Criteria for Generalized Diagonally Dominant Matrices and \textit{M}-Matrices. II

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\textbf{ABSTRACT}

We provide new necessary and sufficient conditions for verifying (strictly) generalized diagonally dominant matrices by applying the inverse of a partitioned matrix and obtain some criteria for identifying (nonsingular) \textit{M}-matrices.

\textbf{1. INTRODUCTION}

Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \). If there is a positive diagonal matrix \( D \) such that \( AD \) is a (strictly) diagonally dominant matrix, then \( A \) is said to be a (strictly) generalized diagonally dominant matrix: briefly, \( A \) is a \textit{GDDM} (SGDDM).

Let \( M(A) = (m_{ij}) \in \mathbb{R}^{n \times n} \), where \( m_{ii} = |a_{ii}|, m_{ij} = -|a_{ij}|, j \neq i, i, j \in N = \{1, 2, \ldots, n\} \), then \( M(A) \) is said to be a comparison matrix of \( A \).

We know that if \( A \) is a strictly diagonally dominant matrix, then \( \det A \neq 0 \) by the Levy-Desplanques theorem. O. Taussky [1] proved that if \( A \) is irreducible and diagonally dominant with \( \sum_{i=1}^{n} |a_{ii}| > \sum_{i=1}^{n} \Lambda_i \), then \( \det A \neq 0 \). P. N. Shivakumar and K. H. Chew [2] showed that if \( |a_{ii}| \geq \Lambda_i, i \in N, j = \{i \mid |a_{ii}| > \Lambda_i, i \in N\} \neq \emptyset \) and there exists a nonzero element chain \( a_{i_1} a_{i_1 i_2} \ldots a_{i_1 p} \) for any \( i \in N - J \), where \( p \in J \), then \( \det A \neq 0 \).

The authors in [3–7] gave some necessary and sufficient conditions for verifying SGDDMs, extending the results in [1] and [2].

In [8] we proved that if \( \Omega = \{ i \mid |a_{ij}| > \Lambda_i = \sum_{j \neq i} |a_{ij}|, \ i \in N \} \neq \emptyset \), \( N_1, N_2 \) are disjoint and such that \( N_1 \cup N_2 = N \), and

\[
(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) \geq \alpha_j \beta_i
\]  

(1.1)

for any \( i \in N_1, j \in N_2 \), where

\[
\alpha_i = \sum_{j \in N_1} |a_{ij}|, \quad \beta_i = \sum_{j \in N_2} |a_{ij}|,
\]

then:

(1) \( A \) is a SGDDM and \( M(A) \) is a nonsingular M-matrix if strict inequality in (1.1) is valid for any pair of indices or \( A \) is irreducible with strict inequality in (1.1) for at least one pair of indices.

(2) \( A \) is not a SGDDM and \( M(A) \) is not a nonsingular M-matrix if all "\( \geq \)" are changed to "\( < \)" in (1.1).

Thus we extended the main results in [1-7].

Let \( A_2 = (m_{ij}), \ i, j \in N_2 \), and \( (P_2)_i = -\sum_{j \in N_1} m_{ij} \) be the \( i \)-th component of \( P_2, i \in N_2 \). Ming-xian Peng [9] proved that if \( \Omega \neq \emptyset \) and if

\[
h_i = \frac{|a_{ii}| - \alpha_i}{\beta_i} \geq (A_2^{-1}P_2)_i = H_j
\]

(1.2)

for any \( i \in \Omega = N_1, j \in N_2 = N - \Omega \), then \( A \) is a GDDM. If strict inequality in (1.2) is valid for all \( i \in N_1, j \in N_2 \), or for any \( i \in J \subseteq \Omega \), one has

\[
h_i = \max_{j \in N_2} H_j,
\]

and for any \( i \in (N - \Omega) \cup J \) there is \( a_{i1}a_{i1i} \cdots a_{iqi} \neq 0 \), where \( q \in \Omega - J \), then \( A \) is a SGDDM. Peng extended the partial results in [1-7].

