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SMOOTH CONCORDANCE OF TOPOLOGICALLY SLICE KNOTS

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§0. INTRODUCTION

THE SET of smooth knots $S^1 \subsetneq S^3$ (up to isotopy) is of great interest to low-dimensional topologists. A standard method for studying this set is to turn it into a group, by means of an equivalence relation called *smooth concordance*. Roughly speaking, two knots K_0 and K_1 are smoothly concordant if there is a smoothly embedded annulus in $S^3 \times I$ with boundary $K_0 \times 0 \cup K_1 \times 1$. The set $\mathscr{K}_{\text{DIFF}}$ of smooth concordance classes of knots is an abelian group under the operation of band-connected sum (respecting string orientations). Elements of the identity class are called *smoothly slice*. A smoothly slice knot in $S^3 = \partial B^4$ is one which bounds a smoothly embedded 2-disk in B^4.

The group $\mathscr{K}_{\text{DIFF}}$ may be studied by forming several quotient groups. In particular, we obtain the *topological concordance group* $\mathscr{K}_{\text{TOP}}$ by allowing flat topological annuli in the above discussion, rather than just smooth ones. Inclusion gives a canonical epimorphism $i: \mathscr{K}_{\text{DIFF}} \to \mathscr{K}_{\text{TOP}}$ whose kernel is the group of smooth concordance classes of knots which are *topologically slice*, i.e., which bound flat topological disks in B^4. The group $\mathscr{K}_{\text{TOP}}$ can be collapsed still further, by modding out the subgroup of *algebraically slice* knots to obtain a group $\mathscr{K}_{\text{ALG}}$. (Specifically, a knot is algebraically slice if it has a Seifert surface whose Seifert form vanishes on a half-dimensional subspace. This is an invariant of topological concordance.)

The group $\mathscr{K}_{\text{ALG}}$ is well-understood. (It contains a free abelian group of infinite rank, as well as 2- and 4-torsion.) We would also like to understand the kernels of the maps $\mathscr{K}_{\text{DIFF}} \to \mathscr{K}_{\text{TOP}} \to \mathscr{K}_{\text{ALG}}$. In higher dimensions, the corresponding kernels vanish; $\mathscr{K}_{\text{DIFF}} = \mathscr{K}_{\text{ALG}}$. In our situation, however, this is false. By means of Casson–Gordon invariants, it has been shown that the kernel of $\mathscr{K}_{\text{TOP}} \to \mathscr{K}_{\text{ALG}}$ contains a free abelian group of infinite rank.

The remaining map $i: \mathscr{K}_{\text{DIFF}} \to \mathscr{K}_{\text{TOP}}$ has resisted any analysis until quite recently, since all available tools have been basically topological. This situation has now changed, due to Donaldson's Theorem [2].

Donaldson has proven that no smooth, closed, simply connected 4-manifold can have a nonstandard definite intersection form. The proof depends heavily on gauge theory, so the result is intrinsically smooth in nature. The theorem implies that in dimension 4, smooth surgery is impossible in many cases where topological surgery can be accomplished by Freedman theory. As a result, many locally flat topological embeddings of 2-spheres in 4-manifolds cannot be smoothed. This observation allows the construction of knots which are topologically slice, but not smoothly slice. (This was first noticed by Casson.) In the present paper, we use this theory to show that $\ker i$ contains elements of infinite order.

We define a smooth concordance invariant, called *kinkiness*, which detects knots with infinite order in $\mathscr{K}_{\text{DIFF}}$. In §1 and §2 we exhibit a large collection of knots with infinite order in $\ker i$. While we do not actually prove that $\ker i$ is larger than $\mathbb{Z}$, it now seems a good conjecture that $\ker i$ contains a free abelian group of infinite rank. In §1, we construct (via an earlier paper [5]) knots with arbitrarily large kinkiness. Since kinkiness gives a lower bound

for the unknotting number, this also shows that there are Alexander polynomial one knots with arbitrarily high unknotting number. In §2, we use a different method to construct an explicit infinite family of "doubled" knots with nonzero kinkiness.

Doubling is a procedure for obtaining new knots from old ones. Freedman has shown that the double of any knot is topologically slice (since it has Alexander polynomial one). Since doubling is compatible with smooth concordance, it defines a function (of sets) $D:\mathscr{K}_{\text{DIFF}} \to \ker i$. The results of §1 imply that D is not onto (and not a homomorphism). In §2, we exhibit knots which may be doubled several times without becoming smoothly slice. In particular, the trefoil knot may be doubled up to 5 times (at least) and still have infinite order in ker i. We also give an infinite family of twists knots which may be doubled up to 7 times and still have infinite order. It seems quite plausible that repeated doubling will never make these knots smoothly slice. (Note that these results depend crucially on the orientation conventions given in §2.)

§1. KINKINESS OF KNOTS

For the remainder of this paper, we work in the smooth category unless otherwise stated.

Two knots $K_0, K_1 : S^1 \to S^3$ are called *concordant* if there is an embedding of an annulus $f : S^1 \times I \subset S^3 \times I$ such that for $t = 0, 1$, $f^{-1}(S^3 \times t) = S^1 \times t$, and $f | S^1 \times t$, as a map from S^1 to S^3, equals K_t. Note that the trivial annulus $K_1 \times \text{id}$ shows that the inverse of K_1 in $\mathscr{K}_{\text{DIFF}}$ is given by K_1 with reversed orientations on both S^1 and S^3 (i.e. the mirror image of K_1 with reversed string orientation).

In [5], an invariant called "kinkiness" was defined for circles in boundaries of 4-manifolds. This was used to distinguish an infinite family of nondiffeomorphic Casson handles. We restate the definition for a knot K in $S^3 = \partial B^4$. Consider all self-transverse immersed disks in B^4 with boundary K. Let $k_+(K)$ be the minimum number of positive kinks (points of self-intersection) occurring in any such disk. Similarly, let $k_-(K)$ be the minimum number of negative kinks in any such disk.

Definition. The *kinkiness* of the knot K is the ordered pair $k(K) = (k_+, k_-)$ as defined above.

Thus, a smoothly slice knot has kinkiness (0, 0). (Note that the converse is false; the figure-eight knot has kinkiness (0, 0) but is not slice.) Kinkiness is easily seen to be an invariant of smooth concordance; hence, it gives a map (of sets) $k : \mathscr{K}_{\text{DIFF}} \to \mathbb{Z} \times \mathbb{Z}$. Note that if we take the mirror image of a knot, k_+ and k_- are interchanged, although k is independent of string orientation.

Let $[K]$ denote the smooth concordance class of K.

THEOREM 1.1. *If $k(K) = (0, n)$, $n \neq 0$, then $[K]$ has infinite order in $\mathscr{K}_{\text{DIFF}}$.*

Proof. Suppose $[K]$ has finite order r, i.e., $r[K] = 0$ $(r \geq 2)$. Then $(r-1)[K] = -[K]$. This is represented by the mirror image of K; hence, $(r-1)[K]$ has kinkiness $(n, 0)$. But $k_+(K) = 0$, so K bounds an immersed disk with no positive kinks. The same is clearly true for the connected sum of $r-1$ copies of K, so $k_+((r-1)[K]) = 0$, not n as claimed above. QED

Similar reasoning shows that if $k(K) = (0, n)$, $n \neq 0$, then the connected sum of K with any knots having $k_+ = 0$ will have infinite order. In §2, we will give an infinite collection of knots in ker i (iterated doubles of twist knots) with kinkiness (0, 1). In this section, we use the results of [5] to prove the following:

THEOREM 1.2. *For any integer $n \geq 0$, there is a knot K in* ker i *with* $k(K) = (0, n)$.

Proof. Fix $n > 0$. In [5] we constructed a Casson handle $CH_{0,n}$ whose first stage kinky handle has exactly n kinks, all negative. $CH_{0,n}$ has the property that any immersed disk in $CH_{0,n}$ spanning the attaching circle necessarily has at least n negative kinks. Furthermore, from the method of construction, it follows that this property is stable under connected sum with $\mathbb{C}P^2$'s; i.e., any spanning disk in $CH_{0,n} \# (\#_m \mathbb{C}P^2)$ must have at least n negative kinks, provided that it is algebraically disjoint from the $\mathbb{C}P^2$'s

Let T_6 be the first 6 stages of $CH_{0,n}$. For each top stage core disk, blow up a $\mathbb{C}P^2$ at each kink. We may then replace the core by an embedded disk. Note that at each negative kink, the new disk will run algebraically twice over the corresponding $\mathbb{C}P^2$, introducing 4 twists in the framing. (See [5] for a discussion of this.) Let T denote T_5 union a tubular neighborhood of each new disk, inside $T_6 \# (\#_m \mathbb{C}P^2)$. T looks like a 6-stage tower with no kinks on the top stage, except that the top stage framings are wrong. In particular, T is diffeomorphic to B^4, but the attaching circle is some nontrivial knot K in ∂T (due to the twisted framings).

The knot K has kinkiness $(0, n)$. In fact, $k_-(K) \geq n$ by previous remarks, since T is embedded in $T_6 \# (\#_m \mathbb{C}P^2) \subset CH_{0,n} \# (\#_m \mathbb{C}P^2)$. But K bounds the first stage core disk of T_5 in T. This has no positive kinks and n negative kinks, so $k(K) = (0, n)$. Furthermore, T_5 is a 5-stage tower, so it contains a flat topological disk spanning K [6]. Thus, K is in ker i.

