An analysis of two variable rational approximants

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ABSTRACT

We present two determinants whose ratio is the Hughes Jones approximant to a power series in two variables. They are generalizations of Jacobi's determinants for Padé approximants. They are useful in certain circumstances when the defining equations are degenerate. We analyze the indeterminacies associated with degenerate approximants, at least one of which is quite different in nature from the degeneracies of the single variable Padé approximants. We are led to suggest a modification of the symmetrizing equations which leads to numerical stability.

1. INTRODUCTION

Padé approximants defined by the Jacobi definition (Jacobi, 1846) always exist in a formal sense and satisfy the identity

$$B^D(z)\,f(z) - A^D(z) = 0(z^{M+N+1}) \qquad (1.1)$$

The only significant point is that polynomials $B^D(z)$ of degree N and $A^D(z)$ of degree M may always be found to satisfy (1.1), and so a Padé approximant (P. A.) to f(z) may always be defined formally. Jacobi's explicit determinantal forms for $A^D(z)$ and $B^D(z)$ are useful if $B^D(z)$ is not identically zero. Reduction of the degree of numerator and denominator of the fraction $A^D(z)/B^D(z)$ is possible in the cases where

(i) $A^D(0) = B^D(0) = 0$. A power of z cancels between numerator and denominator.

(ii) $A^D(z)$ and $B^D(z)$ share a common polynomial factor which cancels from each.

(iii) The coefficient of the highest power of z in $A^D(z)$ or $B^D(z)$ is zero, so that either the numerator has degree less than M or the denominator has degree less than N.

We refer historically to Frobenius (1881) and to Gragg (1972) and Baker (1973) for a discussion of these possibilities. It only follows from (1.1) that

$$f(z) = \frac{A^D(z)}{B^D(z)} + 0(z^{M+N+1}) \qquad (1.2)$$

if $B^D(0) \neq 0$. Consequently, we prefer to use the Baker definition in which all Padé denominators are normalized to unity at $z = 0$ and certain approximants are declared not to exist. The Padé table breaks down into square blocks (Baker 1973, 1974) as shown in Fig. 1.

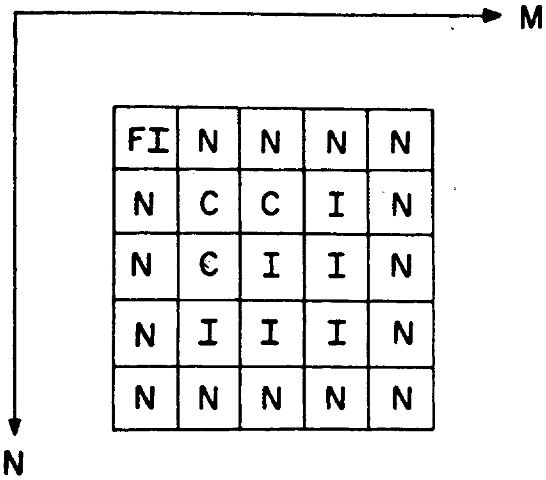

Fig. 1. A block of the Padé table.

For each block, the upper left hand entry, denoted by FI, is nonsingular and has the full indicated order. The other entries, denoted by N, in the upper row and left hand column have property (iii) and are normal. The entries, also denoted by N, in the right hand and lower edges are normal. The inner and upper left triangular block, denoted by C, has approximants with properties (i) and (ii); the equations defining the coefficients are linearly dependent and consistent. The entries in the lower right inner triangular block, denoted by I, cannot exist; the equations defining the coefficients are linearly dependent and inconsistent. We use the concepts behind this categorization to define the degeneracies of the individual two variable approximants.

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Because we require the analogue of (1.2), two variable approximants are always defined subject to the requirement $b_{00} = 1$; any approximants which could not be constructed because the defining equations were linearly dependent were termed degenerate (Hughes Jones and Makinson, 1974). Rational approximants of type $f_{m,m/m,m}(x, y)$ were originally defined by Chisholm (1973). The generalization to simple off diagonal approximants $f_{m,m/n,n}(x, y)$ and their practicality was shown by Graves-Morris, Hughes Jones and Makinson (1974). The further generalization to $f_{m_1,m_2/n_1,n_2}(x,y)$ was made by Hughes Jones (1973).

In section 2, we write down two determinants, $A^D(x, y)$ and $B^D(x, y)$ which satisfy the Jacobi type defining equations for the two variable approximants. They are in fact a minor variant of a special case of Levin's determinants (Levin, 1973) who defined approximants which satisfy identities over very general regions of lattice space. Our definitions are rather more explicit, and so the conditions for the existence of our determinants can be established.