In this paper we prove that if \( \Omega \neq \emptyset \), \( N_1, N_2 \) are disjoint, and \( N_1 \cup N_2 = N \) with

\[
h_i \geq H_j
\]

(1.3)

for any \( i \in N_1, j \in N_2 \), then \( A \) is a GDDM, and we get the results:

(1) If strict inequality in (1.3) is valid for any \( i \in N_1, j \in N_2 \), or if \( J = \{ i \mid h_i > \max_{j \in N_2} H_j, \ i \in N_1 \} \neq \emptyset \) and for any \( i \in N - J \) there is \( a_{i1}a_{i1i} \cdots a_{iqi} \neq 0 \) where \( q \in J \), then \( A \) is a SGDDM.
(2) If $N_1, N_2$ are disjoint and $N_1 \cup N_2 = N$ with
\[ h_i \leq H_j \] (1.4)
for any $i \in N_1, j \in N_2$, then $A$ is not a SGDDM.

If strict inequality in (1.4) is valid for any $i \in N_1, j \in N_2$, or $J = \{i \mid h_i < \min_{j \in N_2} H_j, i \in N_1\} \neq \emptyset$ and for any $i \in N - J$ there is $a_{ii_1}a_{ii_2} \ldots a_{i,q} \neq 0$, where $q \in J$, then $A$ is not a GDDM; so we have extended the main results in [1-9].

2. THE MAIN RESULTS

**Theorem 1.** Let $\Omega = \{i \mid |a_{ii}| > \Lambda_i = \sum_{j \neq i}|a_{ij}|, i \in N\} \neq \emptyset$; $N_1, N_2$ be disjoint and $N_1 \cup N_2 = N$; and $A_2$ be a nonsingular $M$-matrix.

(1) If
\[ \frac{|a_{ii}| - \alpha_i}{\beta_i} > (A_2^{-1}P_2)_j > 0 \] (2.1)
for any $i \in N_1, j \in N_2$ [when $\beta_i = 0, i \in N, (|a_{ii}| - \alpha_i)/\beta_i = +\infty$], then $A$ is a GDDM, and $M(A)$ is an $M$-matrix.

(2) If the strict inequality is valid in (2.1) for any pair of indices or
\[ J = \left\{ i \mid \frac{|a_{ii}| - \alpha_i}{\beta_i} > \max_{j \in N_2} (A_2^{-1}P_2)_j, i \in N_1 \right\} \neq \emptyset, \] (2.2)
and for any $i \in N - J$ there is $a_{ii_1}a_{ii_2} \ldots a_{i,q} \neq 0$ where $q \in J$, then $A$ is a SGDDM, and $M(A)$ is a nonsingular $M$-matrix.

**Proof.** (1): By (2.1) we choose $d$ such that
\[ \min_{i \in N_1} \frac{|a_{ii}| - \alpha_i}{\beta_i} > d \geq \max_{j \in N_2} (A_2^{-1}P_2)_j, \]
and construct
\[ D_1 = \text{diag}(d_i; d_i = d, i \in N_2; d_i - 1, i \in N_1), \]
\[ B = AD_1 = (a_{ij}^{(1)}). \]
When \( i \in N_1 \), we have

\[
|a_{i i}^{(1)}| - \Lambda_i^{(1)} = |a_{i i}| - \alpha_i - d \beta_i \geq |a_{i i}| - \alpha_i - \frac{|a_{i i}| - \alpha_i}{\beta_i} \beta_i = 0.
\]

Let \( B_2 = A_2^d \). When \( i \in N_2 \) we have

\[
(B_2^{-1} P_2)_i = (d^{-1} A_2^{-1} P_2)_i \leq \frac{(A_2^{-1} P_2)_i}{\max_{i \in N_2} (A_2^{-1} P_2)_i} \leq 1.
\]

Let \( B_2^{-1} P_2 = X \leq e = (1, 1, \ldots, 1)^T \), and construct

\[
D_2 = \text{diag}(d_i; d_i = x_i, i \in N_2; d_i = 1, i \in N_1),
\]

\[
C = BD_2 = (a_{ij}^{(2)}).
\]

When \( i \in N_1 \), we have

\[
|a_{i i}^{(2)}| - \Lambda_i^{(2)} = |a_{i i}^{(1)}| - \alpha_i^{(1)} - x_i \beta_i^{(1)} \geq |a_{i i}^{(1)}| - \Lambda_i^{(1)} \geq 0.
\]

