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Tensor products of recurrent hypercyclic semigroups

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ABSTRACT

We study tensor products of strongly continuous semigroups on Banach spaces that satisfy the hypercyclicity criterion, the recurrent hypercyclicity criterion or are chaotic.

1. Introduction and preliminaries

In this note we study tensor products $T(t) \otimes S(t)$ of strongly continuous semigroups $T(t)$ and $S(t)$ acting on Banach spaces X and Y. If α denotes a uniform crossnorm on the (algebraic) tensor product $X \otimes Y$ we denote by $X \tilde{\otimes}_\alpha Y$ the completion of the normed space $(X \otimes Y, \alpha)$. Our main purpose is to show that for strongly continuous semigroups $T(t)$, $S(t)$ satisfying the recurrent hypercyclicity criterion and a uniform crossnorm α on $X \otimes Y$ the semigroup $T(t) \otimes S(t)$ acting on $X \tilde{\otimes}_\alpha Y$ satisfies the recurrent hypercyclicity criterion, too. An important ingredient in the proof of this result is the work by Desch and Schappacher in [4]. Our result is of particular interest when one is working with L^p spaces of the form $L^p(M_1 \times M_2, \mu_1 \otimes \mu_2)$, $p \geqslant 1$, for measure spaces (M_i, μ_i), $i = 1, 2$, as there is a uniform crossnorm α such that $L^p(M_1 \times M_2, \mu_1 \otimes \mu_2) = L^p(M_1, \mu_1) \tilde{\otimes}_\alpha L^p(M_2, \mu_2)$, cf. [3]. Applications of our results to L^p heat semigroups on certain Riemannian manifolds are contained in [7].

Similar results for tensor products of semigroups or operators can be found in [1,9].

1.1. Hypercyclic and recurrent hypercyclic semigroups

A strongly continuous semigroup $T(t)$ on a Banach space X is called *hypercyclic* if there exists $x \in X$ such that its orbit $\{T(t)x\colon\ t \geqslant 0\}$ is dense in X.

If additionally the set of periodic points $\{x \in X\colon\ \exists t > 0 \text{ such that } T(t)x = x\}$ is dense in X, the semigroup $T(t)$ is called *chaotic*.

It is well known that a strongly continuous semigroup $T(t)$ on a separable Banach space X is hypercyclic if and only if it is *topological transitive*, i.e. for any pair of non-empty open subsets $U, V \subset X$ there exists some $t > 0$ with $T(t)U \cap V \neq \emptyset$, cf. [5].

A sufficient condition for hypercyclicity is given by the so-called *hypercyclicity criterion*, cf. [6] for this variant:

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Definition 1.1 *(Hypercyclicity criterion).* A strongly continuous semigroup $T(t)$ on a separable Banach space X satisfies the hypercyclicity criterion if for all non-empty open subsets $U, V, W \subset X$ with $0 \in W$ there exists $t > 0$ such that

$$T(t)U \cap W \neq \emptyset \quad \text{and} \quad T(t)W \cap V \neq \emptyset.$$

(Note that the same t is used in both cases.)

It should be remarked that a strongly continuous semigroup $T(t)$ on X satisfies the hypercyclicity criterion if and only if the semigroup

$$T(t) \times T(t) := \begin{pmatrix} T(t) & 0 \\ 0 & T(t) \end{pmatrix}$$

is hypercyclic on $X \times X$, cf. [6, Theorem 2.5]. This result easily generalizes (with the same proof) to:

Proposition 1.2. *Let $T(t)$ denote a strongly continuous semigroup on a Banach space X that satisfies the hypercyclicity criterion. Then the diagonal semigroup $T^n(t) := T(t) \times \cdots \times T(t)$ is hypercyclic on $X^n := X \times \cdots \times X$ for any natural number $n \geqslant 1$.*

The discrete version of the next corollary can be found in [8, Corollary 6].

