Regularity of local times of random fields

Jiagang Ren a,∗, Xicheng Zhang b

a School of Mathematics and Computational Science, Zhongshan University, Guangzhou, Guangdong 510275, PR China
b Department of Mathematics, Huazhong University of Science and Technology, Wuhan, Hubei 430074, PR China

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Abstract

In this paper, we study the fractional smoothness of local times of general processes starting from the occupation time formula, and obtain the quasi-sure existence of local times in the sense of the Malliavin calculus. This general result is then applied to the local times of N-parameter d-dimensional Brownian motions, fractional Brownian motions and the self-intersection local time of the 2-dimensional Brownian motion, as well as smooth semimartingales.

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1. Introduction

1.1. Background

Let (X, H, μ) be an abstract Wiener space, where X is a separable Banach space, H its Cameron–Martin subspace, and μ the Wiener measure. This will be our basic probability space and its generic elements will be denoted by ω.

Let N ∈ N and set T := [0, 1]^N. Let X := {X(t) ∈ Rd: t ∈ T} be an N-parameter Rd-valued measurable random field on (X, H, μ), Λ := {Λ(t) ∈ R_+: t ∈ T} a positive N-parameter real-

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* Corresponding author.
E-mail addresses: renjg@mail.sysu.edu.cn (J. Ren), hiverzxc@hotmail.com (X. Zhang).
valued process, and \( \nu \) a finite measure on \( (T, \mathcal{B}(T)) \) where \( \mathcal{B}(T) \) is the Borel algebra on \( T \). For any \( S \in \mathcal{B}(T) \), the occupation measure on \( S \) associated to \( (X, \Lambda, \nu) \) is defined by

\[
m_S(\Gamma, \omega) := \int_S 1_{\Gamma}(X(t, \omega)) \Lambda(t, \omega) \nu(dt), \quad \Gamma \in \mathcal{B}(\mathbb{R}^d),
\]

If \( m_S(dx, \omega) \) is absolutely continuous with respect to the Lebesgue measure \( dx, \mu(dx) \)-a.s., then the Radon–Nikodym derivative \( L(S, x, \omega) := m_S(dx, \omega)/dx \) is called the local time of \( (X, \Lambda, \nu) \) on \( S \).

We say that the local time associated to \( (X, \Lambda, \nu) \) exists if for all \( S \in \mathcal{B}(T) \), \( L(S, \cdot, \omega) \) exists for \( \mu \)-almost all \( \omega \). This will be always assumed in the following. We remark that in general, \( (S, x) \mapsto L(S, x, \omega) \) is not a kernel on \( \mathcal{B}(T) \times \mathbb{R}^d \) for \( \mu \)-almost all \( \omega \). However, it is a classical result that one can find a version \( \tilde{L} \) of \( L \) such that \( (S, x) \mapsto \tilde{L}(S, x, \omega) \) is a kernel on \( \mathcal{B}(T) \times \mathbb{R}^d \) for \( \mu \)-almost all \( \omega \) (cf. [7] or see Lemma 2.6). Thereby, by a standard argument, the following occupation time formula holds: for \( \mu \)-almost all \( \omega \), all bounded Borel functions \( f \) on \( T \times \mathbb{R}^d \) and all \( S \in \mathcal{B}(T) \)

\[
\int_S f(s, X(s, \omega)) \Lambda(s, \omega) \nu(ds) = \int_{\mathbb{R}^d} \int_S f(s, x) \tilde{L}(ds, x, \omega).
\]

(1)

It is interesting and the subject of many papers to analyze the regularity of \( L(S, x, \omega) \). For example, on the one hand, the regularity of \( L(\cdot, \cdot, \omega) \) as a function of \( (S, x) \) in the usual sense (i.e., continuity) has been studied extensively (cf. [3,6,7,13,14], etc.); on the other hand, its regularity with respect to \( \omega \) in the sense of Malliavin calculus has also been studied. To explain the latter, let \( D^p_\alpha \) denote the usual Sobolev space on \( (X, \mathcal{H}, \mu) \):

\[
D^p_\alpha := (I - \mathcal{L})^{-\frac{\alpha}{2}} (L^p(X, \mu))
\]

with the norm

\[
\| f \|_{D^p_\alpha} := \| (I - \mathcal{L})^{\frac{\alpha}{2}} f \|_{L^p},
\]

where \( \mathcal{L} \) is the Ornstein–Uhlenbeck operator on \( (X, \mathcal{H}, \mu) \) (cf. [9]). For \( p > 1 \) and \( n \in \mathbb{N} \), an equivalent norm of \( D^p_\alpha \) is given by Meyer’s inequality (cf. [9]):

\[
c^{-1}_{n, p} \| f \|_{D^p_\alpha} \leq \sum_{k=0}^{n} \| \nabla^k f \|_{L^p(X; \mathcal{H}^{\otimes k})} \leq c_{n, p} \| f \|_{D^p_\alpha},
\]

where \( \nabla \) denotes the Malliavin derivative, and the constant \( c_{n, p} > 0 \) only depends on \( p \) and \( n \).

In this respect, Nualart and Vives made the first essay and they proved in [11] that the local time \( L([0, t], x) \) of the standard Brownian motion belongs to \( D^2_\alpha \) for \( \alpha < \frac{1}{2} \). Watanabe in [20] improved this result by proving that actually \( L(t, x) \in D^p_\alpha \) for all \( \alpha < \frac{1}{2} \) and \( p > 1 \). Later, Airault and ourselves in [2] extended Watanabe’s result to certain semimartingales and illustrated the sharpness of this result by looking at the Brownian motion. Following this, Hu and the first named author in [8] proved the quasi-sure existence of local times of smooth semimartingales which means that the occupation time formula (1) holds except on a \((p, r)\)-capacity zero set
provided $p > 1$ and $\alpha < 1/2$. Related works can be found in [16,19,21,22] for semimartingales and [4,5] for fractional Brownian motions. All these results are proved by using either Tanaka’s formula or the chaos expansions.

However, Tanaka’s formula with an Itô stochastic integral term holds essentially only for semimartingales, therefore can hardly be used for processes which are not semimartingales because of the lack of some nice properties such as the moment estimates (BDG inequality) for the corresponding stochastic integrals, and the chaos expansion obviously excludes any possibility of obtaining any results beyond the $L^2$-context.

To get rid of these restrictions, in the present paper we shall only start from the occupation time formula to study the regularity with respect to $\omega$. The advantage of this approach is that we do not need to deal with stochastic integrals and, therefore, the calculus is more convenient and, more importantly, is applicable to much wider classes of processes. Actually, at first, even in the semimartingale case, it turned out that the proof here is simpler and the conditions are relaxed compared with [2]. Secondly, the results in the present paper can be used to deal with local times of processes other than semimartingales, such as $N$-parameter $d$-dimensional Brownian motions, fractional Brownian motions, self-intersection local times, etc. In particular, for fractional Brownian motions we can improve the results of [4,5] from $p = 2$ to all $p > 1$. Thirdly, as such we will have a flexibility in choosing the reference measure $\nu$ which is indispensible in studying some sample path properties of the original process.

Convention. The letter $C$ below with or without subscripts will denote a positive constant, which is unimportant and may change from one line to another line.

1.2. Statements of the main results

We shall work under different sets of assumptions, the first of which is:

(CM$_\beta$) For some $\beta \in (0, 1]$ and any $p > 1$, $S \in \mathcal{B}(T)$, $R > 0$, there exists a constant $C_{p,S,R} > 0$ such that for any $x, y \in \mathbb{R}^d$ with $|x|, |y| \leq R$

$$\|L(S, x) - L(S, y)\|_{L^p} \leq C_{p,S,R} \cdot |x - y|^{\beta}.$$ 

(CD$_n$) For some $n \in \mathbb{N}$ and any $p > 1$

$$\int_T \left( \|X(s)\|_{D^p_n}^p + \|A(s)\|_{D^p_n}^p \right) \, ds < +\infty.$$ 

Now we can state our first result.