When \( i \in N_2 \), we have

\[
|a_{i i}^{(2)}| - \Lambda_i^{(2)} = |a_{i i}^{(2)}| - \beta_i^{(2)} - \alpha_i^{(2)} = (c_2 e)_i - \alpha_i^{(1)}
= (B_2 X)_i - \alpha_i^{(1)} = (P_2)_i - \alpha_i^{(1)} = 0.
\]

Then \( C = AD_1 D_2 - \Lambda D \) satisfies

\[
|a_{i i}^{(2)}| - \Lambda_i^{(2)} \geq 0, \quad i \in N,
\]

so \( A \) is a GDDM. From Lemma 4.1 in [10] we know \( M(A) \) is an \( M \)-matrix.

(2): From (2.2), for any \( i \in N_1, j \in N_2 \) we have

\[
\frac{|a_{i i}| - \alpha_i}{\beta_i} > (A_2^{-1} P_2)_j.
\]

We choose

\[
\min_{i \in N_1} \frac{|a_{i i}| - \alpha_i}{\beta_i} > d > \max_{j \in N_2} (A_2^{-1} P_2)_j
\]
and construct the matrices

\[ D_1 = \text{diag}(d_i|d_i = d, i \in N_2; d_i = 1, i \in N_1), \]

\[ B = AD_1 = \left( a_{ij}^{(1)} \right). \]

When \( i \in N_1 \), we have

\[ |a_{ii}^{(1)}| - \Lambda_i^{(1)} = |a_{ii}| - \alpha_i - d\beta_i > |a_{ii}| - \alpha_i - \frac{|a_{ii}| - \alpha_i}{\beta_i} \beta_i = 0. \]

Let \( B_2 = A_2 d \). When \( i \in N_2 \), we have

\[ \left( B_2^{-1}P_2 \right)_i = \left( d^{-1}A_2^{-1}P_2 \right)_i < \frac{\left( A_2^{-1}P_2 \right)_i}{\max_{i \in N_2} \left( A_2^{-1}P_2 \right)_i} \leq 1. \]

Let \( B_2^{-1}P_2 = x > 0 \), and

\[ y - B_2^{-1}(P_2 + \delta) < \sigma, \quad \delta > 0. \]

Construct

\[ D_2 = \text{diag}(d_i|d_i = y_i, i \in N_2; d_i = 1, i \in N_1), \]

\[ C = BD_2 = \left( a_{ij}^{(2)} \right). \]

When \( i \in N_1 \), we have

\[ |a_{ii}^{(2)}| - \Lambda_i^{(2)} = |a_{ii}^{(1)}| - \alpha_i^{(1)} - y_i \beta_i^{(1)} \geq |a_{ii}^{(1)}| - \Lambda_i^{(1)} > 0; \]

when \( i \in N_2 \), we have

\[ |a_{ii}^{(2)}| - \Lambda_i^{(2)} = \left( |a_{ii}^{(1)}| - \beta_i^{(1)} \right)d - \alpha_i^{(1)} = (c_2 e)_i - \alpha_i^{(1)} \]

\[ = (B_2 y)_i - \alpha_i^{(1)} > (P_2)_i - \alpha_i^{(1)} = 0. \]

Then \( C - AD_1 D_2 - AD \) satisfies

\[ |a_{ii}^{(2)}| - \Lambda_i^{(2)} > 0, \quad i \in N, \]

so \( A \) is a SGDDM. From Theorem 6.2.3 in [10] we know \( M(A) \) is a nonsingular \( M \)-matrix. \[ \blacksquare \]
When

\[ J = \left\{ i \left| \frac{|a_{ii}| - \alpha_i}{\beta_i} > \max_{i \in N_2} \left( A_2^{-1} P_2 \right)_j, i \in N_1 \right\} \right\} \neq \emptyset, \]

just as in the proof above, we can get \( C = AD_1 D_2 = (a_{ij}^{(2)}) \). It satisfies

\[ |a_{ii}^{(2)}| - \Lambda_i^{(2)} > 0, \quad i \in J, \]
\[ |a_{ii}^{(2)}| - \Lambda_i^{(2)} \geq 0, \quad i \in N - J, \]

and for any \( i \in N - J \) there is a nonzero element chain \( a_{ii} a_{i1} \ldots a_{iq} \neq 0 \), \( q \in J \), so \( C \) is a diagonally dominant matrix with a nonzero element chain. From Theorem 5 in [3] we know \( A \) is a SGDDM, and from Theorem 6.2.3 in [10] we know \( M(A) \) is a nonsingular \( M \)-matrix.