If certain determinants are not zero and $B^D(x, y)$ is not identically zero, we will show that $A^D(x,y)/B^D(x,y)$ satisfies the lattice space identities which are generalizations of (1.2) and so the Maclaurin expansions of both function and approximant agree as far as the defining equations indicate.

In section 3, we will explicitly consider degenerate approximants. The two variable case is quite different from the one variable case to the extent that vanishing of the denominator $B^D(x, y)$ at the origin, $x = y = 0$, in the two variable case does not automatically imply that $A^D(x, y)$ and $B^D(x, y)$ have a common factor which cancels. This has implications even when the approximants are defined with the normalization condition $B(0, 0) = 1$. We will find a case when the defining equations are degenerate and consistent, for which an irreducible approximant $A(x, y)/B(x, y)$ exists, with $B(0, 0) = 1$. The Maclaurin expansions of function and approximant agree to the full indicated order, yet the approximant is quite ill-defined. This situation cannot occur for one variable approximants, and we suggest some remedies. We also suggest a weighting for the symmetrizing equations which leads to numerical stability, and compare it with the weighting contemporaneously suggested by Chisholm and Hughes Jones (1975).

2. A DETERMINANTAL FORM

Define notation for Hughes Jones approximants by

$$f_{m_1,m_2/n_1,n_2}(x,y) = \frac{\sum_{\alpha_1=0}^{m_1} \sum_{\alpha_2=0}^{m_2} a_{\alpha_1,\alpha_2} x^{\alpha_1} y^{\alpha_2}}{\sum_{\beta_1=0}^{n_1} \sum_{\beta_2=0}^{n_2} b_{\beta_1,\beta_2} x^{\beta_1} y^{\beta_2}} \tag{2.1}$$

which may be written concisely as

$$f_{\underline{m}/\underline{n}}(\underline{z}) = \sum_{\underline{\alpha}=\underline{0}}^{\underline{m}} a_{\underline{\alpha}} \underline{z}^{\underline{\alpha}} \Big/ \sum_{\underline{\beta}=\underline{0}}^{\underline{n}} b_{\underline{\beta}} \underline{z}^{\underline{\beta}} \tag{2.2}$$

Given a power series,

$$f(x,y) = \sum_{\gamma_1=0}^{\infty} \sum_{\gamma_2=0}^{\infty} c_{\gamma_1,\gamma_2} x^{\gamma_1} y^{\gamma_2} = \sum_{\underline{\gamma}=\underline{0}}^{\infty} c_{\underline{\gamma}} \underline{z}^{\underline{\gamma}} \tag{2.3}$$

we may form approximants by defining

$$E(\underline{z}) = \Big(\sum_{\underline{\gamma}=\underline{0}}^{\infty} c_{\underline{\gamma}} \underline{z}^{\underline{\gamma}}\Big)\Big(\sum_{\underline{\beta}=\underline{0}}^{\underline{n}} b_{\underline{\beta}} \underline{z}^{\underline{\beta}}\Big) - \sum_{\underline{\alpha}=\underline{0}}^{\underline{m}} a_{\underline{\alpha}} \underline{z}^{\underline{\alpha}} \tag{2.4}$$

and equating coefficients to zero over the lattice space $S(\underline{m}/\underline{n})$ which defines the $\underline{m}/\underline{n}$ approximants.

Let $n = \min(n_1, n_2)$

$m = \min(m_1, m_2)$

$l = \min(m, n)$.

Consider the prong emanating from (α, α) where $0 \leqslant \alpha \leqslant l$. For the first branch of the prong, $\alpha_2 =$ fixed $= \alpha$, and α_1 ranges over $m_1 + 1 \leqslant \alpha_1 \leqslant m_1 + n_1 - \alpha$ giving

$$\sum_{\beta_1=0}^{n_1} \sum_{\beta_2=0}^{\alpha} b_{\underline{\beta}}\, c_{\underline{\alpha}-\underline{\beta}} = 0. \tag{2.5a}$$

For the second branch, $\alpha_1 =$ fixed $= \alpha$, α_2 ranges over $m_2 + 1 \leqslant \alpha_2 \leqslant m_2 + n_2 - \alpha$, giving

$$\sum_{\beta_1=0}^{\alpha} \sum_{\beta_2=0}^{n_2} b_{\underline{\beta}}\, c_{\underline{\alpha}-\underline{\beta}} = 0. \tag{2.5b}$$

The symmetrised equation, parametrised by $(\lambda_\alpha, \mu_\alpha)$ (Chisholm and Graves-Morris, 1975) is

$$\lambda_\alpha \sum_{\beta_1=0}^{n_1} \sum_{\beta_2=0}^{\alpha} b_{\underline{\beta}}\, c_{m_1+n_1+1-\alpha-\beta_1,\ \alpha-\beta_2} + \mu_\alpha \sum_{\beta_1=0}^{\alpha} \sum_{\beta_2=0}^{n_2} b_{\underline{\beta}}\, c_{\alpha-\beta_1, m_2+n_2+1-\alpha-\beta_2} = 0 \tag{2.5c}$$

which applies for $1 \leqslant \alpha_1 = \alpha_2 = \alpha \leqslant \ell$. Note that the last prong, $\alpha = \ell$, may have one or more branches missing.

