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The perfect matching polytope and solid bricks ☆

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Abstract

The *perfect matching polytope* of a graph G is the convex hull of the set of incidence vectors of perfect matchings of G. Edmonds (J. Res. Nat. Bur. Standards Sect. B 69B 1965 125) showed that a vector x in $\mathbf{Q}^E$ belongs to the perfect matching polytope of G if and only if it satisfies the inequalities: (i) $x \geqslant 0$ (*non-negativity*), (ii) $x(\partial(v)) = 1$, for all $v \in V$ (*degree constraints*) and (iii) $x(\partial(S)) \geqslant 1$, for all odd subsets S of V (*odd set constraints*). In this paper, we characterize graphs whose perfect matching polytopes are determined by non-negativity and the degree constraints. We also present a proof of a recent theorem of Reed and Wakabayashi.

Keywords: Matchings; The perfect matching polytope; Bricks

1. The perfect matching polytope

For a graph G, as usual, $\mathbf{Q}^E$ denotes the set of all functions from the edge set E o G into the field of rationals. It is convenient to think of the elements of $\mathbf{Q}^E$ as vectors whose coordinates are indexed by the edges of G.

We denote the set of all perfect matchings of a graph G by $\mathcal{M}$ and, for any $M \in \mathcal{M}$, the incidence vector of M by χ^M. The *perfect matching polytope* of G, denoted by $\mathcal{P}oly(G)$, is the convex hull of $\{\chi^M : M \in \mathcal{M}\}$. In a landmark paper, Edmonds [3] gave a linear inequality description of $\mathcal{P}oly(G)$. To present Edmonds' description of $\mathcal{P}oly(G)$, we need

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the notion of a coboundary. For a subset S of the vertex set V, the *coboundary* of S is the set of all edges of G with precisely one end in S. We denote the coboundary of S by $\partial(S)$. (When S is a singleton $\{v\}$, we simply write $\partial(v)$ instead of $\partial(\{v\})$.) We shall refer to any subset of E that is the coboundary of a subset of V as a *cut* of G. For any element x of $\mathbf{Q}^E$ and any cut C of G, we shall denote $\sum_{e \in C} x(e)$ by $x(C)$.

Let M be any perfect matching of G. For any vertex v of G, there is precisely one edge of M incident with v. Thus, for any x in $\mathcal{P}oly(G)$, $x(\partial(v)) = 1$. Also, for any odd subset S of V, $|M \cap \partial(S)|$ must be odd. Thus, $x(\partial(S)) \geqslant 1$.

Theorem 1.1 (*Edmonds,* [3]). *A vector x in Q^E belongs to $\mathcal{P}oly(G)$ of a graph G if and only if it satisfies the following system of linear inequalities*:

$$\begin{array}{rcll} x & \geqslant & 0 & (non\text{-}negativity), \\ x(\partial(v)) & = & 1 \;\; for\ all\ v \in V & (degree\ constraints), \\ x(\partial(S) & \geqslant & 1 \;\; for\ all\ odd\ S \subset V & (odd\ set\ constraints). \end{array}$$

A natural question that arises is whether there are simpler descriptions of $\mathcal{P}oly(G)$ than the one given above. This is indeed the case for bipartite graphs. A vector x in Q^E is *r-regular* if $x(\partial(v)) = r$, for all $v \in V$.

Theorem 1.2. *For a bipartite graph G, a vector x in $\mathbf{Q}^E$ belongs to $\mathcal{P}oly(G)$ if and only if it is non-negative and* 1*-regular*.

There are non-bipartite graphs whose perfect matching polytopes consist precisely of non-negative 1-regular vectors. However, in general, it is not possible to describe the perfect matching polytope of a graph just by the non-negativity and the degree constraints. These observations suggest the following problem:

Problem 1.3. Characterize graphs G for which a vector in $\mathbf{Q}^E$ is in $\mathcal{P}oly(G)$ if and only if it is non-negative and 1-regular.

2. The matching polytope and solid bricks

For a graph G, we denote by $\mathcal{O}(G)$ the set of odd components of G. A set S of vertices of G is a *barrier* of G if $|\mathcal{O}(G - S)| = |S|$. Tutte's perfect matching theorem [6] states that a graph G has a perfect matching if and only if $|\mathcal{O}(G - S)| \leqslant |S|$, for every subset S of $V(G)$.

An edge of a graph is *admissible* if it lies in a perfect matching of the graph. If graph G has a perfect matching then it follows from Tutte's perfect matching theorem that an edge e is admissible if and only if there is no barrier in G containing both ends of e.