Corollary 1.3. *Let $T(t)$ denote a strongly continuous semigroup on a Banach space X that satisfies the hypercyclicity criterion. Then the diagonal semigroup $T^n(t)$ satisfies the hypercyclicity criterion on X^n, $n \geqslant 1$, too.*

Proof. It follows from Proposition 1.2 that the semigroup $T^n(t) \times T^n(t) = T^{2n}(t)$ is hypercyclic, and hence, by [6, Theorem 2.5], the semigroup $T^n(t)$ satisfies the hypercyclicity criterion. $\square$

As in [4] we say that a strongly continuous semigroup satisfies the recurrent hypercyclicity criterion if for open subsets as in Definition 1.1 the set of all times $t > 0$ with $T(t)U \cap W \neq \emptyset$ and $T(t)W \cap V \neq \emptyset$ does not have arbitrarily large holes:

Definition 1.4. A strongly continuous semigroup $T(t)$ on a separable Banach space X satisfies the *recurrent hypercyclicity criterion* if for all non-empty open subsets $U, V, W \subset X$ with $0 \in W$ there exists a constant $L \geqslant 0$ such that each interval $[t, t + L]$ contains an s with

$$T(s)U \cap W \neq \emptyset \quad \text{and} \quad T(s)W \cap V \neq \emptyset.$$

Of course, any semigroup that satisfies the recurrent hypercyclicity criterion is hypercyclic as it satisfies the hypercyclicity criterion.

1.2. Tensor products

For Banach spaces X and Y we denote by $X \otimes Y$ their (algebraic) tensor product. Furthermore, let α be a *tensor norm* (or *uniform crossnorm*) on $X \otimes Y$ (for a definition see [3, 12.1] or [10, 6.1]). Then α is in particular a reasonable crossnorm on $X \otimes Y$ which implies that for $x \in X$ and $y \in Y$ we have

$$\alpha(x \otimes y) = \|x\|_X \cdot \|y\|_Y.$$

If we define for $z \in X \otimes Y$,

$$\pi(z) = \inf\left\{ \sum_{i=1}^{n} \|x_i\|_X \cdot \|y_i\|_Y\colon\ z = \sum_{i=i}^{n} x_i \otimes y_i \right\}$$

this yields a tensor norm and is called *projective norm*. Actually, this norm is the greatest reasonable crossnorm on $X \otimes Y$, i.e. if α is another reasonable crossnorm it follows $\alpha \leqslant \pi$ (cf. [2, p. 64] or [10, Proposition 6.1]). For any norm α on $X \otimes Y$ we denote by $X \tilde{\otimes}_\alpha Y$ the completion of the normed space $(X \otimes Y, \alpha)$.

For bounded operators $T : X \to X$, $S : Y \to Y$, and any uniform crossnorm α the tensor product $T \otimes S$ is a bounded operator on $(X \otimes Y, \alpha)$ by definition of a uniform crossnorm. The unique extension of $T \otimes S$ to $X \tilde{\otimes}_\alpha Y$ is, for simplicity, also denoted by $T \otimes S$.

Similarly, if $T(t) : X \to X$ and $S(t) : Y \to Y$ are strongly continuous semigroups their tensor product $T(t) \otimes S(t)$ is a strongly continuous semigroup on $(X \otimes Y, \alpha)$ for any uniform crossnorm α. To see this, let $z \in X \otimes Y$. Then we have

$$\begin{aligned} \alpha\big(T(t) \otimes S(t)z - z\big) &\leqslant \alpha\big(T(t) \otimes S(t)z - T(t) \otimes Iz\big) + \alpha\big(T(t) \otimes Iz - z\big) \\ &\leqslant \pi\big(T(t) \otimes S(t)z - T(t) \otimes Iz\big) + \pi\big(T(t) \otimes Iz - z\big). \end{aligned}$$

For the first term on the right-hand side it follows if $z=\sum_i x_i \otimes y_i$ is any representation of z,

$$\pi\big(T(t)\otimes S(t)z - T(t)\otimes Iz\big) \leqslant \sum_i \big\|T(t)x_i\big\|_X \cdot \big\|S(t)y_i - y_i\big\|_Y \to 0 \quad \big(t\to 0^+\big).$$

As an analogous argument shows that the second term goes to zero if $t\to 0^+$, it follows that $T(t)\otimes S(t)$ is strongly continuous.

