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Fixed point spaces in primitive actions of simple algebraic groups

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Abstract

Let G be a simple algebraic group of adjoint type acting primitively on an algebraic variety Ω. We study the dimensions of the subvarieties of fixed points of involutions in G. In particular, we obtain a close to best possible function $f(h)$, where h is the Coxeter number of G, with the property that with the exception of a small finite number of cases, there exists an involution t in G such that the dimension of the fixed point space of t is at least $f(h)\dim\Omega$.

1. Introduction

Let G be a simple algebraic group over an algebraically closed field K of arbitrary characteristic $p \geqslant 0$. In this paper we consider primitive actions of G on coset varieties $\Omega = G/H$, where H is a maximal closed subgroup of G.

For $t \in G$, the fixed point space

$$C_\Omega(t) = \{\omega \in \Omega\colon\ \omega t = \omega\}$$

is a subvariety of Ω. In a recent paper [11], Lawther, Liebeck, and Seitz obtained upper bounds for $\dim C_\Omega(t)$ in the case where G is a simple algebraic group of exceptional type acting transitively on Ω, and t is a non-identity element of G. This study was motivated by the notion of *fixed point ratio* in finite group theory. If G is a finite group acting transitively on a set Ω, then the fixed point ratio of $x \in G$ is defined to be the proportion of points

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fixed by x. Such ratios for finite simple groups of Lie type have been studied in a number of papers. In [4–6,16] upper bounds on fixed point ratios are obtained and applied to a number of problems in the case where G is a classical group. For finite simple exceptional groups of Lie type, the reader is referred to [10], where Liebeck, Lawther, and Seitz study fixed point ratios. Using the upper bounds for the corresponding algebraic groups in [11], the authors obtain close to best possible upper bounds.

The study of *lower* bounds for fixed point ratios was initiated by Saxl and Shalev in a paper on the fixity of permutation groups [19]. The fixity f of a finite permutation group G is defined to be the maximal number of fixed points of a non-trivial element of G. In [19], it is shown that if G is a simple primitive permutation group of fixity f, then either $G = PSL_2(q)$ or $Sz(q)$ in their natural permutation actions (of degree $q+1$, q^2+1, respectively), or $|G|$ is bounded by some function of f.

This paper is the first to consider the analogous question on lower bounds for $\dim C_\Omega(t)$ in the context of algebraic groups. Here the natural analogue of the fixed point ratio of $t \in G$ is $\dim C_\Omega(t) - \dim \Omega$. In this paper we shall study the ratio

$$f_\Omega(t) = \frac{\dim C_\Omega(t)}{\dim \Omega}.$$

In particular, we are interested in lower bounds for $\max_{t \in G^\#} f_\Omega(t)$, an analogous notion to that of fixity in finite permutation groups.

Let h denote the Coxeter number of G. In a similar spirit to the result of Saxl and Shalev, we define a close to best possible function $f(h)$ with the property that either there exists an involution $t \in G$ such that $f_\Omega(t) \geqslant f(h)$, or (G, H°) is one of a finite number of possible cases. This is detailed in the following theorem.

Theorem 1. *Let G be a simple algebraic group of adjoint type, over an algebraically closed field K of characteristic $p \geqslant 0$. Let H be either a maximal closed subgroup of G or a finite subgroup of G, and let G act on the coset variety $\Omega = G/H$. Let h denote the Coxeter number of G. Then one of the following holds*:

(i) *there exists an involution $t \in G$ such that*

$$f_\Omega(t) \geqslant \frac{1}{2} - \frac{1}{2h+1};$$

(ii) *either H is finite of odd order, or $(G, H) = (A_1, P_1)$; in both cases, $f_\Omega(t) = 0$ for all involutions $t \in G$*;

(iii) (G, H°) *is listed in Table* 1.1. *In each case there exists an involution $t \in H^\circ$ such that $f_\Omega(t) \geqslant \epsilon$.*

In the statement of Theorem 1, P_i denotes the standard parabolic subgroup of G corresponding to deleting the ith node from the Dynkin diagram (where diagrams are labelled as in Bourbaki [2]). The subgroups $A_2 < D_4$ and $A_2 < B_3$ in Table 1.1 are irreducible embeddings. Also, $A_2, \widetilde{A}_2 < G_2$ are maximal rank subgroups corresponding respectively to long and short A_2 subsystems.