A nontrivial graph is *matching covered* if it is connected and each of its edges is admissible. It is not difficult to see that the search for an answer to Problem 1.3 may be restricted to matching covered graphs. We shall assume that the reader is familiar with the basic notions of this theory. For the terminology and notation not defined here, we refer the reader to [4,1,2].

For a cut $C := \partial(X)$ of G, we denote the graph obtained from G by shrinking the shore $\overline{X}$ to a single vertex $\overline{x}$ by $G\{X; \overline{x}\}$, or simply by $G\{X\}$ if there is no need to refer to the vertex resulting from shrinking $\overline{X}$. We refer to $G\{X\}$ and $G\{\overline{X}\}$ as the *C-contractions of G*.

A cut C of a matching covered graph G is a *separating cut* if both the C-contractions of G are also matching covered, and is *tight* if $|C \cap M| = 1$, for all $M \in \mathcal{M}$. It is easy to show that every tight cut is also separating. But, in general, not every separating cut is tight. A matching covered graph is *solid* if it has no separating cuts other than tight cuts.

A *brace* is a bipartite matching covered graph that has no non-trivial tight cuts and a *brick* is a non-bipartite matching covered graph that has no non-trivial tight cuts. By considering cut contractions with respect to non-trivial tight cuts, any matching covered G graph may be decomposed into bricks and braces. This is known as a *tight cut decomposition* of G. Lovász [4] showed that any two tight cut decompositions of a matching covered graph yield, up to isomorphism, the same list of bricks and braces (except for multiplicities of edges).

A vector $x \in Q^E$ lies in $\mathcal{P}oly(G)$ if and only if the restriction of x to each brick and brace of G lies in the corresponding perfect matching polytope. By Theorem 1.2, the perfect matching polytope of a brace consists of all non-negative 1-regular vectors. In view of this, in seeking an answer to Problem 1.3, we may restrict ourselves to bricks.

One of the basic tools we use is the following binary relation defined on the cuts of a brick. Let C and D be two cuts of a brick G. Cut *D precedes C* if $|M \cap D| \leqslant |M \cap C|$ for each perfect matching M of G. If inequality holds for at least one perfect matching then we say that *D strictly precedes C*. A cut having a certain property is *minimal* with respect to the precedence relation if no other cut having that property strictly precedes it.

Theorem 2.1. *For a brick G, $\mathcal{P}oly(G)$ consists of all non-negative 1-regular vectors if and only if G is solid.*

Proof. Firstly suppose that G is not solid. We wish to show that there is some non-negative 1-regular vector in $\mathbf{Q}^E$ that does not belong to $\mathcal{P}oly(G)$. Since G is nonsolid, it has a nontrivial separating cut C. Let M_0 be a perfect matching of G such that $|M_0 \cap C| > 1$. (Such a perfect matching must exist; otherwise C would be tight.) Also, since C is separating, for every edge e of G, there is a perfect matching M_e of G such that $|M_e \cap C| = 1$. Now let

$$x := \frac{1}{|M_0| - 1}\left(\left(\sum_{e \in M_0} \chi^{M_e}\right) - \chi^{M_0}\right).$$

Clearly the vector x is non-negative, 1-regular with $x(C) < 1$.

Conversely, suppose that G is solid. We wish to prove that every non-negative 1-regular vector in $\mathbf{Q}^E$ belongs to $\mathcal{P}oly(G)$. Assume to the contrary that there is a non-negative 1-regular vector x that does not belong to $\mathcal{P}oly(G)$. Then, by Theorem 1.1 there must exist (nontrivial) odd cuts C with $x(C) < 1$. Let $\mathcal{C}$ denote the set of all cuts C for which $x(C) < 1$ and let $D := \partial(Y)$ be a cut in $\mathcal{C}$ that is minimal with respect to the precedence relation. We shall obtain a contradiction by showing that D is a separating cut.

Consider the D-contraction $G_1 := G\{Y\}$. We wish to show that G_1 is matching covered. If it is not, then there is a subset S of $V(G_1)$ such that either (i) $\mathcal{O}(G_1 - S) > |S|$ or

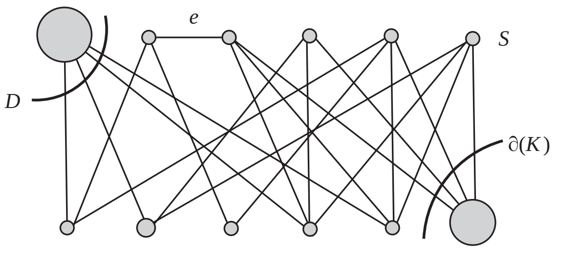

Fig. 1. Finding a separating cut using precedence

(ii) $\mathcal{O}(G_1 - S) = |S|$, but there is an edge e of G_1 with both its end in S (see Fig. 1). In either case, there must be an odd component K of $G_1 - S$ for which $x(\partial(V(K))) < 1$. Such a component is clearly non-trivial. One may verify that $\partial(V(K))$ strictly precedes D, contradicting the choice of D. Therefore G_1 is matching covered. Similarly, $G_2 := G\{\overline{S}\}$ is also matching covered and D is a separating cut. A contradiction. □

The above theorem may also be proved using the well-known characterization of the facets of the matching polytope. From this theorem, we may now deduce the following answer to Problem 1.3.

