JOURNAL OF ALGEBRA 42, 102–120 (1976)

Schur Indices of Some Groups of Lie Type

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Communicated by Walter Feit

Received March 28, 1975; revised August 29, 1975

The representation theory of groups of Lie type has been investigated in several research papers recently, with much attention being directed towards the computation of constituents of induced modules. Our interest here is to start an investigation into the Schur indices of the characters of the groups of Lie type in an attempt to give a more complete description of the representations. The only previous results in this direction are due to Janusz [10], who obtained the local Schur indices of the characters of the group $SL(2, q)$. Our approach concentrates on induced modules, defined via the linear characters of the Sylow p-subgroups (p being the characteristic of the underlying field of the group). Such modules have been studied by Yokonuma [17], and Gelfand and Graev [6], and they have been shown to be multiplicity-free for certain linear characters said to be in general position. We will show that in many circumstances the characters of the induced modules we define are the characters of representations realizable in the rational field. When this is the case, the Schur index of any constituent character of the module divides the multiplicity with which the character appears.

Examination of the character tables of some groups of Lie type confirms that almost all characters do occur exactly once in some induced module, and given favorable conditions in the Borel subgroup, we can deduce that such characters have Schur index 1. To illustrate our methods we have determined the Schur indices of groups of small rank using the currently available character tables, but an attack on the general problem seems to require a better understanding of the construction of the characters of the Lie groups. However, we believe that a knowledge of the character values only on the Sylow p-subgroup may be sufficient to tackle the problem for groups of any rank. Throughout this paper $m_F(X)$ denotes the Schur index of X over the field F. Our findings are presented at the end of the paper.

1. Preliminary Results

The theory of the Schur index is developed in [1, Sect. 70; 9, Sect. 14, Chap. 5]. Initially, we will need none of the deeper results on the Schur index, and will rely on the following property of the index [1, 70.14]:

Lemma 1. *Let F be a representation of a finite group G defined over the rationals and let θ be the character of F. Then if X is an absolutely irreducible constituent of θ and the inner product $(\theta, X) = r \neq 0$, the rational Schur index of X divides r.*

We will also use the following result, which is easily established.

Lemma 2. *Let x be an element of the finite group G and suppose that for each integer m coprime to the order of x, x and x^m are conjugate in G. Then all characters of G take rational values on x.*

2. The General Linear Group

Let G denote the general linear group $GL(n, q)$, where q is a power of the prime p. Our intention is to analyze the structure of an induced character λ^B, where λ is a linear character of a Sylow p-subgroup P of G, and B is the normalizer of P, usually known as a Borel subgroup. We will show that λ^B is the character of a rational representation, and thus the same will be true of λ^G. By computing inner products (X, λ^G), we will obtain information on the Schur indices of the characters of G.

A result of Gelfand and Graev [6, Theorem 1], shows that each character occurs as a constituent of at least one of these induced characters, and so our method theoretically has some applicability. However, the Gelfand–Graev theorem does not hold for all Chevalley groups or the twisted types, a fact which becomes evident even when dealing with the groups of small rank considered in the final section. We will base most of our calculations of Schur indices in this paper on the method to be developed now for the general linear group.

It is well-known that B can be written in the form $B = PH$, where $P \cap H = 1$ and H is isomorphic to n copies of the multiplicative group of $GF(q)$. The next lemma is the key to obtaining our results on the Schur index.

Lemma 3. *Let λ be a linear character of P. Then the induced character λ^B is the character of a rational representation.*

Proof. We will take as a representative for P all those lower triangular matrices whose entries on the main diagonal are 1. H will consist of all diagonal matrices. The derived group of P consists of all those members of P whose entries immediately below the main diagonal are 0, and the derived factor group of P is elementary abelian of order q^{n-1}. To determine the action of H on P/P', we have only to examine the action of H on the entries immediately below the main diagonal of an element of P.

If $h \in H$ has diagonal entries $\lambda_1, ..., \lambda_n$ and $x \in P$ has entries $a_1, ..., a_{n-1}$ below the diagonal, we find that hxh^{-1} has corresponding entries $\lambda_2\lambda_1^{-1}a_1, ...,$ $\lambda_n\lambda_{n-1}^{-1}a_{n-1}$. Let σ be an element of order $p - 1$ in $GF(p)$ and let $m = \text{diag}(1, \sigma, ..., \sigma^{n-1})$, an element of order $p - 1$ in H. We readily see that for any x in P, mxm^{-1} and x^σ are equal modulo P'. Thus if M is the subgroup of B generated by m and P, it follows that each nonidentity element of P/P' is conjugate in M/P' to its $p - 1$ nonidentity powers. In particular, Lemma 2 implies that each character of M/P' takes rational values on P/P'. Now it is easily seen that if λ is any linear character of P, λ^M is irreducible (this follows from [9, 16.13, p. 561] since M/P' is a Frobenius group with kernel P/P'). We also note that λ^M is rational-valued, for it is zero outside P, and we know it must be rational-valued on P.

Finally, we will show that the Schur index of λ^M is 1. If this is true, λ^M is the character of a rational representation and the same must be true of λ^B. Let L be the subgroup of M generated by m. Since $M = PL$, Mackey's theorem [9, 16.9, p. 557], implies that $(\lambda^M)_L$ is the regular representation of L. Thus the trivial representation 1_L of L occurs once in the restriction of the irreducible character λ^M to L, and, by reciprocity, λ^M occurs once in $(1_L)^M$. Lemma 1 implies that λ^M has Schur index 1, and our proof is complete.

Thus we have a criterion for determining the Schur indices of characters of $GL(n, q)$.

THEOREM 1(a). *Let P be a Sylow p-subgroup of $G = GL(n, q)$, where q is a power of p, and let λ be a linear character of P. Then if X is an irreducible constituent of λ^G with multiplicity $r \neq 0$, the rational Schur index of X divides r.*

Proof. We have seen in Lemma 3 that λ^B is a character of a rational representation and the same must be true of λ^G. Our assertion follows from Lemma 1.