QED

Actually, it is sufficient to use the first 3 stages, T_3, in place of T_6 in the above argument. Careful inspection shows that the resulting knot K will be a connected sum of n knots, each of which is a double of a connected sum of twist knots (i.e., the knots K_l, $l \geq 1$, defined in §2). Since Freedman has shown that any double is topologically slice, it follows that K is in ker i. (This also demonstrates that the doubling map $D: \mathscr{K}_{\text{DIFF}} \to \ker i$ cannot be a homomorphism, for K is a connected sum of doubles, but it cannot be concordant to a double unless $n = 0$ or 1.)

§2. DOUBLES OF TWIST KNOTS

In this section, we show that many iterated doubles of twist knots have kinkiness $k = (0, 1)$. We do this modulo the main lemma, which is proven in §3. We also give other applications of the main lemma in this section.

Given a knot K, we define its double, DK, as follows: Consider the solid torus T with inscribed circle C shown in Fig. 1. Map T diffeomorphically to a tubular neighborhood of K, so that (1) the orientation of T is preserved, and (2) an untwisted longitude in ∂T goes to a pushoff of K defining the 0-framing. Define DK to be the image of C (well defined up to isotopy). Let $D^n K$ be the knot obtained from K by n successive doublings.

Note that we require our doubles to be untwisted, and to have negative clasps as in Fig. 1.

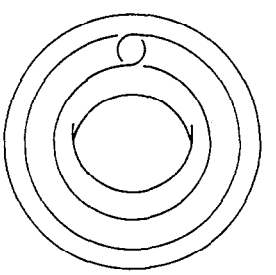

Fig. 1.

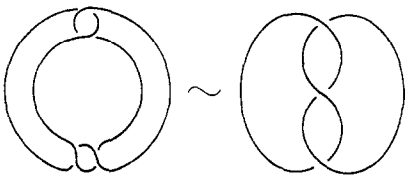

Fig. 2.

(If we consider the figure in $S^3 = \partial B^4$, then C clearly bounds an immersed disk in a small neighborhood of T in B^4. This disk has a single kink, which is negative.) It follows that the kinkiness of any double must be (0, 0) or (0, 1).

We let K_l denote the "l-twisted double" of the unknot. That is, we form K_l from Fig. 1 by putting l right-handed twists in T. Thus, K_1 is the left-handed trefoil knot shown in Fig. 2.

THEOREM 2.1. *The trefoil and its first 5 doubles* (D^nK_1, $0 \leq n \leq 5$) *have kinkiness* $k = (0, 1)$. *Similarly, for* $l > 1$, K_l *and its first 7 doubles* (D^nK_l, $0 \leq n \leq 7$) *have* $k = (0, 1)$.

Thus, all of the above knots D^nK_l have infinite order in $\mathscr{K}_{\text{DIFF}}$. Furthermore, those with $n \geq 1$ all lie in $\ker i$ by Freedman's results. Note that orientations are crucial here, since K_{-2} is the Stevadore's knot, which is smoothly slice. Theorem 2.1 follows from Lemma 2.3, which we will state after developing suitable notation.

Let T_n denote the n-stage tower with a single kink at each stage, and all kinks negative. (For the theory of towers, see [3] or [6].) Construct $\tilde{T}_n$ from T_n and a 4-ball B^4 by gluing T_n onto an unknot in ∂B^4 with framing -1. Now construct $\tilde{T}_{n,l}$ from $\tilde{T}_n$, as follows: Find a standard framed circle on the top stage of T_n, such that attaching a kinky handle along this would yield T_{n+1}. Now add a 2-handle to this curve on $\tilde{T}_n$, with l right twists relative to the standard framing. Thus, $\tilde{T}_{n,0}$ is a -1 Hopf bundle, since the tower collapses to a 2-handle. Note that $H_2(\tilde{T}_n) \cong H_2(\tilde{T}_{n,l}) \cong \mathbb{Z}$, and a generator is given by the core of the first stage of T_n (capped by a disk in B^4).

LEMMA 2.2. $\tilde{T}_{n+1,l}$ *is diffeomorphic to the handlebody obtained by gluing a 2-handle to* B^4 *along the knot* D^nK_l *with framing* -1.

Proof. Recall that T_{n+1} is diffeomorphic to $S^1 \times D^3$, with attaching circle an $(n+1)$-fold Whitehead curve (essentially $D^{n+1}(S^1 \times \text{point in } \partial D^3)$). To get $\tilde{T}_{n+1}$, we glue this onto a 4-ball. Equivalently, we may think of the 4-ball as a 2-handle glued to the attaching curve of T_{n+1}. Now glue on the last handle to obtain $\tilde{T}_{n+1,l}$. This turns $T_{n+1} \approx S^1 \times D^3$ into a 4-ball, but puts l twists into the Whitehead curve, yielding D^nK_l as desired. QED

Figure 3 shows a Kirby calculus picture of this argument, with $n = 0$. The first picture shows $\tilde{T}_1$ (with the 4-ball represented as a 2-handle). In the second picture we have added a

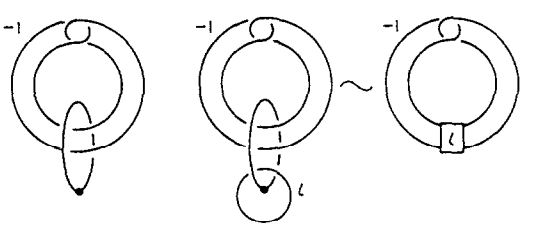

Fig. 3.

handle to get $\tilde{T}_{1,l}$. This handle cancels the 1-handle to yield the desired handlebody on K_l. The pictures for $n > 0$ are similar; simply double the -1-framed curve n times in each picture. (The reader should verify that doubling commutes with the twisting.)

Now, let $M_m = \overline{\mathbb{C}P}^2 \# (\#_m \mathbb{C}P^2)$. Let $e_0, e_1, \ldots, e_m$ be the standard generators of $H_2(M_m)$ (with $e_0 \cdot e_0 = -1$). For convenience, let $N \subset \mathbb{Z} \times \mathbb{Z}$ be the set of pairs (n, l) with $0 \leq n \leq 5$, $l \geq 1$ or $n = 6$, $l \geq 2$ or $n = 7$, $l \geq 3$.

LEMMA 2.3. *For m sufficiently large, $\tilde{T}_8$ has an orientation-preserving embedding in M_m representing the class $x = 3e_0 + \Sigma_{i=1}^{8} e_i$. Similarly, for each (n, l) in N, $\tilde{T}_{n+1,l}$ embeds in M_m (for m large) representing x. These embeddings all have 1-connected complements (except for $\tilde{T}_{8,3}$).*

The proof of this (by direct construction) constitutes §3. The remainder of §2 is devoted to applications. Note that Casson's embedding theorem [1] guarantees the existence of arbitrarily large towers representing x in M_9. Thus, the difficulty in Lemma 2.3 comes from allowing only one kink at each stage.

Proof of Theorem 2.1. First, we show that x (in Lemma 2.3) cannot be represented by a smoothly embedded sphere. Consider the case $m = 8$. Now $x \cdot x = -1$, so x generates an orthogonal summand of the intersection form. Its complement, $x^{\perp}$, has rank = signature $= 8$, so it is positive definite. Furthermore, x is clearly characteristic ($x \cdot y \equiv y \cdot y \pmod 2$ for all y in $H_2(M_8)$) so $x^{\perp}$ is an even form. (In fact, it must be E_8; see [7].) For larger values of m, we merely add $\langle 1 \rangle$ summands to $x^{\perp}$. The E_8 persists, so $x^{\perp}$ is a nonstandard definite form for all $m \geq 8$ (see [7]). But if x were represented by a smooth sphere, we could blow it down as in [5] to get a manifold with form $x^{\perp}$, contradicting Donaldson's Theorem [2].

Now pick (n, l) in N and suppose $D^n K_l$ is slice. By Lemma 2.2, $\tilde{T}_{n+1,l}$ is obtained by gluing a 2-handle to B^4 along $D^n K_l$. The slice disk union the core of the 2-handle gives an embedded sphere in $\tilde{T}_{n+1,l}$ generating H_2. By Lemma 2.3, we now obtain an embedded sphere representing x, which is impossible. Similarly, if $k_-(D^n K_l) = 0$, we have an immersed disk with only positive kinks spanning $D^n K_l$, which gives a sphere with only positive kinks representing x. We can remove these kinks by blowing up $\mathbb{C}P^2$'s as in [5], obtaining an embedded sphere representing x for a larger value of m. Hence, $k_-(D^n K_l) \neq 0$.

Now note that any double of a knot spans an obvious disk with only one kink (as does K_l); thus, $k(D^n K_l) = (0, 1)$. This completes the proof for (n, l) in N. The only remaining case is $D^7 K_2$. This case follows from a modification of Lemma 2.3, which we discuss in section 3.12. QED

COROLLARY 2.4. *The n-fold Whitehead link Wh_n is not smoothly slice for $n \leq 7$.*

The link Wh_n is obtained from the Hopf link by doubling one component n times. Freedman has shown that Wh_n is topologically slice for $n \geq 4$ [4]. (For $n \geq 5$, see [6].)

Proof. Suppose Wh_n is slice for some $n \leq 7$. Then in particular, Wh_7 is slice. By a remark of Casson [1], it follows that Wh_8 may be sliced by a pair of disks D_1 and D_2 with D_1 unknotted in B^4. (This is basically an observation of Gordon that the double of any slice disk for the unknot is itself an unknotted disk.) But $B^4 - (\text{tubular neighborhood of } D_1) \approx S^1 \times D^3 \approx T_8$, and D_2 gives a disk spanning the attaching circle of T_8. Thus, there is an embedded sphere generating $H_2(\tilde{T}_8)$. Lemma 2.3 now gives a sphere representing x in M_m, but we have shown that this violates Donaldson's Theorem. QED

COROLLARY 2.5. *Let $\Sigma_{n,l}$ denote the homology 3-sphere obtained by -1 surgery on $D^n K_l$.*

For (n, l) in N, $\Sigma_{n,l}$ does not bound any positive definite manifold W with $\pi_1(W) = 0$. For $(n, l) \neq (7, 3)$, we may weaken the π_1 condition to: the image of $\pi_1(\partial W) \to \pi_1(W)$ normally generates $\pi_1(W)$.

