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Spectral Representation of the Resolvent of a Discrete Operator

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Let T be a discrete linear operator in a Hilbert space H with spectrum $\sigma(T)=\{\lambda_i\}_{i=1}^{\infty}$, let $R_\lambda(T)$ denote the resolvent of T, and let P_i denote the projection of H onto the generalized eigenspace $\mathcal{N}((\lambda_i I-T)^{m_i})$ along $\mathcal{R}((\lambda_i I-T)^{m_i})$, where m_i is the ascent of the operator $\lambda_i I-T$. In this paper it is shown that

$$R_\lambda(T)=\sum_{i=1}^{\infty}\sum_{j=1}^{m_i}\frac{(-N_i)^{j-1}P_i}{(\lambda-\lambda_i)^j}+\sum_{j=0}^{\infty}(\lambda_0-\lambda)^j R_{\lambda_0}(T_\infty)^{j+1}(I-\mathbb{P}_\infty)$$

in $\mathcal{B}(H)$ for all $\lambda\in\rho(T)$, where N_i is the restriction of $\lambda_i I-T$ to $\mathcal{N}((\lambda_i I-T)^{m_i})$, T_∞ is the restriction of T to $\mathcal{D}(T)\cap\bigcap_{i=1}^{\infty}\mathcal{R}((\lambda_i I-T)^{m_i})$, $\mathbb{P}_\infty=\sum_{i=1}^{\infty}P_i$ (strong convergence), and λ_0 is a fixed but arbitrary point in $\mathbb{C}$. This spectral representation is valid provided there exists $M>0$ such that $\|\sum_{i=1}^{N}P_i\|\leqslant M$, $N=1, 2, ...,$ and generalizes results that apply to self-adjoint, normal, and spectral operators. The results of this paper are applied to represent the resolvent of a differential operator L in $L^2[0,1]$ having infinitely many eigenvalues with ascent $m_i=2$ and are also applied to represent the resolvent of an operator T with $\mathbb{P}_\infty\neq I$. © 1988 Academic Press, Inc.

1. INTRODUCTION

Let T be a discrete linear operator in a Hilbert space H with spectrum $\sigma(T)=\{\lambda_i\}_{i=1}^{\infty}$, and let $R_\lambda(T)$ denote the resolvent of T. If T is normal, then it is well known that

$$R_\lambda(T)=\sum_{i=1}^{\infty}\frac{P_i}{\lambda-\lambda_i} \tag{1.1}$$

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in $\mathscr{B}(H)$ for all $\lambda \in \rho(T)$, where P_i denotes the projection of H onto $\mathcal{N}(\lambda_i I - T)$ along $\mathscr{R}(\lambda_i I - T)$, and where $1/(\lambda - \lambda_i) \in \sigma(R_\lambda(T))$ [7, p. 277]. More generally, if T is spectral, then $R_\lambda(T)$ is spectral [3, p. 2249], and

$$R_\lambda(T) = D_\lambda + Q_\lambda \qquad \text{for all} \quad \lambda \in \rho(T), \tag{1.2}$$

where D_λ is a scalar operator in $\mathscr{B}(H)$ with a representation similar to the one given in (1.1), and Q_λ is a quasinilpotent operator in $\mathscr{B}(H)$ [3, p. 1950]. These results hinge on two essential ingredients: (1) The P_i are orthogonal, or, more generally, the family of all finite sums of the P_i is uniformly bounded, and (2) $P_i x = 0$ for all i implies $x = 0$. In [8] it is shown that there exist discrete operators where both of these properties fail. The purpose of this paper is to present spectral representations for $R_\lambda(T)$ similar to (1.1) and (1.2) by weakening (1) and eliminating (2).

The main result of this paper is summarized as follows: Let $T(\lambda_i) = (\lambda_i I - T)^{m_i}$, where m_i is the ascent of operator $\lambda_i I - T$, and let P_i be the projection of H onto the generalized eigenspace $\mathcal{N}(T(\lambda_i))$ along $\mathscr{R}(T(\lambda_i))$. If there exists a constant $M > 0$ such that $\|\sum_{i=1}^N P_i\| \leqslant M$, $N = 1, 2, ...$, then $H = S_\infty \oplus M_\infty$ (topological direct sum), where $S_\infty = \{x \in H \mid x = \sum_{i=1}^\infty x_i, x_i \in \mathcal{N}(T(\lambda_i))\} = \bar{S}_\infty$ and M_∞ is the zero or infinite-dimensional subspace $\bigcap_{i=1}^\infty \mathscr{R}(T(\lambda_i))$ (see [8]). Let $\mathbb{P}_\infty$ denote the projection of H onto S_∞ along M_∞. Then under the assumption placed on the P_i above, we show that

$$R_\lambda(T) = \sum_{i=1}^\infty \sum_{j=1}^{m_i} \frac{(-N_i)^{j-1} P_i}{(\lambda - \lambda_i)^j} + \sum_{j=0}^\infty (\lambda_0 - \lambda)^j R_{\lambda_0}(T_\infty)^{j+1} (I - \mathbb{P}_\infty) \tag{1.3}$$

in $\mathscr{B}(H)$ for all λ in the resolvent set $\rho(T)$, where N_i is a nilpotent operator on $\mathcal{N}(T(\lambda_i))$, T_∞ is an operator defined on $\mathscr{D}(T) \cap M_\infty$, and λ_0 is a fixed but arbitrary point in the complex plane $\mathbb{C}$. The first series in (1.2) is the sum of all the singular parts (poles) of $R_\lambda(T)$ and completely describes the action of $R_\lambda(T)$ on S_∞. The second series is the analytic part of $R_\lambda(T)$ and describes its action on M_∞. This representation is analogous to the Mittag–Leffler decomposition of a meromorphic function.