The information we have collected so far is sufficient for us to obtain the following positive result concerning the Schur indices of characters of $GL(n, q)$.

THEOREM 2(a). *Let q be a power of the prime p and let X be a character of $G = GL(n, q)$. Then if $X(1)$ is coprime to p, the Schur index of X equals 1.*

Proof. We consider the element c of G whose action on a basis $e_1, ..., e_n$

of the underlying vector space is given by $ce_i = e_i + e_{i+1}$, $1 \leqslant i \leqslant n-1$, $ce_n = e_n$. The order of c is a power of p and c is an example of a so-called regular unipotent element. It has been shown by Simpson in [14, Theorem 1], that any irreducible character X of G takes only the values 0, 1 or -1 on c. Moreover if $X(1)$ is coprime to p, $X(c)$ cannot equal 0.

The centralizer of c in G has order $q^{n-1}(q-1)$ [14, p. 292], and q^{n-1} is the order of the centralizer of c in the Sylow p-subgroup P of G consisting of lower-triangular matrices with 1's on the diagonal. In P we have $\sum_i \theta_i(c)\,\theta_i(c^{-1}) = q^{n-1}$ by the orthogonality relations, where the sum extends over all irreducible characters of P. We note that each linear character λ of P makes a contribution of 1 to this sum. However, since P/P' has order q^{n-1}, there are q^{n-1} different linear characters of P, and their contribution to the sum equals the total. Since the sum consists of nonnegative real numbers, we must have $\theta(c) = 0$ for any nonlinear irreducible character of P.

Let X be any irreducible character of G whose degree is coprime to p. We know that $X(c) = \pm 1$. Let $X_P = \sum a_i\theta_i + \sum b_i\lambda_i$, where the first sum consists of nonlinear characters of P, the second only of linear characters. If the Schur index of X is t, we know from Theorem 1 that t divides each inner product (X, λ_i^G). But since $(X, \lambda_i^G) = b_i$, by reciprocity, t divides each b_i. By our previous arguments, $X(c) = \sum b_i\lambda_i(c) = \pm 1$, and if t is greater than 1, putting $b_i = td_i$, we obtain $\sum d_i\lambda_i(c) = \pm 1/t$. However, the left-hand sum is an algebraic integer, whereas if $t > 1$, $1/t$ is a rational number but not an integer. This is not possible and so $t = 1$, implying that X has Schur index 1.

3. The Special Linear Group $SL(n, q)$

The results we have established about the characters of a Borel subgroup of $GL(n, q)$ are not in general true for a Borel subgroup of $SL(n, q)$. As might be expected, the situation varies according to the value of n and the arithmetic nature of q. We will let B denote, as before, the normalizer of a Sylow p-subgroup P of $SL(n, q)$. P is complemented in B by a subgroup H of order $(q-1)^{n-1}$ and the centralizer of P in B is the center Z of $SL(n, q)$, which has order $(n, q-1)$.

We now consider the characters of B/P'. In general it is not true that all characters take rational values on P/P', nor do all characters have Schur index 1. However, we will show that for $SL(2n+1, q)$ we do have the exact analog of Lemma 3.

Lemma 4. *Let P be a Sylow p-subgroup of $SL(2n+1, q)$ and let B be its normalizer. Then if λ is a linear character of P, λ^B is the character of a rational representation.*

Proof. We follow Lemma 3 and assume P consists of lower triangular matrices. H consists of all diagonal matrices with determinant 1. If σ has order $p-1$ in $GF(p)$, we find that the element $m = \text{diag}(\sigma^{-n}, \sigma^{1-n}, \ldots, \sigma^{n-1}, \sigma^{n})$ has determinant 1 and is thus in H. As in Lemma 3, for any element x of P, mxm^{-1} and x^{σ} are equal modulo P'. The rest of the proof follows Lemma 3.

We now have no trouble in proving:

THEOREM 1(b). *Let P be a Sylow p-subgroup of $G = SL(2n+1, q)$, where q is a power of p, and let λ be a linear character of P. Then if X is an irreducible constituent of λ^G with multiplicity $r \neq 0$, the rational Schur index of X divides r.*

We can also obtain an exact analog of Theorem 2(a).

THEOREM 2(b). *Let q be a power of the prime p and let X be an irreducible character of $G = SL(2n+1, q)$. Then if $X(1)$ is coprime to p, the Schur index of X is 1.*

Proof. We take b to be a regular unipotent element corresponding to the element c introduced in the proof of Theorem 2(a). It is not true that $X(b)$ is necessarily equal to ± 1, but we can modify the proof of Theorem 2(a) to obtain our objective. We note that $X(b)$ is necessarily an integer, for it is not hard to see that b is conjugate to its powers b^r, $(r, p) = 1$, in B and hence in G.

Let $G_1 = GL(2n+1, q)$ and let $X = X_1, \ldots, X_r$ be the distinct G_1-conjugates of X. Since G_1/G is cyclic, it follows from the Clifford theory, [9, Chap. 5, Sect. 17], that there is an irreducible character θ of G_1 with $\theta_G = X_1 + \cdots + X_r$. As the characters $X_1, \ldots, X_r$ are conjugate under automorphisms of G it is easily seen that their Schur indices are equal. Now r divides $|G_1 : G| = q - 1$ and thus, as $\theta(1) = rX(1)$, $\theta(1)$ is also coprime to p.

Let us suppose that the Schur index of X is t. Since Theorem 1(b) holds for G, the argument of Theorem 2(a) may be applied to show that t must divide $X(b)$. But this argument applies to each G_1-conjugate of X, since we know that the conjugates also have Schur index t. We have $\theta(b) = X_1(b) + \cdots + X_r(b) = \pm 1$, by Simpson's theorem. Since t divides each $X_i(b)$, it must divide $\theta(b)$ and consequently $t = 1$, establishing our claim.

