Growth of solutions of second order linear differential equations

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Abstract

This paper is devoted to studying the growth of solutions of equations of type $f'' + h(z)e^{az}f' + Q(z)f = H(z)$ where $h(z)$, $Q(z)$ and $H(z)$ are entire functions of order at most one. We prove four theorems of such type, improving previous results due to Gundersen and Chen.

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1. Introduction and main results

We assume that the reader is familiar with the usual notations and basic results of the Nevanlinna theory [8,11]. We also use basic notions and results of the Wiman–Valiron theory, see [9]. Let now $f(z)$ be a nonconstant meromorphic function in the complex plane. We remark that $\rho(f)$, respectively $\rho_2(f)$ will be used to denote the order, respectively the hyper-order, of $f$. In particular, the hyper-order $\rho_2(f)$ is defined as

$$\rho_2(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log \log r},$$

see [17].

For a set $E \subset R^+$, let $m(E)$, respectively $\lambda(E)$, denote the linear measure, respectively the logarithmic measure, of $E$. By $\chi_E(t)$, we denote the characteristic function of $E$. Moreover, the upper logarithmic density and the lower logarithmic density of $E$ are defined by

$$\log \text{dens}(E) = \limsup_{r \to \infty} \frac{\lambda(E \cap [1, r])}{\log r}, \quad \log \text{dens}(E) = \liminf_{r \to \infty} \frac{\lambda(E \cap [1, r])}{\log r}.$$ 

Observe that $E$ may have a different meaning at different occurrences in what follows.

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We now recall some previous results concerning linear differential equations of type
\[ f'' + e^{-z} f' + Q(z)^2 f = 0, \quad (1.1) \]
where \( Q(z) \) is a transcendental entire function of finite order. In the case of a polynomial \( Q(z) \), properties of solutions of (1.1) have been studied, e.g., in \([2,4,13,16]\). Provided that \( Q(z) \) is an entire function of order even if \( \rho(Q) < 1 \), and where \( a, b \) are complex constants. If \( ab \neq 0 \) and \( \arg a = \arg b \) or if \( a = cb \) for some \( c > 1 \), then all nontrivial solutions \( f \) of (1.2) are of infinite order, see \([3]\). Li and Wang recently investigated the non-homogeneous equation related to (1.1) in the case when \( Q(z) = h(z)e^{bz} \), where \( h(z) \) is a transcendental entire function of order \( \rho(h) < \frac{1}{2} \), and \( b \) is a real constant, see \([14]\).

**Example 1.** The exponential function \( f_0(z) = e^z \) satisfies equation
\[ f'' + e^{-z} f' + Q(z)f = (1 + Q(z))e^z + 1, \]
where \( Q(z) \) can be any entire function. Moreover, choosing \( Q(z) = -1 \) shows that (1.3) may admit a solution of finite order even if \( \rho(H) < 1 \).

**Example 2.** The function \( f_0(z) = e^{z^2} \) satisfies the equation
\[ f'' + e^{-z} f' + Q(z)f = (4z^2 + 2 + 2ze^{-z} + Q(z))e^{z^2}, \]
where the entire function \( Q(z) \) can be arbitrarily chosen.

In this paper, we continue to consider (1.1) in the case of \( \rho(Q) = 1 \). Moreover, we extend our considerations to non-homogeneous equations of type
\[ f'' + A_1(z)e^{a_z} f' + A_0(z)e^{bz} f = H(z), \quad (1.4) \]
where \( A_0(z), A_1(z), H(z) \) are entire functions of order less than one, and \( a, b \in \mathbb{C} \). We now proceed to prove four theorems concerning the growth of solutions of Eq. (1.4):

**Theorem 1.1.** Suppose that \( A_0 \neq 0, A_1 \neq 0, H \) are entire functions of order less than one, and the complex constants \( a, b \) satisfy \( ab \neq 0 \) and \( b \neq a \). Then every nontrivial solution \( f \) of Eq. (1.4) is of infinite order.

**Corollary 1.2.** Suppose that \( Q(z) = h(z)e^{bc} \), where \( h \) is a non-vanishing entire function with \( \rho(h) < 1 \), and the complex constant \( b \) satisfies \( b \neq 0, -1 \). Then every nontrivial solution \( f \) of (1.1) is of infinite order.

**Theorem 1.3.** Suppose that \( A_0 \neq 0, A_1 \neq 0, D_0, D_1, H \) are entire functions of order less than one, and the complex constants \( a, b \) satisfy \( ab \neq 0 \) and \( b/a < 0 \). Then every nontrivial solution \( f \) of equation
\[ f'' + \left( A_1(z)e^{az} + D_1(z)\right)f' + \left( A_0(z)e^{bz} + D_0(z)\right)f = H(z) \]
is of infinite order.

Defining \( Q(z) = -(1 + e^{-z}) \) it is immediate to see that Eq. (1.1) admits a solution \( f_0 = e^z \) of finite order. This prompts us to prove
Theorem 1.4. Let $A_0 \not\equiv 0, A_1 \not\equiv 0$ are entire functions of order less than one, and $\alpha(z)$ is transcendental entire with $\rho(\alpha) < \frac{1}{2}$. Then every nontrivial solution $f$ of

$$f'' + A_1(z)e^{-z}f' + (A_0(z)e^{-z} + \alpha(z))f = 0$$

(1.6)

is of infinite order.

Finally, concerning the case of an entire function $Q$ of order $\rho(Q) = 1$, we consider the equation

$$f'' + h(z)e^{-z}f' + Q(z)f = 0$$

(1.7)

where $h(z)$ is an entire function of order $\rho(h) < \rho(Q) = 1$. By the preceding theorems, every nontrivial solution $f$ of Eq. (1.7) is of infinite order, provided the coefficients $h,Q$ have certain special forms. It is natural to ask about conditions on $Q$, independent of the special form of (1.7), which imply that every nontrivial solution $f$ of (1.7) is of infinite order. As a partial answer, we consider Eq. (1.7) under the condition

$$\lim_{r \to \infty} \frac{T(r, Q)}{\log M(r, Q)} = 1$$

(1.8)

assumed to hold in a set $E \subset [0, +\infty)$ of linear measure large enough:

Theorem 1.5. Suppose that $Q(z)$ and $h(z)$ are entire functions of order $\rho(h) < \rho(Q) = 1$, and that $Q(z)$ satisfies (1.8) in a set $E$ such that $\log \text{dens}(E) > 0$. Then every nontrivial solution $f$ of Eq. (1.7) is of infinite order.