This result applied to many of the operators that appear in mathematical physics, and in particular it applies to many nth-order two-point differential operators in $H = L^2[a, b]$.

The above result is established in Section 3 of this paper following a brief mathematical preliminary section. In Section 4 additional assumptions are placed on T, allowing us to decompose $R_\lambda(T)$ into the sum of a scalar operator and two quasinilpotent operators. Finally, in Section 5 these results are applied to represent the resolvent of a two-point differential operator having infinitely many multiple eigenvalues with $m_i = 2$, and to represent the resolvent of an operator T with $\mathbb{P}_\infty \neq I$.

2. Mathematical Preliminaries

Let H be a complex Hilbert space with inner product $(\, , \,)$ and norm $\| \; \|$, and let $\mathscr{B}(H)$ denote the Banach space of all bounded linear operators defined on H. For any linear operator T in H we denote its domain, range, and null space by $\mathscr{D}(T)$, $\mathscr{R}(T)$, and $\mathscr{N}(T)$, respectively. Let T be a closed, densely defined linear operator in H such that there exists a number λ_0 in its resolvent set $\rho(T)$ for which the resolvent $R_{\lambda_0}(T) = (\lambda_0 I - T)^{-1}$ is compact, i.e., T is a discrete operator in H. Then it is well known that (1) the spectrum $\sigma(T) = \{\lambda_i\}_{i=1}^{\infty}$ is a countable set of eigenvalues having no finite limit points in $\mathbb{C}$, (2) $R_\lambda(T)$ is compact for all $\lambda \in \rho(T)$ with $\sigma(R_\lambda(T)) = \{0, (\lambda - \lambda_i)^{-1}\}_{i=1}^{\infty}$ (if $\dim H < \infty$, then $0 \notin \sigma(R_\lambda(T))$), and (3) the algebraic multiplicity $\nu(\lambda_i)$ of λ_i is finite and equal to $\dim \mathscr{N}((\lambda_i I - T)^{m_i})$, where m_i denotes the ascent of $\lambda_i I - T$ [7, p. 187]. Since T is discrete, it is a Fredholm operator of index 0 with its Fredholm set $\phi(T) = \mathbb{C}$ [9]. Consequently,

$$H = \mathscr{N}(T(\lambda_i)) \oplus \mathscr{R}(T(\lambda_i)) \qquad \text{(topological direct sum)},$$

where $T(\lambda_i) = (\lambda_i I - T)^{m_i}$, $i = 1, 2, ...$ [8]. For $N = 1, 2, ..., \infty$ define the subspaces S_N and M_N by

$$S_N = \left\{ x \in H \mid x = \sum_{i=1}^{N} x_i, \; x_i \in \mathscr{N}(T(\lambda_i)) \right\}$$

and

$$M_N = \bigcap_{i=1}^{N} \mathscr{R}(T(\lambda_i)).$$

Then

$$H = S_N \oplus M_N \qquad \text{(topological direct sum)}, \tag{2.1}$$

$N = 1, 2, ...$. Equation (2.1) also holds when $N = \infty$ iff the sequence of projections $\mathbb{P}_N = \sum_{i=1}^{N} P_i$ of H onto S_N along M_N, where P_i is the projection of H onto $\mathscr{N}(T(\lambda_i))$ along $\mathscr{R}(T(\lambda_i))$, is uniformly bounded in norm by a constant $M > 0$ [8]. It can be shown that $S_N = \{x \in H \mid x = \sum_{i=1}^{N} P_i x\}$, $M_N = \{x \in H \mid P_i x = 0, \; i = 1, 2, ..., N\}$, $N = 1, 2, ..., \infty$, and that M_∞ is either zero or infinite dimensional [3, p. 2295]. When $\|\mathbb{P}_N\| \leqslant M$, $N = 1, 2, ...$, there exists a projection $\mathbb{P}_\infty$ of H onto $S_\infty = \bar{S}_\infty$ along M_∞, such that $\mathbb{P}_\infty x = \sum_{i=1}^{\infty} P_i x$ for all $x \in H$ [8].

The projection P_i is identical with the projection associated with the eigenvalue $(\lambda - \lambda_i)^{-1}$ of $R_\lambda(T)$ [7, p. 187]. Furthermore, it can be shown that $P_i = (1/2\pi i) \int_\Gamma R_\lambda(T)\, d\lambda$, where Γ is a simple closed curve containing λ_i in its interior and $\sigma(T) - \{\lambda_i\}$ in its exterior. From this observation it follows that

$$R_\lambda(T) P_i = P_i R_\lambda(T) \qquad \text{for all} \quad \lambda \in \rho(T) \tag{2.2}$$

[7, p. 178], and hence,

$$R_\lambda(T) \mathbb{P}_N = \mathbb{P}_N R_\lambda(T) \qquad \text{for all} \quad \lambda \in \rho(T), \tag{2.3}$$

$N = 1, 2, \ldots$, and when $\| \mathbb{P}_N \| \leqslant M$, $N = 1, 2, \ldots$,

$$R_\lambda(T) \mathbb{P}_\infty = \mathbb{P}_\infty R_\lambda(T) \qquad \text{for all} \quad \lambda \in \rho(T), \tag{2.4}$$

with similar statements holding for $R_\lambda(T)(I - P_i)$, $R_\lambda(T)(I - \mathbb{P}_N)$, and $R_\lambda(T)(I - \mathbb{P}_\infty)$. A direct consequence of (2.2)–(2.4) is that

$$TP_i x = P_i Tx, \qquad i = 1, 2, \ldots, \tag{2.5}$$

$$T \mathbb{P}_N x = \mathbb{P}_N Tx, \qquad N = 1, 2, \ldots, \tag{2.6}$$

and when $\| \mathbb{P}_N \| \leqslant M$, $N = 1, 2, \ldots$,

$$T \mathbb{P}_\infty x = \mathbb{P}_\infty Tx \tag{2.7}$$

for all $x \in \mathscr{D}(T)$ [7, pp. 172–173]. Note that (2.5)–(2.7) contain the implicit statement that $P_i x$, $\mathbb{P}_N x$ belong to $\mathscr{D}(T)$ whenever x does. Finally, from [8],

$$P_i P_j = \delta_{ij} P_i \qquad \text{for} \quad i, j = 1, 2, \ldots. \tag{2.8}$$