By Theorem 1 and the fact that an irreducible diagonally dominant matrix must be a diagonally dominant matrix with a nonzero element chain, we can get the following results:

**Corollary 1.** If \( \Omega \neq \emptyset, J \neq \phi \), \( A \) is an irreducible matrix, and the inequality is valid in (2.1) then \( A \) is a SGDDM, and \( M(A) \) is a nonsingular \( M \)-matrix.

**Corollary 2.** Let \( \Omega \neq \emptyset \).

1. If there are \( N_1, N_2 \), disjoint with \( N_1 \cup N_2 = N \), such that

\[ (|a_{ii}||d_i|) > a(|a_{jj}| - \beta_j) > 0, (|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) \geq \alpha_j \beta_i \]  

for any \( i \in N_1, j \in N_2 \), then \( A \) is a GDDM, and \( M(A) \) is an \( M \)-matrix.

2. If the strict inequality is valid in (2.3) for any pair of indices or \( A \) is irreducible and the strict inequality is valid in (2.3) for at least one pair of indices, then \( A \) is a SGDDDM, and \( M(A) \) is a nonsingular \( M \)-matrix.

**Proof.** (1): By (2.3), for any \( i \in N_1, j \in N_2 \),

\[ \min_{i \in N_1} \frac{|a_{ii}| - \alpha_i}{\beta_i} \geq \max_{j \in N_2} \frac{\alpha_j}{|a_{jj}| - \beta_j} \geq 0 \]
If we set $A_2 X = P_2$, $x_r = \|X\|_\infty$, then

$$X = A_2^{-1} P_2.$$  

(2.4)

By the $r$th equation in (2.4), we have

$$\sum_{j \in N_2} m_{rj} x_j = (P_2)_r = \alpha_r \geq x_r \sum_{j \in N_2} m_{rj},$$

so

$$\left( A_2^{-1} P_2 \right)_j \leq x_r \leq \frac{\alpha_r}{\sum_{j \in N_1} m_{rq}} = \frac{\alpha_r}{|a_{rr}| - \beta_r} \leq \max_{j \in N_2} \frac{\alpha_j}{|a_{jj}| - \beta_j}.$$ 

for any $i \in N_1, j \in N_2$. By Theorem 1, we know $A$ is a GDDM, and $M(A)$ is an $M$-matrix.

(2): When the strict inequality is valid in (2.3) for any pair of indices, just as in the proof of (1), we can get

$$\left( A_2^{-1} P_2 \right)_j \leq x_r \leq \max_{j \in N_2} \frac{\alpha_j}{|a_{jj}| - \beta_j} < \min_{i \in N_1} \frac{|a_{ii}| - \alpha_i}{\beta_i}.$$ 

By Theorem 1 we know $A$ is a SGDDM, and $M(A)$ is a nonsingular $M$-matrix. When $A$ is irreducible and the strict inequality is valid for at least one pair of indices, just as in the proof of (1), we can get

$$\left( A_2^{-1} P_2 \right)_j \leq x_r \leq \max_{j \in N_2} \frac{\alpha_j}{|a_{jj}| - \beta_j} \leq \min_{i \in N_1} \frac{|a_{ii}| - \alpha_i}{\beta_i},$$

and the strict inequality is valid for at least one pair of indices. By Corollary 1 we know $A$ is a SGDDM, and $M(A)$ is a nonsingular $M$-matrix.

Remark 1. Theorems 2, 3 and Corollary 3 in [9] are exactly Theorem 1 in this paper, where $N_1 = \Omega$, $N_2 = N - \Omega$. Theorems 1, 2 in [8] are exactly Corollary 2(2) in this paper, and Theorems 4, 6 in [4] and Theorem 4 in [9] are precisely Corollary 2(2) in this paper, where $N_1 = \Omega$, $N_2 = N - \Omega$. 


THEOREM 2. Let \( \Omega \neq \emptyset \).