Note that for $n > 0$ (or l even), $\Sigma_{n,l}$ has μ-invariant zero.

Proof. By Lemma 2.2, $\Sigma_{n,l}$ is the boundary of $\tilde{T}_{n+1,l}$. Lemma 2.3 gives an embedding of $\tilde{T}_{n+1,l}$ in M_m. If $\Sigma_{n,l}$ bounded a manifold W as given above, we could remove int $\tilde{T}_{n+1,l}$ from M_m, and replace it by W. This would give a positive definite, 1-connected manifold contradicting Donaldson's Theorem. (Recall that for $(n, l) \neq (7, 3)$, $M_m - \text{int } \tilde{T}_{n+1,l}$ is 1-connected.) QED

Note that $\Sigma_{n,l}$ actually is the oriented boundary of a nonstandard negative definite manifold, namely $M_m - \text{int } \tilde{T}_{n+1,l}$ (with reversed orientation).

Here is a way to interpret these results. Call two homology spheres Σ_1 and Σ_2 equivalent if there is an oriented bordism V between them such that each inclusion $\Sigma_i \to V$ induces a homology isomorphism and a π_1 epimorphism (or alternatively, has image which normally generates $\pi_1(V)$). Equivalence classes of homology spheres form a group under connected sum. The above remark shows that for $(n, l) \in N - (7, 3)$, $\Sigma_{n,l}$ has infinite order in this group. (In fact, any oriented connected sum of these (varying n and l) cannot bound an acyclic manifold with π_1 onto, since it bounds a nonstandard, negative definite, 1-connected manifold.)

§3. PROOF OF LEMMA 2.3

We will prove the lemma by explicitly constructing the desired embeddings, one stage at a time. Thus, we may truncate the argument at any stage to obtain partial results. For example, to show that the double of the trefoil, DK_1, has kinkiness (0, 1), it suffices to embed $\tilde{T}_{2,1}$.

Recall that $\tilde{T}_8 = B^4 \cup T_8$, T_8 the simplest 8-stage tower. We embed this as follows: First we embed B^4 in M_m, m sufficiently large. Then we embed the core of T_8 (a 2-complex) in $M_m - \text{int } B^4$. A regular neighborhood of this will be T_8. The core will be embedded one stage at a time. For the first stage, we must construct an immersed disk c_1 with exactly one kink, which represents the correct homology class x, and intersects ∂B^4 in $\partial c_1 =$ unknot in ∂B^4. A regular neighborhood of c_1 will be k_1, the first stage kinky handle. Note that k_1 will be attached to B^4 with the correct framing, since $x \cdot x = -1$. The second stage core must be a disk c_2 with one kink, which intersects c_1 only at $\partial c_2 =$ a circle in c_1 generating $\pi_1(c_1)$. It will be necessary to check that the corresponding kinky handle k_2 attaches to k_1 with the correct framing. We must then successively construct $c_3, \ldots, c_8$, and verify that each k_n attaches to k_{n-1} with the correct framing. This will give the desired embedding of $\tilde{T}_8$. Note that each kink will necessarily be negative, as required; a positive kink could be eliminated by blowing up a $\mathbb{C}P^2$ as in [5], changing T_8 to a standard 2-handle and contradicting Donaldson's theorem.

The embeddings of $\tilde{T}_{n,l}$ will generally be visible during the above construction. We will merely need to modify $\tilde{T}_{n+1}$ by replacing c_{n+1} by an embedded disk with suitably altered framing.

We will represent $M_m - \text{int } B^4$ as a handlebody: one 0-handle and $m + 1$ 2-handles. The 2-handles represent the standard generators $e_0, \ldots, e_m$ of $H_2(M_m)$; let h_i denote the handle representing e_i. Identify the 0-handle as $B^3 \times I$, with the 2-handles attached to the top face. These will attach to unlinked and unknotted circles, with framings ± 1. (Only h_0 has framing -1). In order to scrutinize embeddings in $B^3 \times I$, we represent the I coordinate by time and observe pictures of successive $B^3 \times$ point slices. We let time increase downward, away from

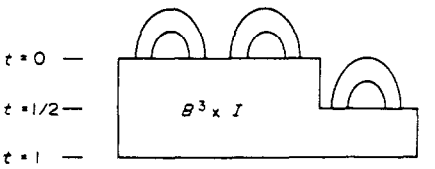

Fig. 4.

the 2-handles. For convenience, we distort the model slightly so that some 2-handles are attached at time $t = \frac{1}{2}$. This is shown schematically in Figure 4.

Our embedded 2-complex will intersect any given 2-handle in a disjoint union of disks parallel to the core of the handle. The boundaries of these disks will be visible at $t = 0$ or $\frac{1}{2}$ as a family of parallel circles, linked together by a ± 1 twist (with -1 only for h_0). We will not draw the handles themselves, but we will assume an excess of them (i.e., m large) so that such collections of circles will always be available when necessary.

We will now begin the explicit construction. We will describe the first few stages carefully to establish the methods involved. Subsequent stages will then be relatively straightforward. A picture of the entire construction appears at the end of this section.

3.1. *Stage* 1. To construct c_1, we begin with 11 oriented disks: three in h_0 and one each in $h_1, \ldots, h_8$. (We attach $h_0, \ldots, h_4$ at $t = 0$, and $h_5, \ldots, h_8$ at $t = \frac{1}{2}$.) The rest of c_1 will lie in the 0-handle, so it will represent the correct homology class $x = 3e_0 + \Sigma_{i=1}^{8} e_i$. The part of c_1 in the 0-handle is shown in Fig. 5. This depicts seven different instants of time, with the pictures ordered from left to right. In the first picture, $t = 0$, we see the oriented boundaries of the disks in $h_0, \ldots, h_4$. As time increases, these are joined together by bands (1-handles) which respect the orientations.

The second picture shows the resulting circle, which isotopes to the left-handed trefoil in the third picture. As time continues to increase, the knot pushes through itself at $t = \frac{1}{4}$ to become an unknot. This creates the kink in c_1, depicted as a large dot in the center picture. At $t = \frac{1}{2}$, the boundaries of the remaining disks appear and trivially band into the circle. We are then left with an unknot at $t = 1$ as required.

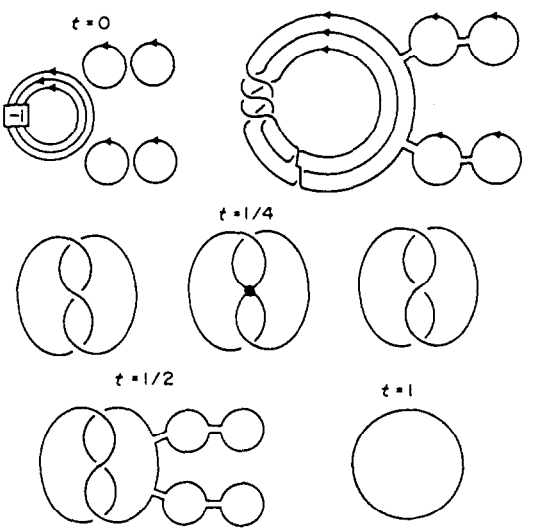

Fig. 5.

Note that the complement of c_1 is 1-connected. To see this, observe that a disk in (for example) h_8 bounds a meridian at $t = \frac{1}{2}$, due to the $+1$ twist in the attaching map. In fact, this gives an embedded S^2 which intersects c_1 exactly once. We will call such an object a *Norman sphere* for c_1, after [8]. Note that this Norman sphere has a twisted normal bundle. We obtain 8 such disjoint spheres from $h_1, \ldots, h_8$.

3.2. *The attaching circle for stage* 2

We must choose carefully the framed circle in ∂k_1 to which we will attach k_2. Now k_1 is a regular neighborhood of c_1; at the time $t = \frac{1}{4}$ (the center picture of Fig. 5) k_1 appears as in Fig. 6. The dashed curve appearing on the boundary is the desired circle. Note that this circle generates $\pi_1(k_1)$, and it is obtained from an embedded circle in c_1 by pushing out to ∂k_1. Thus, attaching k_2 along this curve will yield a 2-stage tower, provided that the framing is correct.

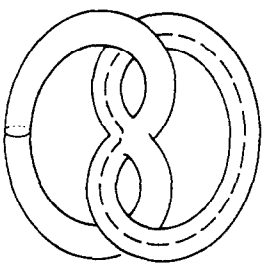

Fig.6.

The correct framing is the one induced by time; i.e., one vector field is tangent to the surface at $t = \frac{1}{4}$ formed by ∂k_1. To prove this, we use the following fact: Attaching a 2-handle to the correct framed curve transforms $(k_1, \partial c_1)$ into $(B^4$, unknot), whereas the incorrect framing results in (B^4, nontrivial knot). Our strategy is to abstractly attach a 2-handle h along the indicated framed curve, then to isotope an unknot onto ∂c_1 within $\partial(k_1 \cup h)$.

Choose a time $t_0 < \frac{1}{4}$, corresponding to the third picture of Fig. 5. If we restrict attention to $t \leq t_0$ and handles $h_0, \ldots, h_4$, then c_1 restricts to an embedded disk, and k_1 restricts to a regular neighborhood of this. Push the disk straight out to ∂k_1, preserving the t coordinate near t_0. The circle C bounding the disk now is clearly an unknot in ∂k_1 (and will remain unknotted when we add h). Figure 7 shows the circle C. (The number of meridinal twists will not concern us.)