**Theorem 1.1.** Assume that (CM$_\beta$) and (CD$_n$) hold for some $\beta \in (0, 1]$ and $n \in \mathbb{N}$. Then for each $S \in \mathcal{B}(T)$ and $x \in \mathbb{R}^d$

$$L(S, x) \in D^p_\alpha, \quad \forall p > 1, \forall \alpha < n\beta / (\beta + d + n).$$

In particular, if (CD$_n$) holds for any $n \in \mathbb{N}$, then

$$L(S, x) \in D^p_\alpha, \quad \forall p > 1, \forall \alpha < \beta.$$
To state our second result on the quasi-sure existence of local times, we need to use the following notion of \((p, r)\)-capacity on the Wiener space introduced in [9].

**Definition 1.2.** Let \(p > 1, r > 0\). For an open set \(O \subset \mathbb{X}\), define
\[
\text{Cap}_{p, r}(O) := \inf \left\{ \| f \|_{D^p_r} : f \in D^p_r, \ f \geq 0 \mu\text{-a.e., } f \geq 1 \text{ on } O \right\},
\]
and for any set \(A \subset \mathbb{X}\), we let
\[
\text{Cap}_{p, r}(A) := \inf \left\{ \text{Cap}_{p, r}(O) : A \subset O \subset \mathbb{X}, \ O \text{ open} \right\}.
\]

A set \(A\) is called an \((\infty, \alpha)\)-slim set if \(\text{Cap}_{p, \alpha}(A) = 0\) for all \(p > 1\), an \((\infty, \alpha^-)\)-slim set if \(\text{Cap}_{p, r}(A) = 0\) for all \(p > 1\) and \(0 < r < \alpha\), and a \((p, \infty)\)-slim set if \(\text{Cap}_{p, r}(A) = 0\) for all \(r \in \mathbb{N}\).

**Remark 1.3.** Naturally an \((\infty, \alpha)\)-slim set is an \((\infty, \alpha^-)\)-slim set. By [8, Theorem 2.15], if \(A\) is an \((\infty, \alpha^-)\)-slim set for some \(\alpha > 0\), then it is also a \((2, \infty)\)-slim set.

**Remark 1.4.** By [9], a set of capacity null is necessarily of measure null but the inverse is false. Hence a property which holds true outside a set of capacity null can be considered finer than the one doing almost surely.

We need the following definition about the quasi-everywhere existence of local times.

**Definition 1.5.** We say that the local time \(L\) associated to \((X, \Lambda, \nu)\) exists \((\infty, \alpha^-)\)-quasi-surely, if there exists an \((\infty, \alpha^-)\)-slim set \(A\) and a version \((\tilde{L}, \tilde{X}, \tilde{\Lambda})\) of \((L, X, \Lambda)\) such that

(i) \(\tilde{X}(s, \omega) = X(s, \omega)\) and \(\tilde{\Lambda}(s, \omega) = \Lambda(s, \omega)\) for \(ds \times \mu\)-almost all \((s, \omega) \in T \times \Omega\).

(ii) For each \(S \in \mathcal{B}(T)\), \(\tilde{L}(S, x, \omega) = L(S, x, \omega)\) for \(dx \times \mu\)-almost all \((x, \omega) \in \mathbb{R}^d \times \Omega\).

(iii) For each \(\omega \in A^c\), \((S, x) \mapsto \tilde{L}(S, x, \omega)\) is a kernel on \(\mathcal{B}(T) \times \mathbb{R}^d\), and
\[
\int_T f(s, \tilde{X}(s, \omega)) \tilde{\Lambda}(s, \omega) \nu(ds) = \int_{\mathbb{R}^d} \int_T f(s, x) \tilde{L}(ds, x, \omega)
\]
holds for every positive Borel function \(f\) on \(T \times \mathbb{R}^d\).

For the quasi-everywhere existence of local times, instead of \((\text{CD}_n)\), we need the following assumption \((\text{CD}'_{n, \gamma})\):

\((\text{CD}'_{n, \gamma})\) For some \(n \in \mathbb{N}\) and any \(p > 1\),
\[
\sup_{s \in T} \left\| X(s) \right\|_{D^p_n}^p + \sup_{s \in T} \left\| \Lambda(s) \right\|_{D^p_n}^p < +\infty
\]
and for some \(\gamma \in (0, 1] \) and all \(t, s \in T\)
\[
\left\| X(t) - X(s) \right\|_{L^p} + \left\| \Lambda(t) - \Lambda(s) \right\|_{L^p} \leq C_p |t - s|^\gamma.
\]
Theorem 1.6. Suppose that \((CM_\beta)\) and \((CD_{n,\gamma}')\) hold for some \(\beta, \gamma \in (0, 1]\) and \(n \in \mathbb{N}\). Let \(\alpha := n\beta / (\beta + d + n) \wedge \gamma\). Then the local time \(L\) associated to \((X, \Lambda, \nu)\) exists \((\infty, \alpha^-)\)-quasi-surely.

This theorem does not give us the regularity of \((S, x) \mapsto L(S, x)\). In order to obtain the quasi-everywhere continuity of \((S, x) \mapsto L(S, x)\) (in a sense to be made precise below), we need to introduce another assumption which is little stronger than \((CM_\beta)\):

\((CM'_{\beta, \delta})\) For some \(\beta, \delta \in (0, 1]\) and any \(p > 1\) and \(R > 0\), there exist constants \(C_p, C_{p, R} > 0\) such that for any \(x, y \in \mathbb{R}^d\) with \(|x|, |y| \leq R\) and \(S \in \mathcal{B}(T)\),

\[
\left\| L(S, x) - L(S, y) \right\|_{L^p} \leq C_{p, R} \cdot m(S)^{\delta} \cdot |x - y|^\beta,
\]

\[
\left\| L(S, 0) \right\|_{L^p} \leq C_p \cdot m(S)^{\delta},
\]

where \(m(S)\) is the Lebesgue measure of \(S\).

Now we can state our third result.

Theorem 1.7. Assume that \((CM'_{\beta, \delta})\) and \((CD_{n,\gamma}')\) hold for some \(\beta, \delta, \gamma \in (0, 1]\) and \(n \in \mathbb{N}\). Set \(\alpha := n\beta / (\beta + d + n) \wedge \gamma\). Then, the local time \(L\) associated to \((X, \Lambda, \nu)\) exists \((\infty, \alpha^-)\)-quasi-surely. Moreover, for any \(r < \alpha\) there exists an \((\infty, r)\)-slim set \(A \subset X\) such that for each \(\omega \in A^c:\n
(i) \ (t, x) \mapsto \tilde{L}(S_t, x, \omega)\) is continuous, where \(S_t := \{s \in T: 0 < s_i \leq t_i, \ i = 1, \ldots, N\}\), and \(t = (t_1, \ldots, t_N) \in T\).

(ii) \(\tilde{L}(M_x(\omega), x, \omega) = 0\) for every \(x \in \mathbb{R}^d\), where \(M_x(\omega) := \{t \in T: \tilde{X}(t, \omega) = x\}\) is the level set at \(x\).

Remark 1.8. If \((CM'_{\beta, \delta})\) is replaced by the following condition:

\((CM''_{\beta, \delta})\) For some \(\beta, \delta \in (0, 1]\) and any \(p > 1\) and \(R > 0\), there exists constant \(C_{p, R} > 0\) such that for any \(x, y \in \mathbb{R}^d\) with \(|x|, |y| \leq R\) and \(t, t' \in T\),

\[
\left\| L(S_t, x) - L(S_{t'}, y) \right\|_{L^p} \leq C_{p, R} \cdot |t - t'|^\delta + |x - y|^\beta,
\]

then by the proof of Lemma 3.3 below, there exists an \((\infty, \alpha^-)\)-slim set \(A \subset X\) such that the conclusions of Theorem 1.7 still hold.

2. Preliminaries on capacities and deterministic occupation times

In this section we collect some results on capacities which will be needed. We refer to [9,17] for more materials.