We turn our attention to the even dimensional special linear group $SL(2n, q)$. When q is odd we will see that results on the Schur index are more complicated than those encountered so far, but when q is even, there are no problems in obtaining analog of Theorems 1(b) and 2(b). For if q is a power of 2, and P is a Sylow 2-subgroup of $SL(2n, q)$, any linear character of P is realizable in the rational field, as P/P' is an elementary abelian 2-group. Thus an analog of Theorem 1(b) is trivial, and the argument of Theorem 2(b) easily gives, for any n.

THEOREM 2(c). *Let q be a power of 2 and let $G = SL(n, q)$. Then any irreducible character of G of odd degree has Schur index* 1.

When dealing with $SL(2n, q)$ for odd values of q, the method just described for investigating Schur indices is no longer so effective. However, when n is odd, we can give a criterion for the existence of characters of Schur index 2.

THEOREM 1(c). *Let q be a power of the prime p, with $q \equiv 1$ (mod 4), and let $G = SL(2n, q)$, n odd. Let P be a Sylow p-subgroup of G and let λ be a nontrivial linear character of P. Then any real-valued irreducible character X of G which occurs in λ^G with odd multiplicity and is nontrivial on the central involution of G has Schur index 2 over the reals, and hence over the rationals.*

Proof. We will assume that P consists of lower triangular matrices. Let w have order 4 in $GF(q)$ and let $h = \operatorname{diag}(w^{-1}, w, \ldots, w^{-1}, w)$. We see that h is in G, normalizes P, and has order 4, with $h^2 = -I$. Following Lemma 3, it is easily seen that for all x in P, hxh^{-1} and x^{-1} are equal modulo P'. Let N be the subgroup generated by P and h. The analysis above implies that all characters of N/P' take real values on P/P'.

Now let λ be a nontrivial linear character of P. It is not hard to see that λ^N consists of two real-valued characters, μ_1 and μ_2, of degree 2. We can assume that the notation is chosen so that $\mu_1(h^2) = 2$, $\mu_2(h^2) = -2$. Using the method of Frobenius and Schur [5, p. 21], it is easily shown that μ_2 must have Schur index 2 over the real numbers. Let X be a real-valued character of G which occurs in λ^G with odd multiplicity, r say, and which satisfies $X(h^2) = -X(1)$. Let $\lambda^N = \mu$. Then since $\lambda^G = \mu^G$, we have $(\mu^G, X) = r$. However, $\mu^G = \mu_1{}^G + \mu_2{}^G$ and all constituents of $\mu_1{}^G$ are trivial on h^2. Thus $(\mu_2{}^G, X) = r$ and correspondingly $X_N = r\mu_2 +$ other characters different from μ_2. Let us suppose that X is realizable in the real field. Then if D is the real representation of G with character X, μ_2 occurs r times in the character of D_N. By Lemma 1, the real index of μ_2 divides r. But we know that the real index of μ_2 is 2 and we have a contradiction. Thus X has Schur index 2 over the reals, and the Brauer–Speiser theorem implies that it has Schur index 2 over the rationals, [3]. This completes the proof.

Although we have not made use of the fact that n is odd in the proof of Theorem 1(c), the theorem is only of interest for odd values of n. For if n is even and $q \equiv 1$ (mod 4), $SL(2n, q)$ has a central element z of order 4 whose square is $-I$. Any character of $SL(2n, q)$ which is nontrivial on $-I$ must be faithful for z, and hence cannot be real-valued. Thus for even values of n, there are no characters satisfying the hypothesis of Theorem 1(c).

However, when n is odd, and $q \equiv 1 \pmod 4$, it would be of interest to know whether all real-valued characters of $SL(2n, q)$ which are nontrivial on $-I$ have Schur index 2.

The following result of the author offers an approach to showing the existence of real-valued characters of Schur index 2 in $SL(2n, q)$ whenever both n and q are odd [7, Theorem 2]. We recall that an element of a group is said to be real if it is conjugate to its inverse and strongly real if it is inverted by an involution.

LEMMA 5. *Let G be a finite group and let $x_1, \ldots, x_r$ be nonconjugate real 2-regular elements of G which are not strongly real. Suppose also that $C(x_i)$ has an abelian Sylow 2-subgroup, $1 \leqslant i \leqslant r$. Then G possesses at least r real-valued irreducible characters of Schur index 2.*

In what follows we will assume that n is odd and greater than 1. This is no restriction as we already have a description of the Schur indices of characters of $SL(2, q)$. We will show that $G = SL(2n, q)$ has classes for which Lemma 5 is applicable. By [9, p. 187, Theorem 7.3], there is a cyclic self-centralizing subgroup S in G of order $q^{2n} - 1/q - 1$. If w is a generator of S, there is an element t in $N(S)$ with $t^{-1}wt = w^q$. Let r be the 2-part of $q + 1$ and let T be a subgroup of order $q^n + 1/r$ in S. As n is odd, T has odd order. Moreover, as $n \geqslant 3$, it may easily be shown that T is an irreducible subgroup. Let x be a generator of T and let $u = t^n$. We have $u^{-1}xu = x^{q^n} = x^{-1}$, as x has order dividing $q^n + 1$, and so x is a real element of G. It may be shown that any involution of $GL(2n, q)$ which inverts x must have n eigenvalues equal to 1 and n equal to -1, and so has determinant equal to -1. We deduce that x is a real 2-regular element which is not strongly real, with $C(x)$ abelian. Now x is conjugate to exactly $2n$ of its powers and so T has $1/2n \cdot \varphi(q^n + 1/r)$ nonconjugate generators, where φ is Euler's function. By Lemma 5, G has at least $(1/2n)\, \varphi(q^n + 1/r)$ real-valued characters of Schur index 2. In particular, we have

PROPOSITION 1. *Let $G = SL(2n, q)$, $q \equiv 1 \pmod 4$, n odd. Then G has at least $(1/2n)\varphi(q^n + 1)$ real-valued irreducible characters of Schur index 2.*

We remark that there is some evidence, based on the Frobenius–Schur involution formula, to indicate that the number of real-valued irreducible characters of Schur index 2 in $SL(2n, q)$ may be some rational polynomial in q of degree n.