We isotope C by letting t increase. Figure 8 shows $t = \frac{1}{4}$. Note that we cannot carry this

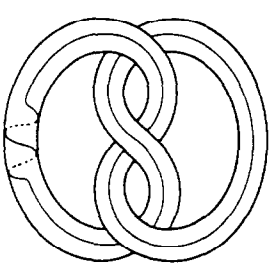

Fig. 7.

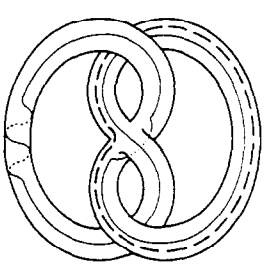

Fig. 8.

isotopy much farther, because of problems which occur near the kink of c_1. We use the 2-handle h to correct these problems.

Note that an arc of C runs parallel to the attaching curve of h (the dashed curve). If we imagine h to be abstractly attached here, we may slide this arc over h (in $\partial(k_1 \cup h)$). Figure 9 shows a neighborhood of the kink. The handle slide is portrayed in the first two pictures. Crosshatching indicates the arc of C which has been slid over h. We have used the fact that h is attached via the framing induced by t; any other framing would cause twisting here involving the time coordinate.

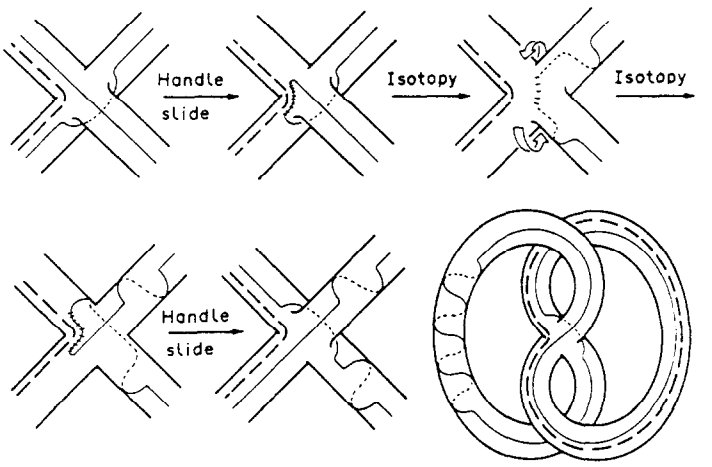

Fig. 9.

In the next two pictures, the crosshatched arc is slid 360° around a vertical circle, as indicated by the arrows. This twists the rest of C around k_1. We then slide the arc back over h. The result is shown in the last picture: We have reversed the crossing near the kink of c_1 (and added several meridinal twists to C).

We next isotope C by increasing t to $\frac{3}{8}$. Now k_1 appears as an unknotted solid torus, with C a (twisted) longitude in the boundary. When we restrict our attention to the remainder of the ambient space ($t \geq \frac{3}{8}$ and $h_5, \ldots, h_8$) c_1 becomes an embedded annulus and k_1 restricts to a regular neighborhood of this. Thus, we may slide C all the way down to $t = 1$, where it still appears as a longitude of a solid torus. (The twisting changes when we go over $h_5, \ldots, h_8$.) The entire solid torus at $t = 1$ lies in ∂k_1, so we may isotope C to the core circle, which is ∂c_1. Hence, ∂c_1 is indeed unknotted in $\partial(k_1 \cup h)$. This completes the proof that we have chosen a correct framed curve for attaching k_2.

3.3. *Stage* 2. In this section we describe the embedding of c_2. The techniques developed here will also be used for most of the higher stages.

Note that at $t = \frac{1}{4}$ there is an embedded disk bounding the required circle with the correct framing; see Fig. 10. Unfortunately, this disk intersects k_1. We fix this with the *Norman trick* (as in [8]). Recall, from section 3.1, the Norman sphere for c_1 associated with handle h_1. We tube this into the given disk, obtaining a new disk D disjoint from k_1 (except along ∂D). Since the Norman sphere represents the homology class e_1, and $e_1 \cdot e_1 = +1$, the framing induced by D will be off by $+1$ twist from the desired one. Thus, thickening D gives us an embedding of $\tilde{T}_{1,1}$.

It is now easy to embed $\tilde{T}_{1,l}$ for $l \geq 2$. To do this, locate $l-1$ disjoint spheres representing $e_9, \ldots, e_{l+7}$ and tube these into D, obtaining a new disk with framing off by l twists. In

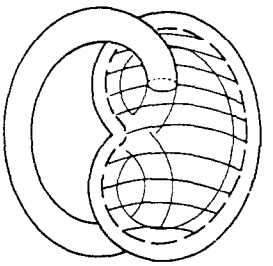

Fig. 10.

general, it is always easy to increase framings in this setting, but difficult to decrease them. For example, an embedding of $\tilde{T}_{n,l}$ automatically gives embeddings of $\tilde{T}_{n,l'}$ for $l' \geq l$.

To embed $\tilde{T}_2$, it is necessary to decrease the framing of D, at the expense of adding a kink. We do this via a trick of Casson [1]. Let τ_1 be the distinguished torus of stage 1; this is embedded near the kink of c_1 and intersects D exactly once. There is a circle C_1 in τ_1 which is a meridian of c_1. Use the Norman sphere for c_1 representing e_2 to make C_1 bound a disk with interior disjoint from c_1, D, and τ_1. Now use this disk to ambiently surger τ_1 into a sphere S. Because the Norman sphere has a twisted normal bundle, the two "parallel" disks used in surgery actually intersect each other once. Thus, S has a single kink. Note that S and τ_1 represent the same class in $H_2(M_m - \operatorname{int} \tilde{T}_1, \partial \tilde{T}_1)$.

Now S intersects D exactly once. We form the connected sum $D \# S$ as follows: Let B be a small 4-ball centered at the intersection point, such that D and S intersect B in a standard pair of transverse disks. Remove these disks, and replace them by an annulus spanning the Hopf link in ∂B. Choose the annulus so that $D \# S$ represents the relative homology class $[D] - [S]$ when D and S are oriented so that $[D] \cdot [S] = +1$. Now the relative homological self-intersection number of $D \# S$ (defined relative to the standard framing given in section 3.2) is given by $[D \# S]^2 = ([D] - [S])^2 = [D]^2 - 2[D] \cdot [S] + [S]^2 = 1 - 2 + 0 = -1$. Thus, if the immersed disk $D \# S$ is thickened to a kinky handle, its framing will be off by -1. We easily correct this by tubing S into a sphere S' representing e_9. The disk $D \# S \# S'$ is the desired c_2.

We show that $M_m - \tilde{T}_2$ is 1-connected: Any loop γ in this space bounds an immersed disk in $M_m - B^4$. We make the disk disjoint from c_2 by finger moves, pushing each intersection across ∂c_2 into c_1. Now remove intersections with c_1, using copies of the Norman sphere representing e_8. We now have the required null homotopy for γ in $M_m - \tilde{T}_2$. The same argument works for $\tilde{T}_{1,l}$. In fact, it will work for all of our constructions through $\tilde{T}_{6,l}$ and $\tilde{T}_7$. We omit further reference to this at these stages. Stage 8 requires further comment, however, because the Norman sphere for e_8 is used in its construction.

3.4. *A Picture of Stage* 2. A neighborhood of the kink in c_1 is depicted in Fig. 11, with the distinguished torus τ_1 shown (dashed curves). Time increases, as usual, from left to right. The

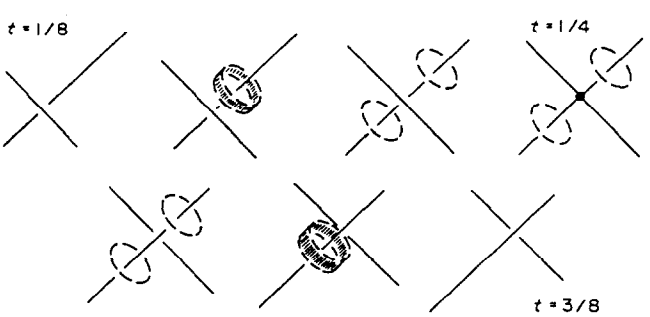

Fig. 11.

torus first appears as an annulus around one sheet of c_1. This splits into two circles which separate around the other sheet of c_1, then converge into another annulus. Thus, τ_1 contains meridians to both sheets of c_1.

We next ambiently surger on the circle C_1 to obtain S. We choose C_1 to be a meridian in the initial annulus of τ_1. The surgery disk runs from C_1 over the handle h_2. This is evident in our picture of S, Fig. 12. At $t = 0$, we see the boundaries of two parallel disks in h_2. As time increases, these become meridians of the trefoil knot formed by c_1. The kink in c_2 occurs in the fourth picture, as the two meridians unlink from each other. We have now described the two "parallel" copies of the surgery disk. Shortly after $t = \frac{1}{8}$, these disks merge into τ_1 (from which the interior of the initial annulus has been removed). Compare the last three pictures with Fig. 11.