Definition 2.1. A function \(f\) defined on \(X\) is said to be \((p, \alpha)\) (respectively \((p, \alpha^-)\))-quasi-continuous if for any \(\varepsilon > 0\), there is an open set \(O_\varepsilon\) such that

\[
\text{Cap}_{p, \alpha}(O_\varepsilon) < \varepsilon \quad \text{(respectively \ Cap}_{p, r}(O_\varepsilon) < \varepsilon \text{ for every } r < \alpha)\]
and the restriction of \( f \) on \( O^c_\delta \) is continuous. Let \( f \) be a \( \mathcal{B}(\mathbb{X}) \)-measurable function. A function \( f^* \) is called a \((p, \alpha)\) (respectively \((p, \alpha^-)\))-redefinition of \( f \) if \( f = f^* \) a.e. and \( f^* \) is \((p, \alpha)\) (respectively \((p, \alpha^-)\))-quasi-continuous.

Given a closed set \( K \), the essential part of \( K \) is defined by

\[
\text{ess}(K) := G^c \cap K,
\]

where \( G \) is the greatest open set with \( \mu(G \cap K) = 0 \).

Obviously \( x \in \text{ess}(K) \) if and only if for every \( \varepsilon > 0 \), \( \mu(B_\varepsilon(x) \cap K) \neq 0 \) where \( B_\varepsilon(x) \) is the \( \varepsilon \)-ball centered at \( x \). In particular, we have the following simple lemma.

**Lemma 2.2.** Let \( K \) be a closed set with \( K = \text{ess}(K) \), and \( A \subset K \) with \( \mu(A) = \mu(K) \). Then for every \( x \in K \) and every \( \varepsilon > 0 \), \( \mu(B_\varepsilon(x) \cap A) \neq 0 \).

The following result is taken from [9, Chapter IV, Section 2].

**Theorem 2.3.** Every function \( f \in D^p_\alpha \) admits a \((p, \alpha)\)-redefinition \( f^* \). Moreover, there exists a decreasing sequence of open sets \( \{O_n, n = 1, 2, \ldots\} \) such that:

1. \( \lim_{n \to \infty} \text{Cap}_{p, \alpha}(O_n) = 0. \)
2. \( O^c_n \) is compact for every \( n \).
3. \( \text{ess}(O^c_n) = O^c_n \) for every \( n \).
4. For every \( n \), \( f \) restricted to \( O^c_n \) is continuous.

The sets \( \{O^c_n, n = 1, 2, \ldots\} \) will be called a continuity net of \( f^* \), or a redefinition net of \( f \).

We also need the following result from [17].

**Theorem 2.4.** Let \( \xi := \{\xi(x) : x \in [0, 1]^d\} \) be a random process. For some \( p > 1 \) and \( r > 0 \), suppose that \( \xi(x) \in D^p_\alpha \) for any \( x \), and there exist constants \( \beta > 0 \) and \( \Xi > 0 \) such that

\[
\|\xi(x) - \xi(y)\|_{D^p_\alpha} \leq \Xi \cdot |x - y|^{d+\beta}.
\]

Then, \( \xi \) admits a \((p, r)\)-quasi-continuous modification \( \tilde{\xi} \) as a \( C([0, 1]^d; \mathbb{R}) \)-valued function, such that for any \( \alpha \in (0, \beta/p) \)

\[
\text{Cap}_{p, r} \left( \sup_{x \neq y \in [0, 1]^d} \frac{|\tilde{\xi}(x) - \tilde{\xi}(y)|}{|x - y|^\alpha} \right) \leq C \cdot \Xi.
\]

Here, for a function \( f \), \( \text{Cap}_{p, r}(f) \) denotes the capacity of \( f \).

**Remark 2.5.** For a set \( A \subset \mathbb{X} \), \( \text{Cap}_{p, r}^p(A) \equiv \text{Cap}_{p, r}(1_A) \), where \( 1_B \) is the indicator function of \( A \).

In the following, we prepare two lemmas related to deterministic occupation times, which are crucial for our proofs below.
Let $\mathbb{M} := \{0, 1\}^N$ denote the space of sequences of 0's and 1's. $\mathbb{M}$ is a compact metric space in its product topology. Let $\mathcal{M}_n$ be the finite $\sigma$-algebra generated by the closed sets

$$M_j = \{(a_1, a_2, \ldots) : a_j = 1\} \subset \mathbb{M}, \quad j = 1, \ldots, n.$$ 

Clearly, $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$. Set $\mathcal{M} := \bigcup_{n \in \mathbb{N}} \mathcal{M}_n$. Then $\sigma(\mathcal{M}) = B(\mathbb{M})$.

Since $T$ is a complete separable metric space under Euclidean metric, $(T, B(T))$ is isomorphic to $(\mathbb{M}, B(\mathbb{M}))$ (cf. [12, Theorem 2.12]), i.e., there exists a one-to-one and onto mapping $\phi : T \mapsto \mathbb{M}$ such that $\phi \in B(T)/B(\mathbb{M})$ and $\phi^{-1} \in B(\mathbb{M})/B(T)$. Moreover, $G := \phi^{-1}(\mathcal{M}) \subset B(T)$ is a countable algebra, and $\sigma(G) = B(T)$.

We need the following classical result. For the reader’s convenience, a standard proof is included here.

**Lemma 2.6.** Let $L : G \times \mathbb{R}^d \mapsto \mathbb{R}_+$ satisfy that for every $S \in G$, $x \mapsto L(S, x)$ is continuous. Let $X : T \mapsto \mathbb{R}^d$ be a measurable function and $\nu : T \mapsto \mathbb{R}_+$ a finite measure. Assume that for any $S \in G$ and every bounded Borel function $f$ on $\mathbb{R}^d$

$$\int_S f(X(t)) \nu(dt) = \int_{\mathbb{R}^d} f(x) L(S, x) \, dx. \quad (3)$$

Then for every $x \in \mathbb{R}^d$, $L(\cdot, x)$ can be uniquely extended to a measure on $B(T)$ (still denoted by $L(\cdot, x)$) such that:

(i) $L : B(T) \times \mathbb{R}^d \mapsto \mathbb{R}_+$ is a kernel, i.e., for each $S \in B(T)$, $x \mapsto L(S, x)$ is a real measurable function, and for each $x \in \mathbb{R}^d$, $S \mapsto L(S, x)$ is a measure on $(T, B(T))$;

(ii) for every positive Borel function $f$ on $T \times \mathbb{R}^d$

$$\int_T f(s, X(s)) \nu(ds) = \int_{\mathbb{R}^d} \int_T f(s, x) L(ds, x). \quad (4)$$

**Proof.** First of all, let us show that for every $x \in \mathbb{R}^d$, the set function $L(\cdot, x)$ is finite additive on $G$. Let $S_1, S_2 \in G$ with $S_1 \cap S_2 = \emptyset$, then by (3)

$$\int_{\mathbb{R}^d} f(x) L(S_1 \cup S_2, x) \, dx = \int_{\mathbb{R}^d} f(x) (L(S_1, x) + L(S_2, x)) \, dx.$$

Since $x \mapsto L(S_1 \cup S_2, x), L(S_1, x)$ and $L(S_2, x)$ are continuous, we have

$$L(S_1 \cup S_2, x) = L(S_1, x) + L(S_2, x), \quad \forall x \in \mathbb{R}^d.$$

Secondly, let us prove that for every $x \in \mathbb{R}^d$, $L(\cdot, x)$ is continuous from above on $G$. Let $S_n \in G \downarrow \emptyset$. If $\lim_{n \to \infty} L(S_n, x) > 0$, then for any $n \in \mathbb{N}$, $S_n \neq \emptyset$, and also $\phi(S_n) \neq \emptyset$. Since
\{\phi(S_n), n \in \mathbb{N}\} \subset \mathcal{M} is a decreasing sequence of closed sets of compact space \(\mathbb{M}\), \(\{\phi(S_n), n \in \mathbb{N}\}\) has the finite intersection property. Hence,
\[
\phi\left(\bigcap_{n \in \mathbb{N}} S_n\right) = \bigcap_{n \in \mathbb{N}} \phi(S_n) \neq \emptyset,
\]
which leads to a contradiction with \(\bigcap_{n \in \mathbb{N}} S_n = \emptyset\).