(1) If there are \( N_1, N_2 \) disjoint with \( N_1 \cup N_2 = N \) and \( A_2 \) is a nonsingular M-matrix such that

\[
(A_2^{-1}P_2)_j > 0, \quad \frac{|a_{ii}| - \alpha_i}{\beta_i} \leq (A_2^{-1}P_2)_j
\]  

(2.5)

for any \( i \in N_1, j \in N_2 \), then \( A \) is not a SGDDM, and \( M(A) \) is not a nonsingular M-matrix.

(2) If the strict inequality is valid in (2.5) for all \( i \in N_1, j \in N_2 \) or if

\[
J = \left\{ \left( \frac{|a_{ii}| - \alpha_i}{\beta_i} < \min_{j \in N_2} (A_2^{-1}P_2)_j, i \in N_1 \right) \right\} \neq \emptyset
\]

and for any \( i \in N - J \) there is \( a_{ii}a_{i_{1,2}} \ldots a_{i_{q,q}} \neq 0, q \in J \), then \( A \) is not a GDDM, and \( M(A) \) is not an M-matrix.

Proof. (1): By (2.5), we choose \( d \) such that

\[
\max_{i \in N_1} \frac{|a_{ii}| - \alpha_i}{\beta_i} \leq d \leq \min_{j \in N_2} (A_2^{-1}P_2)_j.
\]

Construct

\[
D_1 = \text{diag}(d_i), \quad d_i = d, \quad i \in N_2; \quad d_i = 1, \quad i \in N_1,
\]

\[
B = AD_1 = \begin{pmatrix} a_{i_{1,1}}(l) \end{pmatrix}.
\]

When \( i \in N_1 \),

\[
|a_{ii}^{(1)}| - \Lambda_i^{(1)} = |a_{ii}| - \alpha_i - d\beta_i \leq (|a_{ii}| - \alpha_i) - (|a_{ii}| - \alpha_i) = 0.
\]

Let \( B_2 = A_2d \); then when \( i \in N_2 \),

\[
(B_2^{-1}P_2)_i = d^{-1}(A_2^{-1}P_2)_i \geq \frac{(A_2^{-1}P_2)_i}{\min_{i \in N_2}(A_2^{-1}P_2)_i} \geq 1.
\]
We set $B_2 \cdot x = P_2$, $x = B_2^{-1}P_2 \succ e$, and construct
\[
D_2 = \text{diag}(d_i) \mid d_i = x_i, \quad i \in N_2; \quad d_i = 1, \quad i \in N_1,
\]
\[
C = BD_2 = \left( a_{ij}^{(2)} \right).
\]

When $i \in N_1$,
\[
|a_{ii}^{(2)}| - \Lambda_i^{(2)} = |a_{ii}^{(1)}| - \alpha_i^{(1)} - x_i \beta_i^{(1)} \leq |a_{ii}^{(1)}| - \Lambda_i^{(1)} \leq 0;
\]
when $i \in N_2$,
\[
|a_{ii}^{(2)}| - \Lambda_i^{(2)} = (c_2 e)_i - (P_2)_i = (B_2 x)_i - (P_2)_i = 0.
\]

So $A$ is not a SGDDM. From Lemma 6.4.1. in [10], we know $M(A)$ is not a nonsingular $M$-matrix.

(2): If the strict inequality is valid in (2.5) for any pair of indices, we have
\[
\max_{i \in N_1} \frac{|a_{ii}| - \alpha_i}{\beta_i} < d < \min_{j \in N_2} \left( A_2^{-1}P_2 \right)_j
\]

and construct
\[
D_1 = \text{diag}(d_i) \mid d_i = d, \quad i \in N_2; \quad d_i = 1, \quad i \in N_1,
\]
\[
B = AD_1 = \left( a_{ij}^{(1)} \right).
\]

When $i \in N_1$,
\[
|a_{ii}^{(1)}| - \Lambda_i^{(1)} = |a_{ii}| - \alpha_i - d \beta_i < |a_{ii}| - \alpha_i - (|a_{ii}| - \alpha_i) = 0.
\]