Table 1.1

(G, H°)	p	ϵ
(D_4, A_2), (G_2, P_1)	arbitrary	$2/5$
(G_2, P_2)	$\neq 2$	$2/5$
(B_3, A_2)	3	$5/13$
(G_2, A_2), (C_2, P_1)	$\neq 2$	$1/3$
(C_2, P_2)	2	$1/3$
$(G_2, \tilde{A}_2)$	3	$1/3$

Remark 1. The bound in (i) is close to best possible. To see this, note that

$$\frac{|\Sigma^+(G)|}{\dim G} = \frac{1}{2} - \frac{1}{2h+2},$$

where $|\Sigma^+(G)|$ denotes the number of positive roots in the associated root system of G. Let $r = \operatorname{rank} G$. If the subgroup H of G is finite and $t \in H$ is an involution with $\dim t^G$ maximal, then in the most cases we have $\dim t^G = (1/2)(\dim G + r)$ (see [11, 1.5]), and since $\dim G = r(h+1)$, it follows that $f_\Omega(t) = 1/2 - 1/2(h+1)$. It will be shown in Section 4 that there exist examples of arbitrarily large rank where $f_\Omega(t) = 1/2 - 1/2(h+1)$ for every involution t (see Remark 4.1).

Observe that if the Coxeter number h of G is greater than or equal to 3, then $1/2 - 1/(2h+1) > 1/3$. Up to isomorphism, A_1 is the only simple algebraic group such that $h < 3$ (see the table below). If $H < G = A_1$ is either maximal of positive dimension, or finite of even order then with the exception of the case (A_1, P_1), it is possible to find an involution $t \in H$ such that $f_\Omega(t) \geqslant 1/3$ (see Tables 2.4, 3.3.1 and the last paragraph of Lemma 3.6). Thus in view of Theorem 1, we have the following corollary.

Corollary 1. *With G, H, Ω as in Theorem 1, if $(G, H) \neq (A_1, P_1)$, and H is not finite of odd order then there exists an involution $t \in G$ such that*

$$f_\Omega(t) \geqslant \frac{1}{3}.$$

One should note that in many cases we can improve on the lower bound stated in Theorem 1(i). Referring the reader to the tables in Lemmas 3.1–3.3 and 3.6, one observes that in the cases which correspond to these tables, it is possible to establish lower bounds which tend to 1 as the Coxeter number h of G tends to infinity.

For the reader's convenience, we list the values of the Coxeter number h of G, for each type of simple algebraic group G:

G	A_l	B_l	C_l	D_l	G_2	F_4	E_6	E_7	E_8
h	$l+1$	$2l$	$2l$	$2l-2$	6	12	12	18	30

The layout of the paper is as follows. The first section is concerned with various preliminary results from the literature which we shall need for the proof of the theorem.

The key result is the Liebeck/Seitz classification of the maximal subgroups of positive dimension in a simple algebraic group. This is stated in Theorem 2.1 for classical groups, and Theorem 2.2 for those of exceptional type. In Sections 3 and 4, we shall deal with the classical groups, the two parts of Theorem 2.1 dictating the structure of our proof. In Section 5 we turn our attention to the exceptional groups, and complete the proof of Theorem 1.

Notation

The following notation will be used throughout the paper. Let G be a simple algebraic group over K. The fundamental roots in a fundamental system for G are denoted $\alpha_1,\dots,\alpha_l$, with corresponding fundamental dominant weights $\lambda_1,\dots,\lambda_l$. We follow Bourbaki [2] in labelling the Dynkin diagram of G. T_i denotes a torus of rank i. If $\lambda = a_1\lambda_1 + \cdots + a_l\lambda_l$ is a dominant weight then $M(\lambda)$ denotes the irreducible KG-module with high weight λ. If H is a subgroup of G and V is a KG-module then $V \downarrow H$ will denote the restriction of V to H.