Thus, \(\mathcal{L}(\cdot, x)\) is countably additive on \(\mathcal{G}\). By Carathéodory theorem, \(\mathcal{L}(\cdot, x)\) can be uniquely extended to a measure on \(\sigma(\mathcal{G}) = \mathcal{B}(\mathbb{T})\). For \(S \in \mathcal{B}(\mathbb{T})\), the measurability of \(x \mapsto \mathcal{L}(S, x)\) follows from the monotone class theorem. Finally, using the monotone class theorem again, we may find that (4) holds for any positive Borel function \(f\) on \(\mathbb{T} \times \mathbb{R}^d\).

For \(n \in \mathbb{N}\), let \(I_n := \{i 2^{-n}, i = 0, 1, 2, 3, \ldots, 2^n\}\). Let \(D_n\) denote the set of all points in \(\mathbb{T}\) with components in \(I_n\). Let \(\mathcal{H}_n\) be the set of rectangles in \(\mathbb{T}\) with vertexes in \(D_n\) and at least one of which edges has length \(2^{-n}\). Let \(\mathcal{A}\) be the semi-ring generated by \(\mathcal{H} := \bigcup_{n \in \mathbb{N}} \mathcal{H}_n\), i.e., the total of all finite unions of sets in \(\mathcal{H}\). We remark that \(\mathcal{A}\) is countable and \(\mathcal{B}(\mathbb{T}) = \sigma(\mathcal{A})\).

The following lemma is simple.

**Lemma 2.7.** The number of elements in \(\mathcal{H}_n\) is about \(2^{an_n}\) for some \(a_N \geq N\). The mass of each element in \(\mathcal{H}_n\) is less than \(2^{-n}\).

The following lemma plays a key role in the proof of Theorem 1.7 below. Here, the proof is inspired by [7].

**Lemma 2.8.** Let \(\mathcal{L} : \mathcal{A} \times \mathbb{R}^d \mapsto \mathbb{R}_+\) satisfy that for every \(S \in \mathcal{A}\), \(x \mapsto \mathcal{L}(S, x)\) is continuous, and for every \(x \in \mathbb{R}^d\), there are \(N_x \in \mathbb{N}\) and constant \(C_x > 0\) such that for some \(\lambda > 0\) and any \(S \in \bigcup_{n > N_x} \mathcal{H}_n\)
\[
\mathcal{L}(S, x) \leq C_x \cdot \mathfrak{m}(S)^{\lambda},
\]
where \(\mathfrak{m}(S)\) denotes the Lebesgue measure of \(S\).

Let \(\chi : \mathbb{T} \mapsto \mathbb{R}^d\) and \(\nu : \mathbb{T} \mapsto \mathbb{R}_+\) be two measurable functions. Assume that for any \(S \in \mathcal{A}\) and every bounded measurable function \(f\)
\[
\int_S f(\chi(t)) \, \nu(dt) = \int_{\mathbb{R}^d} f(x) \mathcal{L}(S, x) \, dx.
\]

Then for every \(x \in \mathbb{R}^d\), \(\mathcal{L}(\cdot, x)\) can be uniquely extended to a measure on \(\mathcal{B}(\mathbb{T})\) (still denoted by \(\mathcal{L}(\cdot, x)\)) such that:

(i) \(\mathcal{L} : \mathcal{B}(\mathbb{T}) \times \mathbb{R}^d \mapsto \mathbb{R}_+\) is a kernel, and it holds that for every positive Borel function \(f\) on \(\mathbb{T} \times \mathbb{R}^d\)
\[
\int_{\mathbb{T}} f(s, \chi(s)) \, \nu(ds) = \int_{\mathbb{R}^d} \int_{\mathbb{T}} f(s, x) \mathcal{L}(ds, x).
\]
(ii) \((t, x) \mapsto \mathcal{L}(S_t, x)\) is continuous, where \(S_t\) is the rectangle with left corner 0 and right corner \(t \in \mathbb{T}\).

(iii) \(\mathcal{L}(M_x, x) = 0\) for every \(x \in \mathbb{R}^d\), where \(M_x := \{ t \in \mathbb{T} : X(t) = x \}\) is the level set at \(x\).

**Proof.** For each \(x \in \mathbb{R}^d\), the finite additivity of \(\mathcal{L}(\cdot, x)\) on \(\mathscr{A}\) can be verified as in Lemma 2.6. Let us now extend \(\mathcal{L}(\cdot, x)\) to \(\mathcal{B}(\mathbb{T}) = \sigma(\mathscr{A})\).

Set for \(i \in \{1, 2, \ldots, N\}\), \(n \in \mathbb{N}\) and \(m = 1, \ldots, 2^n\)

\[ S_{nmi} := \{ s = (s_1, \ldots, s_N) \in \mathbb{T} : (m - 1)2^{-n} < s_i \leq m2^{-n} \}. \]

Then

\[ S_{nmi} \in \mathcal{H}_n. \]

Fixing \(x \in \mathbb{R}^d\), by (5) we then have that for \(p > 1/\lambda\) and any \(n > N_x\)

\[
\sum_{i=1}^{N} \max_{m=1,\ldots,2^n} \mathcal{L}(S_{nmi}, x) \leq \sum_{i=1}^{N} \left( \sum_{m=1}^{2^n} |\mathcal{L}(S_{nmi}, x)|^p \right)^{1/p} \\
\leq C_x \sum_{i=1}^{N} \left( \sum_{m=1}^{2^n} \mathcal{L}(S_{nmi})^{p\lambda} \right)^{1/p} \\
= C_x \cdot N \cdot 2^{(1-p\lambda)n/p} \downarrow 0, \quad n \to \infty. \tag{8}
\]

For any \(t = (t_1, \ldots, t_N) \in \mathbb{T}, n \in \mathbb{N}\) and \(i \in \{1, 2, \ldots, N\}\), define

\[ S_t := \{ s \in \mathbb{T} : 0 < s_i \leq t_i, \quad i = 1, \ldots, N \}, \]

\[ \tilde{t}^n := (\tilde{t}_1^n, \ldots, \tilde{t}_N^n), \quad \tilde{t}_i^n := \min\{ j2^{-n} : j2^{-n} \geq t_i \}, \]

\[ \bar{t}_i^n := (\bar{t}_1^n, \ldots, \bar{t}_N^n), \quad \bar{t}_i^n := \max\{ j2^{-n} : j2^{-n} < t_i \}. \]

Clearly,

\[ S_{\tilde{t}^n}, S_{\bar{t}^n}, S_{\tilde{t}^n} - S_{\bar{t}^n} \in \mathscr{A}, \quad S_{\tilde{t}^n} \subset S_t \subset S_{\bar{t}^n} \]

and

\[ S_{\bar{t}^n} - S_{\tilde{t}^n} \subset \bigcup_{i=1}^{N} S_{nm_i}, \quad m_i := \bar{t}_i^n2^n. \]

Thus, by (8) we have

\[ \mathcal{L}(S_{\tilde{t}^n} - S_{\tilde{t}^n}, x) = \mathcal{L}(S_{\tilde{t}^n}, x) - \mathcal{L}(S_{\bar{t}^n}, x) \downarrow 0, \quad n \to \infty. \]

So, we may define

\[ \mathcal{L}(S_t, x) := \lim_{n \to \infty} \mathcal{L}(S_{\tilde{t}^n}, x) = \lim_{n \to \infty} \mathcal{L}(S_{\bar{t}^n}, x). \]
From this definition, it is not hard to see that
\[ t \mapsto L(S_t, x) \] is continuous.

It is a classical result that \( L(\cdot, x) \) can be uniquely extended to a measure on \( \mathcal{B}(T) \). Using the monotone class theorem, we know that \( L \) is a kernel on \( \mathcal{B}(T) \times \mathbb{R}^d \) and (7) holds for every positive measurable function \( f \).