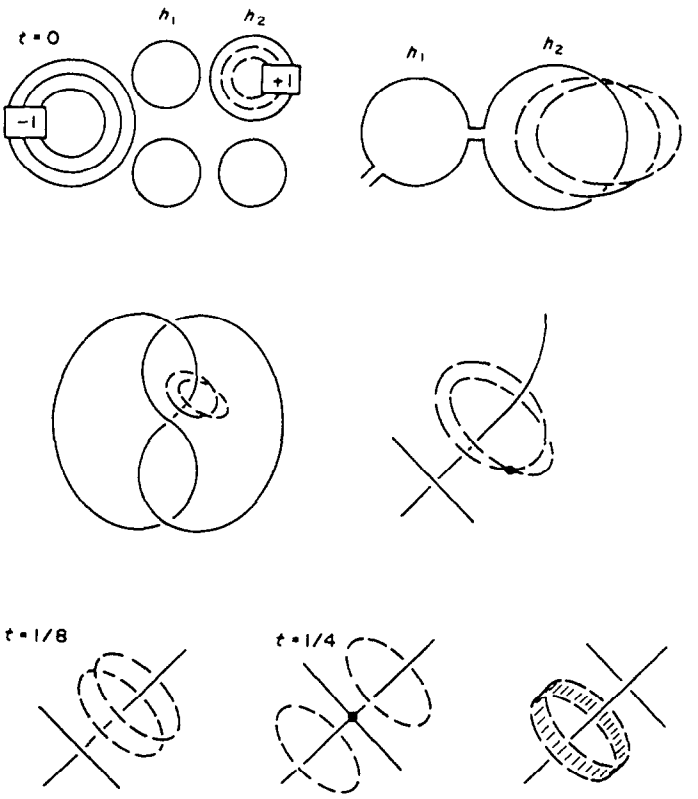

Fig. 12.

We also need a picture of the tubing operation used to form $D \# S$. Before tubing, D and S intersect transversely as in Fig. 13. We now delete a small disk from each surface. (This removes a small segment from each arc in Fig. 13.) Next, we connect the boundary circles by a

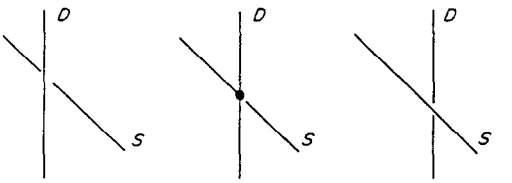

Fig. 13.

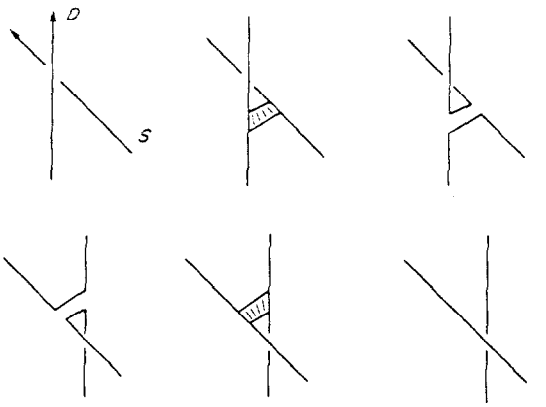

Fig. 14.

small annulus, i.e., a 1-handle and 2-handle. This is shown in Fig. 14. The 1-handle is visible in the second picture, the 2-handle is in the fifth. Note that if D and S are oriented so that $[D]\cdot[S] = +1$ (as in the figure) the resulting surface represents $[D]-[S]$ as desired. (To realize $[D]+[S]$ put a right half-twist in the 1-handle.)

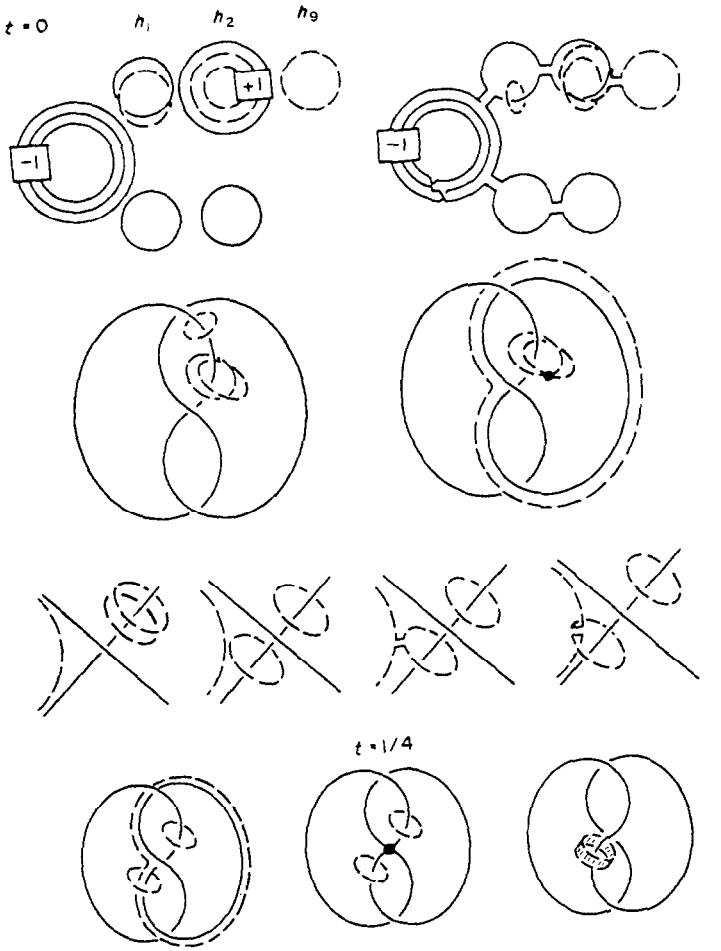

Fig. 15.

The reader is encouraged to draw Fig. 14 with a parallel copy, and notice that two left-handed twists are introduced by this procedure. This is the geometric realization of the algebra in section 3.3 which corrected a framing problem via the term $-2[D]\cdot[S]$.

A complete picture of c_2 is given in Fig. 15. The sphere S is clearly visible. The disk running over h_9 corresponds to the sphere S'. This tubes into S in the second picture. The disk D is also visible: The Norman sphere begins with a disk in h_1, then shrinks to a meridian in the second picture. This meridian continues into the third picture where it tubes into the original disk from which D was constructed (Fig. 10). We have perturbed this disk into general position with respect to time. Thus, we see the meridian gradually expanding until it gets close to the attaching circle (in the fourth picture). In subsequent pictures, D tubes into S (as described above), then merges with c_1 along the correct circle at $t = \frac{1}{4}$.

We showed in section 3.3 that the natural framing of the attaching circle of k_2 agreed with the framing specified in 3.2. This may also be verified directly from the picture: Construct a parallel copy of c_2. At any generic time, this looks like a framing on each circle of c_2. Each disk inside a handle results in one twist, thus at $t = 0$ we have a total of $+4$ twists. Two of these twists are eliminated at the tube between S and D (as described previously). The other two are compensated for by the (negative) kink in c_2. (Recall that we use homology to define framings on kinky handles; thus, $[c_2]\cdot[c_2] = \chi(\nu) + 2\,\text{self}\,c_2 = 2-2 = 0$, as required.) Details are left to the reader.

3.5 *The attaching circle for stage* 3. We now describe the framed circle in ∂k_2 to which we attach k_3. The results of this section will also be useful at later stages, in particular, whenever the kink of the previous stage comes from surgery on a distinguished torus. For this reason, we give several variations of the framed circle.

The fine curves in Fig. 16 represent an annulus A with boundary components $\partial_0 A$ and $\partial_1 A$. The circle $\partial_0 A$ lies in c_2, and $\partial_1 A$ is the circle visible in the last picture. Note that $\partial_0 A$ runs through the kink of c_2, hence, generates $\pi_1(c_2)$. (It is visible in subsequent pictures as a pair of points; these are joined by an arc in the fifth picture.) The annulus A intersects ∂k_2 in a suitable circle. We specify the framing on this by framing all of A. This is achieved by framing $\partial_1 A$ in the last picture. (The normal bundle of $\partial_1 A$ in the slice $t =$ constant is the restriction of the normal bundle of A to $\partial_1 A$.) In fact, the 0-framing on $\partial_1 A$ is correct.

Once we have shown this, we may construct other suitable framed curves as follows: Near $t = \frac{1}{4}$ in Fig. 16, give one circle a 360° twist, then join the circles together as before. This gives a self-diffeomorphism of k_2; hence, it sends the above framed curve to another suitable one. A possible result is shown in Fig. 17. Note that in the third picture, a twist is introduced in the framing of the annulus. Thus, the correct framing on ∂k_2 corresponds to framing -1 on the circle in the last picture.

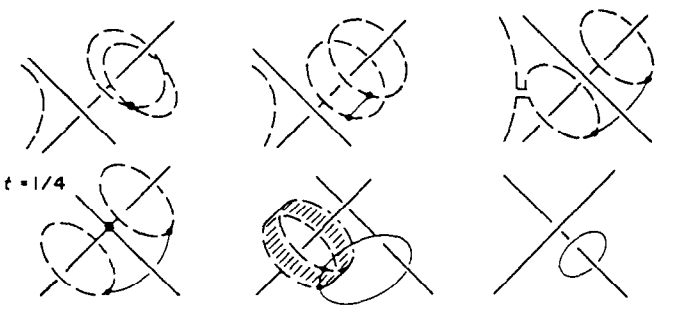

Fig. 16.

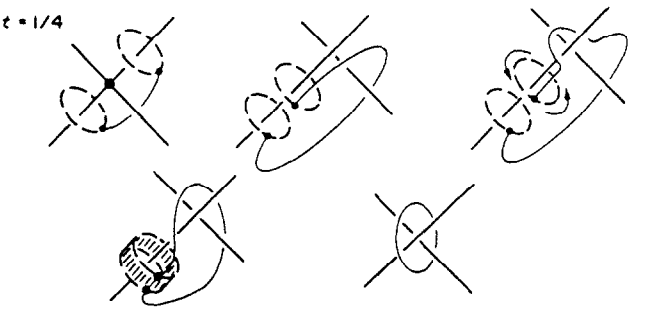

Fig. 17.

