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The Hilbert functions of ACM sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$

Adam Van Tuyl

Department of Mathematical Sciences, Lakehead University, Thunder Bay, ON Canada P7B 5E1

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Abstract

If $\mathbb{X}$ is a set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then the associated coordinate ring $R/I_{\mathbb{X}}$ is an $\mathbb{N}^k$-graded ring. The Hilbert function of $\mathbb{X}$, defined by $H_{\mathbb{X}}(\underline{i}) := \dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{\underline{i}}$ for all $\underline{i} \in \mathbb{N}^k$, is studied. Since the ring $R/I_{\mathbb{X}}$ may or may not be Cohen–Macaulay, we consider only those $\mathbb{X}$ that are ACM. Generalizing the case of $k = 1$ to all k, we show that a function is the Hilbert function of an ACM set of points if and only if its first difference function is the Hilbert function of a multi-graded Artinian quotient. We also give a new characterization of ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$, and show how the graded Betti numbers (and hence, Hilbert function) of ACM sets of points in this space can be obtained solely through combinatorial means.

Keywords: Hilbert function; Points; Multi-projective space; Cohen–Macaulay rings; Partitions

1. Introduction

Let $R = \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$ with $\deg x_{i,j} = e_i$ be the $\mathbb{N}^k$-graded coordinate ring associated to $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. A point $P = \mathcal{P}_1 \times \cdots \times \mathcal{P}_k \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, with $\mathcal{P}_i \in \mathbb{P}^{n_i}$, corresponds to a prime $\mathbb{N}^k$-homogeneous ideal I_P of height $\sum_{i=1}^{k} n_i$ in R. Furthermore, $I_P = (L_{1,1}, \ldots, L_{1,n_1}, \ldots, L_{k,1}, \ldots, L_{k,n_k})$ where $\deg L_{i,j} = e_i$ and $(L_{i,1}, \ldots, L_{i,n_i})$ is the defining ideal of $\mathcal{P}_i \in \mathbb{P}^{n_i}$. If $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then $I_{\mathbb{X}} = \bigcap_{i=1}^{s} I_{P_i}$, where I_{P_i} corresponds to P_i, is the $\mathbb{N}^k$-homogeneous ideal of R associated to $\mathbb{X}$. The ring $R/I_{\mathbb{X}}$ inherits an $\mathbb{N}^k$-graded structure. The Hilbert function of

E-mail address: avantuyl@sleet.lakeheadu.ca.

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$\mathbb{X}$ is then defined by $H_{\mathbb{X}}(\underline{i}) = \dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{\underline{i}}$ for all $\underline{i} \in \mathbb{N}^k$. In this paper we study these Hilbert functions, thereby building upon [5,13].

Each ideal I_{P_i} is also homogeneous with respect to the standard grading, so I_{P_i} defines a linear variety of dimension $k-1$ in $\mathbb{P}^{N-1}$ where $N = \sum_{i=1}^{k}(n_i+1)$. One can therefore take the point of view that our investigation of sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is an investigation of reduced unions of linear varieties with extra hypotheses on the generators to ensure $I_{\mathbb{X}}$ is $\mathbb{N}^k$-homogeneous.

Ideally, we would like to classify those functions that arise as the Hilbert function of a set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. However, besides the case $k = 1$ which is dealt with in [3,4], such a classification continues to elude us. Though some properties of the Hilbert function are known if $k > 1$ (cf. [5,13]), even for sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ this problem remains open.

The proof of the characterization for the case $k = 1$ relies, in part, on the fact that the coordinate ring of any finite set of points in $\mathbb{P}^n$ is always Cohen–Macaulay (CM). However, if $k > 1$, we show how to construct sets of points which fail to be CM. In fact, for each integer $l \in \{1, \ldots, k\}$, we can construct a set of points with depth $R/I_{\mathbb{X}} = l$. The failure of $R/I_{\mathbb{X}}$ to be CM in general provides an obstruction to generalizing the proofs of [3,4].

We therefore restrict our investigation to sets of points that arithmetically Cohen–Macaulay (ACM). With this extra hypothesis on our set of points, we can generalize the proof for the case $k = 1$ as given in [3] to all k. In particular, we show that $H_{\mathbb{X}}$ is the Hilbert function of an ACM set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ if and only if $\Delta H_{\mathbb{X}}$, a generalized first difference function, is the Hilbert function of some $\mathbb{N}^k$-graded Artinian quotient. Our generalization relies on two main ingredients: (1) the existence of a regular sequence in $R/I_{\mathbb{X}}$ such that each element has a specific multi-degree, and (2) the techniques of [9] for lifting monomial ideals.

This characterization is not very satisfactory because it translates our original problem into the open problem of characterizing the Hilbert functions of $\mathbb{N}^k$-graded Artinian quotients. However, we characterize these quotients in the special case $n_1 = \cdots = n_k = 1$, thereby giving a complete description of the Hilbert functions of ACM sets of points in $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$.

In the last two sections we specialize to ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. It was first shown in [5] that the ACM sets of points are characterized by their Hilbert function. We give a new proof of this result, plus a new characterization that depends only upon numerical information describing the set $\mathbb{X}$. We then show that this numerical information also enables us to completely calculate the graded Betti numbers of the minimal free resolution of $I_{\mathbb{X}}$ (and thus, $H_{\mathbb{X}}$) provided $\mathbb{X}$ is ACM. This generalizes the fact that the Hilbert function and Betti numbers of a set of s points in $\mathbb{P}^1$ depend only upon s.

2. Preliminaries: multi-graded rings, Hilbert functions, points

Throughout this paper, **k** denotes an algebraically closed field of characteristic zero. In this section we provide the necessary facts and definitions about multi-graded rings, Hilbert functions, and sets of points in multi-projective spaces. See [10,13] for more on these topics.

Let $\mathbb{N} := \{0, 1, 2, \ldots\}$. For an integer $n \in \mathbb{N}$, we set $[n] := \{1, \ldots, n\}$. We denote $(i_1, \ldots, i_k) \in \mathbb{N}^k$ by $\underline{i}$. We set $|\underline{i}| := \sum_h i_h$. If $\underline{i}, \underline{j} \in \mathbb{N}^k$, then $\underline{i} + \underline{j} := (i_1 + j_1, \ldots, i_k + j_k)$. We write $\underline{i} \geqslant \underline{j}$ if $i_h \geqslant j_h$ for every $h = 1, \ldots, k$. The set $\mathbb{N}^k$ is a semi-group generated by $\{e_1, \ldots, e_k\}$ where $e_i := (0, \ldots, 1, \ldots, 0)$ is the ith standard basis vector of $\mathbb{N}^k$. If $c \in \mathbb{N}$, then $ce_i := (0, \ldots, c, \ldots, 0)$ with c in the ith position.

Set $R = \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, x_{2,0}, \ldots, x_{2,n_2}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$, and induce an $\mathbb{N}^k$-grading on R by setting $\deg x_{i,j} = e_i$. An element $x \in R$ is said to be $\mathbb{N}^k$-*homogeneous* (or simply *homogeneous* if the grading is clear) if $x \in R_{\underline{i}}$ for some $\underline{i} \in \mathbb{N}^k$. If x is homogeneous, then $\deg x := \underline{i}$. An ideal $I = (F_1, \ldots, F_r) \subseteq R$ is an $\mathbb{N}^k$-*homogeneous* (or simply, *homogeneous*) *ideal* if each F_j is $\mathbb{N}^k$-homogeneous.

For every $\underline{i} \in \mathbb{N}^k$, the set $R_{\underline{i}}$ is a finite-dimensional vector space over $\mathbf{k}$. Since a basis for $R_{\underline{i}}$ is the set of all monomials of degree $\underline{i}$,

$$\dim_{\mathbf{k}} R_{\underline{i}} = \binom{n_1 + i_1}{i_1}\binom{n_2 + i_2}{i_2} \cdots \binom{n_k + i_k}{i_k}.$$

If $I \subseteq R$ is a homogeneous ideal, then $S = R/I$ inherits an $\mathbb{N}^k$-graded ring structure if we define $S_{\underline{i}} = (R/I)_{\underline{i}} := R_{\underline{i}}/I_{\underline{i}}$. The numerical function $H_S : \mathbb{N}^k \to \mathbb{N}$ defined by $H_S(\underline{i}) := \dim_{\mathbf{k}}(R/I)_{\underline{i}} = \dim_{\mathbf{k}} R_{\underline{i}} - \dim_{\mathbf{k}} I_{\underline{i}}$ is the *Hilbert function of* S. If $H : \mathbb{N}^k \to \mathbb{N}$ is a numerical function, then the *first difference function of* H, denoted ΔH, is defined by

$$\Delta H(\underline{i}) := \sum_{\underline{0} \leqslant \underline{l} = (l_1, \ldots, l_k) \leqslant (1, \ldots, 1)} (-1)^{|\underline{l}|} H(i_1 - l_1, \ldots, i_k - l_k),$$

where $H(\underline{j}) = 0$ if $\underline{j} \ngeqslant \underline{0}$. If $k = 1$, then our definition agrees with the classical definition.

The $\mathbb{N}^k$-graded polynomial ring R is the coordinate ring of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Let

$$P = [a_{1,0} : \cdots : a_{1,n_1}] \times \cdots \times [a_{k,0} : \cdots : a_{k,n_k}] \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$$

be a point in this space. The ideal of R associated to the point P is the prime ideal $I_P = (L_{1,1}, \ldots, L_{1,n_1}, \ldots, L_{k,1}, \ldots, L_{k,n_k})$ where $\deg L_{i,j} = e_i$ for $j = 1, \ldots, n_i$. If $P_1, \ldots, P_s$ are s distinct points and $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then $I_{\mathbb{X}} = I_{P_1} \cap \cdots \cap I_{P_s}$ where I_{P_i} is the ideal associated to the point P_i. The ring $R/I_{\mathbb{X}}$ then has the following property.

Lemma 2.1 [12, Lemma 3.3]. *Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. For each integer $i \in [k]$ there exists a form $L_i \in R_{e_i}$ such that $\overline{L}_i$ is a nonzero divisor in $R/I_{\mathbb{X}}$.*

We write $H_{\mathbb{X}}$ to denote the Hilbert function $H_{R/I_{\mathbb{X}}}$, and we say $H_{\mathbb{X}}$ is the *Hilbert function* of $\mathbb{X}$. Let $\pi_i : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \to \mathbb{P}^{n_i}$ denote the ith projection morphism. Then $t_i := |\pi_i(\mathbb{X})|$ is the number of distinct ith-coordinates in $\mathbb{X}$. With this notation we have:

Proposition 2.2. *Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with Hilbert function $H_{\mathbb{X}}$.*

(i) [13, Proposition 4.2] *If* $(j_1,\dots,j_i,\dots,j_k)\in\mathbb{N}^k$ *and if* $j_i\geqslant t_i-1$, *then*

$$H_{\mathbb{X}}(j_1,\dots,j_i,\dots,j_k)=H_{\mathbb{X}}(j_1,\dots,t_i-1,\dots,j_k).$$

(ii) [13, Corollary 4.8] $H_{\mathbb{X}}(j_1,\dots,j_k)=s$ *for all* $(j_1,\dots,j_k)\geqslant(t_1-1,t_2-1,\dots,t_k-1)$.