Lastly, we show the joint continuity of \( (t, x) \mapsto L(S_t, x) \). Let \( (t^k, x^k) \to (t, x) \) in \( T \times \mathbb{R}^d \). Since \( L(S_{t_n}, x) \uparrow L(S_t, x) \) as \( n \to \infty \), for any \( \varepsilon > 0 \) there is an \( n_0 \) such that
\[ L(S_t, x) \leq L(S_{t_{n_0}}, x) + \varepsilon. \]

Noticing that \( S_{t_{n_0}} \in \mathcal{A} \) and \( x \mapsto L(S_{t_{n_0}}, x) \) is continuous, we have
\[ L(S_t, x) - \varepsilon \leq \lim_{k \to \infty} L(S_{t_{n_0}}, x^k) \leq \lim \inf_{k \to \infty} L(S_{t_k}, x^k). \]

The second inequality is due to \( S_{t_{n_0}} \subset S_{t_k} \) for sufficiently large \( k \). Hence,
\[ L(S_t, x) \leq \lim \inf_{k \to \infty} L(S_{t_k}, x^k). \]

A similar argument “from the outside” gives
\[ L(S_t, x) \geq \lim \sup_{k \to \infty} L(S_{t_k}, x^k). \]

The continuity of \( (t, x) \mapsto L(S_t, x) \) is thus obtained.

Lastly, (iii) follows from (i) and (ii) (cf. [7, Theorem 6.6]), and the proof is complete. \( \square \)

3. Proofs of the main results

We now give the proofs of the main theorems stated in Section 1. First comes the

3.1. Proof of Theorem 1.1

For the proof we need to introduce spaces \( E^p_{n, \alpha} \) defined by (see e.g. [1] for more details)
\[ E^p_{n, \alpha} := \left\{ u \in L^p : \|u\|_{E^p_{n, \alpha}} := \left( \int_0^1 \left[ \frac{\varepsilon^{-\alpha} K(\varepsilon, u)}{\varepsilon} \right]^p \frac{d\varepsilon}{\varepsilon} \right)^{1/p} < \infty \right\}, \]
where
\[ K(\varepsilon, u) := \inf_{u_1 + u_2 = u} \left\{ \|u_1\|_{L^p} + \varepsilon \|u_2\|_{D^p_n} \right\}. \]

Then we have the following well-known relations (cf. [18,19]): for any \( 1 < p < \infty \) and \( \varepsilon > 0 \),
\[ E^p_{n, \alpha + \varepsilon} \subset D^p_{n\alpha} \subset E^p_{n, \alpha - \varepsilon}. \]
Let \( B_x(r) := \{ y \in \mathbb{R}^d : |y - x| \leq r \} \) for \( r > 0 \) and \( x \in \mathbb{R}^d \). Let \( \varphi \) be a positive \( C^\infty \)-function on \( \mathbb{R}^d \), with support in \( B_0(1) \) and of integral 1. For \( \varepsilon \in (0, 1] \), define the smoothing function
\[
\varphi_\varepsilon(x) := \varepsilon^{-d} \varphi(\varepsilon^{-1} x).
\]
Set for \( S \in \mathcal{B}(T) \) and \( x \in \mathbb{R}^d \)
\[
L_\varepsilon(S, x) := \int_S \varphi_\varepsilon(X(s) - x) A(s) \, ds.
\]
Then, by the occupation time formula (1) we have
\[
L_\varepsilon(S, x) := \int_{\mathbb{R}^d} \varphi_\varepsilon(y - x) L(S, y) \, dy.
\]
Consequently, for fixed \( S \in \mathcal{B}(T) \) and \( x \in \mathbb{R}^d \)
\[
L_\varepsilon(S, x) - L(S, x) = \int_{B_x(\varepsilon)} \varphi_\varepsilon(y) \left( L(S, y) - L(S, x) \right) \, dy.
\]
Hence, by (CM\( \beta \))
\[
\| L_\varepsilon(S, x) - L(S, x) \|_{L^p} \leq \varepsilon^{-d} \int_{B_x(\varepsilon)} \| L(S, y) - L(S, x) \|_{L^p} \, dy
\]
\[
\leq C_p \cdot \varepsilon^{-d} \int_{B_x(\varepsilon)} |y - x|^\beta \, dy
\]
\[
\leq C_p \cdot \varepsilon^\beta.
\]
Let us now estimate the Malliavin derivatives of \( L(S, x) \). For \( n = 1 \), we have by the chain rule
\[
\| \nabla [\varphi_\varepsilon(X(s) - x) \cdot A(s)] \|_{\mathbb{H}}
\]
\[
= \| \varphi_\varepsilon(X(s) - x) \cdot \nabla A(s) + A(s) \sum_{i=1}^d \partial_i \varphi_\varepsilon(X(s) - x) \cdot \nabla X_i(s) \|_{\mathbb{H}}
\]
\[
\leq \varepsilon^{-d} \| \nabla A(s) \|_{\mathbb{H}} + C \varepsilon^{-d-1} \cdot \| \nabla X_s \|_{\mathbb{H}}.
\]
Hence, by (CD\( \alpha \)) and Hölder’s inequality, it is easy to see that
\[
\| L_\varepsilon(S, x) \|_{D_1^p} \leq C \varepsilon^{-d-1}.
\]
For higher order derivatives, by (CD\( n \)) we similarly have
\[ \| L^\varepsilon (S, x) \|_{D^p_n} \leq C \varepsilon^{-d-n}. \] (13)

In (12) and (13), replacing \( \varepsilon \) by \( \frac{\varepsilon^{1/(\beta + d + n)}}{\beta + d + n} \), we then have from the definition of \( K \)-function

\[ K \left( \varepsilon, L(S, x) \right) \leq C \varepsilon^{\beta/(\beta + d + n)}. \]

Thus, for any \( \alpha < \beta/(\beta + d + n) \)

\[ \| L(S, x) \|_{E_{p, \alpha}} = \int_0^1 \left[ \varepsilon^{-\alpha} K \left( \varepsilon, L(S, x) \right) \right]^p \frac{d\varepsilon}{\varepsilon} \leq C \int_0^1 \left[ \varepsilon^{-\alpha} \varepsilon^{\beta/(\beta + d + n)} \right]^p \frac{d\varepsilon}{\varepsilon} < +\infty. \]

The result now follows from (9).

Next we turn to the proof of Theorem 1.6.

3.2. Proof of Theorem 1.6

We will need the following lemma.

**Lemma 3.1.**

(i) Under \((\text{CM}_\beta)\) and \((\text{CD}_n)\), for every \( p > 1 \) and every \( 0 < r < \alpha < n\beta/(\beta + d + n) \), and for any \( S \in \mathcal{B}(T) \) and \( R > 0 \), there exists a constant \( C \) such that for all \( |x|, |y| \leq R \)

\[ \| L(S, x) - L(S, y) \|_{D^p_r} \leq C \cdot |x - y|^{\beta(1 - r/\alpha)}. \] (14)

(ii) Under \((\text{CD}'_{n, \gamma})\), for every \( p > 1 \) and every \( 0 < r < n \), there exists a constant \( C \) such that for all \( t, s \in T \)

\[ \| X(t) - X(s) \|_{D^p_r} + \| \Lambda(t) - \Lambda(s) \|_{D^p_r} \leq C \cdot |t - s|^{\gamma(1 - r/n)}. \]

**Proof.** We only prove (14), the second one is analogue. From the proof of Theorem 1.1, it is easy to see that

\[ \sup_{|x| \leq R} \| L(S, x) \|_{D^p_n} < +\infty. \]

Noting that

\[ D^p_r = [L^p_r, D^p_{\alpha, r}/\alpha], \]

we have by the interpolation theorem (cf. [18]) and \((\text{CM}_\beta)\),
\[ \| L(S, x) - L(S, y) \|_{D_p^r} \leq C \| L(S, x) - L(S, y) \|_{L_p}^{1-r/\alpha} \| L(S, x) - L(S, y) \|_{D_p^r}^{r/\alpha} \]
\[ \leq C \cdot |x - y|^{\beta(1-r/\alpha)} \]

for all $|x|, |y| \leq R$. \(\square\)

Let $C(\mathbb{R}^d, \mathbb{R})$ (also $C(\mathbb{T}, \mathbb{R}^d)$ and $C(\mathbb{T}, \mathbb{R})$) be the continuous functions space. Clearly, $C(\mathbb{T}, \mathbb{R}^d)$ and $C(\mathbb{T}, \mathbb{R})$ are separable Banach spaces, and $C(\mathbb{R}^d, \mathbb{R})$ is a Polish space under the metric

\[ \rho(f_1, f_2) := \sum_{k=1}^{\infty} 2^{-k} \left( \sup_{|x| \leq k} |f_1(x) - f_2(x)| \right) \land 1. \]

Using a suitable localization method, by Lemma 3.1 and Theorem 2.4 we can prove the following.