We conclude this section by showing that whenever q is a square, our previous methods are applicable to the problem of determining Schur indices of characters of $PSL(2n, q)$, for any value of n. The next lemma is in the spirit of previous ones.

LEMMA 6. *Let p be an odd prime and let q be an even power of p. Let P be a Sylow p-subgroup of $G = PSL(2n, q)$ and let λ be a linear character of P. Then λ^G is the character of a rational representation of G.*

Proof. We work in $SL(2n, q)$ initially, taking P to consists of lower triangular matrices. Let σ have order $p - 1$ in $GF(p)$. As q is a square, we can find an element t in $GF(q)$ with $t^2 = \sigma$. Let h be the element diag$(t^{1-2n}, t^{3-2n}, \ldots, t^{2n-3}, t^{2n-1})$ of $SL(2n, q)$. Exactly as in our previous proofs we see that h normalizes P and for x in P, hxh^{-1} and x^σ are equal modulo P'. Now h has order $2(p - 1)$ in $SL(2n, q)$, but as $h^{p-1} = -I$, its image in G has order $p - 1$. Thus in G we can find a subgroup N of the normalizer of P for which each induced character λ^N is irreducible of degree $p - 1$ and realizable in the rationals (just as in the proof of Lemma 3). The proof of the lemma is immediate from this fact.

By considering $PSL(2n, q)$ as a subgroup of the projective general linear group $PGL(2n, q)$ we can imitate the proof of Theorem 2(b) to obtain

THEOREM 2(d). *Let q be an even power of the prime p and let $G = PSL(2n, q)$. Then if X is an irreducible character of G of degree coprime to p, the Schur index of X is* 1.

We can extend this analysis to investigate the local Schur indices of characters of $SL(2n, q)$ whenever q is a square. For, letting B and P denote Borel and Sylow p-subgroups of $G = SL(2n, q)$, the argument of Lemma 6 shows that for any linear character λ of P, λ^B is a rational-valued character. However, λ^B will not be realizable in the rational field (by the argument of Theorem 1(c)).

Now it is not hard to see that any irreducible character X of B/P' remains irreducible as an r-modular character for any prime $r \neq p$, essentially because the restriction of X to P consists of $X(1)$ distinct conjugate linear characters of P, which are all distinct modulo r. A result of the author [8], implies that X has Schur index 1 over the r-adic numbers. Thus as each constituent of λ^B has r-local index 1 and λ^B is rational-valued, λ^B is realizable in the r-adic numbers. The same is true of λ^G. Thus following the argument of Theorem 2(b), we can deduce

THEOREM 2(e). *Let q be an even power of an odd prime p and let $G = SL(2n, q)$. Then if X is an irreducible character of G of degree coprime to p, the only finite prime at which the local index of X can differ from* 1 *is p.*

This theorem is in accordance with the results of [10] for $SL(2, q)$. However, Janusz shows that when q is not a square there are other finite primes at which local indices can exceed 1.

4. The Symplectic Group $Sp(2n, q)$

We will briefly describe how it is possible to apply the methods developed in the previous section to investigate the Schur indices of the characters of the symplectic group. Since the two-dimensional symplectic group is the group $SL(2, q)$, we already have a description of the Schur indices of the characters of $Sp(2, q)$. It is the author's belief that the behavior exhibited by the Schur indices of $Sp(2, q)$ is essentially common to all symplectic groups. This section presents a number of reasons why this should be so. As with the group $SL(2n, q)$, the analysis falls naturally into a number of cases, depending on the nature of q: q even, q odd and a square, $q \equiv 1 \pmod 4$ but not a square, $q \equiv 3 \pmod 4$. Our first result given an initial simplification of the problem of computing Schur indices.

Lemma 7. *Let q be a power of 2 or a power of an odd prime p with $q \equiv 1$* (mod 4). *Then all characters of $G = Sp(2n, q)$ are real-valued and consequently of Schur index at most* 2.

Proof. We must show that each element of G is conjugate to its inverse. This may be established by induction on n, using the conjugacy criterion for elements of the symplectic group given in [11, pp. 36(ii), p. 59]. When q is odd, the essential part of the proof is that -1 is a square in $GF(q)$.

Note 2. When $q \equiv 3 \pmod 4$, Lemma 7 is no longer true. However, we can show that each p-regular class of G is real. Since the number of p-regular classes of G is q^n, and the total number of classes of G is a monic polynomial in q of degree n [11, p. 36(iii)], we can assert that the majority of characters are real-valued.

Our next objective is to give some criteria for computing the Schur index using induced characters. Let V be a $2n$-dimensional symplectic space over $GF(q)$ with basis $e_1, \ldots, e_{2n}$ and form defined by $(e_i, e_j) = \delta(i, 2n + 1 - j)$, $i \leqslant j$. Let G be the group preserving this form and let B and P be Borel and Sylow p-subgroups of G, where p is the prime divisor of q. It is possible to choose P to consist of certain lower triangular matrices with ones on the diagonal. The $2n - 1$ entries immediately below the diagonal of a typical element of x of P take the form $a_1, \ldots, a_{n-1}, a_n, -a_{n-1}, \ldots, -a_1$ where the a_i arbitrary elements of $GF(q)$. If q is odd, the derived group of P consists of those elements of P whose entries immediately below the main diagonal are 0, the derived factor-group being elementary abelian of order q^n.