We set $B_2 = A_2 d$. Then when $i \in N_2$,
\[
(B_2^{-1}P_2)_i = d^{-1}(A_2^{-1}P_2)_i > \frac{(A_2^{-1}P_2)_i}{\min_{i \in N_2}(A_2^{-1}P_2)_i} \geq 1.
\]
We set $B_2 x = P_2$, $\tilde{P} > 0$, such that $B_2 \tilde{P} > 0$, $Y = X - \tilde{P} > \epsilon$, and construct

\[
D_2 = \text{diag}(d_i \mid d_i = y_i, i \in N_2; d_i = 1, i \in N_1),
\]

\[
C = BD_2 = \left( a_{ij}^{(2)} \right).
\]

When $i \in N_1$,

\[
|a_{ii}^{(2)}| - \Lambda_i^{(2)} = |a_{ii}^{(1)}| - \alpha_i - d \beta_i < |a_{ii}^{(1)}| - \Lambda_i^{(1)} < 0.
\]

When $i \in N_2$,

\[
|a_{ii}^{(2)}| - \Lambda_i^{(2)} = (C_2 e)_i - (P_2)_i = (B_2 y)_i - (P_2)_i = (B_2 x)_i - (B_2 \tilde{P})_i < 0,
\]

so $A$ is not GDDM. From Lemma 6.4.1 in [10] we know $M(A)$ is not an $M$-matrix.

If

\[
J = \left\{ i \mid \frac{|a_{ii}| - \alpha_i}{\beta_i} < \min_{j \in N_2} \left( A_2^{-1} P_2 \right)_j, i \in N_1 \right\} \neq \emptyset,
\]

then, as in the proof in (1), we can get that $C = AD_1 D_2 = \left( a_{ij}^{(2)} \right)$ satisfies $|a_{ii}^{(2)}| < \Lambda_i^{(2)}$, $i \in J$, $|a_{ii}^{(2)}| = \Lambda_i^{(2)}$, $i \in N - J$, and for any $i \in N - J$, there is $a_{ii}a_{i1}a_{i2} \ldots a_{iq} \neq 0$, $q \in J$, and there are $M_1 = J$, $M_k = \{ i \mid |a_{ij}^{(2)}| \neq 0, j \in M_{k-1}, |a_{ij}^{(2)}| = 0, j \in M_r, r < k - 1 \}$, $k = 2, 3, \ldots, m$, such that $\bigcup_{k=1}^{m} M_k = N$. We choose

\[
\min_{i \in M_1} \frac{\Lambda_i^{(2)}}{|a_{ij}^{(2)}|} > \delta_1 > 1,
\]

\[
\min_{i \in M_k} \frac{|a_{ij}^{(2)}| - (1 - \delta_{k-1}) r_i^{(2)}}{|a_{ij}^{(2)}|} > \delta_k > 1, \quad k = 2, \ldots, m - 1, \quad \delta_m = 1,
\]

where $r_i^{(2)} = \Sigma_{j \in M_{k-1}} |a_{ij}^{(2)}|$, and construct

\[
D_3 = \text{diag}(d_i \mid d_i = \delta_k, i \in M_k, 1 \leq k \leq m),
\]

\[
G = CD_3 = \left( a_{ij}^{(3)} \right).
\]
When \( i \in M_1 \),

\[ |a_{ii}^{(3)}| - \Lambda_i^{(3)} \leq \delta_1 |a_{ii}^{(2)}| - \Lambda_i^{(2)} < \Lambda_i^{(2)} - \Lambda_i^{(2)} = 0. \]

When \( i \in M_k, k = 2, \ldots, m - 1 \),

\[ |a_{ii}^{(3)}| - \Lambda_i^{(3)} \leq |a_{ii}^{(2)}|\delta_k - (\Lambda_i^{(2)} - r_i^{(2)} + \delta_{k-1}r_i^{(2)}) < |a_{ii}^{(2)}| - \Lambda_i^{(2)} = 0. \]

When \( i \in M_m \),

\[ |a_{ii}^{(3)}| - \Lambda_i^{(3)} = |a_{ii}^{(2)}| - \delta_{m-1}\Lambda_i^{(2)} < |a_{ii}^{(2)}| - \Lambda_i^{(2)} = 0, \]

so \( G \) satisfies \( |a_{ii}^{(3)}| < \Lambda_i^{(3)}, i \in N \). Therefore \( A \) is not a GDDM, and \( M(A) \) is not an \( M \)-matrix.