These equations, 2.5a, b and c, are sufficient to determine all the denominator coefficients if and only if $m \geqslant n$. Otherwise, the extra equations given by the (α, α) prong, where $\ell < \alpha \leqslant n$ are precisely the ones needed for defining the denominator of non-degenerate approximants.

For the first branch, $\alpha_2 =$ fixed $= \alpha$ and α_1 ranges over $\alpha < \alpha_1 \leqslant n_1$,

$$\sum_{\beta_1=0}^{\alpha}\sum_{\beta_2=0}^{\alpha} b_{\underline{\beta}}\, c_{\underline{\alpha}-\underline{\beta}} = 0 \tag{2.5d}$$

For the second branch, α_1 = fixed = α and α_2 ranges over $\alpha < \alpha_2 \leqslant n_2$,

$$\sum_{\beta_1=0}^{\alpha}\sum_{\beta_2=0}^{\alpha_2} b_{\underline{\beta}}\, c_{\underline{\alpha}-\underline{\beta}} = 0 \tag{2.5e}$$

For the equation at the angle of the prong, $\alpha_1=\alpha_2=\alpha$ and

$$\sum_{\beta_1=0}^{\alpha}\sum_{\beta_2=0}^{\alpha} b_{\underline{\beta}}\, c_{\underline{\alpha}-\underline{\beta}} = 0 \tag{2.5f}$$

Again, the last prong may have one or both branches missing. These equations 2.5, together with $b_{00}=1$, determine the denominator coefficients. Let us first consider equations 2.5a, b, and c. We are led to define, for $\alpha > 0$, an array of b coefficients

$$\underline{b}_\alpha^T = (b_{n_1\alpha}, \ldots\ldots, b_{\alpha+1,\alpha}, b_{\alpha,n_2}, \ldots\ldots, b_{\alpha,\alpha+1}, b_{\alpha,\alpha}) \tag{2.6}$$

The superfix T denotes transpose, and the notation emphasizes that a one-dimensional array $\underline{b}_\alpha$ is selected from the original set $\{b_{\underline{\beta}}\}$. The equations (2.5) take on the compact form :

$$D_{00}\,\underline{b}_0 = (0, 0, \ldots\ldots, 0, 0, 1)^T \tag{2.7a}$$

and for $\alpha > 0$,

$$D_{\alpha\alpha}\,\underline{b}_\alpha + \sum_{\beta=0}^{\alpha-1} G_{\alpha\beta}\,\underline{b}_\beta = 0 \tag{2.7b}$$

when the following definitions are made :

$$D_{00} = \begin{bmatrix} C_0^{(1)} & 0 & X_0^{(1)} \\ 0 & C_0^{(2)} & X_0^{(2)} \\ 0 & 0 & 1 \end{bmatrix}$$

and for $\alpha > 0$,

$$D_{\alpha\alpha} = \begin{bmatrix} C_\alpha^{(1)} & 0 & X_\alpha^{(1)} \\ 0 & C_\alpha^{(2)} & X_\alpha^{(2)} \\ \lambda_\alpha X_\alpha^{(1)T} & \mu_\alpha X_\alpha^{(2)T} & Y_\alpha \end{bmatrix}$$

$$C_\alpha^{(1)} = \begin{bmatrix} c_{m_1-n_1+1,0} & c_{m_1-n_1+2,0} & \ldots\ldots & c_{m_1-\alpha,0} \\ c_{m_1-n_1+2,0} & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ c_{m_1-\alpha,0} & \ldots\ldots & & c_{m_1+n_1-2\alpha-1,0} \end{bmatrix}$$

$$C_\alpha^{(2)} = \begin{bmatrix} c_{0,m_2-n_2+1} & c_{0,m_2-n_2+2} & \ldots\ldots & c_{0,m_2-\alpha} \\ c_{0,m_2-n_2+2} & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ c_{0,m_2-\alpha} & & \ldots\ldots & c_{0,m_2+n_2-2} \end{bmatrix}$$

$$X_\alpha^{(1)T} = (c_{m_1-\alpha+1,0} \quad c_{m_1+\alpha+2,0} \quad \ldots\ldots \quad c_{m_1+n_1-2\alpha,}$$

$$X_\alpha^{(2)T} = (c_{0,m_2-\alpha+1} \quad c_{0,m_2-\alpha+2} \quad \ldots\ldots \quad c_{0,m_2+n_2-2}$$

$$Y_\alpha = \lambda_\alpha c_{m_1+n_1-2\alpha+1,0} + \mu_\alpha\, c_{0,m_2+n_2-2\alpha+1}$$