**Lemma 3.2.** For any $p > 1$, $r < n\beta/(\beta + d + n)$ and each $S \in \mathcal{B}(\mathbb{T})$, $L(S, \cdot)$ (respectively $X$ and $\Lambda$) as a $C(\mathbb{R}^d, \mathbb{R})$ (respectively $C(\mathbb{T}, \mathbb{R}^d)$ and $C(\mathbb{T}, \mathbb{R})$) valued random variable, admits a $(p, r)$-redefinition $\tilde{L}(S, \cdot)$ (respectively $\tilde{X}$ and $\tilde{\Lambda}$).

Next we shall follow the method in [9, 2.4.2] to prove the following.

**Lemma 3.3.** Assume that $\text{(CM}_\beta\text{)}$ and $\text{(CD}_n'\gamma\text{)}$ hold for some $\beta, \gamma \in (0, 1]$ and $n \in \mathbb{N}$. Set $\alpha := n\beta/(\beta + d + n) \land \gamma$. Then, for each $S \in \mathcal{B}(\mathbb{T})$, there exist an $(\infty, \alpha^{-})$-slim set $A \subset X$ and versions $(\tilde{L}(S, \cdot), \tilde{X}, \tilde{\Lambda})$ of $(L(S, \cdot), X, \Lambda)$ such that:

(i) $\tilde{X}(\cdot, \omega) = X(\cdot, \omega)$ and $\tilde{\Lambda}(\cdot, \omega) = \Lambda(\cdot, \omega)$ for $\mu$-almost all $\omega \in \Omega$.
(ii) $\tilde{L}(S, x, \omega) = L(S, x, \omega)$ for $dx \times \mu$-almost all $(x, \omega) \in \mathbb{R}^d \times \Omega$.
(iii) For each $\omega \in A^c$, $x \mapsto \tilde{L}(S, x)$ is continuous.
(iv) For each $\omega \in A^c$ and every positive Borel function $f$ on $\mathbb{R}^d$ and

\[ \int_S f(\tilde{X}(s, \omega)) \tilde{\Lambda}(s, \omega) \nu(ds) = \int_{\mathbb{R}^d} f(x) \tilde{L}(S, x, \omega) \nu(dx). \tag{15} \]

**Proof.** In the following proof, we shall fix $S \in \mathcal{B}(\mathbb{T})$.

First of all, since the space of bounded continuous functions is separable, there is a $\mu$-null set $G$ such that for every $\omega \in G^c$ and every bounded continuous function $f$

\[ \int_S f(X(s, \omega)) \Lambda(s, \omega) \nu(ds) = \int_{\mathbb{R}^d} f(x) L(S, x, \omega) \nu(dx). \tag{16} \]

Choose a sequence $p_n \uparrow \infty$ and $r_n \uparrow \alpha$ as $n \uparrow \infty$. By Lemma 3.2 and Theorem 2.4, for each fixed $n \in \mathbb{N}$, let $\tilde{L}_n(S, \cdot)$, $\tilde{X}_n(\cdot)$ and $\tilde{\Lambda}_n(\cdot)$ be the $(p_n, r_n)$-redefinition of $L(S, \cdot)$, $X(\cdot)$ and $\Lambda(\cdot)$, and let $\{O_{m,n}, m \in \mathbb{N}\}$ be a common continuity net for $\tilde{L}_n(S, \cdot)$, $\tilde{X}_n(\cdot)$ and $\tilde{\Lambda}_n(\cdot)$. By choosing $m_n$ sufficiently large we will have

\[ \text{Cap}_{p_n, r_n}(O_{m_n, n}) \leq 2^{-n}. \]
Consider

\[ A_n := \bigcup_{k \geq n} O_{m_k,k}, \]

then

\[ \operatorname{Cap}_{p_n,r_n}(A_n) \leq 2^{-n+1}. \]

Define

\[ Q_n := (\operatorname{ess}(A_n^c))^c, \]

then \( Q_n \) is a decreasing sequence of open sets. By [9, p. 95, 2.1.3] we have

\[ \operatorname{Cap}_{p_n,r_n}(Q_n) = \operatorname{Cap}_{p_n,r_n}(A_n). \]

Denote

\[ A := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} Q_n. \]

By Borel–Cantelli lemma for capacity (cf. [9, 1.2.4]), \( A \) is an \((\infty, \alpha^-)\)-slim set. Furthermore,

\[ Q_n^c \subset A_n^c \subset O_{m_k,n}^c, \quad \text{for } k \geq n. \]

Note that \( \operatorname{ess}(Q_n^c) = Q_n^c \) and for \( \mu \)-almost all \( \omega \) and \( k \geq n \)

\[ \tilde{L}_n(S, \cdot, \omega) = \tilde{L}_{n+k}(S, \cdot, \omega), \]

\[ \tilde{X}_n(\cdot, \omega) = \tilde{X}_{n+k}(\cdot, \omega), \]

\[ \tilde{A}_n(\cdot, \omega) = \tilde{A}_{n+k}(\cdot, \omega), \]

and for every bounded continuous function \( f \) on \( \mathbb{R}^d \) (see (16))

\[ \int_S f(\tilde{X}_n(s, \omega)) \tilde{A}_n(s, \omega) \nu(ds) = \int_{\mathbb{R}^d} f(x) \tilde{L}_n(S, x, \omega) dx. \]

By Lemma 2.2 and the dominated convergence theorem, the above identities hold for every \( \omega \in Q_n^c \).

Hence, we may define for \( \omega \in A^c \)

\[ \tilde{L}(S, \cdot, \omega) := \lim_{n \to \infty} \tilde{L}_n(S, \cdot, \omega), \]

\[ \tilde{X}(\cdot, \omega) := \lim_{n \to \infty} \tilde{X}_n(\cdot, \omega), \]

\[ \tilde{A}(\cdot, \omega) := \lim_{n \to \infty} \tilde{A}_n(\cdot, \omega), \]
and if $\omega \in A$, let them be zero. Then $\tilde{L}$, $\tilde{X}$ and $\tilde{A}$ are the desired versions, since it is plain that (i)–(iii) hold, and (iv) also holds by the monotone class theorem.

Now Theorem 1.6 follows from Lemmas 3.3 and 2.6.

3.3. Proof of Theorem 1.7

The following lemma is similar to Lemma 3.1. The proof is omitted.

**Lemma 3.4.** Under $(CM_{\beta,\delta}')$ and $(CD_n)$, for every $p > 1$ and every $0 < r < \alpha < n\beta/(\beta + d + n)$, and for any $S \in \mathcal{B}(T)$ and $R > 0$, there exists a constant $C$ such that for all $|x|, |y| \leq R$

$$\|L(S, x) - L(S, y)\|_{D^p_R} \leq C \cdot m(S)^{\beta(1-r/\alpha)},$$

$$\|L(S, 0)\|_{D^p_R} \leq C \cdot m(S)^{\beta(1-r/\alpha)}.$$

Now we are in a position to give the proof of Theorem 1.7.

**Proof of Theorem 1.7.** Let $r < n\beta/(\beta + d + n)$ and $\gamma =: \alpha$ be fixed. Choose $\rho \in (r, \alpha)$ and $\lambda \in (0, \delta(1-r/\rho))$. We also pick up $p_0 > 1$ sufficiently large such that

$$p_0 > \left[\frac{d}{(\beta(1-r/\rho))}\right] \vee \left[\frac{a_N}{(\delta(1-r/\rho) - \lambda)}\right],$$

where $a_N$ is from Lemma 2.7, and select

$$\vartheta \in (0, \beta(1-r/\rho) - d/p_0).$$

For every $S \in \mathcal{B}(T)$, let $\tilde{L}(S, \cdot)$ be the version of $L(S, \cdot)$ given in Lemma 3.3. By (17) and Theorem 2.4 we have for each $p \geq p_0$ and $R > 0$

$$\text{Cap}_{p, r} \left( \sup_{x \neq y \in B_0(R)} \frac{|\tilde{L}(S, x) - \tilde{L}(S, y)|}{|x - y|^{\vartheta}} \right) \leq C \cdot m(S)^{p\beta(1-r/\rho)},$$

where $C$ is independent of $S$, but depends on $R$.