By Theorem 2 and the fact that an irreducible matrix must have a strongly connected directed graph we have

**Corollary 3.** If \( \Omega \neq \phi, \Lambda \neq \phi \), \( A \) is an irreducible and the inequality is valid in (2.5), then \( A \) is not a GDDM, and \( M(A) \) is not an \( M \)-matrix.

**Corollary 4.** Let \( \Omega|\phi \),

(1) If for any \( i \in N_1, j \in N_2 \),

\[ 0 < (|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) \leq \alpha_j \beta_i, \tag{2.6} \]

then \( A \) is not a SGDDM, and \( M(A) \) is not a nonsingular \( M \)-matrix.

(2) If the strict inequality is valid in (2.6) for any pair of indices or \( A \) is irreducible and the strict inequality is valid in (2.6) for at least one pair of indices, then \( A \) is not a GDDM, and \( M(A) \) is not an \( M \)-matrix.

**Proof.** (1): By (2.6), for any \( i \in N_1, j \in N_2 \),

\[ \max_{i \in N_1} \frac{|a_{ii}| - \alpha_i}{\beta_i} \leq \min_{j \in N_2} \frac{\alpha_i}{|a_{jj}| - \beta_j}. \]
We set $A_2 X = P_2$; then $X = A_2^{-1} P_2$, $X_r = \min x_i$. Now from the $r$th equation we have

$$
(P_2)_r = \sum_{j \in N_2} m_{rj} x_j \leq x_r \sum_{j \in N_2} m_{rj}
$$

$$
(A_2^{-1} P_2)_j = x_j \geq x_r \geq \frac{(P_2)_r}{\sum_{j \in N_2} m_{rj}} = \frac{\alpha_r}{|a_{rr}| - \beta_r}.
$$

Hence for any $i \in N_1$, $j \in N_2$, we have

$$
(A_2^{-1} P_2)_j \geq \frac{|a_{ii}| - \alpha_i}{\beta_i}.
$$

By Theorem 2 we know $A$ is not a SGDDM, and $M(A)$ is not a nonsingular $M$-matrix.

(2): When the strict inequality is valid in (2.6) for any pair of indices, as in the proof in (1), we have

$$
\min_{j \in N_2} (A_2^{-1} P_2)_j = x_r \geq \frac{\alpha_r}{|a_{rr}| - \beta_r} > \max_{i \in N_1} \frac{|a_{ii}| - \alpha_i}{\beta_i}.
$$

By Theorem 2(2), we know $A$ is not a GDDM, and $M(A)$ is not an $M$-matrix.

When $A$ is an irreducible and the strict inequality is valid in (2.6) for at least one pair of indices, as in the proof in (1), we have

$$
\min_{j \in N_2} (A_2^{-1} P_2)_j = x_r \geq \frac{\alpha_r}{|a_{rr}| - \beta_r} \geq \max_{i \in N_1} \frac{|a_{ii}| - \alpha_i}{\beta_i},
$$

and the strict inequality is valid for at least one pair of indices. By Corollary 3 we know $A$ is not a GDDM, and $M(A)$ is not an $M$-matrix.

**Remark 2.** Theorem 3 in [8] is precisely Corollary 4(1) in this paper, while Theorem 8 in [4] is exactly Corollary 4(1) when $N_1 = \Omega$, $N_2 = N - \Omega$. 


3. EXAMPLE

In this part we give an example to further illustrate the generalizations. Moreover, we provide a method to choose the positively diagonal matrix $D$ which makes $AD$ a strict diagonally dominant matrix.