Figure 18 shows a similar variation. This time we twist the circle in the opposite direction, but the correct framing on the meridian in the last picture is still -1. The difference between Fig. 17 and 18 is that in the latter case, the annulus passes through c_1. (Note that in the absence of c_1, the framing may be changed by 2 simply by throwing an arc of the annulus over one of the circles.) In practice, we must remove the intersection point with a Norman sphere for c_1. The $+1$ twist in the normal bundle of the Norman sphere will cancel the -1 twist in the annulus, so we will be left with a 0-framed meridian like in Fig. 16.

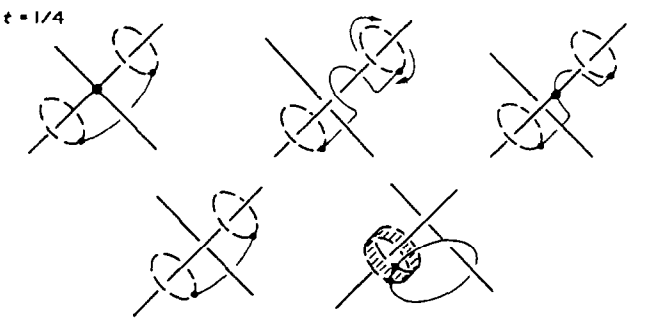

Fig. 18.

To see that the 0-framing in Fig. 16 is correct, recall that by construction $c_2 = D \# S \# S'$. Let S_- denote S with a small puncture. There is an ambient isotopy (moving ∂c_2) which sends c_2 onto S_-. (First shrink away D, sending c_2 onto $S_- \# S'$, then use the puncture to slip the surface off of h_9.) We may assume that this isotopy fixes (pointwise) a neighborhood of the annulus A. (Note that A does not separate c_2.) We may also locate the puncture of S_- at any convenient place on S. Since k_2 is a regular neighborhood of c_2, the isotopy sends k_2 to a regular neighborhood $N(S_-)$. The circle ∂c_2 in ∂k_2 is sent to ∂S_- in $\partial N(S_-)$. It is now sufficient to attach a 2-handle h to $N(S_-)$ along the framed curve in $\partial N(S_-)$ determined by A, then to show that ∂S_- bounds an embedded disk in $\partial(N(S_-) \cup h)$. (Compare with section 3.2.)

Figure 19 shows $N(S_-)$. (Compare with our picture of S, Fig. 12.) Stage 1 has been erased. The two solid tori at $t = 0$ are the attaching regions of 2-handles in h_2. These pass through each other as time increases, exposing the annulus A (fine curve). The puncture of S_- has been put in the final annulus of S (last picture of Fig. 12); thus our last picture shows a neighborhood of a punctured annulus. The dotted curve in the boundary indicates where a 2-handle should be added to get back a neighborhood of the entire annulus; this is the curve ∂S_-.

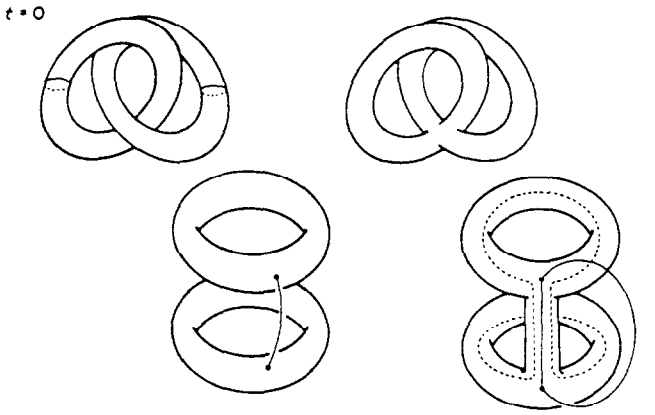

Fig. 19.

The 2-handle h can now be added ambiently. (It will hit stage 1, but this does not concern us now.) Simply turn the annulus A into a disk by spanning $\partial_1 A$, the fine circle in the last picture of Fig. 19, with the obvious disk. Taking a regular neighborhood gives a 2-handle h realizing the correct framing (0-framing in the last picture). To draw h, simply fatten the curves representing A, then cancel the extra 1-handle in the last picture. The result (up to isotopy) is shown in Fig. 20. (The third picture comes from the second by isotopy.) Figure 21 now shows the required disk bounding ∂S_-. The two circles at $t = 0$ bound disks in the boundaries of the 2-handles in h_2. In the third picture, these are connected by a 1-handle. The resulting boundary isotopes to ∂S_-.

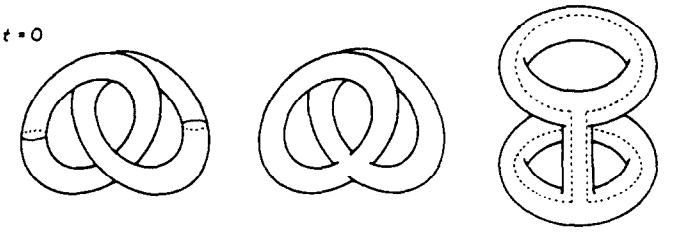

Fig. 20.

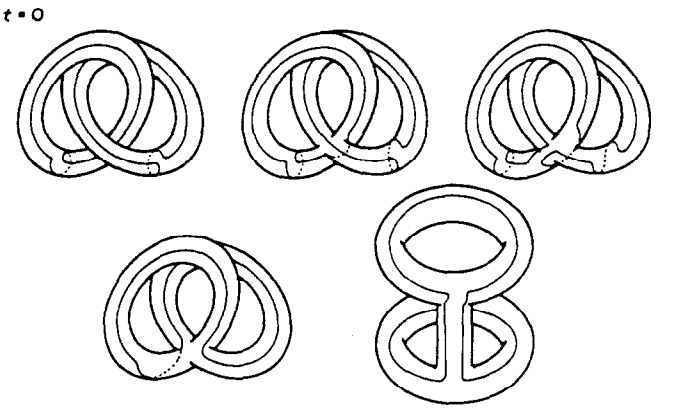

Fig. 21.

3.6 *Stage* 3. This is quite similar to Stage 2. As remarked previously, we may easily cap the attaching circle of Stage 3 by a disk with the correct framing. (Just fill in the meridian shown in Fig. 16.) The disk intersects c_1 once; remove this with the Norman sphere for c_1 representing e_3. We now have an embedding of $\tilde{T}_{2,1}$ (and therefore $\tilde{T}_{2,l}$, $l \geq 1$). (We have thus proven our results about DK_l, $l \geq 1$). We again correct the framing via a distinguished torus. This time we use the torus τ_2, near the kink of c_2. We surger this to an immersed sphere via the Norman sphere for c_2, which runs over the handle h_9 (observe this in Fig. 15). Tubing into a sphere representing e_{10} completes the framing correction.

See Fig. 22 for a complete picture of c_3. (Compare this with Fig. 15). Note that we have made an additional modification shortly after $t = \frac{1}{4}$. We have twisted the attaching circle as in section 3.5 (Fig. 18). We have removed the resulting intersection with c_1, using the Norman sphere representing e_4. As remarked in section 3.5, this cancels the left twist which was introduced. Thus, the framing of k_3 has not been disturbed. This modification appears pointless, but will actually be useful when we construct Stage 4.

3.7 *Stage* 4. For a picture of c_4, we refer the reader to Fig. 24, the big diagram at the end of section 3. In particular, $t < \frac{1}{2}$ (the first page of Fig. 24) covers all of c_2, c_3 and c_4 (as well as the attaching circle of c_5).

The framed attaching circle looks just like that of c_3 (the basic version—Fig. 16). The same argument shows that the 0-framing is correct. This time, however, the obvious disk intersects c_2 (not c_1). Since c_2 has no remaining Norman sphere, we resort to another trick of Casson: A small distinguished torus τ_1' for c_1 hits c_2 exactly once. Surger τ_1' to an immersed sphere via e_4 (as usual), then tube this into the disk to remove the intersection with c_2. This does not affect the framing since τ_1' is disjoint from the disk. Recall that we have already used a Stage 1 torus τ_1 to construct c_2. These two tori interfere slightly. In particular, there is tangling between the attaching circles of Stages 3 and 5. This is corrected by our trick at Stage 3 involving e_4 (as shown immediately after $t = \frac{1}{4}$ in the picture of c_4).

3.8 *Stage* 5. Since c_4 has no obvious Norman sphere, the torus τ_4 is unusable. Thus, we must change our tactics for c_5. We twist the attaching circle in the manner of Fig. 17. This introduces a -1 twist which we remove by tubing into a sphere representing e_{11}. (The handles h_{11}, h_{12} and h_{13} are attached at $t = \frac{1}{2}$). The details are shown in Fig. 24; note the change of notation for c_5 at $t = \frac{1}{2}$. The attaching circle for c_5 forms a Whitehead link with c_1 (at $t = \frac{1}{2}$ and the following picture—the reader should verify this isotopy). This persists without change until $t = \frac{3}{4}$, where c_5 pulls through itself, generating a kink. The link has now become unlinked; we close c_5 with a 2-handle just before $t = 1$. (Note that at $t = \frac{3}{4}$ the 0-framing of c_5 changes to $+2$, due to the compensating twists for the clasp. This corrects for the (negative) kink.)

It is not obvious that c_4 or c_5 can be replaced by embedded disks with twisted framings. We embed $\tilde{T}_{3,1}$ and $\tilde{T}_{4,1}$ by the following trick: Stage 3 could have been constructed by the method which we used for Stage 5. We could have then constructed Stages 4 through 6 by the procedure which we will use for Stages 6 through 8. (Thus the embedding would look identical for $t \geq \frac{1}{2}$, but each stage would be two stories lower.) The new cores of Stages 4 and 5 would then be identical to the actual cores c_6 and c_7. We will show that these may be replaced by embedded disks as desired.