2. Main results

Theorem 2.1. *Let $T(t)$, $S(t)$ denote strongly continuous semigroups on Banach spaces X and Y, respectively, and assume that $T(t)$ satisfies the recurrent hypercyclicity criterion. Furthermore, α denotes a uniform crossnorm on $X\otimes Y$.*

(a) *If $S(t)$ satisfies the hypercyclicity criterion, the semigroup $T(t)\otimes S(t)$ on $X\tilde{\otimes}_\alpha Y$ satisfies the hypercyclicity criterion, too.*
(b) *If $S(t)$ satisfies the recurrent hypercyclicity criterion, the semigroup $T(t)\otimes S(t)$ on $X\tilde{\otimes}_\alpha Y$ satisfies the recurrent hypercyclicity criterion, too.*

Proof. In the following, we use $\|(x,y)\| = \sup\{\|x\|_X, \|y\|_Y\}$ as norm on the product $X\times Y$. Note, that the topology induced by this norm coincides with the usual product topology. As α is a reasonable crossnorm, the canonical bilinear map

$$\psi : \big(X\times Y, \|\cdot\|\big) \to (X\otimes Y, \alpha), \quad (x,y)\mapsto x\otimes y$$

is continuous and has norm $\leqslant 1$ (cf. [2, p. 64]). Hence, for any $n\geqslant 1$, the mapping

$$\psi_n : \begin{cases} X^n\times Y^n \to X\otimes Y, \\ (x_1,\dots,x_n,y_1,\dots,y_n)\mapsto \sum_{k=1}^n \psi(x_k,y_k) \end{cases}$$

is continuous for the norm $\|(x_1,\dots,x_n,y_1,\dots,y_n)\| = \sup\{\|x_k\|_X, \|y_k\|_Y\colon k=1,\dots,n\}$ on $X^n\times Y^n$.

For the proof of part (a) we proceed as follows: Let U,V,W be non-empty open subsets of $X\tilde{\otimes}_\alpha Y$ with $0\in W$. As $X\otimes Y = \operatorname{span}(\psi(X\times Y))$ is dense in $X\tilde{\otimes}_\alpha Y$, we find elements

$$\sum_{k=1}^m x_k\otimes y_k \in U$$

and

$$\sum_{k=1}^n p_k\otimes q_k \in V.$$

Extending one of the sums by zero summands if necessary we may assume $m=n$. Then $\psi_n^{-1}(U)$ and $\psi_n^{-1}(V)$ are non-empty open subsets of $X^n\times Y^n$. Furthermore, $0\in\psi_n^{-1}(W)$.

As $T(t)$ satisfies the recurrent hypercyclicity criterion and $S(t)$ satisfies the hypercyclicity criterion, it follows from [4, Theorem 5.1] that the semigroup

$$\begin{pmatrix} T(t) & 0 \\ 0 & S(t)\end{pmatrix} : X\times Y\to X\times Y$$

satisfies the hypercyclicity criterion and hence, by Corollary 1.3, the semigroup $T^n(t)\times S^n(t)$ satisfies the hypercyclicity criterion, too. Therefore, there exists $t>0$ such that

$$\big(T^n(t)\times S^n(t)\psi_n^{-1}(U)\big)\cap\psi_n^{-1}(W)\neq\emptyset$$

and

$$\big(T^n(t)\times S^n(t)\psi_n^{-1}(W)\big)\cap\psi_n^{-1}(V)\neq\emptyset.$$

As $\psi_n(T^n(t)\times S^n(t)\psi_n^{-1}(U))\subset T(t)\otimes S(t)U$ and $\psi_n(\psi_n^{-1}(V))\subset V$ the proof of part (a) is complete.