For previous related results of similar type, see [12] and [10].

2. Preliminary lemmas

Lemma 2.1. (See [9].) Let $g(z)$ be an entire function of finite order $\rho$, and let $v_k(r)$ be the central index of $g$. Then

$$\rho = \limsup_{r \to \infty} \frac{\log v_k(r)}{\log r}.$$ 

Lemma 2.2. (See [1].) Let $w(z)$ be an entire function of order $\rho(w) = \beta < \frac{1}{2}$. $A(r) = \inf_{|z|=r} \log |w(z)|$ and $B(r) = \sup_{|z|=r} \log |w(z)|$. If $\beta < \alpha < 1$, then

$$\log \text{dens}\{r: A(r) > \cos(\pi \alpha)B(r)\} \geq 1 - \frac{\beta}{\alpha}.$$ 

Lemma 2.3. (See [5].) Let $f(z)$ be a meromorphic function of finite order $\rho$. Given $\zeta > 0$ and $0 < l < \frac{1}{2}$, there exist a constant $K(\rho, \zeta)$ and a set $E_\zeta \subset [0, \infty)$ of lower logarithmic density greater than $1 - \zeta$ such that

$$r \int_{J} \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta < K(\rho, \zeta) \left( l \log \frac{1}{l} \right) T(r, f)$$

(2.1)

for all $r \in E_\zeta$ and for every interval $J \subset [0, 2\pi)$ of length $l$.

Lemma 2.4. Let $f(z)$ be an entire function of finite order $\rho$, and $M(r, f) = |f(re^{i\theta})|$ for every $r$. Given $\zeta > 0$ and $0 < C(\rho, \zeta) < 1$, there exists a constant $0 < l_0 < \frac{1}{2}$ and a set $E_\zeta$ of lower logarithmic density greater than $1 - \zeta$ such that

$$e^{-5\pi} M(r, f)^{1-C(\rho, \zeta)} \leq |f(re^{i\theta})|$$

(2.2)

for all $r \in E_\zeta$ large enough and all $\theta$ such that $|\theta - \theta_r| \leq l_0$. 

For previous related results of similar type, see [12] and [10].
Proof. Restricting log $f(re^{i\theta})$ into its principal branch, i.e. $0 \leq \arg log f(z) < 2\pi$, we start from
\begin{equation}
\log f(re^{i\theta}) = \log f(re^{i\theta}) + \int_{\theta}^{\theta_1} d \log f(re^{i\theta}) = \log f(re^{i\theta}) + ri \int_{\theta}^{\theta_1} \frac{f'(re^{i\theta})}{f(re^{i\theta})} e^{i\theta} d\theta. \tag{2.3}
\end{equation}

Taking moduli of both sides of (2.3), and assuming that $r$ is large enough, we obtain
\[|\log f(re^{i\theta})| \geq |\log f(re^{i\theta})| - r \int_{\theta}^{\theta_1} \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta \geq \log M(r, f) - 2\pi - r \int_{\theta}^{\theta_1} \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta.|]

By Lemma 2.3, there is a set $E_\varsigma$ with $1 - \varsigma \leq \underline{\log dens}(E_\varsigma)$ such that
\[\log M(r, f) - K(\rho, \varsigma) \left( l \log \frac{1}{l} \right) T(r, f) \leq |\log f(re^{i\theta})| + 2\pi \tag{2.4}\]

for all $r \in E_\varsigma$ and $0 < |\theta - \theta_1| = l < \frac{\pi}{2}$, where $K(\rho, \varsigma)$ is a constant depending only on $\rho$ and $\varsigma$. Obviously, there exists $l_0$ such that
\[K(\rho, \varsigma) \left( l \log \frac{1}{l} \right) \leq C(\rho, \varsigma) < 1\]

for all $l < l_0 < \frac{\pi}{2}$. Since $T(r, f) \leq \log M(r, f)$, this and (2.4) implies
\[(1 - C(\rho, \varsigma)) \log M(r, f) \leq |\log f(re^{i\theta})| + 2\pi \leq \sqrt{\log^2 |f(re^{i\theta})| + (3\pi)^2} \leq |\log f(re^{i\theta})| + 5\pi, \tag{2.5}\]

which leads to (2.2).

\[\square\]

Lemma 2.5. Let $f(z)$ and $g(z)$ be two nonconstant entire functions with $\rho(g) < \rho(f) < +\infty$. Given $\varepsilon$ with $0 < 4\varepsilon < \rho(f) - \rho(g)$ and $0 < \delta < 1/4$, there exists a set $E$ with $\log dens(E) > 0$ and a positive constant $r_0$ such that
\[\left| \frac{g(z)}{f(z)} \right| \leq \exp\{-r^{\rho(f) - 2\varepsilon}\} \tag{2.6}\]

for all $z$ such that $r \in E$ is sufficiently large and that $|f(z)| \geq M(r, f)\nu_f(r)^{-\frac{1}{2} + \delta}$.

Proof. Clearly,
\[\nu_f(r) \leq r^{\rho(f) + 1}, \quad \left| g(z) \right| \leq \exp\{r^{\rho(g) + \varepsilon}\} \tag{2.7}\]

for all $r$ sufficiently large. Let then $r_n'$ be a sequence tending to infinity such that
\[\rho(f) = \lim_{n \to \infty} \frac{\log \log M(r_n', f)}{\log r_n'}.
\]

Define $E := \bigcup_{n=1}^{\infty} [r_n', r_n'^{1 + 2\kappa}]$ for $\kappa > 0$. Then
\[\log dens(E) \geq \limsup_{n \to \infty} \frac{\lambda(E \cap [1, r_n'^{1 + 2\kappa}])}{(1 + 2\kappa) \log r_n'} \geq \limsup_{n \to \infty} \frac{\lambda(E \cap [r_n', r_n'^{1 + 2\kappa}])}{(1 + 2\kappa) \log r_n'} = \frac{2\kappa}{1 + 2\kappa} > 0. \tag{2.8}\]

Since $M(r, f)$ is increasing, we have
\[\frac{\log \log M(r, f)}{\log r} \geq \frac{\log \log M(r_n', f)}{(1 + 2\kappa) \log r_n'}\]

for $r \in [r_n', r_n'^{1 + 2\kappa}]$. Therefore, taking $2\rho(f)\kappa = \varepsilon$, we obtain
\[\frac{\log \log M(r, f)}{\log r} \geq \frac{\rho(f)}{1 + 2\kappa} \geq \rho(f)(1 - 2\kappa) = \rho(f) - \varepsilon.\]
This means that for \( r \in E \),
\[
M(r, f) \geq \exp\{r^{\rho(f)} - \epsilon\}. \tag{2.9}
\]
Combining (2.7) and (2.9), we conclude that there exists a positive constant \( r_0 \) such that
\[
\left| \frac{g(z)}{f(z)} \right| \leq r(\rho(f) + 1)\left(1 - \frac{1}{4}\right)\exp\{r^{\rho(f)} - 2\epsilon\}
\]
for all \( z \) satisfying \( |f(z)| \geq M(r, f)\nu f(r) - \frac{1}{4} + \delta \) such that \( r \in E \) is sufficiently large.