B can be written in the form PH, where H consists of diagonal matrices h of the form $\operatorname{diag}(\lambda_1, \ldots, \lambda_n, \lambda_n^{-1}, \ldots, \lambda_1^{-1})$. We find that $h^{-1}xh$, $x \in P$, has entries

$$\lambda_2\lambda_1^{-1}a_1, \ldots, \lambda_n\lambda_{n-1}^{-1}a_{n-1}, \lambda_n^{-2}a_n, \ldots, -\lambda_2\lambda_1^{-1}a_1$$

below the main diagonal and we may thus describe the action of h on P/P'. When q satisfies the congruence $q \equiv 1 \pmod 4$, we can obtain a simple criterion for the Schur index of a faithful irreducible character of G to be 2, for we have

THEOREM 3(a). *Let q be a power of the prime p, with $q \equiv 1 \pmod 4$, and let $G = Sp(2n, q)$. Let P be a Sylow p-subgroup of G and let λ be a linear character of P. Then any faithful irreducible character of G which occurs in λ^G with odd multiplicity has Schur index 2. Any nonfaithful irreducible character of G which occurs with odd multiplicity has Schur index 1 over the reals.*

Proof. We take P to consist of lower triangular matrices, as described earlier. Let σ be an element of order 4 in $GF(q)$ and let h be the element $\mathrm{diag}(\lambda_1, \ldots, \lambda_n, \lambda_n^{-1}, \ldots, \lambda_1^{-1})$ where $\lambda_i = (-1)^i \sigma$, $1 \leqslant i \leqslant n$. Certainly h normalizes P and we find that for any element x of P, hxh^{-1} and x^{-1} are equal modulo P'. Furthermore, $h^2 = -I$. The first statement of the theorem follows from the proof of Theorem 1(c). The second statement follows by noticing that h has order 2 in $PSp(2n, q)$.

By specializing to the case where q is a square, we easily prove

THEOREM 3(b). *Let q be an even power of an odd prime p. Let $G = Sp(2n, q)$ and let P be a Sylow p-subgroup of G. Then if λ is a linear character of P, any nonfaithful irreducible constituent of λ^G which occurs with odd multiplicity has Schur index 1. If X is a faithful irreducible constituent of λ^G which occurs with odd multiplicity, the only finite prime for which the local index of X can differ from 1 is p.*

Proof. Since q is a square, we can easily show that λ^G is rational-valued and that if X is a nonfaithful constituent of λ^G, the Schur index of X divides (X, λ^G). Since we know that the Schur index of X is either 1 or 2, the first statement follows. The second statement follows from arguments identical to those used in the proof of Theorem 2(e).

We will see in the next section that not all irreducible characters of the symplectic group occur as constituents of induced characters λ^G, so Theorems 3(a) and 3(b) are not always applicable. However, we note that our two theorems do apply when λ is in general position, for it follows from Yokonuma's results that λ^G is multiplicity-free. Since the majority of characters of $Sp(2n, q)$ occur in such induced characters, we can even assert that almost all irreducible faithful characters of $Sp(2n, q)$, $q \equiv 1 \pmod 4$, have Schur index 2. We believe that in fact all faithful irreducible characters of this group are of index 2, and will show that this is true for the four dimensional group in the next section.

Lemma 5 may be used to predict the existence of real-valued irreducible characters of Schur index 2 in $Sp(2n, q)$ for any odd value of q. For $Sp(2n, q)$

contains a self-centralizing cyclic subgroup S of order $q^n + 1$. If r is the 2-part of $q^n + 1$, S contains a subgroup T of odd order $q^n + 1/r$, which is always irreducible provided that n is greater than 1. The generators of T provide us with classes satisfying the hypotheses of Lemma 5, and the following result may be proved in a manner similar to Proposition 1.

PROPOSITION 2. *If n is even or $q \equiv 1 \pmod 4$, $Sp(2n, q)$ has at least $(1/2n)\,\varphi(q^n + 1)$ real-valued faithful irreducible characters of Schur index* 2.

ADDENDUM

This part of the paper is independent of the previous sections and is designed to provide a second approach to the problem of determining the Schur indices of the irreducible characters of the classical groups. Using different methods, relying on the Brauer–Speiser theorem, it proves to be easy to show that the Schur index of an irreducible character of $GL(n, q)$ or $U(n, q^2)$ divides 2. Indeed we are led to believe that the Schur index of an irreducible character of any classical group divides 2. However, to decide whether the index of a particular character is equal to 1 or 2 may require methods of the type already developed in Section 1.

The following result illustrates the idea behind our second method.

THEOREM A. *Let G denote either the general linear group $GL(n, q)$ or the general unitary group $U(n, q^2)$. Then the Schur index of any character of G divides* 2.

Proof. We will deal with $GL(n, q)$ first. Let u be the involutory automorphism of G defined by $u(x) = x^*$, where x^* is the transpose–inverse of x, and let H be the split extension of G by u. It is well-known that each element x of G is conjugate to its transpose x^t. Thus we have $w^{-1}xw = x^t$, for some w in G. Applying the automorphism u, we see that x is conjugate to x^{-1} in H. It follows that all characters of H take real values on G.

Now if θ is a real-valued irreducible character of G, the Brauer–Speiser theorem implies that the Schur index of θ divides 2. On the other hand, if θ is an irreducible character of G which is not real-valued, the induced character $\varphi = \theta^H$ is a real-valued irreducible character of H. For since H/G has order 2, φ is either irreducible or is the sum of two distinct extensions θ_1, θ_2 of θ to H. In the latter case, θ_1 is a character of H which cannot be real-valued on G contrary to our earlier deduction. That φ is real-valued follows from the fact that it vanishes outside G and is real-valued on G.