3. Representation of $R_\lambda(T)$

Fix a positive integer N. Then

$$R_\lambda(T) = R_\lambda(T)\, \mathbb{P}_N + R_\lambda(T)(I - \mathbb{P}_N) \qquad \text{for all} \quad \lambda \in \rho(T). \tag{3.1}$$

Equation (2.3) implies that S_N and M_N are invariant subspaces of $R_\lambda(T)$. Thus, the problem of obtaining a representation of $R_\lambda(T)$ is reduced to studying its behavior on S_N and M_N, or equivalently to the study of $R_\lambda(T)\mathbb{P}_N$ and $R_\lambda(T)(I - \mathbb{P}_N)$. We begin by looking at $R_\lambda(T)\mathbb{P}_N$.

For notational purposes we set

$$N_i = (\lambda_i I - T) \mid \mathscr{N}(T(\lambda_i)), \qquad i = 1, 2, \ldots. \tag{3.2}$$

The operator N_i is easily shown to be a continuous nilpotent map of

$\mathcal{N}(T(\lambda_i))$ into $\mathcal{N}(T(\lambda_i))$, satisfying $N_i^{m_i}=0$. On $\mathcal{N}(T(\lambda_i))$ we note that $\lambda I-T=(\lambda-\lambda_i)I+N_i$ and for $\lambda\neq\lambda_i$

$$(\lambda I-T)\left(\sum_{j=1}^{m_i}\frac{(-N_i)^{j-1}}{(\lambda-\lambda_i)^j}\right)=\frac{(-N_i)^{m_i}}{(\lambda-\lambda_i)^{m_i}}+I=I=\left(\sum_{j=1}^{m_i}\frac{(-N_i)^{j-1}}{(\lambda-\lambda_i)^j}\right)(\lambda I-T). \tag{3.3}$$

Equation (3.3) and the fact that $(\lambda I-T)R_\lambda(T)P_i=P_i$ imply that

$$R_\lambda(T)P_i=\sum_{j=1}^{m_i}\frac{(-N_i)^{j-1}P_i}{(\lambda-\lambda_i)^j},\qquad i=1,2,..., \tag{3.4}$$

and hence,

$$R_\lambda(T)\mathbb{P}_N=\sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{(-N_i)^{j-1}P_i}{(\lambda-\lambda_i)^j},\qquad N=1,2,..., \tag{3.5}$$

for all $\lambda\in\rho(T)$.

Before looking at $R_\lambda(T)(I-\mathbb{P}_N)$, we need the following lemmas.

Lemma 3.1. *Let X be a Banach space. Consider the power series $\sum_{n=0}^{\infty}(\lambda-\lambda_0)^n a_n$, where λ, $\lambda_0\in\mathbb{C}$ and $a_n\in X$, $n=1,2,...$.*

(a) *If there exists $a(\lambda)\in X$ such that $\lim_{k\to\infty}\|a(\lambda)-\sum_{n=0}^{k}(\lambda-\lambda_0)^n a_n\|=0$, then $\lim_{n\to\infty}\|(\lambda-\lambda_0)^n a_n\|=0$.*

(b) *If $\limsup_{n\to\infty}\|a_n\|^{1/n}=1/\rho$, then the series is absolutely convergent for $|\lambda-\lambda_0|<\rho$ and divergent for $|\lambda-\lambda_0|>\rho$ (for $1/\rho=0$ set $\rho=\infty$).*

Proof. See [6, p. 96].

Lemma 3.2. *Let T be a discrete operator in a Hilbert space H with $\sigma(T)=\{\lambda_i\}_{i=1}^{\infty}$. Fix $\lambda\in\rho(T)$ and set $d=\inf\{|\lambda_i-\lambda|\mid\lambda_i\in\sigma(T)\}$. Then*

$$r(R_\lambda(T))=\lim_{j\to\infty}\|R_\lambda(T)^j\|^{1/j}=1/d, \tag{3.6}$$

where $r(R_\lambda(T))$ denotes the spectral radius of $R_\lambda(T)$ ($1/d=0$ for $d=\infty$).

Proof. This is immediate from the spectral radius theorem and [7, p. 177].

It is straightforward to show that the conclusion of the last lemma remains true when $\|R_\lambda(T)^j\|^{1/j}$ is replaced by $\|R_\lambda(T)^{j+1}\|^{1/j}$.

Equation (2.6) shows that T commutes with $\mathbb{P}_N$, and hence with $I-\mathbb{P}_N$.

Thus, $T(\mathscr{D}(T) \cap M_N) \subseteq M_N$. In view of this we define $T_N: \mathscr{D}(T) \cap M_N \to M_N$ by

$$T_N = T \mid \mathscr{D}(T) \cap M_N. \tag{3.7}$$

Clearly T_N is a closed linear operator in M_N. From [5, p. 104] it follows that T_N is densely defined in M_N and from [7, p. 178] that $\sigma(T_N) = \{\lambda_i\}_{i=N+1}^{\infty}$. For $\lambda \in \rho(T)$ it is easy to show that

$$R_\lambda(T) \mid M_N = R_\lambda(T_N). \tag{3.8}$$

Since $R_\lambda(T)$ is compact, so is $R_\lambda(T_N)$, and hence, T_N is a discrete operator in the Hilbert space M_N. The next lemma obtains a representation for $R_\lambda(T_N)$ that will be used in representing $R_\lambda(T)(I - \mathbb{P}_N)$.