The next lemma describing the behavior of \( e^{P(z)} \), where \( P(z) \) is a linear polynomial, is a special case of a more general result in [15, p. 254].

**Lemma 2.6.** Suppose that \( P(z) = (\alpha + i\beta)z \), where \( \alpha, \beta \) are real numbers, \( |\alpha| + |\beta| \neq 0 \), and that \( A(z) \neq 0 \) is a meromorphic function with \( \rho(A) < 1 \). Set \( g(z) = A(z)e^{P(z)} \), \( z = re^{i\theta} \), \( \delta(P, \theta) = \alpha \cos(\theta) - \beta \sin(\theta) \). Then for any given \( \epsilon > 0 \), there exists a set \( E \subset (1, +\infty) \) of finite linear measure such that for any \( \theta \in [0, 2\pi) \setminus H \), there is \( R > 0 \) such that for \( |z| = r > R \) and \( r/\in E \), we have

(i) if \( \delta(P, \theta) > 0 \), then
\[
\exp\left((1 - \epsilon)\delta(P, \theta)r\right) < \left| g(re^{i\theta}) \right| < \exp\left((1 + \epsilon)\delta(P, \theta)r\right); \tag{2.10}
\]

(ii) if \( \delta(P, \theta) < 0 \), then
\[
\exp\left((1 + \epsilon)\delta(P, \theta)r\right) < \left| g(re^{i\theta}) \right| < \exp\left((1 - \epsilon)\delta(P, \theta)r\right), \tag{2.11}
\]
where \( H = \{\theta \in [0, 2\pi); \delta(P, \theta) = 0\} \).

**Lemma 2.7.** (See [7].) Let \( f(z) \) be a transcendental meromorphic function of finite order \( \rho \), and let \( \epsilon > 0 \) be a given constant. Then there exists a set \( H \subset (1, \infty) \) that has finite logarithmic measure, such that for all \( z \) satisfying \( |z| \notin H \cup [0, 1] \) and for all \( k, j \), \( 0 \leq j < k \), we have
\[
\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho-1+\epsilon)}. \tag{2.12}
\]

Similarly, there exists a set \( E \subset [0, 2\pi) \) of linear measure zero such that for all \( z = re^{i\theta} \) with \( |z| \) sufficiently large and \( \theta \in [0, 2\pi) \setminus E \), and for all \( k, j \), \( 0 \leq j < k \), the inequality (2.12) holds.

**3. Proof of Theorem 1.1**

Suppose first that \( f \) is a nontrivial solution of (1.4) with \( \rho(f) < \infty \). By [17, Theorem 1.48], we obtain \( \rho(f) \geq 1 \). From (1.4), we have
\[
\frac{f''}{f} + A_1(z)e^{az} \frac{f'}{f} + A_0(z)e^{bz} = \frac{H}{f}. \tag{3.1}
\]
Recalling the Wiman–Valiron theory, for any given \( 0 < \delta < \frac{1}{4} \), there exists a set \( E_1 \) of finite logarithmic measure such that
\[
\frac{f^{(j)}(z)}{f(z)} = \left( \frac{\nu_f(r)}{z} \right)^j (1 + o(1)), \quad j = 1, 2, \tag{3.2}
\]
whenever \( |f(z)| \geq M(r, f)\nu_f(r)^{-\frac{1}{4} + \delta} \), \( r \notin E_1 \). Furthermore, from the definition of the central index, we know that \( \nu_f(r) \to \infty \) as \( r \to \infty \). By Lemma 2.1,
\[
\nu_f(r) \leq r^{\rho(f)+1} \tag{3.3}
\]
for all $r$ sufficiently large. By Lemma 2.7, we have

$$|f^{(j)}(z)| \leq |z|^{j \rho(f) - 1 + \epsilon}, \quad j = 1, 2,$$

(3.4)

for all $z$ satisfying $|z| = r \notin E_2$ where $\lambda(E_2) < \infty$, and $\epsilon$ is any given constant with $0 < 4\epsilon < 1 - \rho(H)$. By Lemma 2.5, there is a set $E_3$ with $\xi = \log \text{dens } E_3 > 0$ such that

$$\frac{v_f(r)^{p-\delta} H(z)}{M(r, f)} \leq \exp \{-r^{1-2\epsilon}\}$$

(3.5)

as soon as $r \in E_3$ is large enough. We may take $\theta_r$ such that $M(r, f) = |f(re^{i\theta_r})|$ for every $r$. By Lemma 2.4, given a constant $0 < C < 1$, there exists a constant $l_0$ and a set $E_4$ with $1 - \frac{\xi}{2} \leq \log \text{dens } (E_4)$ such that

$$e^{-5\pi} M(r, f)^{1-C} \leq |f(re^{i\theta})|$$

(3.6)

for all $r \in E_4$ and $|\theta - \theta_r| \leq l_0$. Recall now that the characteristic functions of $E_3$ and $E_4$ satisfy the relation

$$\chi_{E_3 \cap E_4}(t) = \chi_{E_3}(t) + \chi_{E_4}(t) - \chi_{E_3 \cup E_4}(t).$$

Clearly, $\log \text{dens } (E_3 \cup E_4) \leq 1$. Thus, we get

$$\frac{\xi}{2} \leq \log \text{dens } E_3 + \log \text{dens } (E_4) - \log \text{dens } (E_3 \cup E_4) \leq \log \text{dens } (E_3 \cap E_4).$$

Since $\lambda(E_1 \cup E_2) < \infty$, we have $\log \text{dens } ((E_3 \cap E_4) \setminus (E_1 \cup E_2)) > 0$. Thus, there exists a sequence of points $z_n = r_n e^{i\theta_n}$ with $r_n$ tending to infinity and

$$|f(z_n)| = M(r_n, f), \quad r_n \in (E_3 \cap E_4) \setminus (E_1 \cup E_2).$$

Passing to a subsequence of $\{\theta_n\}$, if needed, we may assume that $\lim_{n \to \infty} \theta_n = \theta_0$.