Let $S_1, S_2, \ldots$ be the sets taken in turn from $\mathcal{H}_1, \mathcal{H}_2, \ldots$, where $\mathcal{H}_i$ is defined before Lemma 2.7. Set

$$K^R_m := \left\{ \omega: \sup_{x \neq y \in B_0(R)} \frac{|\tilde{L}(S_m, x, \omega) - \tilde{L}(S_m, y, \omega)|}{|x - y|^{\vartheta}} \geq m(S_m)^{\lambda} \right\}$$

and

$$K^R := \bigcap_{k=1}^\infty \bigcup_{m=k}^\infty K^R_m.$$

Let us show that $K^R$ is an $(\infty, r)$-slim set for each $R \in \mathbb{N}$. For any $p > p_0$, we then have by Chebyshev's inequality for capacity (cf. [17, (3.5)]) and (19)
\[
\text{Cap}_{p,r}^p(K^R) \leq \lim_{k \to \infty} \text{Cap}_{p,r}^p\left( \bigcup_{m=k}^{\infty} K^R_m \right) \leq \lim_{k \to \infty} \sum_{m=k}^{\infty} \text{Cap}_{p,r}^p(K^R_m) \\
\leq \lim_{k \to \infty} \sum_{m=k}^{\infty} \left[ m(S_m)^{p-\lambda} \text{Cap}_{p,r}^p \left( \sup_{x \neq y \in B_0(R)} \frac{|L(S_m, x) - L(S_m, y)|}{|x - y|^{\rho}} \right) \right] \\
\leq C \lim_{k \to \infty} \sum_{m=k}^{\infty} m(S_m)^{p-\lambda} m(S_m)^{p\delta(1-r/\rho)} \\
\leq C \lim_{k \to \infty} \sum_{m=k}^{\infty} 2^{a \cdot n \cdot m} \cdot 2^{-mp(\delta(1-r/\rho)-\lambda)} = 0,
\]
where the last inequality is due to Lemma 2.7. Define
\[
\mathcal{K} := \bigcup_{R=1}^{\infty} K^R.
\]
Then \(K^R\) is an \((\infty, r)\)-slim set.

Similarly, we define
\[
G_m := \left\{ \omega : \tilde{L}(S_m, 0, \omega) \geq m(S_m)^{\lambda} \right\}.
\]
Then
\[
G := \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} G_m
\]
is also an \((\infty, r)\)-slim set.

Hence, for any \(R \in \mathbb{N}\) and each \(\omega \in K^c \cap G^c\), there exists a \(k(\omega)\) sufficiently large such that for any \(S \in \bigcup_{m>k(\omega)} \mathcal{H}_n\)
\[
\sup_{x \neq y \in B_0(R)} \frac{|\tilde{L}(S, x, \omega) - \tilde{L}(S, y, \omega)|}{|x - y|^{\rho}} \leq m(S)^{\lambda}, \quad (20)
\]
and
\[
\tilde{L}(S, 0, \omega) \leq m(S)^{\lambda}. \quad (21)
\]

The conclusions of Theorem 1.7 now follow from Lemma 2.8 and (20), (21).

4. Examples

4.1. \(N\)-parameter \(d\)-dimensional Wiener process

Now let \((\mathbb{X}, \mathbb{H}, \mu)\) be the classical Wiener space, i.e., \(\mathbb{X}\) consists of all \(\mathbb{R}^d\)-valued continuous functions on \(T\), null at axis, \(\mathbb{H}\) is its Cameron–Martin space consisting of those elements of \(\mathbb{X}\)
which are absolutely continuous and the derivatives are square integrable, and \( \mu \) the Wiener measure. Then the coordinate process, which will be denoted by \( \{W(t), \ t \in \mathbb{T}\} \), is an \( N \)-parameter \( d \)-dimensional Wiener process.

Let \( \varepsilon \in (0,1) \) be fixed, and let \( W_\varepsilon(t) := (W(t_1 + \varepsilon), \ldots, W(t_N + \varepsilon)) \). It is well known that when \( 2N - d > 0 \), the local time \( L_{W_\varepsilon}^S(x) \) of \( (W_\varepsilon, 1, dx) \) on \( S \in \mathcal{B}(\mathbb{T}) \) exists (cf. [6]), i.e.: for any bounded measurable function \( f \) on \( \mathbb{R}^d \),

\[
\int_S f(W_\varepsilon(s)) \, ds = \int_{\mathbb{R}^d} f(x) L_{W_\varepsilon}^S(x) \, dx, \quad \mu\text{-a.s.}
\]

Let \( \beta \in (0, 1 \wedge (2N - d)/2) \) and \( p > 1 \). By [6, (1.12) and (1.13)], we know that for any \( S \in \mathcal{B}(\mathbb{T}) \) and \( x, y \in \mathbb{R}^d \)

\[
\|L_{W_\varepsilon}^S(x)\|_{L^p} \leq C_p \cdot m(S)^{1-d/(2N)},
\]

\[
\|L_{W_\varepsilon}^S(x) - L_{W_\varepsilon}^S(y)\|_{L^p} \leq C_p \cdot m(S)^{1-(d+\beta)/(2N)} \cdot |x - y|^\beta.
\]

Moreover, it is clear that for any \( p > 1, n \in \mathbb{N} \) and \( s, t \in \mathbb{T} \)

\[
\|W(t)\|_{D^n_p} \leq C
\]

and

\[
\|W(t) - W(s)\|_{L^p} \leq C|t - s|^{1/2}.
\]

Hence, we have by Theorems 1.1, 1.6 and 1.7 the following.

**Theorem 4.1.** Let \( 0 < \varepsilon < 1 \) and \( \alpha := 1 \wedge (2N - d)/2 \).

(i) For any \( S \in \mathcal{B}(\mathbb{T}) \) and \( x \in \mathbb{R}^d \), \( L_{W_\varepsilon}^S(x) \in D^p \) provided that \( p > 1 \) and \( r < \alpha \).

(ii) The local time \( L_{W_\varepsilon}^S \) exists in the \((\infty, \alpha^-)\)-quasi-everywhere sense. The corresponding version is denoted by \( \tilde{L}_{W_\varepsilon}^S \).

(iii) For any \( r < \alpha \), there exists an \((\infty, r)\)-slim set \( A \) such that for each \( \omega \in A^c \),

\[
\mathbb{T} \times \mathbb{R}^d \ni (t, x) \mapsto \tilde{L}_{W_\varepsilon}^S(S_t, x, \omega) \in \mathbb{R} \text{ is continuous},
\]

\[
\tilde{L}_{W_\varepsilon}^S(M_x^\varepsilon(\omega), x, \omega) = 0,
\]

where \( S_t := \{s \in \mathbb{T}: s_i \leq t_i, \ i = 1, \ldots, N\} \) and \( M_x^\varepsilon(\omega) := \{t \in \mathbb{T}: W_\varepsilon(t) = x\} \).