LEMMA 3.3. *Let T be a discrete operator in H with $\sigma(T) = \{\lambda_i\}_{i=1}^{\infty}$. For fixed $\lambda_0 \in \rho(T_N)$ set $d = \inf\{|\lambda - \lambda_0| \mid \lambda \in \sigma(T_N)\}$. Then for $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_0| < d$,*

$$R_\lambda(T_N) = \sum_{j=0}^{\infty} (\lambda_0 - \lambda)^j R_{\lambda_0}(T_N)^{j+1} \qquad \text{in } \mathscr{B}(M_N). \tag{3.9}$$

The disk $|\lambda - \lambda_0| < d$ where convergence occurs is maximal.

Proof. Lemmas 3.1 and 3.2 and the comments following them imply that for $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_0| < d$, $\sum_{j=0}^{\infty} (\lambda_0 - \lambda)^j R_{\lambda_0}(T_N)^{j+1}$ converges in $\mathscr{B}(M_N)$, and that the disk where convergence occurs is maximal. Thus, it remains only to show that equality holds in (3.9).

Fix $x \in \mathscr{D}(T_N)$, and let λ be such that $|\lambda - \lambda_0| < d$. Note that $\lambda I - T_N = (\lambda - \lambda_0)I + (\lambda_0 I - T_N)$. Lemma 3.1 implies that

$$\begin{aligned} &\left(\sum_{j=0}^{\infty} (\lambda_0 - \lambda)^j R_{\lambda_0}(T_N)^{j+1}\right)(\lambda I - T_N)x \\ &\quad = \lim_{k \to \infty} \left[-\sum_{j=0}^{k} (\lambda_0 - \lambda)^{j+1} R_{\lambda_0}(T_N)^{j+1} x + \sum_{j=0}^{k} (\lambda_0 - \lambda)^j R_{\lambda_0}(T_N)^j x \right] \\ &\quad = \lim_{k \to \infty} [Ix - (\lambda_0 - \lambda)^{k+1} R_{\lambda_0}(T_N)^{k+1} x] = Ix. \end{aligned} \tag{$*$}$$

Now take any $x \in M_N$. Let λ be such that $|\lambda - \lambda_0| < d$ and note that

$$(\lambda I - T_N)\left(\sum_{j=0}^{k} (\lambda_0 - \lambda)^j R_{\lambda_0}(T_N)^{j+1}\right)x = Ix - (\lambda_0 - \lambda)^{k+1} R_{\lambda_0}(T_N)^{k+1} x.$$

The operator $\lambda I - T_N$ is closed since T_N is closed. This fact together with Lemma 3.1 and the last equation gives

$$\lim_{k \to \infty} (\lambda I - T_N)\left(\sum_{j=0}^{k} (\lambda_0 - \lambda)^j R_{\lambda_0}(T_N)^{j+1} x\right) = Ix. \qquad (**)$$

Lines (*) and (**) prove (3.9). ∎

The results of these last few pages yield

THEOREM 3.4. *Let T be a discrete operator in H with $\sigma(T) = \{\lambda_i\}_{i=1}^{\infty}$, and for a fixed positive integer N set $T_N = T | \mathscr{D}(T) \cap M_N$. Then $\sigma(T_N) = \{\lambda_i\}_{i=N+1}^{\infty}$ and for $\lambda \in \rho(T) \subseteq \rho(T_N)$:*

(i) $R_\lambda(T) = R_\lambda(T)\mathbb{P}_N + R_\lambda(T)(I - \mathbb{P}_N)$,

(ii) $R_\lambda(T)\mathbb{P}_N = \sum_{i=1}^{N} \sum_{j=1}^{m_i} ((-N_i)^{j-1} P_i/(\lambda - \lambda_i)^j)$, $\quad N_i = (\lambda_i I - T) | \mathscr{N}(T(\lambda_i))$,

(iii) $R_\lambda(T)(I - \mathbb{P}_N) = R_\lambda(T_N)(I - \mathbb{P}_N)$.

Also, for fixed $\lambda_0 \in \rho(T_N)$ and for $d = \inf\{|\xi - \lambda_0| \mid \xi \in \sigma(T_N)\}$,

$$R_\lambda(T)(I - \mathbb{P}_N) = \sum_{j=0}^{\infty} (\lambda_0 - \lambda)^j R_{\lambda_0}(T_N)^{j+1}(I - \mathbb{P}_N) \qquad \text{in } \mathscr{B}(H) \qquad (3.10)$$

for all $\lambda \in \rho(T)$ with $|\lambda - \lambda_0| < d$.

This theorem provides a complete spectral representation of $R_\lambda(T)$ in the case of $\sigma(T) = \{\lambda_i\}_{i=1}^{N}$, but only a partial representation when $\sigma(T) = \{\lambda_i\}_{i=1}^{\infty}$. In the following we show that under the assumption $\| \mathbb{P}_N \| \leqslant M$ for $N = 1, 2, \ldots$, the results of Theorem 3.4 are still valid when N is replaced by ∞. This assumption is necessary to ensure the existence of $\mathbb{P}_\infty$ as a continuous operator defined on all of H.