Remark 2.3. Fix an integer $i\in[k]$, and fix $k-1$ integers in $\mathbb{N}$, say $j_1,\dots,j_{i-1},j_{i+1},\dots,j_k$. Set $\underline{j}_l:=(j_1,\dots,j_{i-1},l,j_{i+1},\dots,j_k)$ for each integer $l\in\mathbb{N}$. Then Proposition 2.2(i) can be interpreted as saying the sequence $\{H_{\mathbb{X}}(\underline{j}_l)\}$ becomes constant. In fact, $H_{\mathbb{X}}(\underline{j}_l)=H_{\mathbb{X}}(\underline{j}_{t_i-1})$ for all $l\geqslant t_i-1$.

Proposition 2.4 [13, Proposition 3.2]. *Let* $\mathbb{X}$ *be a finite set of points in* $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ *with Hilbert function* $H_{\mathbb{X}}$. *Fix an integer* $i\in[k]$. *Then the sequence* $H=\{h_j\}$, *where* $h_j:=H_{\mathbb{X}}(je_i)$, *is the Hilbert function of* $\pi_i(\mathbb{X})\subseteq\mathbb{P}^{n_i}$.

We end this section with some comments on the depth and Krull dimension of $R/I_{\mathbb{X}}$. Let $\mathbf{m}:=\bigoplus_{\underline{0}\neq\underline{j}\in\mathbb{N}^k}R_{\underline{j}}=(x_{1,0},\dots,x_{k,n_k})$ be the maximal ideal of R. If $I\subseteq R$ is an $\mathbb{N}^k$-homogeneous ideal, then recall that we say a sequence $F_1,\dots,F_r$ of elements is a *regular sequence modulo* I if and only if

(i) $(I,F_1,\dots,F_r)\subseteq\mathbf{m}$,
(ii) $\overline{F}_1$ is not a zero divisor in R/I, and
(iii) $\overline{F}_i$ is not a zero divisor in $R/(I,F_1,\dots,F_{i-1})$ for $1<i\leqslant r$.

The *depth of* R/I, written depth R/I, is the length of the maximal regular sequence modulo I.

Because each prime ideal I_{P_i} has height $\sum_{i=1}^k n_i$, it follows that K-dim $R/I_{\mathbb{X}}=k$, the number of projective spaces. This result, coupled with Lemma 2.1, implies $1\leqslant$ depth $R/I_{\mathbb{X}}\leqslant k$. Thus, every set of points in $\mathbb{P}^n$ has depth $R/I_{\mathbb{X}}=1$. If $k\geqslant 2$, the value for depth $R/I_{\mathbb{X}}$ is not immediately clear. In fact, for each integer $l\in[k]$ we can construct a set of points in $\mathbb{X}$ such that depth $R/I_{\mathbb{X}}=l$. We begin with a lemma.

Lemma 2.5. *Fix a positive integer* k. *Denote by* X_1 *and* X_2 *the two points*

$$X_1:=[1:0]\times[1:0]\times\cdots\times[1:0]\quad\textit{and}\quad X_2:=[0:1]\times[0:1]\times\cdots\times[0:1]$$

in $\mathbb{P}^1\times\cdots\times\mathbb{P}^1$ (k *times*). *If* $\mathbb{X}:=\{X_1,X_2\}$, *then* depth $R/I_{\mathbb{X}}=1$.

Proof. The defining ideal of $\mathbb{X}$ is $I_{\mathbb{X}}=I_{X_1}\cap I_{X_2}=(\{x_{a,0}x_{b,1}\mid 1\leqslant a\leqslant k,1\leqslant b\leqslant k\})$ in the $\mathbb{N}^k$-graded ring $R=\mathbf{k}[x_{1,0},x_{1,1},x_{2,0},x_{2,1},\dots,x_{k,0},x_{k,1}]$. Since $x_{1,0}+x_{1,1}$ does not vanish at either point, it suffices to show that every nonzero element of $R/(I_{\mathbb{X}},x_{1,0}+x_{1,1})$ is a zero divisor.

So, set $J=(I_{\mathbb{X}},x_{1,0}+x_{1,1})$ and suppose that $0\neq\overline{F}\in R/J$. Without loss of generality, we can take F to be $\mathbb{N}^k$-homogeneous. We write F as $F=F_0+F_1x_{1,0}+F_2x_{1,0}^2+\cdots$, where $F_i\in\mathbf{k}[x_{1,1},x_{2,0},\dots,x_{k,1}]$. Since $x_{1,0}x_{1,1}\in I_{\mathbb{X}}$, it follows that $x_{1,0}^2=x_{1,0}(x_{1,0}+$

$x_{1,1}) - x_{1,0}x_{1,1} \in J$. Hence, we can assume that $F = F_0 + F_1x_{1,0}$. The element $x_{1,0} \notin J$. Although $x_{1,0}, F \notin J$, we claim that $Fx_{1,0} \in J$. Indeed, for each integer $1 \leqslant b \leqslant k$, $x_{1,0}x_{b,1} \in I_{\mathbb{X}} \subseteq J$. Furthermore, for $1 \leqslant a \leqslant k$, the element $x_{1,0}x_{a,0} = x_{a,0}(x_{1,0} + x_{1,1}) - x_{a,0}x_{1,1} \in J$. Hence, each term of $F_0x_{1,0}$ is in J, so $F_0x_{1,0} \in J$. But then $Fx_{1,0} = F_0x_{1,0} + F_1x_{1,0}^2 \in J$. So, every $0 \neq \overline{F} \in R/J$ is a zero divisor because it is annihilated by $\bar{x}_{1,0}$. □

Proposition 2.6. *Fix a positive integer k, and let $n_1, \ldots, n_k$ be k positive integers. Then, for every integer $l \in [k]$, there exists a set of points $\mathbb{X}$ in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with* depth $R/I_{\mathbb{X}} = l$.

Proof. For every $l \in [k]$, we construct a set with the desired depth. Denote $P_i := [1:0:\cdots:0] \in \mathbb{P}^{n_i}$ for $1 \leqslant i \leqslant k$ and $Q_i := [0:1:0:\cdots:0] \in \mathbb{P}^{n_i}$ for $1 \leqslant i \leqslant k$. Fix an $l \in [k]$ and let X_1 and X_2 be the following two points of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$:

$$X_1 := P_1 \times P_2 \times \cdots \times P_k \quad \text{and} \quad X_2 := P_1 \times P_2 \times \cdots \times P_{l-1} \times Q_l \times \cdots \times Q_k.$$

If we set $\mathbb{X}_l := \{X_1, X_2\}$, we claim that depth $R/I_{\mathbb{X}_l} = l$. The defining ideal of $\mathbb{X}_l$ is

$$I_{\mathbb{X}_l} = \begin{pmatrix} x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{l-1,1}, \ldots, x_{l-1,n_{l-1}}, \\ x_{l,2}, \ldots, x_{l,n_l}, \ldots, x_{k,2}, \ldots, x_{k,n_k}, \\ \{x_{a,0}x_{b,1} \mid l \leqslant a \leqslant k,\ l \leqslant b \leqslant k\} \end{pmatrix}.$$

It follows that $R/I_{\mathbb{X}_l} \cong S/J$, where

$$S/J = \frac{\mathbf{k}[x_{1,0}, x_{2,0}, x_{3,0}, \ldots, x_{l-1,0}, x_{l,0}, x_{l,1}, x_{l+1,0}, x_{l+1,1}, \ldots, x_{k,0}, x_{k,1}]}{(\{x_{a,0}x_{b,1} \mid l \leqslant a \leqslant k, l \leqslant b \leqslant k\})}.$$

The indeterminates $x_{1,0}, x_{2,0}, \ldots, x_{l-1,0}$ give rise to a regular sequence in S/J. Thus, depth $R/I_{\mathbb{X}_l} =$ depth $S/J \geqslant l-1$. Set $K = (J, x_{1,0}, \ldots, x_{l-1,0})$. Then

$$S/K \cong \frac{\mathbf{k}[x_{l,0}, x_{l,1}, x_{l+1,0}, x_{l+1,1}, \ldots, x_{k,0}, x_{k,1}]}{(\{x_{a,0}x_{b,1} \mid l \leqslant a \leqslant k, l \leqslant b \leqslant k\})}.$$

The ring S/K is then isomorphic to the $\mathbb{N}^{k-l+1}$-graded coordinate ring of the two points $\{[1:0] \times [1:0] \times \cdots \times [1:0], [0:1] \times [0:1] \times \cdots \times [0:1]\}$ in $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ ($(k-l+1)$ times). From Lemma 2.5 we have depth $S/K = 1$, and hence, depth $R/I_{\mathbb{X}_l} = l - 1 + 1 = l$. □

3. The Hilbert functions of ACM sets of points

For an arbitrary set of points $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, depth $R/I_{\mathbb{X}} \leqslant$ K-dim $R/I_{\mathbb{X}}$. If the equality holds, the coordinate ring is Cohen–Macaulay (CM), and the set of points are said to be arithmetically Cohen–Macaulay (ACM). We now investigate the Hilbert functions of those sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ that are also ACM. Under this extra hypothesis, we can generalize the characterization of the Hilbert functions of sets of points in $\mathbb{P}^n$ as found in [3].

We begin with a preparatory lemma.

Lemma 3.1. *Let $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a finite set of points, and suppose $L_1, \ldots, L_t$, with $t \leqslant k$ and $\deg L_i = e_i$, give rise to a regular sequence in $R/I_{\mathbb{X}}$. Then there exists a positive integer l such that $(x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{t,0}, \ldots, x_{t,n_t})^l \subseteq (I_{\mathbb{X}}, L_1, \ldots, L_t)$.*

Proof. Set $J_i := (I_{\mathbb{X}}, L_1, \ldots, L_i)$ for $i = 1, \ldots, t$. Since $L_1, \ldots, L_t$ form a regular sequence on $R/I_{\mathbb{X}}$, for each $i = 1, \ldots, t$ we have a short exact sequence with degree $(0, \ldots, 0)$ maps:

$$0 \longrightarrow (R/J_{i-1})(-e_i) \xrightarrow{\times \overline{L}_i} R/J_{i-1} \longrightarrow R/J_i \longrightarrow 0,$$

where we set $J_0 := I_{\mathbb{X}}$. From the exact sequences we derive the following formula:

$$\dim_{\mathbf{k}}(R/J_t)_{\underline{i}} = \sum_{\underline{0} \leqslant (j_1, \ldots, j_t) \leqslant \underline{1}} (-1)^{(j_1 + \cdots + j_t)} \dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{i_1 - j_1, \ldots, i_t - j_t, i_{t+1}, \ldots, i_k},$$

where we set $\dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{\underline{j}} = 0$ if $\underline{j} \ngeqslant \underline{0}$.

