilon_1, \varepsilon_2)) \in \mathcal{F}$ is contained in $F(\alpha_1, \ldots, \alpha_t, \varepsilon_1) \cap F(\beta_1, \ldots, \beta_t, \varepsilon_2)$ it suffices to show that each $F(\alpha_1, \ldots, \alpha_t, \varepsilon)$ is not empty.

There exists an extension of $f: H \mapsto \mathbb{T}$ to $\chi: G \mapsto \mathbb{T}$ such that $\chi \in \widehat{G_d}$. By Lemma 2.2 there is a $\chi' \in \widehat{G}$ such that $\|\chi'(\alpha_i) - \chi(\alpha_i)\| \leqslant \varepsilon$ for $i = 1, \ldots, t$. Hence $\chi' \in F(\alpha_1, \ldots, \alpha_t, \varepsilon) \neq \emptyset$.

ad (b): Let $\alpha \in H$ and assume that U is a neighbourhood of $f(\alpha)$. There exists an $\varepsilon > 0$ such that $\|\xi - f(\alpha)\| < \varepsilon$ implies $\xi \in U$. $\chi \in F(\alpha, \varepsilon) \in \mathcal{F}$ implies $\|\chi(\alpha) - f(\alpha)\| < \varepsilon$ proving $\mathcal{F}\text{-}\lim_\chi \chi(\alpha) = f(\alpha)$.

ad (c): Let $\beta \notin H$ and $F \in \mathcal{F}$ be arbitrary. We will show that there exist $\chi_1, \chi_2 \in F$ such that $\|\chi_1(\beta) - \chi_2(\beta)\| \geqslant 1/4$. $F \in \mathcal{F}$ implies that there exist $\alpha_1, \ldots, \alpha_t \in H$ and $\varepsilon > 0$ such that $F(\alpha_1, \ldots, \alpha_t, \varepsilon) \subseteq F$. Note that there is a $\chi' \in \widehat{G_d}$ with $\chi'(h) = 0$ for all $h \in H$ and $\chi'(\beta) \geqslant 1/3$. By Lemma 2.2 there exists a $\chi \in \widehat{G}$ such that $\|\chi(\alpha_i)\| < \varepsilon/2$ for $i = 1, \ldots, t$ and $\chi(\beta) > 1/4$. Pick $\chi_1 \in F(\alpha_1, \ldots, \alpha_t, \varepsilon/2) \subseteq F$ arbitrary and let $\chi_2 = \chi + \chi_1$. Then χ_2 is also in F and $\|\chi_2(\beta) - \chi_1(\beta)\| = \|\chi(\beta)\| > 1/4$.

(3) Let $H = \{\alpha_t, t \in \mathbb{N}\}$. The proof of (2) shows that for each $n \in \mathbb{N}$ there is a $\chi_n \in \widehat{G}$ such that $\|\chi_n(\alpha_i) - f(\alpha_i)\| < 1/n$ for $i = 1, \ldots, n$. The sequence $(\chi_n)_{n=1}^\infty$ has the desired properties.

(4) If G is metrizable and H is countable we know by Theorem 3.1 that there exists a sequence $(\widetilde{\chi}_n)_{n=1}^\infty$ in $\widehat{G}$ such that

$$\|\widetilde{\chi}_n(\alpha)\| \to 0 \quad \text{iff} \quad \alpha \in H.$$

Let furthermore $(\chi_n)_{n=1}^\infty$ be as in (3) and define $\chi'_{2n} := \chi_n$ and $\chi'_{2n+1} := \chi_n + \widetilde{\chi}_n$. Then $\chi'_n(\alpha) \to f(\alpha)$ for all $\alpha \in H$.

Conversely, for $\beta \notin H$ the sequence $(\chi'_n(\beta))_{n=1}^\infty$ cannot converge: If $\chi'_n(\beta) \to c$ for some $c \in \mathbb{T}$, then $\widetilde{\chi}_n(\beta) = \chi'_{2n+1}(\beta) - \chi'_{2n}(\beta) \to 0$. Hence $\beta \in H$, contradiction. □

We want to apply Theorem 5.1 to Question 5.4 in [2] which, in our notation, reads as follows. Let $(c_n)_{n=1}^\infty$ be a sequence in $\mathbb{Z}$. Are the subsequent conditions (i) and (ii) equivalent?