Then it follows that

$$\det D_{\alpha\alpha} = \lambda_\alpha \det C_{\alpha-1}^{(1)} \det C_\alpha^{(2)} + \mu_\alpha \det C_\alpha^{(1)} \det C_{\alpha-1}^{(2)}$$

$G_{\alpha\beta}$ is quite complicated to define. We write it in block form

$$G_{\alpha\beta} = \begin{bmatrix} G_{\alpha\beta}^{11} & G_{\alpha\beta}^{12} & G_{\alpha\beta}^{13} \\ G_{\alpha\beta}^{21} & G_{\alpha\beta}^{22} & G_{\alpha\beta}^{23} \\ G_{\alpha\beta}^{31} & G_{\alpha\beta}^{32} & G_{\alpha\beta}^{33} \end{bmatrix}$$

Let $m(r \times s)$ denote the class of matrices with r rows and s columns. We label the elements of a matrix by $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, s$. From 2.5, 2.6, we are led to define the elements of each block of $G_{\alpha\beta}$:

$G_{\alpha\beta}^{11} \in m(n_1-\alpha \times n_1-\beta)$; $(G_{\alpha\beta}^{11})_{ij} = c_{m_1-n_1+i+j-1,\alpha-\beta}$

$G_{\alpha\beta}^{12} \in m(n_1-\alpha \times n_2-\beta)$; $(G_{\alpha\beta}^{12})_{ij} = c_{m_1-\beta+i,\alpha-n_2-1+j}$

$G_{\alpha\beta}^{13} \in m(n_1-\alpha \times 1)$; $(G_{\alpha\beta}^{13})_{ij} = c_{m_1-\beta+i,\,\alpha-\beta}$

$G_{\alpha\beta}^{21} \in m(n_2-\alpha \times n_1-\beta)$; $(G_{\alpha\beta}^{21})_{ij} = c_{\alpha-n_1-1+j,m_2-\beta+i}$

$G_{\alpha\beta}^{22} \in m(n_2-\alpha \times n_2-\beta)$; $(G_{\alpha\beta}^{22})_{ij} = c_{\alpha-\beta,m_2-n_2+i+j-1}$

$G_{\alpha\beta}^{23} \in m(n_2-\alpha \times 1)$; $(G_{\alpha\beta}^{23})_{ij} = c_{\alpha-\beta,m_2-\beta+i}$

$G_{\alpha\beta}^{31} \in m(1 \times n_1-\beta)$; $(G_{\alpha\beta}^{31})_{ij} = \lambda_\alpha c_{m_1-\alpha+j,\alpha-\beta} + \mu_\alpha c_{\alpha-n_1-1+j,m_2+n_2-\alpha-\beta+1}$

$$G^{32}_{\alpha\beta} \in m(1 \times n_2-\beta); \qquad (G^{32}_{\alpha\beta})_{ij} = \lambda_\alpha c_{m_1+n_1-\alpha-\beta+1,\alpha-n_2-1+j} + \mu_\alpha c_{\alpha-\beta,m_2-\alpha+j}$$

$$G^{33}_{\alpha\beta} = \lambda_\alpha c_{m_1+n_1-\alpha-\beta+1,\alpha-\beta} + \mu_\alpha c_{\alpha-\beta,m_2+n_2-\alpha-\beta+1}$$

If any index of $c_{i,j}$ is negative, then the element is understood as zero. If $n > m$, then the prong equations for $\ell < \alpha \leqslant n$ are needed, corresponding to (2.5d, e, f). A possible configuration is shown in Fig.2 The cross-hatched area shows the subspace of the $n_1 \times n_2$ lattice space where the extra equations are needed. The lines with arrows show the branches associated with $\underline{b}_\alpha$ in (2.6). The following definitions are motivated by (2.5d, e, f) and (2.6). For $\alpha > \ell$.