Throughout the rest of this section we assume that there is a constant $M > 0$ such that $\| \mathbb{P}_N \| \leqslant M$, $N = 1, 2, \ldots$. From this assumption it follows that there exists a projection $\mathbb{P}_\infty$ from H onto $S_\infty = \bar{S}_\infty$ along M_∞, where $\mathbb{P}_\infty x = \sum_{i=1}^{\infty} P_i x$ for all $x \in H$. Clearly

$$R_\lambda(T) = R_\lambda(T)\mathbb{P}_\infty + R_\lambda(T)(I - \mathbb{P}_\infty) \qquad \text{for all } \quad \lambda \in \rho(T). \qquad (3.11)$$

From (2.4) it follows that S_∞ and M_∞ are invariants of $R_\lambda(T)$, implying that we can obtain a representation of $R_\lambda(T)$ by obtaining representations of $R_\lambda(T)\mathbb{P}_\infty$ and $R_\lambda(T)(I - \mathbb{P}_\infty)$.

Equation (2.7) implies that $T(\mathscr{D}(T) \cap M_\infty) \subseteq M_\infty$. Define $T_\infty \colon \mathscr{D}(T) \cap M_\infty \to M_\infty$ by

$$T_\infty = T | \mathscr{D}(T) \cap M_\infty. \qquad (3.12)$$

As in the case of T_N, T_∞ is a closed linear operator in M_∞. We claim that $\mathscr{D}(T) \cap M_\infty$ is dense in M_∞. To see this, let $x \in M_\infty$. Since $\mathscr{D}(T)$ is dense in H, there exists a sequence $\{x_i\}_{i=1}^\infty$ in $\mathscr{D}(T)$ such that $x_i \to x$. Also, since T commutes with $\mathbb{P}_\infty$, it commutes with $I - \mathbb{P}_\infty$, and hence, $\{(I - \mathbb{P}_\infty)x_i\}_{i=1}^\infty$ belongs to $\mathscr{D}(T) \cap M_\infty$. The continuity of $I - \mathbb{P}_\infty$ implies that $(I - \mathbb{P}_\infty)x_i \to (I - \mathbb{P}_\infty)x = x$, thereby showing that $\overline{\mathscr{D}(T) \cap M_\infty} = M_\infty$. Thus, T_∞ is densely defined in M_∞. From a result in [9] it follows that $\lambda I - T_\infty$ maps $\mathscr{D}(T) \cap M_\infty$ 1–1 onto M_∞ for all $\lambda \in \mathbb{C}$, and hence, $\sigma(T_\infty) = \varnothing$. It can also be shown that

$$R_\lambda(T) \mid M_\infty = R_\lambda(T_\infty) \qquad \text{for all} \quad \lambda \in \rho(T). \tag{3.13}$$

Since $R_\lambda(T)$ is compact, so is $R_\lambda(T_\infty)$, and hence, T_∞ is a discrete operator in the Hilbert space M_∞.

The next two lemmas provide the results necessary to represent $R_\lambda(T)\mathbb{P}_\infty$ and $R_\lambda(T)(I - \mathbb{P}_\infty)$. The first is the analogue of Lemma 3.3 and is proved in a similar fashion. A proof of the second can be found in [2, p. 8].

LEMMA 3.5. *Let T be a discrete operator in H with $\sigma(T) = \{\lambda_i\}_{i=1}^\infty$. Then for $\lambda_0 \in \rho(T_\infty) = \mathbb{C}$,*

$$R_\lambda(T_\infty) = \sum_{j=0}^{\infty} (\lambda_0 - \lambda)^j R_{\lambda_0}(T_\infty)^{j+1} \qquad \text{in } \mathscr{B}(M_\infty) \tag{3.14}$$

for all $\lambda \in \mathbb{C}$.

LEMMA 3.6. *Let X be a Banach space with $A, A_i \in \mathscr{B}(X)$, $i = 1, 2, \ldots$. If $A_i \to A$ pointwise and if $K \in \mathscr{B}(X)$ is compact, then $A_i K \to AK$ in $\mathscr{B}(X)$.*

We now are in a position to state and prove the main result of this section and of this paper.

THEOREM 3.7. *Let T be a discrete operator in H with $\sigma(T) = \{\lambda_i\}_{i=1}^\infty$. Assume there exists a constant $M > 0$ such that $\|\mathbb{P}_N\| \leqslant M$, $N = 1, 2, \ldots$. Set $T_\infty = T \mid \mathscr{D}(T) \cap M_\infty$. Then $R_\lambda(T) = R_\lambda(T)\mathbb{P}_\infty + R_\lambda(T)(I - \mathbb{P}_\infty)$ for $\lambda \in \rho(T)$ with*

$$R_\lambda(T)\mathbb{P}_\infty = \sum_{i=1}^{\infty} \sum_{j=1}^{m_i} \frac{(-N_i)^{j-1} P_i}{(\lambda - \lambda_i)^j} \qquad \text{in } \mathscr{B}(H), \tag{3.15}$$

where $N_i = (\lambda_i I - T) \mid \mathscr{N}(T(\lambda_i))$, and with

$$R_\lambda(T)(I - \mathbb{P}_\infty) = \sum_{j=0}^{\infty} (\lambda_0 - \lambda)^j R_{\lambda_0}(T_\infty)^{j+1}(I - \mathbb{P}_\infty) \qquad \text{in } \mathscr{B}(H), \tag{3.16}$$

where λ_0 is a fixed element of $\rho(T_\infty) = \mathbb{C}$.