3.9 *Stage* 6. Isotope c_5 onto the part of c_5 at $t \geq \frac{5}{8}$. Let N be the ambient space for $t \geq \frac{5}{8}$, with an open regular neighborhood of c_1 deleted. Thus, N is clearly diffeomorphic to $S^1 \times D^3$. At $t = \frac{5}{8}$, c_5 wraps around this in a Whitehead curve. Our pictures of c_5 at $t \geq \frac{5}{8}$ are the same as

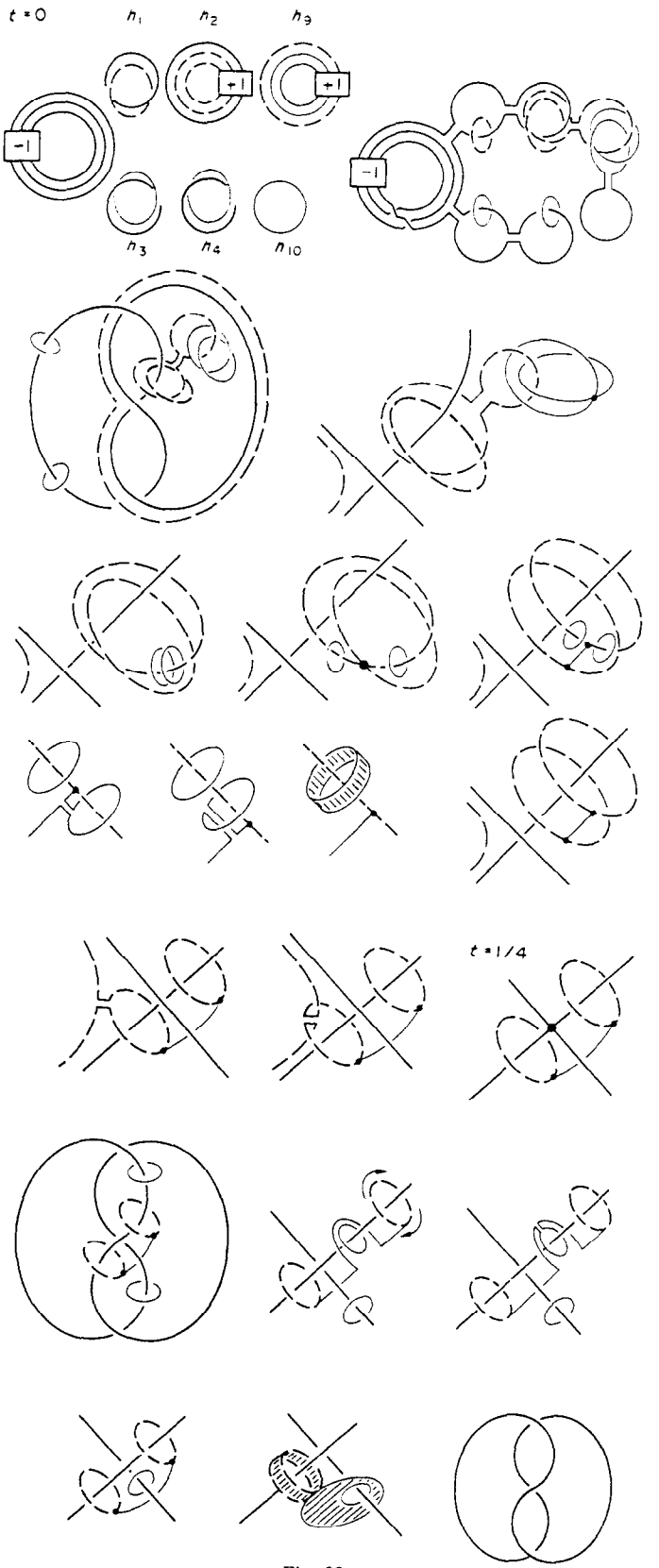

Fig. 22

those of the core of the kinky handle $k \approx S^1 \times D^3$ shown in Fig. 23. Hence, $(N, c_5) \approx (k, \text{core})$, N is a regular neighborhood of c_5, and standard kinky handle theory shows that a 0-framed meridian of c_1 is a suitable attaching curve for c_6. In Fig. 24, this meridian is visible just before time $t = \frac{3}{4}$, when it merges into c_5.

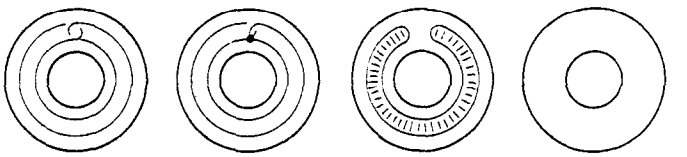

Fig. 23.

We now construct c_6 in the usual way: First obtain $\tilde{T}_{5,1}$ by tubing into the Norman sphere representing e_5. Then correct the framing with the torus τ_5 (surgered over h_{11}) and the e_{12} sphere.

3.10 *Stage* 7. The attaching curve is standard (twisted as in Fig. 17). The resulting circle links c_5 and c_1 as Borromean rings. We unlink this with two Norman spheres for c_1; see the last four pictures of Fig. 24. The Norman spheres change the framing from -1 (due to the twist in the attaching circle) to $+1$. Thus, we have embedded $\tilde{T}_{6,1}$. We correct, as usual, with a distinguished torus τ_6.

3.11 *Stage* 8. We begin with the usual (untwisted) attaching curve, a 0-framed meridian of c_6. Note that c_6 has no remaining Norman spheres. Instead, we tube into a sphere representing e_5, which intersects both c_6 and c_1. We remove the intersection with c_1, using the e_8 Norman sphere for c_1 (see $t = \frac{2}{3}$). We have now introduced two twists in the framing, so we have an embedding of $\tilde{T}_{7,2}$. Fortunately, the distinguished torus technique is capable of cancelling two twists; simply forget the extra sphere which was used on previous occasions.

Recall, from Section 3.3, the argument that $\tilde{T}_n$ has 1-connected complement. The argument fails for $\tilde{T}_8$, since we have used up all Norman spheres for c_1. We may easily correct this, however. Given a disk bounding a loop γ, first remove intersections with $\tilde{T}_2$, as we did previously. The disk now runs over h_8 many times, so we have introduced intersections with c_8. Next, push all intersections with $\tilde{T}_8$ down to c_3 by finger moves. We may now remove these intersections using the Norman sphere for c_3 (representing e_{10}). Our disk is now disjoint from $\tilde{T}_8$. Thus, $\tilde{T}_8$ (and similarly $\tilde{T}_{7,2}$) has 1-connected complement.

3.12 *Stage* 9. We conclude section 3 by discussing embeddings of $\tilde{T}_{8,l}$. (These are not drawn in Fig. 24.) The attaching circle for the new disk is a 0-framed meridian of c_7 (by the usual argument). Since c_7 has no Norman sphere left, we use a sphere representing e_6, introducing an intersection with c_1. Remove this via a sphere for e_3. The resulting intersection with c_3 can be eliminated by the Norman sphere representing e_{10}. We have now obtained an embedding of $\tilde{T}_{8,3}$. The complement is unlikely to be 1-connected, since there are no remaining Norman spheres. However, if we also tube the top stage into a sphere representing e_{14}, we obtain an embedding of $\tilde{T}_{8,4}$ which has a Norman sphere for the top stage. This has 1-connected complement. (Given a disk bounding γ, remove it from $\tilde{T}_8$ as before, then separate it from the top stage via the Norman sphere.)

Finally, we show how to prove Theorem 2.1 for D^7K_2, without an embedding of $\tilde{T}_{8,2}$. Tube c_1 into a sphere representing e_{15}. This gives an embedding of a manifold $\bar{T}_8$, representing the homology class $x + e_{15}$. The manifold $\bar{T}_8$ is constructed the same way as $\tilde{T}_8$,

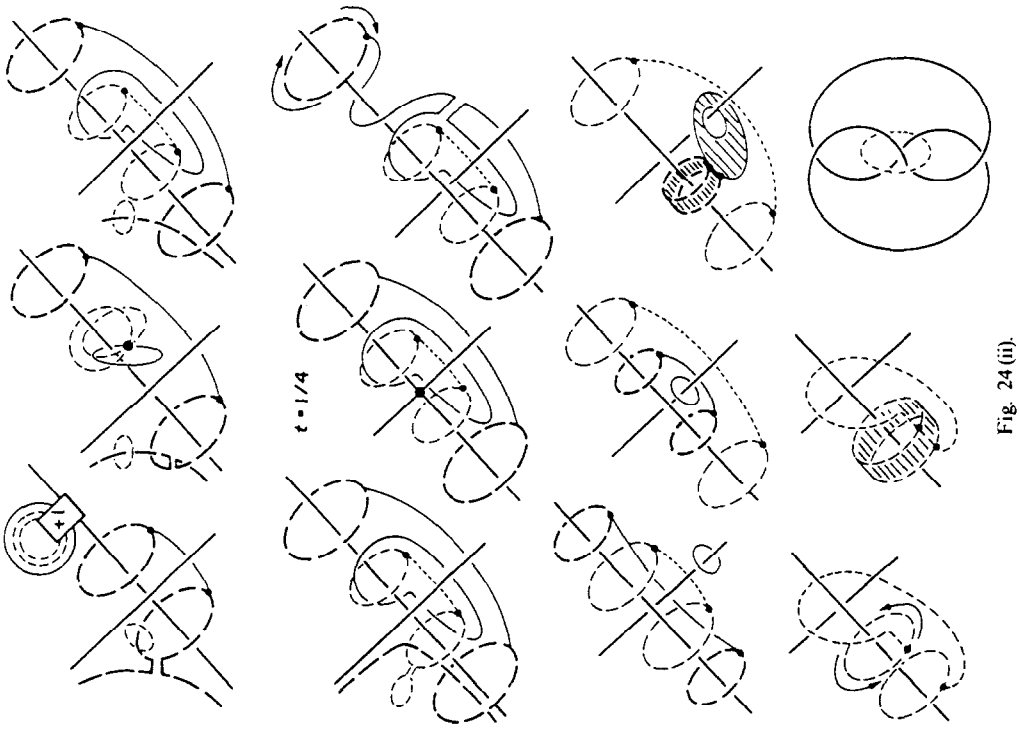

Fig. 24 (ii).