We now discuss three cases separately.

**Case 1.** First assume that $\delta(az, \theta_0) > 0$. From the continuity of $\delta(az, \theta)$, we have

$$\frac{1}{2} \delta(az, \theta_0) < \delta(az, \theta_n) < \frac{3}{2} \delta(az, \theta_0)$$

(3.7)

for sufficiently large $n$. From (2.10), we deduce that

$$\exp \left\{ \frac{1 - \epsilon}{2} \delta(az, \theta_0) r_n \right\} \leq |A_1(z_n) e^{az_n}| \leq \exp \left\{ \frac{3(1 + \epsilon)}{2} \delta(az, \theta_0) r_n \right\}$$

(3.8)

for all $n$ sufficiently large. From (3.1), we have

$$\left| \frac{f'(z_n)}{f(z_n)} + \frac{A_0(z_n)}{A_1(z_n)} e^{(b-a)z_n} \right| \leq \frac{e^{-az_n}}{A_1(z_n)} \left( \frac{f''(z_n)}{f(z_n)} + \frac{|H(z_n)|}{M(r_n, f)} \right).$$

(3.9)

We now divide our consideration in Case 1 in three subcases:

**Subcase 1.1.** We first assume that $\theta_0$ satisfies $\eta := \delta((b - a)z, \theta_0) > 0$. From the continuity of $\delta((b - a)z, \theta)$, we also have

$$\frac{1}{2} \delta((b - a)z, \theta_0) \leq \delta((b - a)z, \theta_n) \leq \frac{3}{2} \delta((b - a)z, \theta_0)$$

for sufficiently large $n$. Again from (2.10), we obtain that

$$\exp \left\{ \frac{1 - \epsilon}{2} \eta r_n \right\} \leq \frac{A_0(z_n)}{A_1(z_n)} \left| e^{(b-a)z_n} \right| \leq \exp \left\{ \frac{3(1 + \epsilon)}{2} \eta r_n \right\},$$

(3.10)

when $n$ is large. Substituting (3.2), (3.3) and (3.5) into (3.9), we obtain

$$\left| \frac{v_f(r_n)}{z_n} (1 + o(1)) + \frac{A_0(z_n)}{A_1(z_n)} e^{(b-a)z_n} \right| \leq \frac{r_n 2^{\rho(f)+1}}{A_1(z_n) e^{az}}$$

(3.11)
for sufficiently large $n$. Considering (3.8), clearly
\[ \frac{r_n^{2\rho(f)+1}}{A_1(z_n)e^{a z_n}} e^{\frac{1}{2} \delta(a z, \theta_0) r_n} \leq \exp \left\{ -\frac{(1-2\varepsilon)}{2} \delta(a z, \theta_0) r_n \right\}. \]

Combining this with (3.3), (3.10) and (3.11), we conclude that
\[ \exp \left\{ \frac{(1-\varepsilon)}{2} \eta r_n \right\} \leq \left| A_0(z_n) e^{(b-a)z_n} + \frac{v_f(r_n)}{r_n} - \frac{v_f(r_n)}{r_n} \right| \leq \exp \left\{ -\frac{(1-2\varepsilon)}{2} \delta(a z, \theta_0) r_n \right\} + 2r_\rho r_n \leq 3r_\rho, \]
a contradiction.

**Subcase 1.2.** Next assume that $\eta := \delta((b-a)z, \theta_0) < 0$. Then from (2.11), for $n$ large enough, we deduce that
\[ \exp \left\{ \frac{3(1+\varepsilon)}{2} \eta r_n \right\} \leq \left| A_0(z_n) e^{(b-a)z_n} \right| \leq \exp \left\{ \frac{(1-\varepsilon)}{2} \eta r_n \right\}. \] (3.12)

It follows from (3.11) and (3.12) that
\[ \frac{v_f(r_n)}{r_n} (1 + o(1)) \leq \exp \left\{ \frac{(1-\varepsilon)}{2} \eta r_n \right\} + \exp \left\{ -\frac{(1-2\varepsilon)}{2} \delta(a z, \theta_0) r_n \right\} \]
as $n \to \infty$. This implies that
\[ v_f(r_n) \to 0, \quad n \to \infty, \]
which is impossible.

**Subcase 1.3.** Assume finally that $\eta := \delta((b-a)z, \theta_0) = 0$. Here, (3.6) may be used to construct another sequence of points $z_n^* = r_n e^{i\theta_n^*}$ with $\lim_{n \to \infty} \theta_n^* = \theta_0^*$ such that $\eta_1 := \delta((b-a)z, \theta_0^*) > 0$. Indeed, we may suppose, without loss of generality, that
\[
\delta((b-a)z, \theta) > 0, \quad \theta \in (\theta_0 + 2k\pi, \theta_0 + (2k+1)\pi),
\]
\[
\delta((b-a)z, \theta) < 0, \quad \theta \in (\theta_0 + (2k-1)\pi, \theta_0 + 2k\pi),
\]
with $k \in \mathbb{Z}$. When $n$ is large enough, we have $|\theta_0 - \theta_n| \leq l_0$. Choose now $\theta_n^*$ such that $\frac{l_0}{2} \leq \theta_n^* - \theta_n \leq l_0$. Then
\[ \theta_n + \frac{l_0}{2} \leq \theta_n^* \leq \theta_n + l_0, \]
and
\[ \theta_0 + \frac{l_0}{2} \leq \theta_0^* \leq \theta_0 + l_0. \] (3.13)

For sufficiently large $n$, we have (3.6) for $z_n^*$, and $\eta_1 := \delta((b-a)z, \theta_0^*) > 0$. Therefore,
\[ \left| \frac{H(z_n^*)}{f(z_n^*)} \right| \leq \frac{v_f(r_n)^{1-\delta} M(r_n, H)}{e^{-5\gamma} M(r_n, f)^{1-C}} \]
and
\[ \exp \left\{ \frac{(1-\varepsilon)}{2} \eta_1 r_n \right\} \leq \left| A_0(z_n^*) e^{(b-a)z_n^*} \right| \leq \exp \left\{ \frac{3(1+\varepsilon)}{2} \eta_1 r_n \right\}. \] (3.14)