Proof. We know that $R_\lambda(T)$ is compact and that $\mathbb{P}_N \to \mathbb{P}_\infty$ pointwise. Therefore, Lemma 3.6 can be applied, together with (2.2) and (3.4), to show that

$$\mathbb{P}_\infty R_\lambda(T) = \sum_{i=1}^{\infty} P_i R_\lambda(T) = \sum_{i=1}^{\infty} R_\lambda(T) P_i = \sum_{i=1}^{\infty} \sum_{j=1}^{m_i} \frac{(-N_i)^{j-1} P_i}{(\lambda - \lambda_i)^j}$$

in $\mathscr{B}(H)$. Equation (3.16) is immediate from (3.13) and (3.14). ∎

Theorem 3.7 yields a complete spectral representation of $R_\lambda(T)$ when $\sigma(T) = \{\lambda_i\}_{i=1}^{\infty}$, provided $\|\mathbb{P}_N\| \leqslant M$, $N = 1, 2, \ldots$. As a concluding remark we note that the convergence of the two series appearing in the theorem is uniform on compact subsets of $\rho(T)$.

4. Representation of $R_\lambda(T)$ Based on Spectral Operator-Type Assumptions

Throughout this section we assume the stronger assumption that the family of all finite sums of the projections P_i is uniformly bounded in norm by a constant $M > 0$. This assumption is one of two made in the theory of discrete spectral operators, the other being that $M_\infty = \{0\}$, which we do not assume.

To simplify the statement of later results, we make the following definition.

Definition 4.1. A discrete operator T with $\sigma(T) = \{\lambda_i\}_{i=1}^{\infty}$ is *spectral-like* iff the family of all finite sums of the projections P_i is uniformly bounded in norm by a positive constant M.

We now show that T being spectral-like allows more to be said about the representation of $R_\lambda(T)$. We start with

Lemma 4.2. *Let T be a spectral-like operator in H. Then for every $x \in S_\infty$*

$$\frac{\|x\|^2}{4M^2} \leqslant \sum_{i=1}^{\infty} \|P_i x\|^2 \leqslant 4M^2 \|x\|^2, \tag{4.1}$$

where the constant M is as in Definition 4.1.

Proof. Fix a positive integer N. Let $F = F(N)$ denote the family of all mappings from $\{1, 2, \ldots, N\}$ into $\{-1, 1\}$. Let $\gamma \in F$ and denote the value of γ at $j \in \{1, 2, \ldots, N\}$ by γ_j. Let $x \in S_\infty$ and set

$$x(\gamma) = \sum_{j=1}^{N} \gamma_j P_j x, \qquad x_+(\gamma) = \sum_{\gamma_j > 0} P_j x, \qquad x_-(\gamma) = \sum_{\gamma_j < 0} P_j x.$$

Then

$$x_+(\gamma)+x_-(\gamma)=\sum_{j=1}^{N} P_j x \equiv x_N,$$

and

$$x_+(\gamma)-x_-(\gamma)=x(\gamma).$$

Note that F has 2^N elements, that

$$\|x_N\|^2+\|x(\gamma)\|^2=2\,\|x_+(\gamma)\|^2+2\,\|x_-(\gamma)\|^2, \tag{$*$}$$

and that

$$\frac{1}{2^N}\sum_{\gamma\in F}\|x(\gamma)\|^2=\sum_{j=1}^{N}\|P_j x\|^2 \tag{$**$}$$

[4, p. 334]. Let $A=\{j\,|\,\gamma_j>0\}$ and $B=\{j\mid\gamma_j<0\}$. Then $(*)$ implies that

$$\begin{aligned}\|x(\gamma)\|^2&\leqslant 2\,\|x_+(\gamma)\|^2+2\,\|x_-(\gamma)\|^2\\ &=2\left\|\sum_{j\in A}P_j x_N\right\|^2+2\left\|\sum_{j\in B}P_j x_N\right\|^2\\ &\leqslant 4M^2\,\|x_N\|^2.\end{aligned}$$

Consequently,

$$\frac{1}{2^N}\sum_{\gamma\in F}\|x(\gamma)\|^2=\sum_{j=1}^{N}\|P_j x\|^2\leqslant 4M^2\,\|x_N\|^2.$$

We note that there exists $\gamma\in F$ such that

$$\|x(\gamma)\|^2\leqslant\sum_{j=1}^{N}\|P_j x\|^2,$$

for otherwise

$$\|x(\gamma)\|^2>\sum_{j=1}^{N}\|P_j x\|^2 \qquad \text{for all} \quad \gamma\in F,$$

implying that

$$\frac{1}{2^N}\sum_{\gamma\in F}\|x(\gamma)\|^2>\sum_{j=1}^{N}\|P_j x\|^2,$$

contradicting (∗∗). Now (∗) implies that for his γ

$$\begin{aligned}\|x_N\|^2 &\leqslant 2\,\|x_+(\gamma)\|^2 + 2\,\|x_-(\gamma)\|^2\\ &= \Big\|\sum_{j\in A} P_j x(\gamma)\Big\|^2 + 2\,\Big\|\sum_{j\in B} P_j x(\gamma)\Big\|^2\\ &\leqslant 4M^2\,\|x(\gamma)\|^2\\ &\leqslant 4M^2 \sum_{j=1}^{N} \|P_j x\|^2.\end{aligned}$$

Thus,

$$\frac{\|x_N\|^2}{4M^2} \leqslant \sum_{j=1}^{N} \|P_j x\|^2 \leqslant 4M^2\,\|x_N\|^2. \quad \blacksquare$$

COROLLARY 4.3. *Let T be a spectral-like operator in H with $\sigma(T) = \{\lambda_i\}_{i=1}^{\infty}$. Then the function*

$$\|x\|_* = \left(\sum_{i=1}^{\infty} \|P_i x\|^2\right)^{1/2}, \qquad x \in S_\infty, \tag{4.2}$$

is a norm on S_∞. Furthermore, S_∞ is complete under $\|\cdot\|_*$ with

$$\frac{\|x\|_*}{2M} \leqslant \|x\| \leqslant 2M\,\|x\|_*. \tag{4.3}$$

Proof. Lemma 4.2 shows that $\|\cdot\|_*$ is well defined and implies (4.3). The fact that $\|\cdot\|_*$ is a norm follows from the triangle inequality in l^2 together with the result that $S_\infty \cap M_\infty = \{0\}$. The completeness of S_∞ under $\|\cdot\|_*$ is a trivial consequence of (4.3). $\blacksquare$