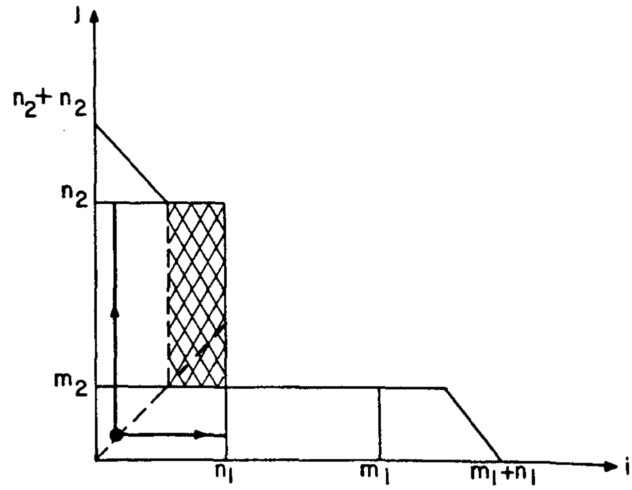

Fig. 2. The lattice space $S(\underline{m}/\underline{n})$ of coefficients c_{ij} for an approximant with $l = m < n$.

$$D_{\alpha\alpha} = \begin{bmatrix}
0 & 0 \cdots\cdots c_{0,0} & & & c_{1,0} \\
0 & \ddots \quad \vdots & & & \vdots \\
\vdots & c_{0,0} \quad \vdots & & 0 & \vdots \\
\vdots & \ddots \quad \vdots & & & \vdots \\
c_{0,0} & \cdots\cdots\cdots c_{n_1-\alpha-1,0} & & & c_{n_1-\alpha,0} \\
 & & 0 & 0 \cdots\cdots c_{0,0} & c_{0,1} \\
 & & 0 & \quad\cdot^{\cdot^{\cdot}} & \vdots \\
 & 0 & & c_{0,0} & \vdots \\
 & & & \cdot^{\cdot^{\cdot}} \quad & \vdots \\
 & & c_{0,0} & \cdots\cdots c_{0,n_2-\alpha-1} & c_{0,n_2-\alpha} \\
 & 0 & & 0 & c_{0,0}
\end{bmatrix}$$

In this case, $|\det D_{\alpha\alpha}| = |c_{0,0}^{\,n_1+n_2-2\alpha+1}|$

$$G^{11}_{\alpha\beta} \in m(n_1-\alpha \times n_1-\beta); \quad (G^{11}_{\alpha\beta})_{ij} = c_{\alpha+i-n_1+j-1,\alpha-\beta}$$

$$G^{12}_{\alpha\beta} \in m(n_1-\alpha \times n_2-\beta), \quad (G^{12}_{\alpha\beta})_{ij} = c_{\alpha+i-\beta,\alpha-n_2+j-1}$$

$$G^{13}_{\alpha\beta} \in m(n_1-\alpha \times 1), \quad (G^{13}_{\alpha\beta})_{i} = c_{\alpha+i-\beta,\alpha-\beta}$$

$$G^{21}_{\alpha\beta} \in m(n_2-\alpha \times n_1-\beta), \quad (G^{21}_{\alpha\beta})_{ij} = c_{\alpha-n_1+j-1,\alpha+i-\beta}$$

$$G^{22}_{\alpha\beta} \in m(n_2-\alpha \times n_2-\beta), \quad (G^{22}_{\alpha\beta})_{ij} = c_{\alpha-\beta,\alpha+i-n_2+j-1}$$

$$G^{23}_{\alpha\beta} \in m(n_2-\alpha \times 1), \quad (G^{23}_{\alpha\beta})_{i} = c_{\alpha-\beta,\alpha+i-\beta}$$

$$G^{31}_{\alpha\beta} \in m(1 \times n_1-\beta), \quad (G^{31}_{\alpha\beta})_{j} = c_{\alpha-n_1+j-1,\alpha-\beta}$$

$$G^{32}_{\alpha\beta} \in m(1 \times n_2-\beta), \quad (G^{32}_{\alpha\beta})_{j} = c_{\alpha-\beta,\alpha-n_2+j-1}$$

$$G^{33}_{\alpha\beta} = c_{\alpha-\beta,\,\alpha-\beta}$$

Then the equations (2.7) for the denominator coefficients are the key of the solution process. The prong method ensures that (2.7) has block lower triangular form. Frobenius method of solution of 2.7 is used, in the same way as for Padé approximants.

We write the denominator

$$B(x,y) = \sum_{\underline{\beta}=\underline{0}}^{\underline{n}} b_{\beta_1,\beta_2}\, x^{\beta_1}\, y^{\beta_2} \qquad (2.8)$$

and numerator

$$A(x,y) = \sum_{\underline{\alpha}=\underline{0}}^{\underline{m}} a_{\alpha_1,\alpha_2}\, x^{\alpha_1}\, y^{\alpha_2} \qquad (2.9)$$

Since the defining equations of (2.4) give $a_{\underline{\alpha}}$ as a linear combination of $b_{\underline{\beta}}$, both numerator and denominator are linear in $b_{\underline{\beta}}$ and follow from the solution of (2.7).