By the proof of Lemma 2.5, we may assume that
\[ M(r_n, f) \geq \exp\{r_\rho(f)^{1-\varepsilon}\}. \]
Therefore, previous estimates may be combined to result in
\[
\frac{|H(z_n^*)|}{f(z_n^*)} \leq r_n^{(\rho(f)+1)(\frac{1}{2} - \delta)} \exp[r_n^{\rho(H)+\epsilon}] \exp[r_n^{\rho(f) - \frac{1}{2} \epsilon}] \leq \exp[-r_n^{1-2\epsilon}]
\] (3.15)
for all \( n \) large enough. Taking now \( l_0 \) small enough, we have \( \delta(az, \theta_0^*) > 0 \) by the continuity of \( \delta(az, \theta) \). This yields
\[
\exp\left\{ \frac{(1 - \epsilon)}{2} \delta(az, \theta_0) r_n \right\} \leq |A_1(z_n^*) e^{az}| \leq \exp\left\{ \frac{3(1 + \epsilon)}{2} \delta(az, \theta_0) r_n \right\}.
\] (3.16)
Substituting (3.4) and (3.15) into (3.9), we have
\[
(3.19)
\]
for the sequence of points \( z \).

Case 2. Suppose now that \( \delta(az, \theta_0) < 0 \). Then from the continuity of \( \delta(az, \theta) \) and (2.11), we have
\[
\exp\left\{ \frac{3(1 + \epsilon)}{2} \delta(az, \theta_0) r_n \right\} \leq |A_1(z_n^*) e^{az}| \leq \exp\left\{ \frac{(1 - \epsilon)}{2} \delta(az, \theta_0) r_n \right\}
\] (3.17)
for all \( n \) sufficiently large. From (3.1), we have
\[
\left\| \frac{f''(z_n^*)}{f(z_n^*)} + A_0(z_n^*) e^{bz_n} \right\| \leq \left\| A_1(z_n^*) e^{az} \right\| \left\| \frac{f'(z_n^*)}{f(z_n^*)} \right\| + \frac{|H(z_n)|}{M(r_n, f)}
\] (3.18)
as \( n \to \infty \). Again, we have to treat three subcases separately.

Subcase 2.1. Assume first that \( \delta(bz, \theta_0) > 0 \). From the continuity of \( \delta(bz, \theta) \) and (2.10), we deduce that
\[
\exp\left\{ \frac{(1 - \epsilon)}{2} \delta(bz, \theta_0) r_n \right\} \leq |A_0(z_n^*) e^{bz}| \leq \exp\left\{ \frac{3(1 + \epsilon)}{2} \delta(bz, \theta_0) r_n \right\}
\] (3.19)
for \( n \) large enough. Substituting (3.2), (3.3), (3.5) and (3.17) into (3.18) results in
\[
\left\| \frac{v_f(r_n)}{z_n^*} \right\|^2 (1 + o(1)) + A_0(z_n) e^{bz_n} \leq \exp\left\{ -r_n^{1-2\epsilon} \right\}.
\] (3.20)
Combining this with (3.3) and (3.19), we have
\[
\exp\left\{ \frac{(1 - \epsilon)}{2} \delta(bz, \theta_0) r_n \right\} \leq \exp\left\{ -r_n^{1-3\epsilon} \right\} + r_n^{2\rho(f)} \leq 2r_n^{2\rho(f)},
\]
a contradiction.

Subcase 2.2. We assume that \( \delta(bz, \theta_0) < 0 \). By continuity of \( \delta(bz, \theta) \) and (2.11), we now have
\[
\exp\left\{ \frac{3(1 + \epsilon)}{2} \delta(bz, \theta_0) r_n \right\} \leq |A_0(z_n) e^{bz}| \leq \exp\left\{ \frac{(1 - \epsilon)}{2} \delta(bz, \theta_0) r_n \right\}
\] (3.21)
for all \( n \) sufficiently large. It follows from (3.17), (3.18), (3.21) and Lemma 2.5 that \( v_f(r_n) \to 0 \) as \( n \to \infty \), which is impossible.

Subcase 2.3. Suppose that \( \delta(bz, \theta_0) = 0 \). Arguing similarly as in Subcase 1.3, we may again construct another sequence of points \( z_n^* \) satisfying \( \frac{1}{2} \leq |\theta_n^* - \theta_0| \leq l_0 \) such that \( \delta(az, \theta_0^*) < 0 \) < \( \delta(bz, \theta_0^*) \) where \( \theta_0^* = \lim_{n \to \infty} \theta_n^* \). Replacing \( \delta(az, \theta_0) \) with \( \delta(az, \theta_0^*) \) in (3.17) and \( \delta(bz, \theta_0) \) with \( \delta(bz, \theta_0^*) \) in (3.19), respectively, we obtain (3.17) and (3.19) for the sequence of points \( z_n^* \). Arguing as in Subcase 1.3, we also have (3.15) for the points \( z_n^* \). Similarly as before, we get
\[
|A_0(z_n^*) e^{bz_n^*}| \leq |A_1(z_n^*) e^{az}| r_n^{\rho(f)+\epsilon} + \exp\left\{ -r_n^{1-2\epsilon} \right\} + r_n^{2\rho(f)+\epsilon}
\]
for all \( n \) sufficiently large. A contradiction follows by combining this inequality with (3.17) and (3.19) for \( z_n^* \).
Case 3. In this final case, we suppose that $\delta(az, \theta_0) = 0$. We discuss three subcases according to $\delta(bz, \theta_0)$ as follows.

Subcase 3.1. Suppose that $\delta(bz, \theta_0) > 0$. By an argument similar to that in Subcase 1.3, we can choose another sequence of points $z_n^* = r_n e^{i\theta_n^*}$ with $\theta_n^* = \lim_{n\to\infty} \theta_n^*$ and $l_0 \leq |n_0 - \theta_n^*| \leq l_0$ such that $z_n^*$ satisfies (3.15) and $\delta(az, \theta_n^*) < 0 < \delta(bz, \theta_n^*)$. Similarly as in Subcase 2.3, a contradiction follows as $n \to \infty$.