It should be noted that

$$(x, y)_* = \sum_{i=1}^{\infty} (P_i x, P_i y), \qquad x, y \in S_\infty, \tag{4.4}$$

defines an inner product on S_∞ with $\|x\|_* = (x, x)_*^{1/2}$, $x \in S_\infty$. Furthermore, if $x \in \mathcal{N}(T(\lambda_i))$ and $y \in \mathcal{N}(T(\lambda_j))$, then

$$(x, y)_* = \sum_{k=1}^{\infty} (P_k x, P_k y) = \sum_{k=1}^{\infty} (P_k P_i x, P_k P_j y) = \delta_{ij}(x, y)$$

by (2.8), thereby showing that the generalized eigenspaces $\mathcal{N}(T(\lambda_i))$, $\mathcal{N}(T(\lambda_j))$ are orthogonal under this inner product when $i \neq j$.

If $R_\lambda(T)$ is assumed to be a Hilbert–Schmidt operator, then $\sum_{i=1}^{\infty} 1/|\lambda-\lambda_i|^2<\infty$ [1, p. 194]. With this fact in hand, we present the next theorem, which provides a "Jordan-canonical form" decomposition of $R_\lambda(T)$.

THEOREM 4.4. *Let T be a spectral-like operator in H with $\sigma(T)=\{\lambda_i\}_{i=1}^{\infty}$, and assume $R_\lambda(T)$ is Hilbert–Schmidt for all $\lambda\in\rho(T)$. Then*

$$R_\lambda(T)=D_\lambda+Q_\lambda+R_\lambda(T)(I-\mathbb{P}_\infty) \qquad \textit{for all} \quad \lambda\in\rho(T), \tag{4.5}$$

where

$$\text{(a)} \quad D_\lambda=\sum_{i=1}^{\infty}\frac{P_i}{\lambda-\lambda_i} \qquad \textit{in } \mathscr{B}(H),$$

$$\text{(b)} \quad Q_\lambda=\sum_{i=1}^{\infty}\sum_{j=2}^{m_i}\frac{(-N_i)^{j-1}P_i}{(\lambda-\lambda_i)^j} \qquad \textit{in } \mathscr{B}(H).$$

Furthermore, D_λ, Q_λ, and $R_\lambda(T)(I-\mathbb{P}_\infty)$ are compact operators on H with $\sigma(D_\lambda)=\sigma(R_\lambda(T))$ and $\sigma(Q_\lambda)=\sigma(R_\lambda(T)(I-\mathbb{P}_\infty))=\{0\}$.

Proof. Fix $\lambda\in\rho(T)$. Define $D_N\colon H\to H$ by $D_N=\sum_{i=1}^{N}(P_i/(\lambda-\lambda_i))$, $N=1, 2, \ldots$. Since $\mathscr{R}(D_N)$ is finite dimensional, D_N is compact. If $x\in H$ and $k\geqslant l$, then Corollary 4.3 implies that

$$\|(D_k-D_l)x\|^2=\left\|\sum_{i=l+1}^{k}\frac{P_ix}{\lambda-\lambda_i}\right\|^2$$
$$\leqslant 4M^2\left\|\sum_{i=l+1}^{k}\frac{P_ix}{\lambda-\lambda_i}\right\|_*^2\leqslant 4M^4\|x\|^2\sum_{i=l+1}^{k}\frac{1}{|\lambda-\lambda_i|^2},$$

and hence, by the comments preceding this theorem, $\{D_N\}_{N=1}^{\infty}$ is a Cauchy sequence in $\mathscr{B}(H)$. Therefore, there exists a compact operator $D_\lambda\in\mathscr{B}(H)$ such that

$$D_\lambda=\sum_{i=1}^{\infty}\frac{P_i}{\lambda-\lambda_i}. \tag{$*$}$$

If H is infinite dimensional, then it is clear that $0\in\sigma(D_\lambda)$. Let $x\in\mathscr{N}(\lambda_iI-T)$ with $x\neq 0$. Then $D_\lambda x=(\lambda-\lambda_i)^{-1}x$, so $(\lambda-\lambda_i)^{-1}\in\sigma(D_\lambda)$, i.e., $\sigma(R_\lambda(T))\subseteq\sigma(D_\lambda)$. For the reverse inclusion suppose $\xi\in\sigma(D_\lambda)$ with $\xi\neq 0$. Then there exists $x\neq 0$ such that $D_\lambda x=\xi x$. From ($*$) it is clear that $x\in S_\infty$, so there exists an integer l such that $P_lx\neq 0$. This implies that $P_lD_\lambda x=P_lx/(\lambda-\lambda_l)=\xi P_lx$, i.e., $\xi=(\lambda-\lambda_i)^{-1}$. Thus, $\sigma(D_\lambda)=\sigma(R_\lambda(T))$.