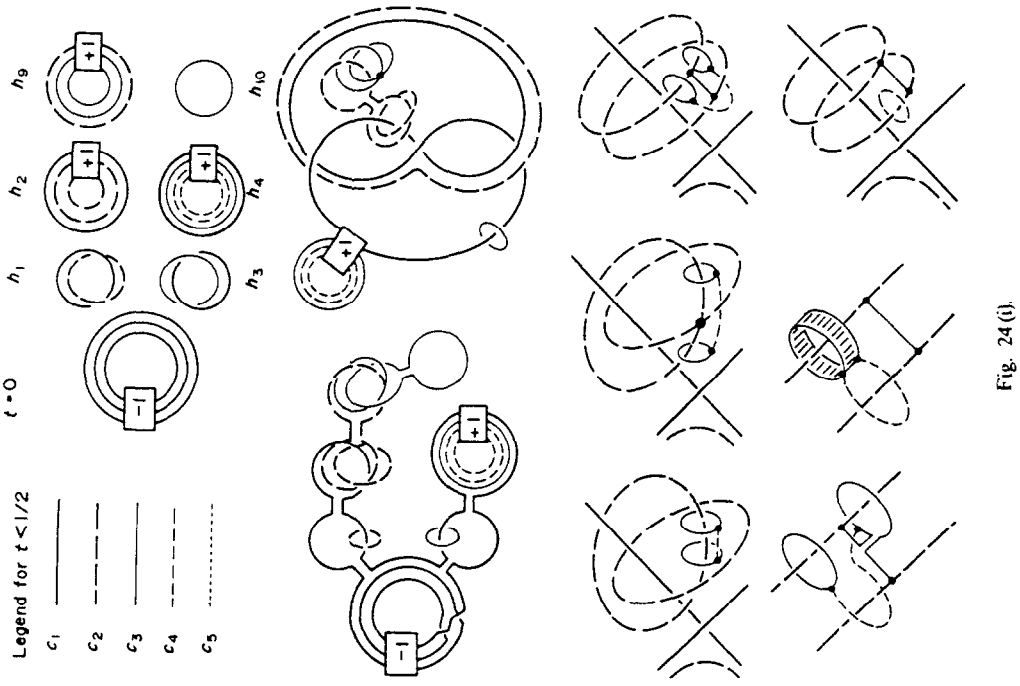

Fig. 24 (i).

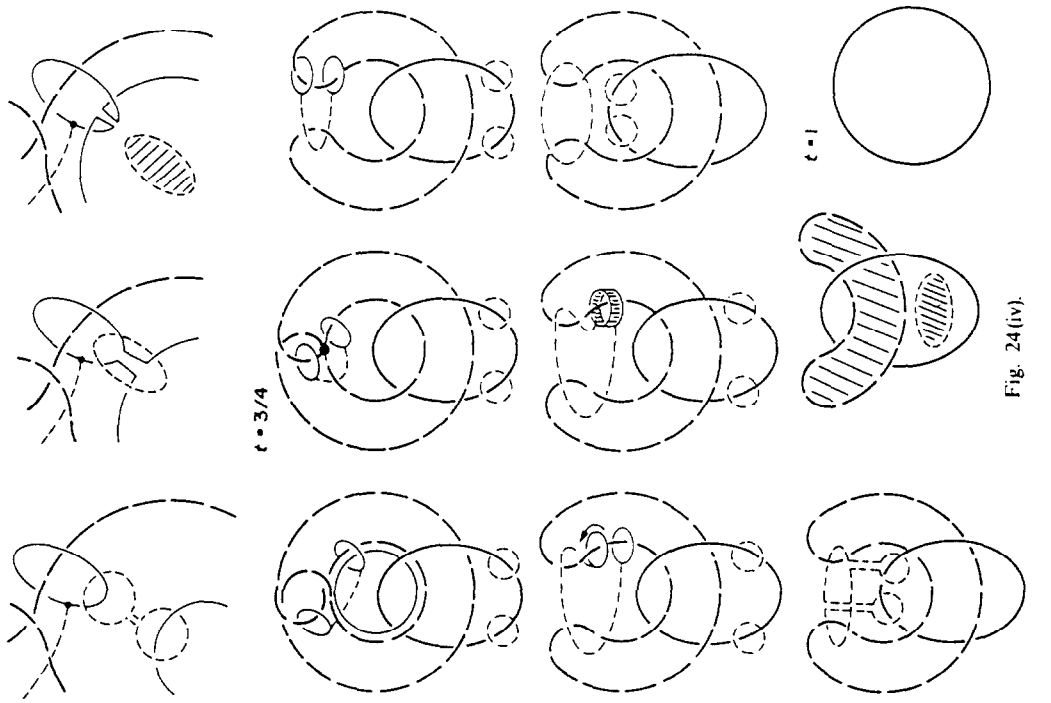

Fig. 24 (iv).

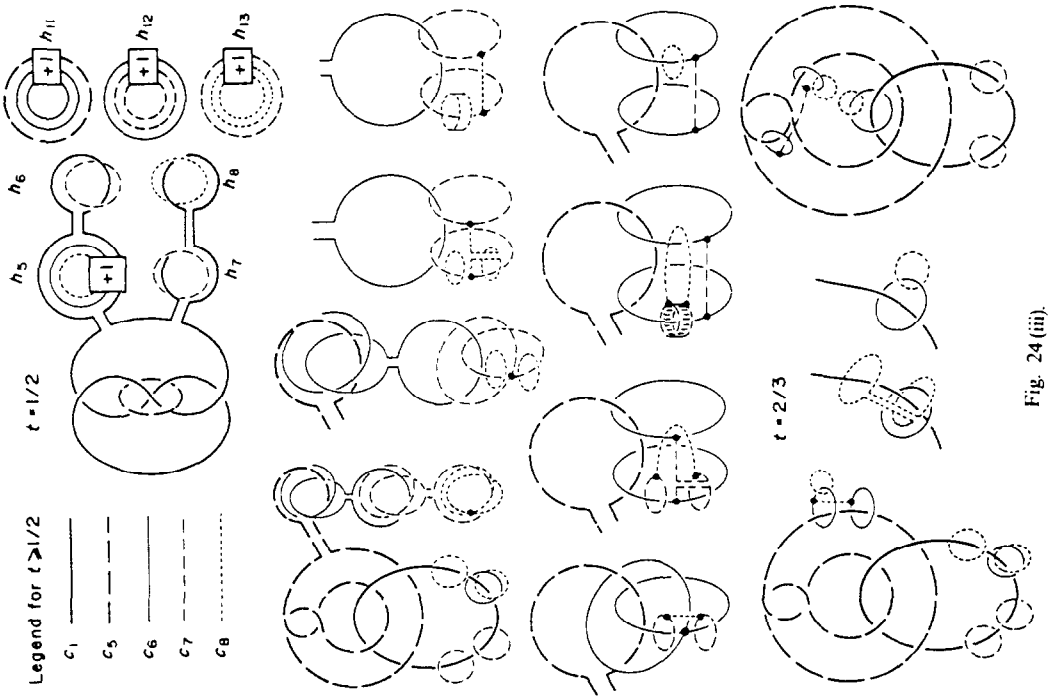

Fig. 24 (iii).

except that T_8 is glued to B^4 with framing 0. We cap $\bar{T}_8$ with a disk with framing 2, and call the result $\bar{T}_{8,2}$: Simply tube into the e_6 sphere (as for $\bar{T}_{8,3}$), and remove the intersection with the first stage via e_{15}. The result has 1-connected complement, due to the Norman sphere for c_3 representing e_{10}. Now as in section 2, if the knot D^7K_2 were slice, we would obtain an embedded sphere S in $\bar{T}_{8,2}$ representing $x+e_{15}$. Furthermore, a meridian of S could be homotoped into $\partial\bar{T}_{8,2}$ in $\bar{T}_{8,2}-S$. This implies that M_m-S is 1-connected. Surgery on S now contradicts Donaldson's theorem.

§4. CONCLUDING REMARKS

It seems likely that with some clever tricks, several more stages could be inserted into the tower of section 3. This, of course, would sharpen the results in section 2. A more interesting question is this: Can the procedure be continued indefinitely? The main defect of our method is that each stage uses up more Norman spheres than it creates (with the notable exception of Stage 5, which adds one). Thus, we need either (1) a method without this defect, or (2) settings in which we can start Stage 1 with arbitrarily large numbers of Norman spheres. The ultimate conjecture in this context is that no $\bar{T}_n$ or $\bar{T}_{n,l}$ ($l \geq 1$) has an embedded sphere (or immersed sphere with only positive kinks) generating its homology. This would imply, for example, that no iterated double of K_l ($l \geq 1$) is smoothly slice.

Another line of inquiry would be to try to attach the top stage embedded disks to framed curves which are knotted in ∂T_n. The effect of this would be to replace K_l in section 2 by the twisted double of the given knot. An interested reader is invited to experiment with these ideas.

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