Using the further definitions

$$F = \det D'_{00} \prod_{\alpha=1}^{n} \det D_{\alpha\alpha}$$

$\widetilde{D}_{00} = (D_{00}$, with the last row deleted)

$\widetilde{D}'_{00} = (\widetilde{D}_{00}$, with the last column deleted)

$$\underline{z}_\alpha = (x^{n_1}y^{\alpha}, \ldots, x^{\alpha+1}y^{\alpha}, x^{\alpha}y^{n_2}, \ldots, x^{\alpha}y^{\alpha+1}, x^{\alpha}y^{\alpha})$$

$$\underline{Z}_\alpha = (\zeta_{n_1,\alpha}, \ldots, \zeta_{\alpha+1,\alpha}, \zeta_{\alpha,n_2}, \ldots, \zeta_{\alpha,\alpha+1}, \zeta_{\alpha,\alpha})$$

where

$$\zeta_{\sigma,\tau} = \sum_{\mu=\sigma}^{m_1} \sum_{\nu=\tau}^{m_2} c_{\mu-\sigma,\nu-\tau}\, x^{\mu} y^{\nu},$$

we solve 2.7, 2.8 and 2.9 when $F \neq 0$:

$$A(x,y) = A^D(x,y)\,(-)^{n_1 n_2}/F$$

$$B(x,y) = B^D(x,y)\,(-)^{n_1 n_2}/F$$

$$f_{\underline{m}/\underline{n}}(x,y) = A^D(x,y)/B^D(x,y) \qquad (2.10a)$$

$$B^D(x,y) = \begin{vmatrix} \widetilde{D}_{00} & & & \\ G_{10} & D_{11} & & 0 \\ G_{20} & G_{21} & D_{22} & \\ \vdots & & & \\ G_{n0} & G_{n1} & G_{n2} \ldots & D_{nn} \\ \underline{z}_0 & \underline{z}_1 & \underline{z}_2 & \ldots\ \underline{z}_n \end{vmatrix} \qquad (2.10b)$$

$$A^D(x,y) = \begin{vmatrix} \widetilde{D}_{00} & & & \\ G_{10} & D_{11} & & 0 \\ G_{20} & G_{21} & D_{22} & \\ \vdots & & & \\ G_{n0} & G_{n1} & G_{n2} \ldots & D_{nn} \\ \underline{Z}_0 & \underline{Z}_1 & \underline{Z}_2 & \ldots\ \underline{Z}_n \end{vmatrix} \qquad (2.10c)$$

These equations give explicit solutions of the two variable rational approximants when F is non-zero, which is to say that the approximants are non-degenerate. The generalization to N-dimensions is straightforward.

3. DEGENERATE APPROXIMANTS

To define two variable rational approximants, we seek solutions of 2.5 with $b_{00} = 1$. Then the Maclaurin coefficients of both function and approximant agree over the lattice space $S(\underline{m}/\underline{n})$.

The approximant always exists and is unique provided

$$F = \prod_{\alpha=0}^{n} \det D_{\alpha\alpha} \neq 0.$$

Such approximants are termed non-degenerate. We must consider the cases which lead to $F = 0$ and so to degenerate approximants. The alternatives, which are not exclusive, are :

Type 1 degeneracy : $\det D'_{00} = 0$

Type 2 degeneracy : $\det D_{\alpha\alpha} = 0$ for some α, $1 \leqslant \alpha \leqslant n$.

Type 1 degeneracy occurs when either of the P.As. $[m_1/n_1]f(z_1, 0)$ or $[m_2/n_2]f(0, z_2)$ is degenerate; to be specific, we assume that at least the $[m_1/n_1]$ approximant to $f(z_1, 0)$ is degenerate. This means that the Padé equations which determine

$$b_{1,0}, b_{2,0}, \ldots, b_{n_1,0}$$

are linearly dependent. The equations must be either inconsistent (type 1, INC) or consistent (type 1, CON).

3.1. Type 1, INC

We say, in this case, that $f_{\underline{m}/\underline{n}}(z)$ does not exist, because an $[\underline{m}/\underline{n}]$ cannot be found. In this instance, the determinantal solution (2.10) may be useful. The coefficient of x^0y^0 vanishes in numerator and denominator, and inspection of (2.10b and c) shows that at least a factor x cancels between numerator and denominator.

Suppose that $x^i y^j$ is the greatest monomial divisor of numerator and denominator. Provided the weights are chosen self consistently and $A^D(x, y)$ and $B^D(x,y)$ are not both identically zero, their ratio is the true $[m_1-i, m_2-j/n_1-i, n_2-j]$ approximant to $f(x, y)$. Of course, the weights associated with formation of each approximant must be chosen self consistently, so that the basic approximation problem is unchanged. The following examples show in detail how these situations may occur.