For $N=1,2,\ldots$ define $Q_N\colon H\to H$ by $Q_N=\sum_{i=1}^{N}\sum_{j=2}^{m_i}(-N_i)^{j-1}P_i/(\lambda-\lambda_i)^j$, where we insert 0 in the sum for the terms with $m_i = 1$. Consider the sequence $\{Q_N\}_{N=1}^{\infty}$ of compact operators in $\mathscr{B}(H)$. From Theorem 3.4 we see that $Q_N = R_\lambda(T)\mathbb{P}_N - D_N$. Thus, for any k, l we have

$$\|Q_k - Q_l\| \leqslant \|R_\lambda(T)(\mathbb{P}_k - \mathbb{P}_l)\| + \|D_k - D_l\|.$$

The results from the first paragraph of this proof and Lemma 3.6 imply that $\{Q_N\}_{N=1}^{\infty}$ is a Cauchy sequence in $\mathscr{B}(H)$. Thus, there is a compact operator $Q_\lambda \in \mathscr{B}(H)$ such that

$$Q_\lambda = \sum_{i=1}^{\infty} \sum_{j=2}^{m_i} \frac{(-N_i)^{j-1} P_i}{(\lambda - \lambda_i)^j}. \qquad (**)$$

Clearly $\sigma(Q_\lambda) \neq \varnothing$. Suppose there is a nonzero $\xi \in \sigma(Q_\lambda)$. Then there exists a nonzero $x \in S_\infty$ and a projection P_l such that $P_l Q_\lambda x = \xi P_l x \neq 0$. If we note that $Q_\lambda P_l x = P_l Q_\lambda x$, then it is clear that $Q_\lambda P_l x = (Q_\lambda P_l) P_l x = \xi P_l x$, i.e., $\xi \in \sigma(Q_\lambda P_l)$. Since $(Q_\lambda P_l)^{m_l} = 0$, $Q_\lambda P_l$ is nilpotent, and hence, $\sigma(Q_\lambda P_l) = \{0\}$. This leads to a contradiction. Therefore, $\sigma(Q_\lambda) = \{0\}$.

Clearly $R_\lambda(T)(I - \mathbb{P}_\infty)$ is compact because $R_\lambda(T)$ is compact. Also, $R_\lambda(T)(I - \mathbb{P}_\infty) = R_\lambda(T_\infty)(I - \mathbb{P}_\infty)$. Since $\sigma(T_\infty) = \varnothing$, $\sigma(R_\lambda(T_\infty)) = \{0\}$, and hence, $\sigma(R_\lambda(T)(I - \mathbb{P}_\infty)) = \{0\}$. ∎

It should be noted that (4.5) generalizes (1.2) and that (4.5)(b) provides a representation for Q_λ previously unavailable.

5. Applications

In $H = L^2[0, 1]$ define the second-order differential operator L by

$$\mathscr{D}(L) = \{u \in H^2[0, 1] \mid u'(0) + u'(1) = 0, u(0) = 0\}, \qquad Lu = -u'',$$

where $H^2[0, 1]$ denotes the subspace of H consisting of all functions $u \in C^1[0, 1]$ with u' absolutely continuous on $[0, 1]$ and $u'' \in H$. Then L is a discrete operator in H [11]. Furthermore, $\sigma(L) = \{\lambda_i\}_{i=1}^{\infty}$, where $\lambda_i = [(2i-1)\pi]^2$, with $\nu(\lambda_i) = 2 = m_i$, $i = 1, 2, \ldots$. It is shown in [10] that the family of all finite sums of the projections P_i is uniformly bounded in norm by the constant $M = 6$, that $S_\infty = H$, that $M_\infty = \{0\}$, and in [11] that $R_\lambda(L)$ is Hilbert–Schmidt. Thus, for all $\lambda \in \rho(L)$

$$R_\lambda(L) = \sum_{i=1}^{\infty} \frac{P_i}{\lambda - \lambda_i} + \sum_{i=1}^{\infty} \frac{(L - \lambda_i I) P_i}{(\lambda - \lambda_i)^2} \quad \text{in } \mathscr{B}(H). \qquad (5.1)$$

The operator L in this example comes from a class of differential operators which has not previously been studied, the distinguishing feature being the presence of infinitely many multiple eigenvalues. These operators will be the subject of a future paper [10].

In the product Hilbert space $H = L^2[0, 1] \times L^2[0, 1]$, with the standard inner product and norm, define the linear operator T by

$$\mathscr{D}(T) = \mathscr{D}(K) \times \mathscr{D}(L), \qquad T(u, v) = (Ku, Lu),$$

where K and L are the differential operators in $L^2[0, 1]$ defined by

$$\mathscr{D}(K) = \{u \in H^2[0, 1] \mid u(0) = u(1) = 0\}, \qquad Ku = -u'',$$

and

$$\mathscr{D}(L) = \{u \in H^2[0, 1] \mid u(0) = u'(0) = 0\}, \qquad Lu = -u'',$$

respectively. It can be shown that T is an spectral-like operator in H with $\sigma(T) = \{\lambda_i\}_{i=1}^{\infty}$, where $\lambda_i = (i\pi)^2$ for $i = 1, 2, \ldots$. It can also be shown that $R_\lambda(T)$ is a Hilbert–Schmidt operator for all $\lambda \in \rho(T)$, that $S_\infty = \{(u, 0) \in H \mid u \in L^2[0, 1]\} = \bar{S}_\infty$, that $M_\infty = \{(0, v) \in H \mid v \in L^2[0, 1]\}$, that $m_i = 1$ for all i, and that $P_i(u, v) = ((u, \phi_i)\, \phi_i, 0)$, where $\phi_i(t) = \sqrt{2}\sin(i\pi t)$, $i = 1, 2, \ldots$. Thus, for all $\lambda \in \rho(T)$

$$R_\lambda(T) = \sum_{i=1}^{\infty} \frac{P_i}{\lambda - \lambda_i} + R_\lambda(T)(I - \mathbb{P}_\infty) \qquad \text{in } \mathscr{B}(H), \tag{5.2}$$

where $\mathbb{P}_\infty(u, v) = (u, 0)$ for all $(u, v) \in H$.

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