For each integer $j = 1, \ldots, t$, set $t_j := |\pi_j(\mathbb{X})|$. By Proposition 2.2, if $i_j \geqslant t_j$, then $\dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{i_j e_j} = \dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{(i_j - 1)e_j}$. This fact, coupled with above formula, implies that $\dim_{\mathbf{k}}(R/J_t)_{t_j e_j} = \dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{t_j e_j} - \dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{(t_j - 1)e_j} = 0$. Thus $R_{t_j e_j} = (J_t)_{t_j e_j}$, or equivalently, $(x_{j,0}, \ldots, x_{j,n_j})^{t_j} \subseteq (I_{\mathbb{X}}, L_1, \ldots, L_t)$. Since this is true for each integer $1 \leqslant j \leqslant t$, there exists an integer $l \gg 0$ such that $(x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{t,0}, \ldots, x_{t,n_t})^l \subseteq J_t$, as desired. □

Proposition 3.2. *Suppose that $\mathbb{X}$ is an ACM set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Then there exist elements $\overline{L}_1, \ldots, \overline{L}_k$ in $R/I_{\mathbb{X}}$ such that $L_1, \ldots, L_k$ give rise to a regular sequence in $R/I_{\mathbb{X}}$ and $\deg L_i = e_i$.*

Proof. The existence of a regular sequence of length k follows from the definition of a CM ring. The nontrivial part of this statement is the existence of a regular sequence whose elements have specific multi-degrees.

By Lemma 2.1 there exists a form $L_1 \in R_{e_1}$ such that $\overline{L}_1$ is a nonzero divisor of $R/I_{\mathbb{X}}$. To complete the proof it is enough to show for each $t = 2, ..., k$ there exists an element $L_t \in R_{e_t}$ such that $\overline{L}_t$ is a nonzero divisor of the ring $R/(I_{\mathbb{X}}, L_1, \ldots, L_{t-1})$.

Set $J := (I_{\mathbb{X}}, L_1, \ldots, L_{t-1})$ and let $J = Q_1 \cap \cdots \cap Q_r$ be the ideal's primary decomposition. For each i set $\wp_i := \sqrt{Q_i}$. Since $\bigcup_{i=1}^{r} \overline{\wp}_i$ is the set of zero divisors of R/J, we want to show that $\bigcup_{i=1}^{r} (\wp_i)_{e_t} \subsetneq R_{e_t}$. Since R_{e_t} is a vector space over an infinite field, R_{e_t} cannot be expressed as the finite union of proper subvector spaces. Thus, it suffices to show $(\wp_i)_{e_t} \subsetneq R_{e_t}$ for each i.

So, suppose there is $i \in [r]$ such that $(\wp_i)_{e_t} = R_{e_t}$, or equivalently, $(x_{t,0}, \ldots, x_{t,n_t}) \subseteq \wp_i$. By Lemma 3.1 there exists $l \in \mathbb{N}^+$ such that $(x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{t-1,0}, \ldots, x_{t-1,n_{t-1}})^l \subseteq J \subseteq Q_i$. It follows that

$$(x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{t,0}, \ldots, x_{t,n_t}) \subseteq \wp_i.$$

Because the prime ideal $\wp_i$ also contains $I_{\mathbb{X}} = I_{P_1} \cap \cdots \cap I_{P_s}$, where I_{P_j} is the prime ideal associated to $P_j \in \mathbb{X}$, we can assume, after relabeling, $I_{P_1} \subseteq \wp_i$. Let $\wp := I_{P_1} + (x_{1,0}, \ldots, x_{t,n_t})$. Since $I_{P_1} = (L_{1,1}, \ldots, L_{1,n_1}, \ldots, L_{k,1}, \ldots, L_{k,n_k})$ where $\deg L_{m,n} = e_m$,

$$\wp = (x_{1,0}, \ldots, x_{t,n_t}, L_{t+1,1}, \ldots, L_{t+1,n_{t+1}}, \ldots, L_{k,1}, \ldots, L_{k,n_k}) \subseteq \wp_i.$$

Thus $\mathrm{ht}_R(\wp_i) \geqslant \mathrm{ht}_R(\wp) = (\sum_{i=1}^k n_i) + t$, where $\mathrm{ht}_R(I)$ denotes the height of I.

From the identity $\mathrm{ht}_R(J) = \text{K-dim}\, R - \text{K-dim}\, R/J$ we calculate the height of J:

$$\mathrm{ht}_R(J) = \left(\sum_{i=1}^k n_i + 1\right) - \big(k - (t-1)\big) = \left(\sum_{i=1}^k n_i\right) + (t-1).$$

Since $\mathbb{X}$ is ACM, R/J is CM, and hence the ideal J is height unmixed, i.e., all the associated primes of J have height equal to $\mathrm{ht}_R(J)$. But $\wp_i$ is an associated prime of J with $\mathrm{ht}_R(\wp_i) > \mathrm{ht}_R(J)$. This contradiction implies our assumption $(\wp_i)_{e_t} = R_{e_t}$ cannot be true. □

Remark 3.3. If $S = \mathbf{k}[x_0, \ldots, x_n]$ is an $\mathbb{N}^1$-graded ring with $I \subseteq S$ such that S/I is CM, then a maximal regular sequence can be chosen so that each element is homogeneous [1, Proposition 1.5.11]. However, as stated in [11] (but no example is given), it is not always possible to pick a regular sequence that respects the multi-grading. For example, let $S = \mathbf{k}[x, y]$ with $\deg x = (1, 0)$ and $\deg y = (0, 1)$ and $I = (xy)$. Then S/I is CM, but all homogeneous elements of S/I, which have the form $\overline{cx}^a$ or $\overline{cy}^b$ with $c \in \mathbf{k}$, are zero divisors. Note that $\bar{x} + \bar{y}$ is a nonzero divisor, but not homogeneous. The fact that a homogeneous regular sequence can be found in a multi-graded ring is thus a very special situation.

We extend the notion of a graded Artinian quotient in the natural way.

Definition 3.4. A homogeneous ideal I in the $\mathbb{N}^k$-graded ring R is an *Artinian ideal* if any of the following equivalent statements hold:

(i) $\text{K-dim}\, R/I = 0$.
(ii) $\sqrt{I} = \mathbf{m} = (x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k})$.
(iii) For each integer $i \in [k]$, $H_{R/I}(le_i) = 0$ for all $l \gg 0$.

A ring $S = R/I$ is an $\mathbb{N}^k$*-graded Artinian quotient* if the homogeneous ideal I is an Artinian.

Corollary 3.5. *Let $\mathbb{X}$ be an ACM set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with Hilbert function $H_{\mathbb{X}}$. Then*

$$\Delta H_{\mathbb{X}}(i_1, \ldots, i_k) := \sum_{\underline{0} \leqslant \underline{l} = (l_1, \ldots, l_k) \leqslant (1, \ldots, 1)} (-1)^{|\underline{l}|} H_{\mathbb{X}}(i_1 - l_1, \ldots, i_k - l_k),$$

where $H_{\mathbb{X}}(\underline{i}) = 0$ *if* $\underline{i} \ngeqslant \underline{0}$, *is the Hilbert function of some* $\mathbb{N}^k$*-graded Artinian quotient of the ring* $\mathbf{k}[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k}]$.

Proof. By Proposition 3.2, there exists k forms $L_1, \ldots, L_k$ that give rise to a regular sequence in $R/I_{\mathbb{X}}$ with $\deg L_i = e_i$. After making a linear change of variables in the $x_{1,i}$'s, a linear change of variables in the $x_{2,i}$'s, etc., we can assume that $L_i = x_{i,0}$.

The ideal $(I_{\mathbb{X}}, x_{1,0}, \ldots, x_{k,0})/(x_{1,0}, \ldots, x_{k,0})$ is isomorphic to an ideal J of the ring $S = \mathbf{k}[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k}]$. Set $A := S/J$, and so

$$A \cong \frac{R/(x_{1,0}, \ldots, x_{k,0})}{(I_{\mathbb{X}}, x_{1,0}, \ldots, x_{k,0})/(x_{1,0}, \ldots, x_{k,0})} \cong \frac{R}{(I_{\mathbb{X}}, x_{1,0}, \ldots, x_{k,0})}.$$

The ring A is Artinian because there exists $l \gg 0$ by Lemma 3.1 such that $\mathbf{m}^l \subseteq (I_{\mathbb{X}}, x_{1,0}, \ldots, x_{k,0})$.

It therefore remains to compute the Hilbert function of A. Set $J_i = (I_{\mathbb{X}}, x_{1,0}, \ldots, x_{i,0})$ for $i = 1, \ldots, k$. For each $i = 1, \ldots, k$ we have a short exact sequence with degree $(0, \ldots, 0)$ maps:

$$0 \longrightarrow (R/J_{i-1})(-e_i) \xrightarrow{\times \bar{x}_{i,0}} R/J_{i-1} \longrightarrow R/J_i \longrightarrow 0,$$

where $J_0 := I_{\mathbb{X}}$. From the k short exact sequences we have that

$$H_{R/J_k}(\underline{i}) = \Delta H_{\mathbb{X}}(\underline{i}) := \sum_{\underline{0} \leqslant \underline{l} = (l_1, \ldots, l_k) \leqslant (1, \ldots, 1)} (-1)^{|\underline{l}|} H_{\mathbb{X}}(i_1 - l_1, \ldots, i_k - l_k),$$

where $H_{\mathbb{X}}(\underline{i}) = 0$ if $\underline{i} \ngeqslant \underline{0}$. This completes the proof since $A \cong R/J_k$. □

The remainder of this section is devoted to showing that the necessary condition in Corollary 3.5 is also sufficient. To demonstrate this converse, we describe how to *lift* an ideal.

Definition 3.6. Let $R = \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$ and let $S = \mathbf{k}[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k}]$ be $\mathbb{N}^k$-graded rings. Let $I \subseteq R$ and $J \subseteq S$ be $\mathbb{N}^k$-homogeneous ideals. Then we say I is a *lifting of J to R* if

(i) I is radical in R;
(ii) $(I, x_{1,0}, \ldots, x_{k,0})/(x_{1,0}, \ldots, x_{k,0}) \cong J$;
(iii) $x_{1,0}, \ldots, x_{k,0}$ give rise to a regular sequence in R/I.

Our plan is to lift a monomial ideal of S to an $\mathbb{N}^k$-homogeneous ideal I of R, using the techniques and results of [9], so that I is the ideal of a reduced set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. We make a brief digression to introduce the relevant content of [9].

Suppose that S and R are as in Definition 3.6, but for the moment, we only assume that they are $\mathbb{N}^1$-graded. For each indeterminate $x_{i,j}$ with $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant n_i$, choose infinitely many linear forms $L_{i,j,l} \in \mathbf{k}[x_{i,j}, x_{1,0}, x_{2,0}, \ldots, x_{k,0}]$ with $l \in \mathbb{N}^+$. We only assume that the coefficient of $x_{i,j}$ in $L_{i,j,l}$ is not zero. The infinite matrix A, where

$$A := \begin{bmatrix} L_{1,1,1} & L_{1,1,2} & L_{1,1,3} & \cdots \\ \vdots & \vdots & \vdots & \\ L_{1,n_1,1} & L_{1,n_1,2} & L_{1,n_1,3} & \cdots \\ \vdots & \vdots & \vdots & \\ L_{k,1,1} & L_{k,1,2} & L_{k,1,3} & \cdots \\ \vdots & \vdots & \vdots & \\ L_{k,n_k,1} & L_{k,n_k1,2} & L_{k,n_k,3} & \cdots \end{bmatrix}$$

is called a *lifting matrix*. By using the lifting matrix, we associate to each monomial $m = x_{1,1}^{a_{1,1}} \cdots x_{1,n_1}^{a_{1,n_1}} \cdots x_{k,1}^{a_{k,1}} \cdots x_{k,n_k}^{a_{k,n_k}}$ of the ring S the element

$$\overline{m} = \left[\prod_{i=1}^{n_1}\left(\prod_{j=1}^{a_{1,i}} L_{1,i,j}\right)\right] \cdots \left[\prod_{i=1}^{n_k}\left(\prod_{j=1}^{a_{k,i}} L_{k,i,j}\right)\right] \in R.$$

Depending upon our choice of $L_{i,j,l}$'s, $\overline{m}$ may or may not be $\mathbb{N}^k$-homogeneous. However, $\overline{m}$ is homogeneous. If $J = (m_1, \ldots, m_r)$ is a monomial ideal of S, then we use I to denote the ideal $(\overline{m}_1, \ldots, \overline{m}_r) \subseteq R$. The following properties, among others, relate R/I and S/J.