(i) There exists a precompact Abelian group $G \supseteq \mathbb{Z}$ such that $c_n \to h$ in G and $\langle h\rangle \cap \mathbb{Z} = \{0\}$.
(ii) There exists an infinite subgroup $A \leqslant \mathbb{T}$ such that $c_n\alpha \to 0$ holds for all $\alpha \in A$.

Conditions (i) is obviously equivalent to (i′) below:

(i′) There exists a group compactification (ι, G) of $\mathbb{Z}$ such that ι is 1–1, $\iota(c_n) \to h$ in G and $\langle h\rangle \cap \iota(\mathbb{Z}) = \{0\}$.

We remark first that (ii) implies (i′): Let A be the infinite subgroup such that $c_n\alpha \to 0$ holds for all $\alpha \in A$. Define a group compactification (G, ι) were $\iota : \mathbb{Z} \to \mathbb{T}^A$ is given by $n \mapsto (n\alpha)_{\alpha\in A}$ and $G = \overline{\iota(\mathbb{Z})} \subseteq \mathbb{T}^A$. Obviously (ι, G) is a compactification of $\mathbb{Z}$ and since A is infinite, ι is 1–1. Then $\iota(c_n) \to 0 =: h$ in G and in particular $\langle h\rangle \cap \iota(\mathbb{Z}) = \{0\}$.

To see that the converse does not hold, pick $\alpha, \beta \in \mathbb{T}$, such that α and β are linearly independent over the rationals. Define a homomorphism $f : \langle\alpha\rangle \to \langle\beta\rangle, n\alpha \mapsto n\beta$. By Theorem 5.1.4 choose a sequence $(c_n)_{n=1}^\infty$ in $\mathbb{Z}$ such that $c_n\alpha \to f(\alpha) = \beta$ and $(c_n\gamma)_{n=1}^\infty$ does not converge for $\gamma \in \mathbb{T} \setminus \langle\alpha\rangle$.

Then $\iota : \mathbb{Z} \to \mathbb{T}, n \mapsto n\alpha$ gives rise to a group compactification of $\mathbb{Z}$. Put $h := \beta$ such that $\iota(c_n) = c_n\alpha \to \beta$. Since α and β were chosen to be linearly independent we have $\langle h\rangle \cap \iota(\mathbb{Z}) = \langle\beta\rangle \cap \langle\alpha\rangle = \{0\}$. Thus (i′) is valid. On the other hand (ii) fails since $c_n\gamma \to 0$ only for $\gamma = 0$.

For a different type of counterexample fix a prime p and consider the p-adic integers $\mathbb{Z}_p$. Choose an arbitrary sequence $(k_n)_{n=1}^\infty$ in $\{0, 1, \ldots, p-1\}$ which contains infinitely many nonzero elements and satisfies $k_1 = 1$. Using this, put for each $n \in \mathbb{N}$, $c_{2n} = \sum_{i=1}^n k_i p^i$ and let $c_{2n+1} = c_{2n} + p^n$. Then

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} c_{2n} = \sum_{i=1}^\infty k_i p^i =: h \in \mathbb{Z}_p \setminus \mathbb{Z}.$$

Hence $kh \in \mathbb{Z}_p \setminus \mathbb{Z}$ for all $k \in \mathbb{Z} \setminus \{0\}$, so (i) holds.

Next pick $\alpha \in \mathbb{T}$ such that $\lim_{n\to\infty} c_n\alpha = 0$. It follows that also $\lim_{n\to\infty}(c_{2n+1} - c_{2n})\alpha = p^n\alpha = 0$, so $\alpha = \frac{a}{p^l} + \mathbb{Z}$ for some $l \in \mathbb{N}$ and $a \in \{0, 1, \ldots, p^l - 1\}$. But then, for all $n \geqslant l$

$$\|c_{2n}\alpha\| = \left\|\left(\sum_{i=1}^n k_i p^i\right)\frac{a}{p^l}\right\| = \left\|\left(\sum_{i=1}^{l-1} k_i p^i\right)\frac{a}{p^l}\right\|.$$

Since $k_1 = 1$ the last term is a nonzero constant whenever $a \neq 0$. Hence $\alpha = 0$ and (ii) fails.

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