Example 1

Consider the $[2,2/2,2]$ approximant to $f(x, y) =$ $= c_{\alpha\beta} x^{\alpha} y^{\beta}$ with $c_{00} = c_{20} = 1; c_{10} = c_{30} = -1;$ $c_{40} \neq 1;\ c_{01}\, c_{03} - c_{02}^2 \neq 0.$

The equations attached to the first prong are

$$-b_{20} + b_{10} - b_{00} = 0$$

$$b_{20} - b_{10} + c_{40}\, b_{00} = 0$$

$$b_{00} = 1$$

$$c_{01}\, b_{02} + c_{02}\, b_{01} + c_{03}\, b_{00} = 0$$

$$c_{02}\, b_{02} + c_{03}\, b_{01} + c_{04}\, b_{00} = 0$$

These equations are inconsistent. However, if we let $b_{00} = 0$ in the context of determinantal solutions, then it follows that

$$b_{00} = b_{01} = b_{02} = 0$$

and from the numerator equations that

$$a_{00} = a_{01} = a_{02} = 0.$$

Thus we expect $f_{2,2/2,2}(x, y)$ to reduce to $f_{1,2/1,2}(x, y)$ for suitable weights. Let $\lambda_1, \mu_1, \lambda_2, \mu_2$ be the weights associated with $f_{2,2/2,2}(x, y)$ and $\tilde{\lambda}_1, \tilde{\mu}_1$ be the weights associated with $f_{1,2/1,2}(x, y)$. Inspection of the remaining equations shows consistency for the choice

$$\lambda_1 = 0, \mu_1 = 1,\ \lambda_2 = 0, \mu_2 = 1,\ \tilde{\lambda}_1 = 1, \tilde{\mu}_1 = 0.$$

Example 2

This is the same as example 1, except that $c_{01}\, c_{03} - c_{02}^2 = 0$. This case is so degenerate that the determinantal method fails because $A^D(x,y) = B^D(x,y) = 0$, as is evident by substitution in (2.10b and c).

3.2. Type 1, CON

In this case, one may set up to the equations so that two determinants vanish, representing the two conditions on the coefficients of the given power series. The conditions imply (Baker, 1974) that $[m_1-k/n_1-k]f(x,0)$ has, for some $k > 0$, a Maclaurin expansion which agrees with $f(x, 0)$ up to order $x^{m_1+n_1}$.

Indeed, numerator and denominator of $f_{m_1-k,\,0/n_1-k,\,0}(x, 0)$ may be multiplied by any polynomial in x of degree k to give a superficially different approximant which might be called $f_{m_1 0/n_1 0}(x,0)$.

There is no intrinsic difficulty with one variable approximants, but the genuine two variable approximants behave differently from Padé approximants, as the following example shows.

Example 3

$$f(x, y) = 1 - x + x^2 - x^3 + c_{01} y + c_{11}\, xy + \dots.$$

and form

$$f_{1,1/2,0} = \frac{a_{00} + a_{10} x + a_{01} y + a_{11} xy}{1 + b_{10} x + b_{20} x^2}$$

The equations are

$$b_{20} - b_{10} = -1$$

$$-b_{20} + b_{10} = 1$$

$$a_{00} = 1$$

$$a_{01} = b_{10} - 1$$

$$a_{01} = c_{01}$$

$$a_{11} = c_{01}\, b_{10} + c_{11}$$

The equations for the denominator coefficients are dependent and consistent, and we may say that b_{10} is undetermined, and the others are given in terms of b_{10}

$$a_{10} = b_{20} = b_{10} - 1,\quad a_{00} = 1$$

$$a_{01} = c_{01},\quad a_{11} = c_{01}\ b_{10} + c_{11}$$

Hence

$$f_{1,1/2,0}(x, y) = \frac{[1 + (b_{10} - 1)x][1 + c_{01} y] + xy\,(c_{11} + c_{01})}{(1 + x)\,[1 + (b_{10} - 1)\,x]}$$

We see that $f_{1,1/2,0}$ satisfies the S(1,1/2,0) lattice identities and yet is indeterminate in a real sense : unless $c_{11} + c_{01} = 0$, the approximant takes almost all values if b_{10} is quite arbitrary. This type of situation does not occur for Padé approximants.

No serious problem is presented by this type of ambiguity, although the orderly analysis is destroyed. If a lower order approximant satisfies the identities, the problem is solved ipso facto. This occurs in the example above, when $c_{11} + c_{01} = 0$ and then

$f_{0,1/20}(x,y) = \dfrac{1 + c_{01}\, y}{1 + x}$ satisfies the S(1,1/2,0) identities. If no lower order approximant satisfies the identities, one must use more coefficients of the original series to determine the approximants. In the example above, the equation corresponding to x^2y is

$$(c_{01} + c_{11})\, b_{10} = c_{01} - c_{21}$$

which always has a solution, since $c_{01} + c_{11} \neq 0$.