Proposition 3.7 [9, Corollary 2.10]. *Let $J \subseteq S$ be a monomial ideal, and let I be the ideal constructed from J using any lifting matrix. Then*

(i) *S/J is CM if and only if R/I is CM;*
(ii) *$(I, x_{1,0}, \ldots, x_{k,0})/(x_{1,0}, \ldots, x_{k,0}) \cong J$;*
(iii) *$x_{1,0}, \ldots, x_{k,0}$ give rise to a regular sequence in R/I.*

We now consider the lifting of a monomial ideal using the lifting matrix $\mathcal{A} := [L_{i,j,l}]$, where

$$L_{i,j,l} = x_{i,j} - (l-1)x_{i,0} \quad \text{for } 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n_i,\ \text{and } l \in \mathbb{N}^+.$$

The lifting matrix $\mathcal{A}$ associates to every monomial $m = x_{1,1}^{a_{1,1}} \cdots x_{1,n_1}^{a_{1,n_1}} \cdots x_{k,1}^{a_{k,1}} \cdots x_{k,n_k}^{a_{k,n_k}}$ of S the following $\mathbb{N}^k$-homogeneous form of R:

$$\overline{m} = \left[\prod_{i=1}^{n_1}\left(\prod_{j=1}^{a_{1,i}} \big(x_{1,i} - (j-1)x_{1,0}\big)\right)\right] \cdots \left[\prod_{i=1}^{n_k}\left(\prod_{j=1}^{a_{k,i}} \big(x_{k,i} - (j-1)x_{k,0}\big)\right)\right].$$

Thus, if I is constructed from a monomial ideal $J \subseteq S$ using $\mathcal{A}$, I is $\mathbb{N}^k$-homogeneous. In fact, using $\mathcal{A}$, the ideal I is a lifting of J to R. To prove this statement, we need

Lemma 3.8 [9, Corollary 2.18]. *Let $J \subseteq S$ be a monomial ideal and let I be constructed from J using any lifting matrix. If $J = Q_1 \cap \cdots \cap Q_r$ is the primary decomposition of J, then $I = \overline{Q}_1 \cap \cdots \cap \overline{Q}_r$ where $\overline{Q}_i$ is the ideal generated by the lifting of the generators of Q_i.*

Proposition 3.9. *Let $J \subseteq S$ be a monomial ideal and let I be the ideal constructed from J using the lifting matrix $\mathcal{A}$. Then I is a lifting of J to R.*

Proof. Since Proposition 3.7 is true for any lifting matrix, it suffices to show that I is radical. Let $J = Q_1 \cap \cdots \cap Q_r$ be the primary decomposition of J. Since J is a monomial ideal, then by [9, Remark 2.19] we have that each Q_i is a complete intersection of the form

$$Q_i = \left(x_{1,i_{1,1}}^{a_{1,i_{1,1}}}, \ldots, x_{1,i_{1,p_1}}^{a_{1,i_{1,p_1}}}, \ldots, x_{k,i_{k,1}}^{a_{k,i_{k,1}}}, \ldots, x_{k,i_{k,p_k}}^{a_{k,i_{k,p_k}}}\right)$$

with $a_{j,i_{j,l}} \geqslant 1$ for each variable that appears in Q_i. Using the lifting matrix $\mathcal{A}$ we then have

$$\overline{Q}_i = \left(\prod_{l=1}^{a_{1,i_{1,1}}} L_{1,i_{1,1},l}, \ldots, \prod_{l=1}^{a_{1,i_{1,p_1}}} L_{1,i_{1,p_1},l}, \ldots, \prod_{l=1}^{a_{k,i_{k,1}}} L_{k,i_{k,1},l}, \ldots, \prod_{l=1}^{a_{k,i_{k,p_k}}} L_{k,i_{k,p_k},l}\right),$$

where $L_{i,j,l} = x_{i,j} - (l-1)x_{i,0}$. But then $\overline{Q}_i$ is a reduced complete intersection. It then follows from Lemma 3.8 that I must be radical. □

We now describe the zero set of the lifted ideal I. For each

$$(\underline{\alpha}_1, \ldots, \underline{\alpha}_k) := \big((a_{1,1}, \ldots, a_{1,n_1}), \ldots, (a_{k,1}, \ldots, a_{k,n_k})\big) \in \mathbb{N}^{n_1} \times \cdots \times \mathbb{N}^{n_k},$$

set $X_1^{\underline{\alpha}_1} \cdots X_k^{\underline{\alpha}_k} := x_{1,1}^{a_{1,1}} \cdots x_{1,n_1}^{a_{1,n_1}} \cdots x_{k,1}^{a_{k,1}} \cdots x_{k,n_k}^{a_{k,n_k}}$. If P is the set of all monomials of S including the monomial 1, then there exists a bijection between P and $\mathbb{N}^{n_1} \times \cdots \times \mathbb{N}^{n_k}$ given by the map $X_1^{\underline{\alpha}_1} \cdots X_k^{\underline{\alpha}_k} \leftrightarrow (\underline{\alpha}_1, \ldots, \underline{\alpha}_k)$. To each tuple $(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)$ we associate the point $\overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)} \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, where

$$\overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)} := [1 : a_{1,1} : a_{1,2} : \cdots : a_{1,n_1}] \times \cdots \times [1 : a_{k,1} : a_{k,2} : \cdots : a_{k,n_k}].$$

Note that if $m = X_1^{\underline{\alpha}_1} \cdots X_k^{\underline{\alpha}_k} \in P$ and if $\overline{m}$ is constructed from $X_1^{\underline{\alpha}_1} \cdots X_k^{\underline{\alpha}_k}$ using the lifting matrix $\mathcal{A}$, then $\overline{m}(\overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)}) \neq 0$. In fact, it follows from our construction that $\overline{m}(\overline{(\underline{\beta}_1, \ldots, \underline{\beta}_k)}) = 0$ if and only if some coordinate of $(\underline{\beta}_1, \ldots, \underline{\beta}_k)$ is strictly less than some coordinate of $(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)$.

If J is a monomial ideal of S, then let N be the set of monomials in J. The elements of $M := P \backslash N$ are representatives for a $\mathbf{k}$-basis of the $\mathbb{N}^k$-graded ring S/J. Set

$$\overline{M} := \left\{\overline{(\underline{\beta}_1, \ldots, \underline{\beta}_k)} \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \;\middle|\; X_1^{\underline{\beta}_1} \cdots X_k^{\underline{\beta}_k} \in M\right\}.$$

Let $\mathbf{I}(\overline{M})$ denote the $\mathbb{N}^k$-homogeneous ideal associated to $\overline{M}$. If $m = X_1^{\underline{\alpha}_1} \cdots X_k^{\underline{\alpha}_k} \in J$ is a minimal generator, then for each $X_1^{\underline{\beta}_1} \cdots X_k^{\underline{\beta}_k} \in M$ there exists at least one coordinate of $(\underline{\beta}_1, \ldots, \underline{\beta}_k)$ that is strictly less then some coordinate of $(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)$. So $\overline{m}(\overline{(\underline{\beta}_1, \ldots, \underline{\beta}_k)}) = 0$ for all $X_1^{\underline{\beta}_1} \cdots X_k^{\underline{\beta}_k} \in M$, and hence the lifted ideal $I \subseteq \mathbf{I}(\overline{M})$. On the other hand, $\overline{M} = \mathbf{V}(I)$, the zero set of I, so by the $\mathbb{N}^k$-graded analog of the Nullstellensatz (cf. [13, Theorem 2.3]) we have $\mathbf{I}(\overline{M}) \subseteq \sqrt{I}$. Because I is radical by Proposition 3.9, we have just shown.

Lemma 3.10. *Let $J \subseteq S$ be a monomial ideal and let I be the ideal constructed from J using the lifting matrix $\mathcal{A}$. Then, with the notation as above, $I = \mathbf{I}(\overline{M})$.*

We come to the main result of this section.

Theorem 3.11. *Let $H : \mathbb{N}^k \to \mathbb{N}$ be a numerical function. Then H is the Hilbert function of an ACM set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ if and only if the first difference function*

$$\Delta H(i_1, \ldots, i_k) = \sum_{\underline{0} \leqslant \underline{l} = (l_1, \ldots, l_k) \leqslant (1, \ldots, 1)} (-1)^{|\underline{l}|} H(i_1 - l_1, \ldots, i_k - l_k),$$

where $H(\underline{i}) = 0$ if $\underline{i} \ngeq \underline{0}$, is the Hilbert function of some $\mathbb{N}^k$-graded Artinian quotient of $S = \mathbf{k}[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k}]$.

Proof. Because of Corollary 3.5, we only need to show one direction. So, if ΔH is the Hilbert function of some $\mathbb{N}^k$-graded Artinian quotient of S, then there exists an $\mathbb{N}^k$-homogeneous ideal $J \subseteq S$ with $\Delta H(\underline{i}) = H_{S/J}(\underline{i})$ for all $\underline{i} \in \mathbb{N}^k$. By replacing J with its leading term ideal, we can assume that $J = (m_1, \ldots, m_r)$ is a monomial ideal of S.

Let $I \subseteq R$ be the ideal constructed from J using the lifting matrix $\mathcal{A}$. By Proposition 3.9, the ideal $J \cong (I, x_{1,0}, \ldots, x_{k,0})/(x_{1,0}, \ldots, x_{k,0})$ where $x_{1,0}, \ldots, x_{k,0}$ give rise to a regular sequence in R/I. Because $\deg x_{i,0} = e_i$, we have k short exact sequences with degree $(0, \ldots, 0)$ maps:

$$0 \longrightarrow (R/J_{i-1})(e_i) \xrightarrow{\times \overline{x}_{i,0}} R/J_{i-1} \longrightarrow R/J_i \longrightarrow 0,$$

where $J_i := (I, x_{1,0}, \ldots, x_{i,0})$ for $i = 1, \ldots, k$ and $J_0 := I$. Furthermore,

$$S/J \cong \frac{R/(x_{1,0}, \ldots, x_{k,0})}{(I, x_{1,0}, \ldots, x_{k,0})/(x_{1,0}, \ldots, x_{k,0})} \cong R/(I, x_{1,0}, \ldots, x_{k,0}) = R/J_k.$$

Then, using the k short exact sequences to calculate the $H_{R/I}$, we find that $H = H_{R/I}$.