2. Preliminary results

Let G be a simple algebraic group over K. When G is classical we shall write $G = Cl(V) \in \{SL(V), Sp(V), SO(V)\}$, where V is the natural module. As Theorem 1 is stated for adjoint groups, any element t whose square is scalar is said to be an involution in the classical group.

We now introduce some notation which will be used throughout the paper. We use $[M_1,\dots,M_n]$ to denote the block diagonal matrix with the matrices $M_1,\dots,M_n$ down the diagonal, and $[J_2^m]$ will represent the $2m \times 2m$ block diagonal matrix with m Jordan 2-blocks on the diagonal. At times we shall also use the notation $[-jI_a, jI_{n-a}]$, where a is odd and it is understood that $j \in K$ satisfies $j^n = -1$. Similarly, i will always denote a field element such that $i^2 = -1$.

If $G = C_l$ or D_l, then $\{e_1, f_1, \dots e_l, f_l\}$ will denote respectively a standard symplectic or orthogonal basis of the natural module V and all matrices are written with respect to this specific ordering. However, it will also be necessary to consider the ordering $\{e_1,\dots,e_l, f_1,\dots,f_l\}$, and any matrix A written with respect to this ordering will be denoted by $[A]^{\diamondsuit}$. When $p = 2$, there exists an abstract isomorphism $\psi : SO_{2l+1} \to Sp_{2l}$ which is also a homomorphism of algebraic groups (see [21, Theorem 28]). Therefore, we shall only consider the case $G = B_l$ when p is odd. We order our orthogonal basis as $\{e_1, f_1,\dots,e_l, f_l, x\}$, where x is non-singular. For a full description of these bases, see [9, §2.5].

In Sections 3 and 4, we shall make much use of the notation and main result of [14]. In order to state this, we first define six collections of maximal subgroups H of a simple classical algebraic group $G = Cl(V)$.

Class $\mathcal{C}_1$: *Subspace stabilisers*. Subgroups $H = G_U$, where U is a totally singular or non-degenerate proper non-zero subspace of V. In the case $(G, p) = (SO(V), 2)$, we also allow the case where U is non-singular of dimension 1.

Class $\mathcal{C}_2$: *Stabilisers of orthogonal decompositions*. Here $H = G_{\{V_1,\dots,V_t\}}$, where $V = \bigoplus_{i=1}^t V_i$, $t > 1$, and the subspaces V_i are mutually orthogonal and isometric.

Class $\mathcal{C}_3$: *Stabilisers of totally singular decompositions*. Here we have $G = Sp(V)$ or $SO(V)$ and $H = G_{\{W,W'\}}$, where $V = W \oplus W'$ and W, W' are maximal totally singular subspaces. Note that if $G = SO(V)$ and $\dim V \equiv 2 \pmod 4$ then H is not maximal and hence we exclude this case.

Class $\mathcal{C}_4$: *Tensor product subgroups*. In this case either $V = V_1 \otimes V_2$ with $\dim V_i > 1$ and $H = N_G(Cl(V_1) \circ Cl(V_2))$ acting naturally on the tensor product, or $V = \bigotimes_{i=1}^k V_i$ with $k > 1$, the V_i mutually isometric and $H = N_G(\prod Cl(V_i))$, again acting naturally. See Lemma 3.4 for the specific details on which classical subgroups appear as factors.

Class $\mathcal{C}_5$: *Finite local groups*. We have $H = N_G(R)$, where R is an irreducible q-group of symplectic type, for a prime q different from p. Each subgroup in this class is finite.

Class $\mathcal{C}_6$: *Classical subgroups*. These are the subgroups $N_G(Sp(V))$ and $N_G(SO(V))$ in $G = SL(V)$, and $N_G(SO(V))$ in $G = Sp(V)$ when $p = 2$.