Subcase 3.2. Suppose next that $\delta(bz, \theta_0) < 0$. By the definition of $\delta(P, \theta)$ in Lemma 2.6, we may define

$$\delta'(az, \theta) := -\alpha \sin \theta - \beta \cos \theta = \delta(az, \theta + \pi/2)$$

where $a = \alpha + i\beta$. Since $a \neq 0$, we have $\delta'(az, \theta_0) \neq 0$. For $z_n' = r_n e^{i\theta_n'}$ satisfying $0 < |\theta_n' - \theta_0| \leq l_0$, we know that $z_n'$ satisfies (3.15) and $\delta(az, \theta_n') \neq 0$. By continuity of $\delta(bz, \theta)$, we may assume that $\delta(bz, \theta_n') < 0 < \delta(az, \theta_n')$ for a suitable $l_0$, $0 < \theta_n' - \theta_0 \leq l_0$. Then $\delta'(az, \theta_0) > 0$, which means that for a suitable $l_0$, $\delta'(az, \theta_0) \leq \delta'(az, \theta) \leq \frac{3}{2} \delta'(az, \theta_0), \quad \theta \in (\theta_0, \theta_0 + l_0)$.

Since we have chosen $z_n$ such that $|f(z_n)| = M(r_n, f)$ and $\theta_n \to \theta_0$ as $n \to \infty$, we have $|f(r_n e^{i\theta_0})| \geq M(r_n, f) v_f(r_n)^{-\frac{1}{2}+\delta}$ for $n$ sufficiently large. From (3.1), we have

$$\left| \frac{f'(z_n')}{f(z_n')} \right| \leq \left| \frac{1}{A_1(z_n')} e^{-az_n'} \right| \left( |A_0(z_n') e^{b z_n'}| + \left| \frac{f''(z_n')}{f(z_n')} \right| + \left| \frac{H(z_n')}{f(z_n')} \right| \right).$$  \hspace{1cm} \text{(3.22)}$$

By Lemma 2.6, we have

$$\exp\{-(1 + \varepsilon)\delta(az, \theta_n')\} \leq \left| \frac{1}{A_1(z_n')} e^{-az_n'} \right| \leq \exp\{-(1 - \varepsilon)\delta(az, \theta_n')\}$$  \hspace{1cm} \text{(3.23)}$$

and

$$\exp\{(1 + \varepsilon)\delta(bz, \theta_n')\} \leq |A_0(z_n') e^{b z_n'}| \leq \exp\{(1 - \varepsilon)\delta(bz, \theta_n')\}$$  \hspace{1cm} \text{(3.24)}$$

for all $n$ sufficiently large. Making use of (3.4), (3.15), (3.23) and (3.24), (3.22) implies that

$$\left| \frac{f'(r_n e^{i\theta})}{f(r_n e^{i\theta})} \right| \leq \exp\{-(1 - 2\varepsilon)\delta(az, \theta) r_n\}.$$$$

As $\theta_n'$ may be taken arbitrarily in $(\theta_0, \theta_0 + l_0)$, for sufficiently large $r_n$, we in fact obtained

$$\left| \frac{f'(r_n e^{i\theta})}{f(r_n e^{i\theta})} \right| \leq \exp\{-(1 - 2\varepsilon)\delta(az, \theta) r_n\}, \quad \theta \in (\theta_0, \theta_0 + l_0).$$  \hspace{1cm} \text{(3.25)}$$

Therefore, for $\theta \in (\theta_0, \theta_0 + l_0)$, we have

$$\xi(r_n, \theta) = r_n \theta_0^\theta \left| \frac{f'(r_n e^{i\theta})}{f(r_n e^{i\theta})} \right| d\theta \leq r_n \theta_0^\theta e^{-\eta_1(\theta) r_n} d\theta = \frac{1}{\eta_1(\theta)} e^{-\eta_2(\theta) r_n} d(\eta_2(\theta) r_n),$$

where $\eta_1(\theta) = (1 - 2\varepsilon)\delta'(az, \theta), \quad \eta_2(\theta) = (1 - 2\varepsilon)\delta(az, \theta)$. Since $\delta(az, \theta) > 0$ for all $\theta \in (\theta_0, \theta_0 + l_0)$, it is easy to see

$$0 \leq \xi(r_n, \theta) \leq \frac{2}{(1 - 2\varepsilon)\delta'(az, \theta_0)} (e^{-\eta_2(\theta_0) r_n} - e^{-\eta_2(\theta) r_n}).$$

This leads to

$$0 \leq \xi(r_n, \theta) \leq \frac{2}{\eta_1(\theta_0)}$$  \hspace{1cm} \text{(3.26)}$$

for all $n$ large enough. By the proof of Lemma 2.4, we have

$$\log|f(r_n e^{i\theta_0})| - \xi(r_n, \theta) \leq \log|f(r_n e^{i\theta})| + 2\pi.$$
It follows from this and (3.26) that
\[ v_f(r_n)^{-\frac{1}{4} + \delta} M(r_n, f) = \exp\{ -2\pi - 2/\eta_1(\theta_0) \} v_f(r_n)^{-\frac{1}{4} + \delta} M(r_n, f) \leq |f(r_n e^{i\theta})| \]  
(3.27)
for \( \theta \in (\theta_0, \theta_0 + l_0) \), where \( 0 < \delta' < \delta < \frac{1}{4} \). Therefore, we can choose another sequence of points \( z_n^* = r_n e^{i\theta_n^*} \) with \( \theta_n^* = \frac{l_0}{2} + \theta_0 \) such that \( z_n^* \) satisfies (3.15). Furthermore, from (3.27), when \( n \) is sufficiently large, \( z_n^* \) satisfies (3.2). Thus, (3.2) and (3.25) imply that \( v_f(r_n) \to 0 \) as \( n \to \infty \), which is impossible.

When \( \delta(bz, \theta_n') < 0 < \delta(az, \theta_n') \) for \(-l_0 < \theta_n' - \theta_0 < 0\), clearly \( \xi(r_n, \theta) \leq 0 \) for all \( \theta \in (\theta_0 - l_0, \theta_0) \). Therefore, we similarly get
\[ v_f(r_n)^{-\frac{1}{4} + \delta} M(r, f) = \exp\{ -2\pi \} v_f(r_n)^{-\frac{1}{4} + \delta} M(r, f) \leq |f(r_n e^{i\theta})| \]  
(3.28)
for \( \theta \in (\theta_0, \theta_0 + l_0) \), where \( 0 < \delta' < \delta < \frac{1}{4} \), a contradiction.

**Subcase 3.3.** Finally, suppose that \( \delta(bz, \theta_0) = 0 \). We now have \( a/b = c \in \mathbb{R}, c \neq 0, 1 \), and so \( az = cbz, \quad (b - a)z = (1 - c)bz \).