For the proof of part (b) let $U,V,W\subset X\tilde{\otimes}_\alpha Y$ be non-empty open subsets with $0\in W$. As in part (a) we find $n\in\mathbb{N}$ such that the sets $\psi_n^{-1}(U)$, $\psi_n^{-1}(V)$, and $\psi_n^{-1}(W)$ are non-empty open subsets of $X^n\times Y^n$ with $0\in\psi_n^{-1}(W)$. Since both semigroups $T(t)$ and $S(t)$ satisfy the recurrent hypercyclicity criterion, it follows from [4, Corollary 5.6] that the semigroup $T^n(t)\times S^n(t)$ satisfies the recurrent hypercyclicity criterion, too. One can now conclude as in the proof of part (a) that the semigroup $T(t)\otimes S(t)$ satisfies the recurrent hypercyclicity criterion. □

Corollary 2.2. *Let $T(t)$, $S(t)$ denote chaotic semigroups on Banach spaces X and Y. If α denotes a uniform cross norm on the algebraic tensor product $X\otimes Y$ the semigroup $T(t)\otimes S(t)$ on $X\tilde{\otimes}_\alpha Y$ satisfies the recurrent hypercyclicity criterion.*

Proof. This follows directly from Theorem 2.1 since any chaotic semigroup satisfies the recurrent hypercyclicity criterion, cf. [4, Corollary 6.2]. □

In order to state the next corollary, we need some preparation. Let T denote a bounded operator on a Banach space X. T is called chaotic, if—similar to the case of semigroups—there is $x \in X$ whose orbit $\{T^n x\colon\ n \in \mathbb{N}\}$ is dense in X and if the set of periodic points $\{x \in X\colon\ \exists n \in \mathbb{N}$ such that $T^n x = x\}$ is dense in X as well.

Corollary 2.3. *Let $T(t)$, $S(t)$ denote strongly continuous semigroups on Banach spaces X and Y and α a uniform crossnorm.*

(a) *If there is $t_0 > 0$ such that $T(t_0)$ is a chaotic operator and $S(t_0)$ has a dense set of periodic points, the semigroup $T(t) \otimes S(t)$ is chaotic.*
(b) *If there are $p_1, p_2, q_1, q_2 \in \mathbb{N}$ such that $T(p_1/q_1)$ and $S(p_2/q_2)$ are chaotic the tensor product $T(t) \otimes S(t)$ is chaotic.*

Proof. We first prove (a). From [9, Corollary 1.12] it follows that the operator $T(t_0) \otimes S(t_0)$ is chaotic and hence, the semigroup $T(t) \otimes S(t)$ is chaotic.

To show (b) we first remark that both semigroups $T(t)$ and $S(t)$ are chaotic and hence their tensor product satisfies the recurrent hypercyclicity criterion. It remains to show the density of the periodic points. Lets denote by P_1 (resp. P_2) the set of periodic points of the operator $T(p_1/q_1)$ (resp. $S(p_2/q_2)$). These are dense linear spaces and hence, $P_1 \otimes P_2$ is dense in $X \tilde{\otimes}_\alpha Y$. Furthermore, if $x \otimes y \in P_1 \otimes P_2$ there exist $n, m \in \mathbb{N}$ with $T(p_1/q_1)^n x = T(np_1/q_1)x = x$, $S(p_2/q_2)^m y = S(mp_2/q_2)y = y$, and $np_1/q_1 = mp_2/q_2 =: t$. Then we have

$$T(t) \otimes S(t)(x \otimes y) = T(t)x \otimes S(t)y = x \otimes y$$

and $x \otimes y$ is therefore a periodic point of $T(t) \otimes S(t)$. Since

$$P_1 \otimes P_2 = \operatorname{span}\{x_k \otimes y_k\colon\ x_k \in P_1,\ \ y_k \in P_2\}$$

and as with $x_k \otimes y_k$, $k = 1, \ldots, n$, also $\sum_{k=1}^n x_k \otimes y_k$ is a periodic point, $P_1 \otimes P_2$ consists only of periodic points. □

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