If N is the set of monomials in J, then $M = P \backslash N$ is a finite set of monomials because S/J is Artinian. By Lemma 3.10, I is the ideal of the finite set of points $\overline{M} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Finally, by Proposition 3.7 the set $\overline{M}$ is ACM because S/J is Artinian, and hence, CM. $\square$

Since characterizing the Hilbert functions of ACM sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is equivalent to characterizing the Hilbert functions of $\mathbb{N}^k$-graded Artinian quotients of S, Theorem 3.11 translates one open problem into another open problem because we do not have a theorem like Macaulay's theorem [7] for $\mathbb{N}^k$-graded rings if $k > 1$. However, as shown below, there is a Macaulay-type theorem for $\mathbb{N}^k$-graded quotients of $\mathbf{k}[x_{1,1}, x_{2,1}, \dots, x_{k,1}]$. As a consequence, we can explicitly describe all the Hilbert functions of ACM sets of points in $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (k times) for any positive integer k.

So, suppose that $S = \mathbf{k}[x_1, \dots, x_k]$ and $\deg x_i = e_i$, where e_i is the ith standard basis vector of $\mathbb{N}^k$. We prove a stronger result by characterizing the Hilbert functions of all quotients of S, not only the Artinian quotients.

Theorem 3.12. *Let $S = \mathbf{k}[x_1, \dots, x_k]$ with $\deg x_i = e_i$, and let $H : \mathbb{N}^k \to \mathbb{N}$ be a numerical function. Then there exists a homogeneous ideal $I \subsetneq S$ with Hilbert function $H_{S/I} = H$ if and only if*

(i) $H(0, \dots, 0) = 1$,
(ii) $H(\underline{i}) = 1$ *or* 0 *if* $\underline{i} > \underline{0}$, *and*
(iii) *if* $H(\underline{i}) = 0$, *then* $H(\underline{j}) = 0$ *for all* $\underline{j} \geqslant \underline{i}$.

Proof. If $I \subsetneq S$ is an $\mathbb{N}^k$-homogeneous ideal such that $H_{S/I} = H$, then condition (i) is simply a consequence of the fact that $I \subsetneq S$. Statement (ii) follows from the inequality $0 \leqslant H_{S/I}(\underline{i}) \leqslant \dim_{\mathbf{k}} S_{\underline{i}} = 1$. Finally, if $H_{S/I}(\underline{i}) = 0$, then $x_1^{i_1} \cdots x_k^{i_k} \in I$, or equivalently, $S_{\underline{i}} \subseteq I$ because $x_1^{i_1} \cdots x_k^{i_k}$ is a basis for $S_{\underline{i}}$. So, if $\underline{j} \geqslant \underline{i}$, then $S_{\underline{j}} \subseteq I$, and hence, $H_{S/I}(\underline{j}) = 0$, thus proving (iii).

Conversely, suppose that H is a numerical function satisfying (i)–(iii). If $H(\underline{i}) = 1$ for all $\underline{i} \in \mathbb{N}^k$, then the ideal $I = (0) \subseteq S$ has the property that $H_{S/I} = H$.

So, suppose $H(\underline{i}) \neq 1$ for all $\underline{i}$. Set $\mathcal{I} := \{\underline{i} \in \mathbb{N}^k \mid H(\underline{i}) = 0\}$. Note that $\mathcal{I} \neq \mathbb{N}^k$ because $\underline{0} \notin \mathcal{I}$. Let I be the ideal $I := \langle \{x_1^{i_1} \cdots x_k^{i_k} \mid \underline{i} \in \mathcal{I}\}\rangle$ in S. We claim that $H_{S/I}(\underline{i}) = H(\underline{i})$ for all $\underline{i} \in \mathbb{N}^k$. It is immediate that $H_{S/I}(\underline{0}) = H(\underline{0}) = 1$. Moreover, if $H(\underline{i}) = 0$, then $H_{S/I}(\underline{i}) = 0$ because $x_1^{i_1} \cdots x_k^{i_k} \in I_{\underline{i}} \subseteq I$, i.e., $S_{\underline{i}} \subseteq I$.

So, we need to check that if $H(\underline{i}) = 1$, then $H_{S/I}(\underline{i}) = 1$. Suppose $H_{S/I}(\underline{i}) = 0$. This implies that $x_1^{i_1} \cdots x_k^{i_k} \in I$. But because $\underline{i} \notin \mathcal{I}$, there is a monomial $x_1^{j_1} \cdots x_k^{j_k} \in I$ with $\underline{j} \in \mathcal{I}$, such that $x_1^{j_1} \cdots x_k^{j_k}$ divides $x_1^{i_1} \cdots x_k^{i_k}$. But this is equivalent to the statement that $\underline{j} \leqslant \underline{i}$. But this contradicts hypothesis (iii). So $H_{S/I}(\underline{i}) = 1$. □

Using Theorem 3.12 and the definition of an $\mathbb{N}^k$-graded Artinian quotient, we then have:

Corollary 3.13. *Let $S = \mathbf{k}[x_1, \dots, x_k]$ with $\deg x_i = e_i$, and let $H : \mathbb{N}^k \to \mathbb{N}$ be a numerical function. Then H is the Hilbert function of an $\mathbb{N}^k$-graded Artinian quotient of S if and only if*

(i) $H(0, \dots, 0) = 1$,
(ii) $H(\underline{i}) = 1$ *or* 0 *if* $\underline{i} > (0, \dots, 0)$,

(iii) *if* $H(\underline{i}) = 0$, *then* $H(\underline{j}) = 0$ *for all* $\underline{j} \geqslant \underline{i}$, *and*
(iv) *for each* $i \in [k]$ *there exists an integer* t_i *such that* $H(t_i e_i) = 0$.

Corollary 3.14. *Let* $H : \mathbb{N}^k \to \mathbb{N}$ *be a numerical function. Then* H *is the Hilbert function of an ACM set of distinct points in* $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (k *times*) *if and only* ΔH *satisfies conditions* (i)–(iv) *of Corollary* 3.13.

Remark 3.15. It follows from the previous corollaries that H is the Hilbert function of an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ if and only if

(i) $\Delta H(i, j) = 1$ or 0,
(ii) if $\Delta H(i, j) = 0$, then $\Delta H(i', j') = 0$ for all $(i', j') \in \mathbb{N}^2$ with $(i', j') > (i, j)$, and
(iii) there exists integers t and r such that $\Delta H(t, 0) = 0$ and $\Delta H(0, r) = 0$.

Giuffrida, Maggioni, and Ragusa proved precisely this result in [5, Theorems 4.1 and 4.2]. We investigate ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ in further detail in the next two sections.

4. ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$

If $\mathbb{X}$ is an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1$, then by Theorem 3.11 the function $\Delta H_{\mathbb{X}}$ is the Hilbert function of a bigraded Artinian quotient of $\mathbf{k}[x_1, y_1]$. In [5] it was shown that the converse of this statement is also true, thereby classifying the ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. In this section we revisit this result by giving a new proof of this characterization that depends only upon numerical information describing $\mathbb{X}$.

We begin with a brief digression to introduce some needed combinatorial results. Recall that a tuple $\lambda = (\lambda_1, \ldots, \lambda_r)$ of positive integers is a *partition* of an integer s, denoted $\lambda \vdash s$, if $\sum \lambda_j = s$ and $\lambda_i \geqslant \lambda_{i+1}$ for each i. If $\lambda \vdash s$, then the *conjugate* of λ is the tuple $\lambda^* = (\lambda_1^*, \ldots, \lambda_{\lambda_1}^*)$ where $\lambda_i^* := \#\{\lambda_j \in \lambda \mid \lambda_j \geqslant i\}$. Moreover, λ^* is also a partition of s.

To any partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash s$ we can associate the following diagram: on an $r \times \lambda_1$ grid, place λ_1 points on the first line, λ_2 points on the second, and so on. The resulting diagram is called the *Ferrers diagram* of λ. For example, suppose $\lambda = (4, 4, 3, 1, 1) \vdash 13$. Then the Ferrers diagram is

$$\begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & & & \\ \bullet & & & \end{array}.$$

The conjugate of λ can be read off the Ferrers diagram by counting the number of dots in each column as opposed to each row. In this example, $\lambda^* = (5, 3, 3, 2) \vdash 13$.

The following lemma, whose proof is a straightforward combinatorial exercise, describes some of the relations between a partition and its conjugate.

Lemma 4.1. *Let $\alpha = (\alpha_1, \ldots, \alpha_n) \vdash s$ and $\beta = (\beta_1, \ldots, \beta_m) \vdash s$. If $\alpha^* = \beta$, then*

(i) $\alpha_1 = |\beta|$ *and* $\beta_1 = |\alpha|$,
(ii) *if* $\alpha' = (\alpha_2, \ldots, \alpha_n)$ *and* $\beta' = (\beta_1 - 1, \ldots, \beta_{\alpha_2} - 1)$, *then* $(\alpha')^* = \beta'$.

Let $\mathbb{X}$ denote a set of reduced points in $\mathbb{P}^1 \times \mathbb{P}^1$, and associate to $\mathbb{X}$ two tuples $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$ as follows. Let $\pi_1(\mathbb{X}) = \{P_1, \ldots, P_t\}$ be the t distinct first coordinates in $\mathbb{X}$. Then, for each $P_i \in \pi_1(\mathbb{X})$, let $\alpha_i := |\pi_1^{-1}(P_i)|$, i.e., the number of points in $\mathbb{X}$ which have P_i as its first coordinate. After relabeling the α_i so that $\alpha_i \geqslant \alpha_{i+1}$ for $i = 1, \ldots, t-1$, we set $\alpha_{\mathbb{X}} = (\alpha_1, \ldots, \alpha_t)$. Analogously, for each $Q_i \in \pi_2(\mathbb{X}) = \{Q_1, \ldots, Q_r\}$, we let $\beta_i := |\pi_2^{-1}(Q_i)|$. After relabeling so that $\beta_i \geqslant \beta_{i+1}$ for $i = 1, \ldots, r-1$, we set $\beta_{\mathbb{X}} = (\beta_1, \ldots, \beta_r)$. So, by construction, $\alpha_{\mathbb{X}}, \beta_{\mathbb{X}} \vdash s = |\mathbb{X}|$. Note that $|\alpha_{\mathbb{X}}| = |\pi_1(\mathbb{X})|$ and $|\beta_{\mathbb{X}}| = |\pi_2(\mathbb{X})|$.