Let $\varphi = \varphi_1, \ldots, \varphi_r$ be the distinct algebraic conjugates of φ. Since φ is real-valued, the Brauer–Speiser theorem implies that $\psi = 2(\varphi_1 + \cdots + \varphi_r)$ is the character of a rational representation of H. Thus ψ_G is the character of a rational representation of G. We note that θ occurs exactly twice in ψ_G, since φ is the unique character of H whose restriction to G contains θ. Lemma 1 now implies that $m_Q(\theta)$ is at most 2, as required.

The proof for the unitary group is essentially the same. We take G to consist of all elements x which satisfy $x\sigma(x)^t = 1$, where σ is the involution of $GF(q^2)$. With this presentation of G, if x is in G, x^t is also in G. Since σ defines an involutory automorphism of G, we may form the split extension, H, of G by σ. In H, we have $x^\sigma = x^*$ for each element x of G. However, since x and x^t are known to be conjugate in the full linear group, we deduce from a result of Wall [11], that they are conjugate in G. Thus x and x^{-1} are conjugate in H and the rest of the proof is identical with that for $GL(n, q)$.

5. Specific Calculations

We will use character tables to compute the Schur indices of all characters of $GL(n, q)$, $n \leqslant 4$, $SL(3, q)$, $PSU(3, q^2)$, Ree groups of type G_2, Suzuki groups, and $Sp(4, q)$, when q is even or an even power of an odd prime. For the sake of brevity, all computations of inner products have been omitted.

Theorem 4. *Let G be any of the groups $GL(n, q)$, where $n \leqslant 4$. Then any character of G has Schur index* 1.

Proof. Our sources for character tables are [15] and [20] (also [12], by using the device of changing q to $-q$). Let q be a power of p, and let B, P be Borel and Sylow p-subgroups of G. Let λ be a linear character of P in general position. When $G = GL(2, q)$, we find that all nonlinear characters of G are constituents of λ^G, with multiplicity 1, of course. For $GL(3, q)$, we find that λ^G contains all characters of G whose degrees are polynomials in q of degree 3. The only nonlinear characters of G which remain are those of degree $q^2 + q$ and $q^2 + q + 1$. However, if μ is any linear character of P which has $q - 1$ conjugates in B, μ^G contains these characters with multiplicity 1. Thus Theorem 1(a) gives the result.

We now turn our attention of $GL(4, q)$. To compute inner products, we must assign elements of P to the four conjugacy classes of p-elements of G, but we omit details of this routine matter. We find that λ^G contains all characters of G whose degrees are polynomials in q of degree 6. We also note that Steinberg has proved in [19], for any general linear group, that all constituents of $1_B{}^G$ are realizable in the rational numbers. For $GL(4, q)$, these are the characters of Table 9 in [20]. The tensor products of these characters with linear characters of G must also be of Schur index 1. As we know that all characters of G of degree coprime to p have Schur index 1, the only characters we have not accounted for are those of degree $\alpha = q(q+1)^2(q^2+1)$ and $\beta = q(q^2+1)(q^2+q+1)$. However, we find by taking inner products that characters of degree α and β are contained exactly once in a certain induced character μ^G, where μ is a linear character of P which has $(q-1)^2$ conjugates in B. Once more, Theorem 1(a) may be invoked to complete the proof.

An investigation of the Schur indices of characters of $SL(3, q)$ follows the pattern established in Theorem 4.

THEOREM 5. *Any character of $G = SL(3, q)$ has Schur index* 1.

Proof. We simply find that every character of G occurs exactly once in some character λ^G induced from a linear character of a Sylow p-subgroup of G. Theorem 1(b) gives us the result.

We continue our investigation by computing the Schur indices of the characters of the simple groups $PSU(3, q^2)$.

THEOREM 6. *Let G be the group $PSU(3, q^2)$. All irreducible characters of G have Schur index* 1 *except for a rational-valued character of degree $q^2 - q$. This has Schur index* 2 *and the corresponding division algebra component of the rational group algebra has nonzero Hasse invariants only at the infinite prime and p, where p is the prime divisor of q.*

Proof. We first consider the case when $(q+1, 3) = 1$. Let P be a Sylow p-subgroup of G and B its normalizer. P/P' is elementary abelian of order q^2 and B/P' is a Frobenius group whose complement H is cyclic of order $q^2 - 1$ [9, p. 242]. It follows that if λ is any nontrivial linear character of P, λ has $q^2 - 1$ conjugates and λ^B is irreducible. Since λ^B is just the character of the regular representation on H, λ^B has Schur index 1 and is the character of a rational representation. Using the character table in [15], we find that all but two nonlinear characters of G contain λ exactly once on restriction to P. The familiar argument of previous pages implies that these characters have Schur index 1.

The two remaining nonlinear characters have degrees $q^2 - q + 1$ and $q^2 - q$. The former contains the trivial representation of P exactly once, and so has Schur index 1. However, the character X of degree $q^2 - q$ contains no linear character of P. The method of Frobenius and Schur previously discussed shows that $m_R(X) = 2$, and, as we have noted before, the Brauer–Speiser theorem gives $m_Q(X) = 2$. We note that X_B is irreducible. Let $\theta = X_P$. We discover that θ is a sum of $q - 1$ irreducible characters of degree q, these being all the nonlinear characters of P. As the representation theory for P over algebraically closed fields of characteristic different from

p is classical, it follows easily that X_B, and hence X, must be irreducible as an r-modular character for any prime $r \neq p$. The result of the author [8], shows that if a complex irreducible character remains irreducible as an r-modular character, then its Schur index over the r-adic number is 1. It follows that X has Schur index 1 over the r-adic numbers for $r \neq p$. But a theorem of Hasse implies that there are two distinct primes for which X has Schur index 2. We already know that the infinite prime is one of these, and evidently the only other possibility is p.