If \( c < 0 \), we may take \( l_0 \) small enough such that \( \delta(bz, \theta) < 0 < \delta(az, \theta) \), provided that either \( \theta \in (\theta_0, \theta_0 + l_0) \) or \( (\theta_0 - l_0, \theta_0) \). By an argument similar to that in Subcase 3.2, we have (3.25) and (3.27). Then by Wiman–Valiron theory, we get \( v_f(r_n) \to 0 \) as \( n \to \infty \), a contradiction.

If \( 0 < c < 1 \), we similarly obtain \( \delta((b - a)z, \theta) > 0 \) and \( \delta(az, \theta) > 0 \), provided that either \( \theta \in (\theta_0, \theta_0 + l_0) \) or \( (\theta_0 - l_0, \theta_0) \), for some \( l_0 \) small enough. By an argument similar to that in Subcase 1.3, a contradiction follows.

Finally, if \( c > 1 \), we obtain \( \delta((b - a)z, \theta) < 0 < \delta(az, \theta) \) for either \( \theta \in (\theta_0, \theta_0 + l_0) \) or \( (\theta_0 - l_0, \theta_0) \). Furthermore, \( z_n' = r_n e^{i\theta_n'} \) satisfies (3.15), provided \( \theta_n' \in (\theta_0, \theta_0 + l_0) \) or \( (\theta_0 - l_0, \theta_0) \). Hence, (3.1) implies that
\[ |f'(z_n')| = \left| \frac{A_0(z_n') e^{(b - a)z_n'}}{A_1(z_n')} + \frac{1}{A_1(z_n')} e^{-a'z_n'} \left( \frac{H(z_n')}{f(z_n')} \right) \right| \leq |f(z_n')| \]
Similarly as in Subcase 3.2, we get (3.25) and (3.27). By a standard Wiman–Valiron argument, a contradiction again follows.

**4. Proof of Theorem 1.3**

Suppose that \( f \) is a nontrivial solution of (1.6) with finite order. By [17], Theorem 1.48 again, \( \rho(f) \geq 1 \). Rewrite now (1.6) as
\[ \frac{f''}{f} + (A_1(z) e^{az} + D_1(z)) f' f + (A_0(z) e^{bz} + D_0(z)) = \frac{H(z)}{f}. \]  
(4.1)
Since \( q = \max(\rho(D_0), \rho(D_1)) < 1 \), we have
\[ |D_j(z)| \leq \exp\{ \rho^{q + e} \}, \quad j = 0, 1, \]  
(4.2)
for any \( e \) such that \( 0 < 3e < 1 - q \). Similarly as in the proof of Theorem 1.1, we may choose a sequence of points \( z_n = r_n e^{i\theta_n}, r_n \to \infty \), such that \( \lim_{n \to \infty} \theta_n = \theta_0 \) and that
\[ |f(z_n)| = M(r_n, f), \quad r_n \in (E_3 \cap E_4) \setminus (E_1 \cup E_2). \]
In particular, the sequence of points \( z_n \) satisfies (3.2)–(3.6). Since \( a/b = c < 0 \), there are three case to be discussed, according to the signs of \( \delta(az, \theta_0) \) and \( \delta(bz, \theta_0) \).

**Case 1.** First assume that \( \delta(bz, \theta_0) < 0 < \delta(az, \theta_0) \). By Lemma 2.6, and the continuity of \( \delta(az, \theta) \) and \( \delta(bz, \theta) \), we deduce that
\[ \exp\left\{ \frac{1 - e}{2} \delta(az, \theta_0) r_n \right\} \leq |A_1(z_n) e^{az_n}| \leq \exp\left\{ \frac{3(1 + e)}{2} \delta(az, \theta_0) r_n \right\} \]  
(4.3)
and
\[ \exp \left\{ \frac{3(1 + \varepsilon)}{2} \delta(bz, \theta_0)r_n \right\} \leq |A_0(z_n)e^{bza_n}| \leq \exp \left\{ \frac{1 + \varepsilon}{2} \delta(bz, \theta_0)r_n \right\} \] (4.4)
for all \( n \) sufficiently large. From (4.1), we get
\[ \left| \left( A_1(z_n)e^{aza_n} + D_1(z_n) \right) \frac{f'(z_n)}{f(z_n)} \right| \leq \left| \frac{H(z_n)}{M(r_n, f)} \right| + \left| \frac{f''(z_n)}{f(z_n)} \right| + \left| A_0(z_n)e^{bza_n} + D_0(z_n) \right|. \] (4.5)
Combining (4.2) with (4.3) and (4.4), respectively, we conclude that
\[ |A_0(z_n)e^{bza_n} + D_0(z_n)| \leq \exp \left\{ r^{\theta + 2\varepsilon} \right\} \]
and
\[ \exp \left\{ \frac{1 - 2\varepsilon}{2} \delta(az, \theta_0)r_n \right\} \leq |A_1(z_n)e^{aza_n} + D_1(z_n)| \]
provided \( n \) is large enough. Substituting these estimates with (3.2) and (3.5) into (4.5), we obtain
\[ v_f(r_n) \leq 2r_n \exp \left\{ - \frac{1 - 2\varepsilon}{2} \delta(az, \theta_0)r_n \right\} \left( 2 \left( \frac{v_f(r_n)}{r_n} \right)^2 + \exp \{ r^{\theta + 2\varepsilon} \} \right) \]
for \( n \) large enough. Considering (3.3), this leads to \( v_f(r_n) \to 0 \), which is a contradiction.

Case 2. Suppose next that \( \delta(az, \theta_0) < 0 < \delta(bz, \theta_0) \). Similarly as in Case 1, we obtain by Lemma 2.6 that
\[ \exp \left\{ \frac{3(1 + \varepsilon)}{2} \delta(az, \theta_0)r_n \right\} \leq |A_1(z_n)e^{aza_n}| \leq \exp \left\{ \frac{1 + \varepsilon}{2} \delta(az, \theta_0)r_n \right\} \] (4.6)
and
\[ \exp \left\{ \frac{1 - \varepsilon}{2} \delta(bz, \theta_0)r_n \right\} \leq |A_0(z_n)e^{bza_n}| \leq \exp \left\{ \frac{3(1 + \varepsilon)}{2} \delta(bz, \theta_0)r_n \right\} \] (4.7)
for all \( n \) sufficiently large. It now follows from (4.1) that
\[ |A_0(z_n)e^{bza_n} + D_0(z_n)| \leq \left| \frac{H(z_n)}{M(r_n, f)} \right| + \left| \frac{f''(z_n)}{f(z_n)} \right| + \left| \left( A_1(z_n)e^{aza_n} + D_1(z_n) \right) \frac{f'(z_n)}{f(z_n)} \right|. \] (4.8)
Combining (4.2) with (4.6) and (4.7), respectively, we obtain
\[ |A_1(z_n)e^{aza_n} + D_1(z_n)| \leq \exp \left\{ r^{\theta + 2\varepsilon} \right\} \]
and
\[ \exp \left\{ \frac{1 - \varepsilon}{2} \delta(bz, \theta_0)r_n \right\} \leq |A_0(z_n)e^{bza_n} + D_0(z_n)| \]
for \( n \) large enough. Substituting these inequalities with (3.4) and (3.5) into (4.8), we conclude that
\[ \exp \left\{ \frac{1 - \varepsilon}{2} \delta(bz, \theta_0)r_n \right\} \leq r_n^2 \rho(f) \exp \left\{ r_n^{\theta + 2\varepsilon} \right\}, \]
which is impossible.