**Example.** Let

$$A = \begin{pmatrix}
7 & 4 & 2i & 2i \\
\frac{7}{8} & 3 & i & 1 \\
\frac{7}{4} & i & 3 & i \\
7i/4 & i & 1 & 3
\end{pmatrix}.$$ 

Obviously,

$$M(A) = \begin{pmatrix}
7 & -\frac{4}{7} & -2 & -2 \\
-\frac{7}{8} & 3 & -1 & -1 \\
-\frac{7}{4} & -1 & 3 & -1 \\
-\frac{7}{4} & -1 & -1 & 3
\end{pmatrix}.$$ 

For $N_1 = \Omega = \{1, 2\}, N_2 = N - N_1 = \{3, 4\}$ we have

$$A_2 = \begin{pmatrix}
3 & -1 \\
-1 & 3
\end{pmatrix}, \quad A_2^{-1} = \begin{pmatrix}
\frac{3}{8} & \frac{1}{8} \\
\frac{1}{8} & \frac{3}{8}
\end{pmatrix}, \quad P_2 = \begin{pmatrix}
\frac{11}{4} \\
\frac{11}{4}
\end{pmatrix},$$

$$A_2^{-1}P_2 = \begin{pmatrix}
\frac{11}{8} \\
\frac{11}{8}
\end{pmatrix},$$

and

$$\frac{|a_{11}| - \alpha_1}{\beta_1} = \frac{45}{28}, \quad \frac{|a_{22}| - \alpha_2}{\beta_2} = \frac{17}{16},$$

$$(A_2^{-1}P_2)_i = \frac{11}{8} > \frac{|a_{22}| - \alpha_2}{\beta_2} = \frac{17}{16}.$$
So $A$ satisfies neither the conditions of the main theorem in [9] nor the conditions of the main theorems in [1-8], but $A$ satisfies Theorem 1(2) of this paper, for $N_1 = \{1\}$, $N_2 = \{2, 3, 4\}$. In fact, if we choose $d = \frac{49}{32}$ and construct

$$D_1 = \text{diag}(1, \frac{49}{32}, \frac{49}{32}, \frac{49}{32}),$$

then

$$B - (a_{ij}^{(1)}) = AD_1 = \begin{pmatrix}
7 & \frac{7}{8} & 49i/16 & 49i/16 \\
7i/8 & \frac{147}{32} & 49i/32 & \frac{49}{32} \\
\frac{7}{4} & 49i/32 & \frac{147}{32} & 49i/32 \\
7i/4 & 49i/32 & \frac{49}{32} & \frac{147}{32}
\end{pmatrix},$$

$$B_2^1P_2 = A_2^1d^1P_2 = \begin{pmatrix}
\frac{6}{7} \\
1 \\
1
\end{pmatrix}.$$

Construct

$$D_2 = \text{diag}(1, \frac{6}{7}, 1, 1),$$

$$C = (a_{ij}^{(2)}) = BD_2 = \begin{pmatrix}
7 & \frac{12}{16} & 49i/16 & 49i/16 \\
7i/8 & \frac{126}{32} & 49i/32 & \frac{49}{32} \\
\frac{7}{4} & 42i/32 & \frac{147}{32} & 49i/32 \\
7i/4 & 42i/32 & \frac{49}{32} & \frac{147}{32}
\end{pmatrix}.$$

$C$ is an irreducible diagonally dominant matrix, $|a^{(2)}_{11}| = 7 > \Lambda^{(2)}_1 = \frac{110}{16}$, $|a^{(2)}_{ii}| = \Lambda^{(2)}_i$, and $a^{(2)}_{ii} \neq 0$, $i = 2, 3, 4$. Then $M_1 = \{1\}$, $M_2 = \{2, 3, 4\}$. We choose $\Lambda^{(2)}_1/|a^{(2)}_{11}| = \frac{110}{122} < \frac{111}{122} = \delta_1 < 1$. Construct

$$D_3 = \text{diag}(\frac{111}{122}, 1, 1, 1).$$
DIAGONALLY DOMINANT MATRICES

Then

\[ \tilde{A} - \left( a_{ij}^{(3)} \right) = CD - \begin{pmatrix}
\frac{111}{16} & \frac{12}{16} & 49i/16 & 49i/16 \\
111i/128 & \frac{126}{32} & 49i/32 & 49i/32 \\
\frac{111}{64} & 42i/32 & \frac{147}{32} & 49i/32 \\
111i/64 & 42i/32 & \frac{49}{32} & \frac{147}{32}
\end{pmatrix}. \]

Obviously \( \tilde{A} \) satisfies \( |a_{ii}^{(3)}| > A_i^{(3)}, i \in N \), so \( \tilde{A} \) is a strictly diagonally dominant matrix.

Clearly if we change the rows into the columns in the matrices, the corresponding results are still true.

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