We write the Hilbert function $H_{\mathbb{X}}$ as an infinite matrix (m_{ij}) where $m_{ij} := H_{\mathbb{X}}(i,j)$. Proposition 2.4 gives

$$m_{i,0} = \begin{cases} i+1 & 0 \leqslant i \leqslant t-1, \\ t & i \geqslant t, \end{cases} \quad \text{and} \quad m_{0,j} = \begin{cases} i+1 & 0 \leqslant i \leqslant r-1, \\ r & i \geqslant r, \end{cases}$$

because $\pi_1(\mathbb{X}) = \{P_1, \ldots, P_t\} \subseteq \mathbb{P}^1$ and $\pi_2(\mathbb{X}) = \{Q_1, \ldots, Q_r\} \subseteq \mathbb{P}^1$. This fact, combined with Proposition 2.2, implies that $H_{\mathbb{X}}$ has the form

$$H_{\mathbb{X}} = \begin{bmatrix} 1 & 2 & \cdots & r-1 & \mathbf{r} & r & \cdots \\ 2 & & & & \mathbf{m_{1,r-1}} & m_{1,r-1} & \cdots \\ \vdots & & * & & \vdots & \vdots & \\ t-1 & & & & \mathbf{m_{2,r-1}} & m_{2,r-1} & \cdots \\ \mathbf{t} & \mathbf{m_{t-1,1}} & \cdots & \mathbf{m_{t-1,r-2}} & \mathbf{s} & s & \cdots \\ t & m_{t-1,1} & \cdots & m_{t-1,r-2} & s & s & \\ \vdots & \vdots & & \vdots & \vdots & & \ddots \end{bmatrix}, \tag{1}$$

where the values denoted by $(*)$ need to be calculated. Set

$$B_C = (m_{t-1,0}, \ldots, m_{t-1,r-1}) \quad \text{and} \quad B_R = (m_{0,r-1}, \ldots, m_{t-1,r-1}).$$

From our description of $H_{\mathbb{X}}$, we see that if we know the values in the tuples B_C and B_R, we will know all but a finite number of values of $H_{\mathbb{X}}$. As shown below, the tuples B_C and B_R can be computed directly from the tuples $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$ defined above. If λ is a tuple, then we shall abuse notation and write $\lambda_i \in \lambda$ to mean that λ_i is a coordinate of λ.

Proposition 4.2 [13, Proposition 5.11]. *Let $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a finite set of points.*

(i) *If $B_C = (m_{t-1,0}, \ldots, m_{t-1,r-1})$ where $m_{t-1,j} = H_{\mathbb{X}}(t-1,j)$ for $j = 0, \ldots, r-1$, then*

$$m_{t-1,j} = \#\{\alpha_i \in \alpha_{\mathbb{X}} \mid \alpha_i \geqslant 1\} + \#\{\alpha_i \in \alpha_{\mathbb{X}} \mid \alpha_i \geqslant 2\} + \cdots + \#\{\alpha_i \in \alpha_{\mathbb{X}} \mid \alpha_i \geqslant j+1\}.$$

(ii) *If* $B_R = (m_{0,r-1}, \dots, m_{t-1,r-1})$ *where* $m_{j,r-1} = H_{\mathbb{X}}(j, r-1)$ *for* $j = 0, \dots, t-1$, *then*

$$m_{j,r-1} = \#\{\beta_i \in \beta_{\mathbb{X}} \mid \beta_i \geqslant 1\} + \#\{\beta_i \in \beta_{\mathbb{X}} \mid \beta_i \geqslant 2\} + \cdots + \#\{\beta_i \in \beta_{\mathbb{X}} \mid \beta_i \geqslant j+1\}.$$

We can rewrite this result more compactly using the language of combinatorics introduced above. If $p = (p_1, \dots, p_k)$, then we write Δp to denote the tuple $\Delta p := (p_1, p_2 - p_1, \dots, p_k - p_{k-1})$.

Corollary 4.3 [13, Corollary 5.13]. *Let* $\mathbb{X}$ *be a finite set of points in* $\mathbb{P}^1 \times \mathbb{P}^1$. *Then*

(i) $\Delta B_C = \alpha^*_{\mathbb{X}}$,
(ii) $\Delta B_R = \beta^*_{\mathbb{X}}$.

Remark 4.4. In [13] the tuple $B_{\mathbb{X}} = (B_C, B_R)$ was called the *border* of the Hilbert function.

Recall that $\Delta H_{\mathbb{X}}$, the *first difference function* of $H_{\mathbb{X}}$, is defined by

$$\Delta H_{\mathbb{X}}(i, j) := H_{\mathbb{X}}(i, j) - H_{\mathbb{X}}(i-1, j) - H_{\mathbb{X}}(i, j-1) + H_{\mathbb{X}}(i-1, j-1),$$

where $H_{\mathbb{X}}(i, j) = 0$ if $(i, j) \not\geqslant (0, 0)$. The entries of $\alpha^*_{\mathbb{X}}$ and $\beta^*_{\mathbb{X}}$ can then be read from $\Delta H_{\mathbb{X}}$.

Corollary 4.5. *Let* $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ *be a finite set of points, and set* $c_{i,j} := \Delta H_{\mathbb{X}}(i, j)$. *Then*

(i) *for every* $0 \leqslant j \leqslant r-1 = |\pi_2(\mathbb{X})| - 1$,

$$\alpha^*_{j+1} = \sum_{h \leqslant |\pi_1(\mathbb{X})|-1} c_{h,j},$$

where α^*_{j+1} *is the* $(j+1)$*th entry of* $\alpha^*_{\mathbb{X}}$, *the conjugate of* $\alpha_{\mathbb{X}}$,
(ii) *for every* $0 \leqslant i \leqslant t-1 = |\pi_1(\mathbb{X})| - 1$,

$$\beta^*_{i+1} = \sum_{h \leqslant |\pi_2(\mathbb{X})|-1} c_{i,h},$$

where β^*_{i+1} *is the* $(i+1)$*th entry of* $\beta^*_{\mathbb{X}}$.

Proof. Use Proposition 4.2 and the identity $H_{\mathbb{X}}(i, j) = \sum_{(h,k) \leqslant (i,j)} c_{h,k}$ to compute α^*_{j+1}:

$$\begin{aligned}\alpha^*_{j+1} &= H_{\mathbb{X}}(t-1, j) - H_{\mathbb{X}}(t-1, j-1) \\ &= \sum_{(h,k) \leqslant (t-1,j)} c_{h,k} - \sum_{(h,k) \leqslant (t-1,j-1)} c_{h,k} = \sum_{h \leqslant t-1 = |\pi_1(\mathbb{X})|-1} c_{h,j}.\end{aligned}$$

The proof for the second statement is the same. □

Lemma 4.6. *Let $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a finite set of points, and suppose that $\alpha^*_{\mathbb{X}} = \beta_{\mathbb{X}}$. Let P be a point of $\pi_1(\mathbb{X})$ such that $|\pi_1^{-1}(P)| = \alpha_1$. Set $\mathbb{X}_P := \pi_1^{-1}(P)$. Then $\pi_2(\mathbb{X}_P) = \pi_2(\mathbb{X})$.*

Proof. Since $\mathbb{X}_P \subseteq \mathbb{X}$, we have $\pi_2(\mathbb{X}_P) \subseteq \pi_2(\mathbb{X})$. Now, by our choice of P, $|\pi_2(\mathbb{X}_P)| = \alpha_1$. But since $|\pi_2(\mathbb{X})| = |\beta_{\mathbb{X}}|$ and $\alpha^*_{\mathbb{X}} = \beta_{\mathbb{X}}$, from Lemma 4.1 we have $|\pi_2(\mathbb{X})| = |\beta_{\mathbb{X}}| = \alpha_1 = |\pi_2(\mathbb{X}_P)|$, and hence $\pi_2(\mathbb{X}_P) = \pi_2(\mathbb{X})$. □

Proposition 4.7. *Suppose that $\mathbb{X}$ is a set of $s = tr$ points in $\mathbb{P}^1 \times \mathbb{P}^1$ such that $\alpha_{\mathbb{X}} = (r, \ldots, r)$ (t times) and $\beta_{\mathbb{X}} = (t, \ldots, t)$ (r times). Then $\mathbb{X}$ is a complete intersection, and the graded minimal free resolution of $I_{\mathbb{X}}$ is given by*

$$0 \longrightarrow R(-t, -r) \longrightarrow R(-t, 0) \oplus R(0, -r) \longrightarrow I_{\mathbb{X}} \longrightarrow 0.$$

Proof. Because $|\alpha_{\mathbb{X}}| = t$ and $|\beta_{\mathbb{X}}| = r$, $\pi_1(\mathbb{X}) = \{P_1, \ldots, P_t\}$ and $\pi_2(\mathbb{X}) = \{Q_1, \ldots, Q_r\}$, where $P_i, Q_j \in \mathbb{P}^1$. Since $|\mathbb{X}| = tr$, $\mathbb{X} = \{P_i \times Q_j \mid 1 \leqslant i \leqslant t, 1 \leqslant j \leqslant r\}$. Hence, if $I_{P_i \times Q_j} = (L_{P_i}, L_{Q_j})$ is the ideal associated to the point $P_i \times Q_j$, then the defining ideal of $\mathbb{X}$ is

$$I_{\mathbb{X}} = \bigcap_{i,j} (L_{P_i}, L_{Q_j}) = (L_{P_1} L_{P_2} \cdots L_{P_t}, L_{Q_1} L_{Q_2} \cdots L_{Q_r}).$$

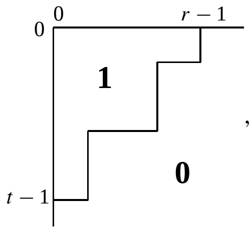
Since $\deg L_{P_1} L_{P_2} \cdots L_{P_t} = (t, 0)$ and $\deg L_{Q_1} L_{Q_2} \cdots L_{Q_r} = (0, r)$, the two generators form a regular sequence on R, and hence, $\mathbb{X}$ is a complete intersection. The graded minimal free resolution is then given by the *Koszul resolution*, taking into consideration that $I_{\mathbb{X}}$ is bigraded. □

We now come to the main result of this section.

Theorem 4.8. *Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with Hilbert function $H_{\mathbb{X}}$. Then the following are equivalent*:

- (i) *$\mathbb{X}$ is ACM,*
- (ii) *$\Delta H_{\mathbb{X}}$ is the Hilbert function of an $\mathbb{N}^2$-graded Artinian quotient of $\mathbf{k}[x_1, y_1]$,*
- (iii) *$\alpha^*_{\mathbb{X}} = \beta_{\mathbb{X}}$.*

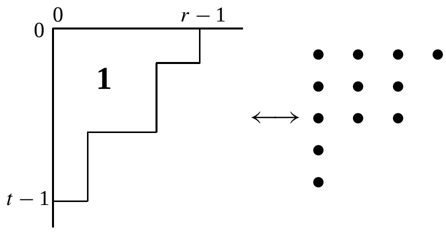
Proof. The implication (i) ⇒ (ii) is Corollary 3.5. So, suppose that (ii) holds. Because $\Delta H_{\mathbb{X}}$ is the Hilbert function of an $\mathbb{N}^2$-graded Artinian quotient of $\mathbf{k}[x_1, y_1]$, Corollary 3.14, Remark 3.15, and matrix (1) give

$\Delta H_{\mathbb{X}} =$ 

,

where $t = |\pi_1(\mathbb{X})|$ and $r = |\pi_2(\mathbb{X})|$. We have written $\Delta H_{\mathbb{X}}$ as an infinite matrix whose indexing starts from zero rather than one.