Case 3. Finally, we have to assume that \( \delta(az, \theta_0) = \delta(bz, \theta_0) = 0 \). Similarly as in Subcase 1.3 of the proof of Theorem 1.1, we may again use (3.6) to construct a sequence of points \( z_n^* = r_ne^{i\theta_n^*} \) with \( \lim_{n \to \infty} \theta_n^* = \theta_0^* \) such that \( \delta(az, \theta_0^*) < 0 \) and that (3.15) holds for \( z_n^* \). Indeed, we may assume, without loss of generality, that
\[ \delta(az, \theta_0) > 0, \quad \theta \in (\theta_0 + 2k\pi, \theta_0 + (2k + 1)\pi), \quad \delta(bz, \theta_0) < 0, \quad \theta \in (\theta_0 + (2k - 1)\pi, \theta_0 + 2k\pi), \]
for all \( k \in \mathbb{Z} \). Provided \( n \) is large enough, we have \( |\theta_0 - \theta_n| \leq l_0 \). Choosing now \( \theta_0^* \) such that \( \frac{l_0}{2} \leq \theta_n^* - \theta_0^* \leq l_0 \), then \( \theta_0 - l_0 \leq \theta_0^* \leq \theta_0 - \frac{l_0}{2} \) and so \( \delta(az, \theta_0^*) < 0 \). Since now \( \delta(bz, \theta_0^*) > 0 \), a contradiction follows as in Case 2 above.
5. Proof of Theorem 1.4

Suppose that \( f \) is a nontrivial solution of (1.6) with finite order. Observing that \( \rho(\alpha) < \frac{1}{2} \rho(\alpha) + \frac{1}{4} < 1 \), we may apply Lemma 2.2 to find a set \( E_1 \) with \( \log \text{dens} E_1 \geq \frac{1-2\rho(\alpha)}{1+2\rho(\alpha)} > 0 \) such that

\[
\left| \alpha(z) \right| \geq M(r, \alpha)^\gamma, \quad \gamma = \cos \left( \frac{2\rho(\alpha) + 1}{4} - \frac{\pi}{4} \right),
\]

for all \( |z| = r \in E_1 \). We may choose a sequence of points \( z_n = r_n e^{i\theta_0} \) such that (3.4) and (5.1) apply at the same time and that \( \delta(-z, \theta_0) < 0 \). Rewriting

\[
\frac{f''(z)}{f(z)} + A_1(z)e^{-z} \frac{f'(z)}{f(z)} + A_0(z)e^{-z} + \alpha(z) = 0,
\]

we may apply (2.11) to conclude that

\[
\max \left( |A_1(z_n)e^{-z_n}|, |A_0(z_n)e^{-z_n}| \right) \leq \exp \left\{ \frac{1 - \varepsilon}{2} \delta(-z, \theta_0) r_n \right\}.
\]

Combining (5.3) with (5.1) and (5.2), we obtain

\[
M(r_n, \alpha)^\gamma \leq |\alpha(z_n)| \leq \left| \frac{f''(z_n)}{f(z_n)} \right| + \left| A_1(z_n)e^{-z_n} \frac{f'(z_n)}{f(z_n)} \right| + |A_0(z_n)e^{-z_n}| \leq r_n^{2\rho(f)}.
\]

This implies that \( \alpha(z) \) is a polynomial, a contradiction.

6. Proof of Theorem 1.5

Suppose that \( f \) is a transcendental solution of (1.7) of finite order. Given \( 0 < d < \frac{1}{4} \), define

\[
K_r := \left\{ \theta \mid \log |Q(re^{i\theta})| \leq (1 - d) \log M(r, Q) \right\}.
\]

Since \( Q(z) \) satisfies the condition (1.8) in a set \( E \) of upper logarithmic density \( \zeta > 0 \), it is not difficult to see that the linear measure \( m(K_r) \) approaches to zero as \( r \to \infty \) through \( r \in E \). Indeed,

\[
T(r, Q) \leq \left( 1 - \frac{m(K_r)}{2\pi} \right) \log M(r, Q) + (1 - 2d) \frac{m(K_r)}{2\pi} \log M(r, Q)
\]

for \( r \in E \) sufficiently large. This now results in a contradiction with (1.8), if \( m(K_r) \) does not approach to zero as \( r \to \infty \). By Lemma 2.4, we may now choose a sequence of points \( z_n = r_n e^{i\theta_0} \) such that \( z_n \) satisfies (2.12) and

\[
(1 - 2d) \log M(r, Q) < \log |Q(re^{i\theta})| \]

at the same time and, moreover, \( \delta(-z, \theta_0) < 0 \). Then, by (2.11),

\[
\exp \left\{ \frac{3(1 + \varepsilon)}{2} \delta(-z, \theta_0) r_n \right\} \leq |h(z_n)e^{-z_n}| \leq \exp \left\{ \frac{1 - \varepsilon}{2} \delta(-z, \theta_0) r_n \right\}
\]

for all \( n \) large enough. Writing now Eq. (1.7) in the form

\[
\frac{f''(z)}{f(z)} + h(z)e^{-z} \frac{f'(z)}{f(z)} + Q(z) = 0
\]

and substituting (2.12), (6.2) and (6.3) into (6.4), we obtain

\[
M(r_n, Q)^{1-2d} \leq |Q(r_n e^{i\theta_0})| \leq r_n^{2(\rho(f)+\varepsilon)},
\]

and therefore

\[
M(r_n, Q) \leq r_n^{4(\rho(f)+\varepsilon)}.
\]

This is a contradiction, as \( Q(z) \) is transcendental entire.
References