An alternative is to exploit the hypothesis of maximal analyticity lying behind Padé methods, and fix the parameters by maximizing the distance of the nearest moveable singularity from the origin. In the example, $b_{10} = 1$ sends the singularity to infinity.

3.3. Type 2 degeneracies

We now suppose that the denominator coefficients of the first prong $(b_{n_1,0},\ldots, b_{1,0}, b_{0,n_2},\ldots, b_{0,1})$ have been determined and subsequent prongs have to be calculated. Again we may separate the cases where the equations at stage α are consistent (type 2 CON) and inconsistent (type 2 INC). We have assumed tacitly that $\lambda_\alpha : \mu_\alpha$ is given and the problem is that $\det D_{\alpha\alpha} = 0$.

2, INC

The determinantal form may reduce and give a lower order approximant. However, we must say that the requested approximant does not exist.

2, CON

Either a lower order approximant is valid to the requested order, or else the ambiguity is removed by using more coefficients of f(x,y) or by imposing maximal analyticity, as previously described.
For type 2 degeneracies, we suggest a different and almost certainly better approach. The arbitrariness of $\lambda_\alpha : \mu_\alpha$ in the definition of the approximants is embarrassing, because there is always a choice which makes any prong of any approximant indeterminate. The procedure suggested here is expected to give better results than the choice $\lambda_\alpha = \mu_\alpha = 1$ used hitherto.
Since

$$\det D_{\alpha\alpha} = \lambda_\alpha \det C^{(1)}_{\alpha-1} \det C^{(2)}_\alpha + \mu_\alpha \det C^{(1)}_\alpha \det{}^{(2)}_{\alpha-1},$$

there is a unique value of $\lambda_\alpha : \mu_\alpha$ which causes the equations to be degenerate; we seek an "opposite" value which maximizes $|\det D_{\alpha\alpha}|$ and hopefully moves spurious singularities of the approximants as far from the origin as possible. If $|\det D_{\alpha\alpha}|$ is maximized subject to the constraint $|\lambda_\alpha|^2 + |\mu_\alpha|^2 = 1$, we find the choice

$$\lambda_\iota : \mu_\alpha = [\det C^{(1)}_{\alpha-1} \det C^{(2)}_\alpha]^* : [\det C^{(1)}_\alpha \det C^{(2)}_{\alpha-1}]^* \tag{3.1}$$

If $m_1 = m_2$ and $n_1 = n_2$, which includes Chisholm approximants and simple off-diagonal approximants, this scheme reduces to $\lambda_\alpha = \mu_\alpha$ for exactly symmetric functions. For antisymmetric functions, namely ones for which $f(x, y) = -f(y, x)$ the scheme reduces to $\lambda_\alpha = -\mu_\alpha$ Thus the scheme (3.1) gives the best values in these two cases. This choice of $\lambda_\alpha : \mu_\alpha$ gives a scheme which is always well defined if the Padé table of $f(x, 0)$ and $f(0, y)$ are normal, and also caters for some cases where some of the determinants in the C-table vanish.
If we consider only the Chisholm approximants $f_{m,m/m,m}(x, y)$, then this choice of $\lambda_\alpha : \mu_\alpha$ gives invariance under the homographic transformations

$$u = \frac{Ax}{1 + Bx}, \quad v = \frac{Cx}{1 + Dx} \quad \text{with } |A| = |C|$$

More explicitly, this means that if $f(x, y) = g(u, v)$ then $f_{m,m/m,m}(x, y) = g_{m,m/m,m}(u, v)$.
This particular choice of $\lambda_\alpha : \mu_\alpha$ is designed to give the approximants numerical stability. It is clear that the theorem of Chisholm and Graves-Morris (1975) takes on a stronger form if this choice is used *ab initio.*
The choice of Chisholm and Hughes Jones (1975), namely

$$\lambda_\alpha : \mu_\alpha = \det C^{(1)}_\alpha \det C^{(2)}_{\alpha-1} : \det C^{(1)}_{\alpha-1} \det C^{(2)}_\alpha \tag{3.2}$$

should be contrasted with (3.1). This would normally be the preferred choice for a sequence of Chisholm approximants which are not degenerate, because the authors show that it is invariant under independent homographic transformations of the variables. Since both choices (3.1) and (3.2) give identical results for symmetric and antisymmetric functions, at least for simple off diagonal approximants, which is the preferred choice in the general case is an open question.

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