By Corollary 4.5 the number of 1's in the $(i-1)$th row of $\Delta H_{\mathbb{X}}$ for each integer $1 \leqslant i \leqslant t$ is simply the ith coordinate of $\beta_{\mathbb{X}}^*$. Similarly, the number of ones in the $(j-1)$th column of $\Delta H_{\mathbb{X}}$ for each integer $1 \leqslant j \leqslant r$ is the jth coordinate of $\alpha_{\mathbb{X}}^*$. Now $\Delta H_{\mathbb{X}}$ can be identified with the Ferrers diagram of $\beta_{\mathbb{X}}^*$ by associating to each 1 in $\Delta H_{\mathbb{X}}$ a dot in the Ferrers diagram in the natural way, i.e.,



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By using the Ferrers diagram and Corollary 4.5, it is now straightforward to calculate that the conjugate of $\beta_{\mathbb{X}}^*$ is $(\beta_{\mathbb{X}}^*)^* = \beta_{\mathbb{X}} = \alpha_{\mathbb{X}}^*$, and so (iii) holds.

To demonstrate that (iii) implies (i), we proceed by induction on the tuple $(|\pi_1(\mathbb{X})|, |\mathbb{X}|)$. For any positive integer s, if $(|\pi_1(\mathbb{X})|, |\mathbb{X}|) = (1, s)$, then $\alpha_{\mathbb{X}} = (s)$ and $\beta_{\mathbb{X}} = (1, \ldots, 1)$ (s times), and so $\alpha_{\mathbb{X}}^* = \beta_{\mathbb{X}}$. Then by Proposition 4.7, $\mathbb{X}$ is a complete intersection and hence ACM.

So, suppose that $(|\pi(\mathbb{X})), |\mathbb{X}|) = (t, s)$ and that the result holds true for all $\mathbb{Y} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ with $\alpha_{\mathbb{Y}}^* = \beta_{\mathbb{Y}}$ and $(t, s) >_{\mathrm{lex}} (|\pi_1(\mathbb{Y})|, |\mathbb{Y}|)$, where $>_{\mathrm{lex}}$ is the lexicographical ordering on $\mathbb{N}^2$.

Suppose that P_1 (after a possible relabeling) is the element of $\pi_1(\mathbb{X})$ such that $|\pi_1^{-1}(P_1)| = \alpha_1$. Let L_{P_1} be the form of degree $(1, 0)$ that vanishes at P_1. By abusing notation, we also let L_{P_1} denote the $(1, 0)$-line in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by L_{P_1}.

Set $\mathbb{X}_{P_1} := \mathbb{X} \cap L_{P_1} = \pi_1^{-1}(P_1)$ and $\mathbb{Z} := \mathbb{X} \backslash \mathbb{X}_{P_1}$. It follows that $\alpha_{\mathbb{Z}} = (\alpha_2, \ldots, \alpha_t)$ and $\beta_{\mathbb{Z}} = (\beta_1 - 1, \ldots, \beta_{\alpha_2} - 1)$. Now $(t, s) >_{\mathrm{lex}} (|\pi_1(\mathbb{Z})|, |\mathbb{Z}|)$, and moreover, $\alpha_{\mathbb{Z}}^* = \beta_{\mathbb{Z}}$ by Lemma 4.1. Thus, by the induction hypothesis, $\mathbb{Z}$ is ACM.

Suppose that $\pi_2(\mathbb{X}) = \{Q_1, \ldots, Q_r\}$. Let L_{Q_i} be the degree $(0, 1)$ form that vanishes at $Q_i \in \pi_2(\mathbb{X})$ and set $F := L_{Q_1} L_{Q_2} \cdots L_{Q_r}$. Because $\alpha_{\mathbb{X}}^* = \beta_{\mathbb{X}}$, from Lemma 4.6 we have $\pi_2(\mathbb{X}_{P_1}) = \pi_2(\mathbb{X})$. So, $\mathbb{X}_{P_1} = \{P_1 \times Q_1, \ldots, P_1 \times Q_r\}$, and hence $I_{\mathbb{X}_{P_1}} = \bigcap_{i=1}^r (L_{P_1}, L_{Q_i}) = (L_{P_1}, F)$. Furthermore, if $P \times Q \in \mathbb{Z}$, then $Q \in \pi_2(\mathbb{Z}) \subseteq \pi_2(\mathbb{X})$, and thus $F(P \times Q) = 0$. Therefore $F \in I_{\mathbb{Z}}$. Because F is in $I_{\mathbb{Z}}$ and is also a generator of $I_{\mathbb{X}_{P_1}}$, we can show:

Claim. *Let $I = L_{P_1} \cdot I_{\mathbb{Z}} + (F)$. Then $I = I_{\mathbb{X}}$.*

Proof. Since $I_{\mathbb{X}} = I_{\mathbb{Z}\cup\mathbb{X}_{P_1}} = I_{\mathbb{Z}} \cap I_{\mathbb{X}_{P_1}}$, we will show $I_{\mathbb{Z}} \cap I_{\mathbb{X}_{P_1}} = L_{P_1} \cdot I_{\mathbb{Z}} + (F)$. So, suppose that $G = L_{P_1}H_1 + H_2F \in L_{P_1} \cdot I_{\mathbb{Z}} + (F)$ with $H_1 \in I_{\mathbb{Z}}$ and $H_2 \in R$. Because L_{P_1} and F are in $I_{\mathbb{X}_{P_1}}$, we have $G \in I_{\mathbb{X}_{P_1}}$. Since $H_1, F \in I_{\mathbb{Z}}$, $G \in I_{\mathbb{Z}}$. Thus $G \in I_{\mathbb{Z}} \cap I_{\mathbb{X}_{P_1}}$.

Conversely, let $G \in I_{\mathbb{Z}} \cap I_{\mathbb{X}_{P_1}}$. Since $G \in I_{\mathbb{X}_{P_1}}$, $G = L_{P_1}H_1 + FH_2$. We need to show that $H_1 \in I_{\mathbb{Z}}$. Because $G, F \in I_{\mathbb{Z}}$, we also have $L_{P_1}H_1 \in I_{\mathbb{Z}}$. But for every $P \times Q \in \mathbb{Z}$, $P \neq P_1$, and thus $L_{P_1}(P \times Q) \neq 0$. Hence $L_{P_1}H_1 \in I_{\mathbb{Z}}$ if and only if $H_1(P \times Q) = 0$ for every $P \times Q \in \mathbb{Z}$. □

We note that $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is ACM if and only if the variety $\widetilde{\mathbb{X}} \subseteq \mathbb{P}^3$ defined by $I_{\mathbb{X}}$, considered as a homogeneous ideal of $R = \mathbf{k}[x_0, x_1, y_0, y_1]$, is ACM. As a variety of $\mathbb{P}^3$, $\widetilde{\mathbb{X}}$ is a curve since K-dim $R/I_{\mathbb{X}} = 2$. Let $\widetilde{\mathbb{Z}}$ denote the curve of $\mathbb{P}^3$ defined by $I_{\mathbb{Z}}$, considered also as a homogeneous ideal of R. The claim implies that the curve $\widetilde{\mathbb{X}}$ is a basic double link of $\widetilde{\mathbb{Z}}$. Since the Cohen–Macaulay property is preserved under linkage (see [8, Theorem 3.2.3] and following remark), $\widetilde{\mathbb{X}}$ is an ACM curve of $\mathbb{P}^3$, or equivalently, $\mathbb{X}$ is an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. □

Remark 4.9. Giuffrida et al. [5, Theorem 4.1] demonstrated the equivalence of statements (i) and (ii) of Theorem 4.8 via different means. Our contribution is to show that the ACM sets of points are also characterized by $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$. This result has been extended in [6] to characterize ACM fat point schemes in $\mathbb{P}^1 \times \mathbb{P}^1$.

Corollary 4.10. *Let $\mathbb{X}$ be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with $\alpha_{\mathbb{X}} = (\alpha_1, \dots, \alpha_t)$, and $\pi_1(\mathbb{X}) = \{P_1, \dots, P_t\}$. Suppose (after a possible relabeling) that $|\pi_1^{-1}(P_i)| = \alpha_i$. Set*

$$\mathbb{X}_i := \mathbb{X} \Big\backslash \big\{\pi_1^{-1}(P_1) \cup \cdots \cup \pi_1^{-1}(P_i)\big\} \quad \text{for } 0 \leqslant i \leqslant t-1,$$

where $\mathbb{X}_0 := \mathbb{X}$. If $\mathbb{X}$ is ACM, then, for each integer $0 \leqslant i \leqslant t-1$, $\mathbb{X}_i$ is ACM with $\alpha_{\mathbb{X}_i} = (\alpha_{i+1}, \alpha_{i+2}, \dots, \alpha_t)$.

Proof. It is sufficient to show that for each $i = 0, \dots, t-2$, if $\mathbb{X}_i$ is ACM, then $\mathbb{X}_{i+1}$ is ACM. Since $\mathbb{X}_{i+1} = \mathbb{X}_i \backslash \{\pi_1^{-1}(P_{i+1})\}$, $\mathbb{X}_{i+1}$ is constructed from $\mathbb{X}_i$ by removing the α_{i+1} points of $\mathbb{X}_i$ which have P_{i+1} as its first coordinate. The tuple $\beta_{\mathbb{X}_{i+1}}$ is constructed from $\beta_{\mathbb{X}_i}$ by subtracting 1 from α_{i+1} coordinates in $\beta_{\mathbb{X}_i}$. But because $\alpha^*_{\mathbb{X}_i} = \beta_{\mathbb{X}_i}$, we have $|\beta_{\mathbb{X}_i}| = \alpha_{i+1}$ and thus $\beta_{\mathbb{X}_{i+1}} = (\beta_1 - 1, \dots, \beta_{\alpha_{i+1}} - 1) = (\beta_1 - 1, \dots, \beta_{\alpha_{i+2}} - 1)$. But by Lemma 4.1, $\alpha^*_{\mathbb{X}_{i+1}} = \beta_{\mathbb{X}_{i+1}}$, and hence $\mathbb{X}_{i+1}$ is ACM by Theorem 4.8. □

5. The Betti numbers of ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$

If $\mathbb{X}$ is a set of s points in $\mathbb{P}^1$, then it is well known that the graded Betti numbers, and consequently, the Hilbert function of $\mathbb{X}$, can be determined solely from $|\mathbb{X}| = s$. If we restrict to ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$, we can extend this result to show that the graded Betti numbers in the minimal free resolution of $\mathbb{X}$ (and hence, $H_{\mathbb{X}}$) can be computed directly from the tuples $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$ introduced in the previous section which numerically describe $\mathbb{X}$.