**4.2. Self-intersection local time of Brownian motion on the plane**

Let \( \{W(t): t \in [0, 1]\} \) be a 2- or 3-dimensional Brownian motion. Define a random fields \( X: [0, 1]^2 \mapsto \mathbb{R}^d, \ d = 2 \) or 3 by

\[
X(t, s) := W(t) - W(s), \quad t, s \in [0, 1].
\]
Set for $\varepsilon > 0$

$$T_\varepsilon := \{(t, s) \in [0, 1]^2 : |t - s| \geq \varepsilon\}.$$ 

It is well known that for any $S \in B(T_\varepsilon)$, the local time $L^X(S, x)$ of $(X, 1, dx)$ exists, and is called the self-intersection local time of Brownian motion on the plane (cf. [15]). That is to say that for any bounded measurable function $f$ on $\mathbb{R}^d$, $d = 2$ or 3

$$\int \int_S f(W(t) - W(s)) \, dt \, ds = \int_{\mathbb{R}^d} f(x)L^X(S, x) \, dx.$$ 

Let $\gamma < 1 - 4/d$ and $\beta < 1/(d - 1)$. From [15, p. 336] we know that for any $p \in \mathbb{N}$

$$\|L^X(S, 0)\|_p \leq C \cdot m(S)\gamma,$$

$$\|L^X(S, x) - L^X(S, y)\|_p \leq C \cdot m(S)^\gamma \cdot |x - y|^{\beta}.$$ 

Moreover, it is clear that for any $p > 1$, $n \in \mathbb{N}$ and $s, t, s', t' \in [0, 1]$

$$\|X(t, s) - X(t', s')\|_{L^p} \leq C(|t - t'|^{1/2} + |s - s'|^{1/2})$$

and

$$\|X(t, s)\|_{D^p_n} < +\infty.$$ 

Hence, we have by Theorems 1.1, 1.6 and 1.7 the following result which recovers the related one in [20].

**Theorem 4.2.** Let $0 < \varepsilon < 1$ and $\alpha := 1/(d - 1)$, $d = 2$ or 3.

(i) For any $S \in B(T_\varepsilon)$ and $x \in \mathbb{R}^d$, $L^X(S, x) \in D^p_r$ provided that $p > 1$ and $r < \alpha$.

(ii) The local time $L$ exists in the $(\infty, \alpha^-)$-quasi-everywhere sense. The corresponding version is denoted by $\tilde{L}$.

(iii) For any $r < \alpha$, there exists an $(\infty, r)$-slim set $A$ such that for each $\omega \in A^c$,

$$T_\varepsilon \times \mathbb{R}^d \ni (t, x) \mapsto \tilde{L}^X(S_t, x, \omega) \in \mathbb{R}$$

is continuous,

and

$$\tilde{L}^X(M^\varepsilon(\omega), x, \omega) = 0,$$

where $S_t := \{s \in T_\varepsilon : s_i \leq t_i, \ i = 1, 2\}$ and $M^\varepsilon(\omega) := \{(t, s) \in T_\varepsilon : W_t - W_s = x\}.$
4.3. Fractional Brownian motion

Here we assume that $N = d = 1$. Let $H \in (0, 1)$, the fractional Brownian motion with Hurst parameter $H \in (0, 1)$ is defined by

$$B_H(t) := \int_0^t K_H(t, s) \, dW(s),$$

where

$$K_H(t, s) := c_H(t - s)^{H - \frac{1}{2}} + c_H \int_s^t (r - s)^{H - \frac{3}{2}} \left(1 - \frac{r}{s}\right)^{\frac{1}{2} - H} \, dr,$$

$c_H$ is a constant decided by the condition

$$\mathbb{E}[B_H(t)B_H(s)] = \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H}).$$

Obviously, $t \mapsto B_H(t)$ is a Gaussian process, and by Berman’s result (cf. [3]), its local time $L^B_H(t, x) := L^B_H([0, t], x)$ of $(B_H, 1, dx)$ exists for any $t \in (0, 1)$.

Let $\beta < 1 \wedge \left(\frac{1}{2H} - \frac{1}{2}\right)$, $0 < \delta < 1 - H(1 + 2\beta)$ and $p > 1$. Then by [3, (8.6)], we have for any $t \in [0, 1]$ and $x, y \in \mathbb{R}$

$$\|L^B_H(t, x) - L^B_H(t, y)\|_{L^p} \leq C_p \cdot t^\delta \cdot |x - y|^{\beta},$$

$$\|L^B_H(t, 0)\|_{L^p} \leq C_p \cdot t^\delta.$$  

Noticing that

$$\nabla B_H(t) = \int_0^t K_H(t, s) \, ds,$$

we have that for any $n \in \mathbb{N}$, $p > 1$ and $t \in [0, 1]$

$$\|B_H(t)\|_{D^p_n} \leq C_H.$$

Moreover, it is not hard to see that

$$\|B_H(t) - B_H(s)\|_{L^p} \leq C_H |t - s|^H.$$

Hence, we have by Theorems 1.1, 1.6 and 1.7

**Theorem 4.3.** Let $\alpha := 1 \wedge \left(\frac{1}{2H} - \frac{1}{2}\right)$.

(i) For any $S \in \mathcal{B}(\mathbf{T})$ and $x \in \mathbb{R}$, $L^B_H(S, x) \in D^p_r$ provided that $p > 1$ and $r < \alpha$. 

(ii) The local time $L^{B_H}$ exist in the $(\infty, \alpha^-)$-quasi-everywhere sense. The corresponding version is denoted by $\tilde{L}^{B_H}$.

(iii) For any $r < \alpha$, there exists an $(\infty, r)$-slim set $A$ such that for each $\omega \in A^c$,

$$[0, 1] \times \mathbb{R} \ni (t, x) \mapsto \tilde{L}^{B_H}(t, x, \omega) \in \mathbb{R} \text{ is continuous,}$$

$$\tilde{L}^{B_H}(M^c_x(\omega), x, \omega) = 0,$$

where $M_x(\omega) := \{t \in [0, 1]: B_H(t) = x\}$.

This result improves the one in [5] which claims that $L^{B_H}(t, x) \in D^2_r$ for $r < 1 \wedge (\frac{1}{2H} - \frac{1}{2})$.

4.4. Smooth semimartingale

Let $X$ be a one-dimensional smooth semimartingale in the sense of Malliavin–Nualart [10], i.e.,

$$Y(t) := Y_0 + \int_0^t M(s) \, dW(s) + \int_0^t N(s) \cdot |M(s)|^2 \, ds, \quad t \in [0, 1],$$

where $Y_0 \in \mathbb{R}$, $s \mapsto M(s), N(s)$ are measurable and adapted processes satisfying that $M(s), N(s) \in \bigcap_{p,n>1} D^p_n$ and for any $p > 1$ and $n \in \mathbb{N}$

$$\int_0^1 (\|M(s)\|_{D^p_n}^p + \|N(s)\|_{D^p_n}^p) \, ds < +\infty.$$

Let $L_Y(t, x)$ be the local time of semimartingale defined by Tanaka’s formula,

$$|Y(t) - x| = |X_0 - x| + \int_0^t \text{sgn}(Y(s) - x) M(s) \, dW(s)$$

$$+ \int_0^t \text{sgn}(Y(s) - x) N(s) |M(s)|^2 \, ds + 2L_Y(t, x),$$

which is just the local times $\hat{L}_Y([0, t], x)$ associated with $(Y, \Lambda, dx)$, where $\Lambda(t) = |M(t)|^2$.

It is not hard to check that $(CM''_{1/2, 1/2})$ and $(CD'_{n, 1/2})$ hold for any $n \in \mathbb{N}$. Finally, we have by Theorems 1.1 and 1.6, and Remark 1.8 the following.

**Theorem 4.4.** Let $\alpha = 1/2$.

(i) For any $t \in [0, 1]$ and $x \in \mathbb{R}$, $L_Y(t, x) \in D^p_r$ provided that $p > 1$ and $r < \alpha$.

(ii) The local times $L_Y^\alpha$ exist in the $(\infty, \alpha^-)$-quasi-everywhere sense. The corresponding version is denoted by $\tilde{L}_Y$. 
There exists an \((\infty, \alpha^-)\)-slim set \(A\) such that for each \(\omega \in A^c\), the mapping \((t, x) \mapsto \tilde{L}(t, x, \omega)\) is continuous and \(\tilde{L}^Y(M^x_\omega(x, \omega), x, \omega) = 0\) for every \(x \in \mathbb{R}\), where \(M^x_\omega := \{t \in [0, 1]: \tilde{Y}(t, \omega) = x\}\) is the level set at \(x\), \(\tilde{Y}\) is the version of \(Y\) in Theorem 1.7.

This result generalizes the ones in [2] and [8].

References