When 3 divides $q + 1$, the analysis proceeds along the lines above. B/P' is still a Frobenius group, but the complement is of order $q^2 - 1/3$. Each nonidentity element of P/P' is conjugate in B/P' to its $p - 1$ nonidentity powers.

In this case, if λ is a nontrivial linear character of P, λ^B is irreducible, rational-valued and of Schur index 1. Three linear characters of P are required to construct induced characters which contain all but two nonlinear characters with multiplicity 1. The remaining nonlinear characters can be handled by the previous methods. This completes the proof.

Our next investigation concerns the four-dimensional symplectic group $Sp(4, q)$. The character theory varies according to whether q is even or odd. We first examine the case where q is odd and, following the discussion of Section 4, prove

THEOREM 7. *Let q be a power of an odd prime satisfying $q \equiv 1 \pmod 4$. Then all faithful characters of $G = Sp(4, q)$ have Schur index 2. All irreducible characters of $G_1 = PSp(4, q)$ have Schur index 1 over the reals. If in addition q is a square, all characters of G_1 have Schur index 1 over the rationals, and the only finite prime where the local index of a faithful character of G may differ from 1 is p.*

Proof. We use the character table given in [18], and begin by calculating the number of involutions in the two groups. In addition to the central involution of G, there is another class of involutions with centralizer $SL(2, q) \times SL(2, q)$. Thus if we include the identity element, there are $2 + q^2(q^2 + 1)$ involutions in G. In G_1 we obtain $1 + q^2(q^2 + 1)/2$ involutions from those in G. A further class of involutions is obtained by considering any element of order 4 in G whose square equals $-I$. The centralizer of such an element is $GL(2, q)$ and in G_1 we obtain $q^3(q^3 + q^2 + q + 1)/2$ more involutions. This accounts for all involutions of G.

The results of [18] give us the degrees of the characters of G_1. A straightforward calculation shows that the sum of these degrees equals the number of involutions in G_1. The Frobenius–Schur theorem implies that all characters of G_1 have Schur index 1 over the reals. We also find that the sum of the degrees of the faithful irreducible characters of G equals $(q^6 + q^5 + q^3 -$

$q^2 - 2)/2$, which is the difference in the number of involutions in the two groups. Once more the Frobenius–Schur theorem may be invoked to show that all the faithful characters have Schur index 2 over the reals, and hence over the rationals.

Theorem 3(b) is applicable if we assume from now on that q is a square. We take P to be a Sylow p-subgroup of G, B^* to be a Borel subgroup of G and B its image in G_1. Up to conjugacy in B^*, there are two linear characters of P, λ_1 and λ_2, which are in general position and these each have $(q-1)^2/2$ conjugates in B^*. The induced characters $\lambda_1{}^G$ and $\lambda_2{}^G$ are distinct and, of course, multiplicity-free. We find that the constituents of $\lambda_1{}^G$ or $\lambda_2{}^G$ are precisely those characters whose degrees are polynomials in q of degree 4.

Further computations of inner products can be used to show that all other characters of G also occur exactly once in suitable induced characters μ^G, where μ is a linear character of P, with the exception of the character $\theta = \theta_{10}$ of Srinivasan's list. Assuming that we can show θ has Schur index 1, our theorem follows from Theorem 3(b). The character θ contains no linear characters in its restriction to P. However, we find that θ_B is still irreducible and θ_P is a sum of $(q-1)^2/2$ distinct characters of P of degree q. Thus if φ is an irreducible constituent of θ_P, φ^B is irreducible, rational-valued and equals θ_B. If $B = PH$ where H is abelian of order $(q-1)^2/2$, Mackey's theorem [9, 16.9, p. 557], shows that $(\varphi^B)_H$ is the character of $\varphi(1)$ copies of the regular representation of H. By reciprocity θ_B occurs in $(1_H)^B$ $\varphi(1)$ times and Lemma 1 implies that the Schur index of θ_B divides $\varphi(1)$, a power of p. But θ_B has Schur index 1 or 2 as it is rational-valued, by the Brauer–Speiser theorem, and we deduce that its Schur index is 1. Since $(\theta_B)^{G_1}$ contains θ once, and θ_B is realizable in the rationals, θ is itself realizable in the rationals, and this finishes our proof.

To investigate the Schur indices of characters of $G = Sp(4, q)$ when q is a power of 2, we use Enomoto's character table given in [2]. A Sylow 2-subgroup P of G is a split extension of an elementary abelian group of order q^3 by an elementary abelian group of order q. It is not hard to show that all characters of P are realizable in the rational field (this is true of any split extension of an elementary abelian 2-group by another elementary abelian 2-group). We shall use this fact to prove

THEOREM 8. *All characters of $G = Sp(4, q)$, q even, have Schur index* 1.

Proof. We know that all characters of G are real-valued and consequently those of odd degree have Schur index 1. Since the majority of characters of G are of odd degree we only have a few characters to check. We find that if λ is a linear character of P in general position, λ^G contains all characters of G whose degrees are polynomials in q of degree 4. This leaves only the characters

θ_1, θ_2, θ_3 and θ_5 to check. All but θ_5 are contained with multiplicity 1 in induced characters μ^G, for suitable linear characters μ of P.

When $q = 2$, θ_5 is linear, corresponding to the fact that $Sp(4, 2)$ is not simple. For $q \geqslant 4$, we find that θ_5 contains no linear character of P in its restriction to P. However, $(\theta_5)_P$ consists of $(q-1)^2$ distinct characters of P of degree $q/2$. If φ is a constituent of $(\theta_5)_P$, we know φ is realizable in the rationals and since θ_5 occurs once in φ^G, θ_5 has Schur index 1.

The author believes that if P is a Sylow 2-subgroup of $Sp(2n, q)$, where q is even, then the rational field is a splitting field for P. If this is true, it may prove to be of use in investigating the Schur indices of characters of $Sp(2n, q)$, just as we saw it was for $Sp(4, q)$.