Theorem 2.2. *The perfect matching polytope $\mathcal{P}oly(G)$ of a matching covered graph G consists of non-negative and* 1*-regular vectors if and only if G is either bipartite or has at most one brick and that brick is solid.*

3. A characterization of solid bricks

If a brick G is non-solid, a separating cut of G serves as a succinct certificate for demonstrating that G is non-solid. Thus, the problem of deciding whether or not an input brick G is solid is in $co - \mathcal{NP}$. We do not know if this decision problem is in $\mathcal{NP}$. In this section, we shall present the following attractive characterization of non-solid bricks that may be helpful in constructing examples of solid bricks. This was suggested to us by Bruce Reed and Yoshiko Wakabayashi.

Theorem 3.1 (*Reed and Wakabayashi, 2003*). *A brick G has a separating cut if and only if it has two disjoint odd circuits C_1 and C_2 such that $G - (V(C_1) \cup V(C_2))$ has a perfect matching.*

The proof of the above theorem that we shall present here is based on the following result proved in [2].

Theorem 3.2. *Let G be a brick that has a separating cut. Then there exist disjoint subsets X and Y of V such that*:

1. *the graphs $G_1 := G\{X\}$ and $G_2 := G\{Y\}$ are bricks*, *and*

2. *the graph G' obtained from G by shrinking X to a vertex x and Y to a vertex y is a bipartite matching covered graph containing x and y in different parts of its bipartition.*

A graph G is *critical* if $G - v$ has a perfect matching for each $v \in V$. Clearly, any odd circuit is critical. In fact, any nontrivial critical graph G has an ear-decomposition $(G_1, G_2, \ldots, G_r)$ such that G_1 is an odd circuit and, for $2 \leqslant i \leqslant r$, G_i is obtained from G_{i-1} by adding an odd ear. We therefore have:

Lemma 3.3 (*Lovász [4]*). *Every non-trivial critical graph G contains an odd circuit C such that $G - V(C)$ has a perfect matching.*

In addition to the above lemma, we require the following lemma. We shall refer to a subset X of V such that $G[X]$ is critical as a *critical set*.

Lemma 3.4 (*Lovász and Plummer [5]*). *If we delete one vertex from each part of any bipartite matching covered graph, the resulting graph has a perfect matching.*

A graph G is *bicritical* if $G - \{x, y\}$ has a perfect matching, for any two distinct vertices x and y of G. If G is bicritical then $G - x$ is critical for any vertex x of G. It is easy to see that every brick is bicritical.

Theorem 3.5. *A brick G is non-solid if and only if there exist two disjoint non-trivial critical subsets X and Y of $V(G)$ such that $G - (X \cup Y)$ has a perfect matching.*

Proof. Firstly, suppose that G has a non-trivial separating cut. Then, by Theorem 3.2, there exist two subsets X and Y of V satisfying the conditions in the statement of that Theorem. Since $G\{X\}$ and $G\{Y\}$ are bricks, each of X and Y is non-trivial and critical. And, by Lemma 3.4, $G - (X \cup Y)$ has a perfect matching.

Conversely, suppose that G is a brick and it has disjoint non-trivial critical sets X and Y such that $G - (X \cup Y)$ has a perfect matching, M. For each vertex v of X, let $M(v)$ denote a perfect matching of $G[X] - v$. Likewise, for each vertex v of Y, let $N(v)$ denote a perfect matching of $G[Y] - v$. Let

$$x := \frac{1}{|X| - 1} \sum_{v \in X} \chi^{M(v)} + \frac{1}{|Y| - 1} \sum_{v \in Y} \chi^{N(v)} + \chi^{M}.$$

Note that x is non-negative and 1-regular. Moreover, X is odd and $x(\partial(X)) = 0$. By Theorem 1.2, $x \notin \mathcal{P}oly(G)$. By Theorem 2.1, G is not solid. $\square$

We note that underlying the above proof there is an algorithm. Given any pair of disjoint critical subsets X and Y of V such that $G - (X \cup Y)$ has a perfect matching, this algorithm can be used to find a separating cut of G.

In view of Lemma 3.3, Theorem 3.1 is a corollary of Theorem 3.5.

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