We require the following notation to describe the minimal free resolution. Suppose that $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a set of points with $\alpha_{\mathbb{X}} = (\alpha_1, \ldots, \alpha_t)$. Define the following sets:

$$C_{\mathbb{X}} := \{(t,0),(0,\alpha_1)\} \cup \{(i-1,\alpha_i) \mid \alpha_i - \alpha_{i-1} < 0\},$$
$$V_{\mathbb{X}} := \{(t,\alpha_t)\} \cup \{(i-1,\alpha_{i-1}) \mid \alpha_i - \alpha_{i-1} < 0\}.$$

We take $\alpha_{-1} = 0$. With this notation, we have

Theorem 5.1. *Suppose that $\mathbb{X}$ is an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Let $C_{\mathbb{X}}$ and $V_{\mathbb{X}}$ be constructed from $\alpha_{\mathbb{X}}$ as above. Then the graded minimal free resolution of $I_{\mathbb{X}}$ is given by*

$$0 \longrightarrow \bigoplus_{(v_1,v_2)\in V_{\mathbb{X}}} R(-v_1,-v_2) \longrightarrow \bigoplus_{(c_1,c_2)\in C_{\mathbb{X}}} R(-c_1,-c_2) \longrightarrow I_{\mathbb{X}} \longrightarrow 0.$$

Proof. We proceed by induction on the tuple $(|\pi_1(\mathbb{X})|, |\mathbb{X}|)$. If s is any integer, and $(|\pi_1(\mathbb{X}), |\mathbb{X}|) = (1,s)$, then $\alpha_{\mathbb{X}} = (s)$ and $\beta_{\mathbb{X}} = (1,\ldots,1)$ (s times). The conclusion follows from Proposition 4.7 since $C_{\mathbb{X}} = \{(1,0),(0,s)\}$ and $V_{\mathbb{X}} = \{(1,s)\}$.

So, suppose $(|\pi_1(\mathbb{X}), |\mathbb{X}|) = (t,s)$ with $t > 1$. Then

$$\alpha_{\mathbb{X}} = (\underbrace{\alpha_1,\ldots,\alpha_1}_{l}, \alpha_{l+1},\ldots,\alpha_t),$$

i.e., $\alpha_{l+1} < \alpha_1$, but $\alpha_l = \alpha_1$. If $l = t$, then $\mathbb{X}$ is a complete intersection and the resolution is given by Proposition 4.7. The conclusion now follows because $C_{\mathbb{X}} = \{(l,0),(0,\alpha_1)\}$ and $V_{\mathbb{X}} = \{(l,\alpha_1)\}$.

If $l < t$, let $P_1,\ldots,P_l$ be the l points of $\pi_1(\mathbb{X})$ that have $|\pi_1^{-1}(P_i)| = \alpha_1$. Set $\mathbb{Y} = \pi_1^{-1}(P_1) \cup \cdots \cup \pi_1^{-1}(P_l)$. Because $\mathbb{X}$ is ACM, $\alpha_1 = |\beta_{\mathbb{X}}|$, and hence, $\mathbb{Y} = \{P_i \times Q_j \mid 1 \leqslant i \leqslant l,\ Q_j \in \pi_2(\mathbb{X})\}$. So, $\alpha_{\mathbb{Y}} = (\alpha_1,\ldots,\alpha_1)$ and $\beta_{\mathbb{Y}} = (l,\ldots,l)$, and thus $\mathbb{Y}$ is a complete intersection. In fact, $I_{\mathbb{Y}} = (L_{P_1}\cdots L_{P_l}, L_{Q_1}\cdots L_{Q_{\alpha_1}})$ where L_{P_i} is the form of degree $(1,0)$ that vanishes at all the points of $\mathbb{P}^1 \times \mathbb{P}^1$ which have P_i as their first coordinate, and L_{Q_i} is the form of degree $(0,1)$ that vanishes at all points $P \times Q \in \mathbb{P}^1 \times \mathbb{P}^1$ such that $Q = Q_i$.

Let $F := L_{P_1}\cdots L_{P_l}$ and $G := L_{Q_1}\cdots L_{Q_{\alpha_1}}$. By Proposition 4.7, the resolution of $I_{\mathbb{Y}}$ is

$$0 \longrightarrow R(-l,-\alpha_1) \xrightarrow{\phi_2} R(-l,0) \oplus R(0,-\alpha_1) \xrightarrow{\phi_1} I_{\mathbb{Y}} \longrightarrow 0,$$

where $\phi_1 = [F\ G]$ and $\phi_2 = \left[\begin{smallmatrix} G \\ -F \end{smallmatrix}\right]$. Let $\mathbb{Z} := \mathbb{X}\backslash\mathbb{Y}$. Since $\pi_2(\mathbb{Z}) \subseteq \pi_2(\mathbb{X})$, it follows that $G = L_{Q_1}\cdots L_{Q_{\alpha_1}} \in I_{\mathbb{Z}}$. Hence, $\operatorname{im}\phi_2 \subseteq I_{\mathbb{Z}}(-l,0) \oplus R(0,-\alpha_1)$. □

Claim. $I_{\mathbb{X}} = F \cdot I_{\mathbb{Z}} + (G)$.

Proof. By construction, $\mathbb{X} = \mathbb{Z} \cup \mathbb{Y}$. Hence, we want to show that $I_{\mathbb{Z}} \cap I_{\mathbb{Y}} = F \cdot I_{\mathbb{Z}} + (G)$. The proof now follows as in the proof of the Claim in Theorem 4.8. □

From the above resolution for $I_{\mathbb{Y}}$, the claim, and the fact that $\operatorname{im}\phi_2 \subseteq I_{\mathbb{Z}}(-l, 0) \oplus R(0, -\alpha_1)$, we have the following short exact sequence of R-modules:

$$0 \longrightarrow R(-l, -\alpha_1) \xrightarrow{\phi_2} I_{\mathbb{Z}}(-l, 0) \oplus R(0, -\alpha_1) \xrightarrow{\phi_1} I_{\mathbb{X}} = F \cdot I_{\mathbb{Z}} + (G) \longrightarrow 0,$$

where ϕ_1 and ϕ_2 are as above.

By Corollary 4.10 the set $\mathbb{Z}$ is ACM with $\alpha_{\mathbb{Z}} = (\alpha_{l+1}, \ldots, \alpha_t)$. Therefore, the induction hypothesis holds for $\mathbb{Z}$. With the above short exact sequence, we can use the *mapping cone construction* (see [14, Section 1.5]) to construct a resolution for $I_{\mathbb{X}}$. In particular, we get

$$\begin{aligned} 0 \longrightarrow \Bigg[\bigoplus_{(v_1,v_2)\in V_{\mathbb{Z}}} R\big(-(v_1+l), -v_2\big)\Bigg] \oplus R(-l, -\alpha_1) \\ \longrightarrow \Bigg[\bigoplus_{(c_1,c_2)\in C_{\mathbb{Z}}} R\big(-(c_1+l), -c_2\big)\Bigg] \oplus R(0, \alpha_1) \longrightarrow I_{\mathbb{X}} \longrightarrow 0. \end{aligned}$$

Since the resolution has length 2, and because $\mathbb{X}$ is ACM, the resolution of $I_{\mathbb{X}}$ cannot be made shorter by the Auslander–Buchsbaum formula (cf. [14, Theorem 4.4.15]).

To show that this resolution is minimal, it is enough to show that no tuple in the set $\{(c_1+l, c_2) \mid (c_1, c_2) \in C_{\mathbb{Z}}\} \cup \{(0, \alpha_1)\}$ is in the set $\{(v_1+l, v_2) \mid (v_1, v_2) \in V_{\mathbb{Z}}\} \cup \{(l, \alpha_1)\}$. By the induction hypothesis, we can assume that no $(c_1, c_2) \in C_{\mathbb{Z}}$ is in $V_{\mathbb{Z}}$, and hence, if $(c_1+l, c_2) \in \{(c_1+l, c_2) \mid (c_1, c_2) \in C_{\mathbb{Z}}\}$, then (c_1+l, c_2) is not in $\{(v_1+l, v_2) \mid (v_1, v_2) \in V_{\mathbb{Z}}\}$. If $(c_1+l, c_2) = (l, \alpha_1)$ for some $(c_1, c_2) \in C_{\mathbb{Z}}$, then this implies that $(0, \alpha_1)$. But this contradictions the induction hypothesis. Similarly, if $(0, \alpha_1) \in \{(v_1+l, v_2) \mid (v_1, v_2) \in V_{\mathbb{Z}}\}$, this implies $(-l, \alpha_1) \in V_{\mathbb{Z}}$, which is again a contradiction of the induction hypothesis. So the resolution above is minimal.

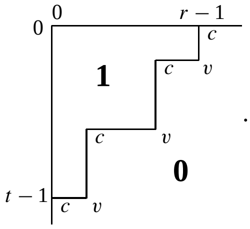
To complete the proof we only need to verify that

(i) $C_{\mathbb{X}} = \{(c_1+l, c_2) \mid (c_1, c_2) \in C_{\mathbb{Z}}\} \cup \{(0, \alpha_1)\}$,
(ii) $V_{\mathbb{X}} = \{(v_1+l, v_2) \mid (v_1, v_2) \in V_{\mathbb{Z}}\} \cup \{(l, \alpha_1)\}$.

Because the verification of these statements is tedious, but elementary, we omit the details. □

Remark 5.2. It was shown in [5, Theorem 4.1] that the graded Betti numbers for an ACM set of points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ could be determined via the first difference function $\Delta H_{\mathbb{X}}$, i.e.,

$\Delta H_{\mathbb{X}} =$



.

An element of $C_{\mathbb{X}}$, which [5] called a *corner* of $\Delta H_{\mathbb{X}}$, corresponds to a tuple (i, j) that is either $(t, 0)$, $(0, \alpha_1) = (0, r)$, or has the property that $\Delta H_{\mathbb{X}}(i, j) = 0$, but $\Delta H_{\mathbb{X}}(i-1, j) = \Delta H_{\mathbb{X}}(i, j-1) = 1$. We have labeled the corners of $\Delta H_{\mathbb{X}}$ with a c in the above diagram. An element of $V_{\mathbb{X}}$ is a *vertex*. A tuple (i, j) is called a vertex if $\Delta H_{\mathbb{X}}(i, j) = \Delta H_{\mathbb{X}}(i-1, j) = \Delta H_{\mathbb{X}}(i, j-1) = 0$, but $\Delta H_{\mathbb{X}}(i-1, j-1) = 1$. We have labeled the vertices of $\Delta H_{\mathbb{X}}$ with a v in the above diagram. Besides giving a new proof for the resolution of an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1$, we have shown that the graded Betti numbers can be computed directly from the tuple $\alpha_{\mathbb{X}}$.

Using the resolution as given in Theorem 5.1 we can compute $H_{\mathbb{X}}$ directly from $\alpha_{\mathbb{X}}$ for any ACM set of points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$. Formally:

Corollary 5.3. *Let $\mathbb{X}$ be an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with $\alpha_{\mathbb{X}} = (\alpha_1, \ldots \alpha_t)$. Then*

$$H_{\mathbb{X}} = \begin{bmatrix} 1 & 2 & \cdots & \alpha_1 - 1 & \alpha_1 & \alpha_1 & \cdots \\ 1 & 2 & \cdots & \alpha_1 - 1 & \alpha_1 & \alpha_1 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 1 & 2 & \cdots & \alpha_2 - 1 & \alpha_2 & \alpha_2 & \cdots \\ 1 & 2 & \cdots & \alpha_2 - 1 & \alpha_2 & \alpha_2 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 1 & 2 & \cdots & \alpha_3 - 1 & \alpha_3 & \alpha_3 & \cdots \\ 1 & 2 & \cdots & \alpha_3 - 1 & \alpha_3 & \alpha_3 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix} + \cdots + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 1 & 2 & \cdots & \alpha_t - 1 & \alpha_t & \alpha_t & \cdots \\ 1 & 2 & \cdots & \alpha_t - 1 & \alpha_t & \alpha_t & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Acknowledgments

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