We conclude our investigations by calculating the Schur indices of Suzuki and Ree groups. Our findings are simply

THEOREM 9. *Let G be a simple Suzuki group or Ree of type G_2. Then all characters of G have Schur index* 1.

Proof. Let G be a Suzuki group defined over $GF(q)$, where q is an odd power of 2, and let $2q = r^2$. Let P be a Sylow 2-subgroup of G, and let B be its normalizer. P has order q^2 and P/P' is elementary abelian of order q. Using the character table of G in [21], we find that if λ is a nontrivial linear character of P, λ^G contains all nonlinear characters of G once, with the exception of characters X_1, X_2 of degree $r(q-1)/2$. Since λ is realizable in the rationals, we have already shown that all but two characters of G have Schur index 1.

We deal with X_1 and X_2 next. Any nonlinear irreducible character of P has degree $r/2$ and is defined over $Q(i)$. If X is such a nonlinear character, X has $q-1$ conjugates in B and X^B is irreducible. We can obtain two characters of B by the induction process and they are the restrictions to B of X_1 and X_2. Let σ and τ be these characters of B. Now σ and τ are algebraic conjugates (and so are X_1, X_2), with $Q(\sigma) = Q(i)$. Thus there must be an irreducible constituent X of σ_P whose algebraic conjugate X^1 occurs in τ_P. Since P is a 2-group, and X is not real-valued, a result of Roquette shows that X must have Schur index 1, for only real-valued characters of 2-groups have Schur index greater than 1 [14; or 5, p. 77]. Thus there is a rational representation of P whose character $\theta = X + X^1$. In this case, $\theta^B = \sigma + \tau$ is the character of a rational representation and so is θ^G. Since $(X_1, \theta^G) = (\sigma^G, X_1) + (\tau^G, X_1) = (\sigma, X_1)_B + (\tau, X_1)_B = 1$, X_1, and hence X_2, has Schur index 1, as required.

We turn now to the Ree groups, where we encounter more difficulties. The character table of a group of Ree type in [22] serves to compute inner products of characters. Let G be a Ree group defined over the field $GF(q)$, where $q = 3^{2k+1}$, $k \geqslant 1$. Let P be a Sylow 3-subgroup of G, and let B be its normalizer. B/P' is a Frobenius group with kernel P/P' of order q. Thus, by previous arguments, if λ is any nontrivial character of P, λ^B is the character of an absolutely irreducible rational representation, and any constituent of λ^G which occurs just once has Schur index 1.

Using the character table we find that only the following nonlinear characters of G do not satisfy $(X, \lambda^G) = 1$: characters 2 and 5-10. However character 2 occurs once in $1_P{}^G$ and so has index 1. None of the others contains a linear character of P on restriction to P, and we must employ more elaborate methods to deal with these characters.

We first note that a Sylow 2-subgroup of G is elementary abelian. Thus a result of [4] implies that the 2-part of the Schur index of each character is trivial. We now proceed to show that the 3-part of the index is also trivial. Let X be any irreducible character of G, and let 3^r be the 3-part of its Schur index. By a theorem of Brauer [1, 70.28], there exists a subgroup M of G and an irreducible character θ of M such that 3^r is the 3-part of the Schur index of θ. Furthermore, M has a normal cyclic 3-complement N. The 2-part of the order of N is at most 2, as N is cyclic and the Sylow 2-subgroups of G have exponent 2. If 2 divides the order of N, the involution of N must be central in M, and M is then a direct product of a group of order 2 and one of odd order. Thus, as far as properties of the Schur index are concerned, we may as well assume that M has odd order.

Now it is known that the centralizer of any 3-element of G is either a 3-group or has order $2 \times$ power of 3. As we are assuming that M has odd order, it follows that M is either a Frobenius group with kernel N and complement a Sylow 3-subgroup, or M has order coprime to 3, or is a 3-group. Since, by a theorem of Roquette, [14], the Schur indices of characters of 3-groups are always 1, r must be 0 in the third case, as it must also be in the second. In the first case, we can also show that the Schur indices of the characters of M are all 1. For irreducible characters of M have the form λ^M, where λ is an irreducible (linear) character of N, or they are irreducible characters of M/N [9, 16.13, p. 561]. As M/N is a 3-group, its characters have Schur index 1 by [14]. Finally, a character λ^M has Schur index 1, for by Mackey's theorem, its restriction to a Sylow 3-subgroup H of M is the regular representation of H, and we know this implies that λ^M has Schur index 1. Thus, whatever the structure of M, all its characters have Schur index 1.

To complete the proof, we note that if X is any one of characters 5–10, X has p-defect 0 for each prime p distinct from 2 or 3 dividing $X(1)$. It follows from a result of Solomon [16], that the Schur index of X is coprime to p. Since it is also coprime to 2 and 3, the Schur index of X is 1, and our proof is now complete.

In conclusion, we present our findings of this section in

THEOREM. (1) *The Schur index of any irreducible character of the following groups is always* 1: $GL(2, q)$, $GL(3, q)$, $GL(4, q)$, $PSp(4, q^2)$, *q odd*, $PSp(4, 2^a)$ $SL(3, q)$, *Suzuki groups* ${}^2B_2(q)$, *Ree groups* ${}^2G_2(q)$, $PSL(4, q^2)$ (*this follows easily from Lemma* 6 *and the proof of Theorem* 4).

(2) $PSU(3, q^2)$ *has a single irreducible character of Schur index* 2. *All other irreducible characters have Schur index* 1.

(3) *If* $q \equiv 1 \pmod 4$, *all faithful irreducible characters of* $Sp(4, q)$ *have Schur index* 2, *and all irreducible characters of* $PSp(4, q)$ *are real-valued and of Schur index* 1 *over the reals.*

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