Theorem 2.1 [14, Theorem 1]. *Let $G = Cl(V)$ be a classical simple algebraic group over an algebraically closed field of arbitrary characteristic, and let H be a closed subgroup of G. Let $\mathcal{C}(G)$ denote the collection $\bigcup_i \mathcal{C}_i$ of subgroups of G. Then one of the following holds*:

(i) *H is contained in a member of $\mathcal{C}(G)$*;
(ii) *modulo scalars, H is almost simple, and the quasisimple subgroup $E(H)$ is irreducible on V. Furthermore, if H is infinite, then $E(H)$ is tensor-indecomposable on V.*

As we remarked in the introduction, when G is classical we shall use this theorem to prove Theorem 1 in two stages, beginning with the case where our maximal subgroup H is a member of one of the collections $\mathcal{C}_i$. Our approach when H is not in $\mathcal{C}(G)$ is less direct and we need to appeal to some recent results [17] concerning the irreducible representations of simple algebraic groups in prime characteristic.

The study of maximal closed subgroups of exceptional simple algebraic groups dates back to the fundamental work of Dynkin, and the problem of classifying all such subgroups of positive dimension has only recently been solved for arbitrary algebraically closed fields. The following result is due to Liebeck and Seitz.

Theorem 2.2 [12, Corollary 2.1(i)]. *Let G be a simple algebraic group of exceptional type over an algebraically closed field of characteristic $p \geqslant 0$. The maximal closed subgroups of positive dimension in G are as follows*:

(a) *maximal parabolic subgroups*;
(b) *maximal reductive subgroups of maximal rank* (*see Table* 2.1);
(c) *$N_G(X)$, with X as in Table* 2.2;
(d) *$G = E_7$, $p \neq 2$ and $H = (2^2 \times D_4).S_3$*;
(e) *$G = E_8$, $p \neq 2, 3, 5$ and $H = A_1 \times S_5$.*

Table 2.1
Maximal reductive subgroups $N_G(M)$ of maximal rank

G	M
E_8	$D_8, A_1E_7, A_8, A_2E_6, A_4^2, D_4^2, A_2^4, A_1^8, T_8$
E_7	$A_1D_6, A_7, A_2A_5, A_1^3D_4, A_1^7, E_6T_1, T_7$
E_6	$A_1A_5, A_2^3, D_4T_2, T_6$
F_4 $(p \neq 2)$	$B_4, D_4, A_1C_3, A_2\widetilde{A}_2$
F_4 $(p = 2)$	above, plus duals
G_2	$A_1\widetilde{A}_1, A_2, \widetilde{A}_2$ $(p = 3)$

Table 2.2

G	X
E_8	A_1 (3 classes, $p \geqslant 23, 29, 31$ resp.), B_2 $(p \geqslant 5)$, G_2F_4, A_1A_2 $(p \neq 2, 3)$, $A_1G_2^2$ $(p \neq 2)$
E_7	A_1 (2 classes, $p \geqslant 17, 19$ resp.), A_2 $(p \geqslant 5)$, A_1F_4, A_1^2 $(p \neq 2, 3)$, A_1G_2 $(p \neq 2)$, G_2C_3
E_6	A_2 $(p \neq 2, 3)$, G_2 $(p \neq 7)$, F_4, C_4 $(p \neq 2)$, A_2G_2
F_4	A_1 $(p \geqslant 13)$, G_2 $(p = 7)$, A_1G_2 $(p \neq 2)$
G_2	A_1 $(p \geqslant 7)$

For use in Section 4, we need some results on the self-dual irreducible representations of the simple algebraic group $G = SL_n$.

Proposition 2.3. *If $G = SL_n$ and $\rho : G \to GL(V)$ is a non-trivial self-dual irreducible representation of minimal degree then one of the following holds*:

(i) $G = SL_2$ (*in which case every irreducible representation is self-dual*);
(ii) $G = SL_4$ *and* $V = \bigwedge^2 U$, *where U is the natural* 4-*dimensional G-module*;
(iii) $G = SL_6$ *and* $V = \bigwedge^3 U$, *where U is the natural* 6-*dimensional G-module*;
(iv) ρ *is the adjoint representation of G*.

Proof. In [17, Theorem 5.1], Lübeck lists all p-restricted irreducible SL_n-modules for $n \geqslant 13$ whose dimension is at most $(n-1)^2/8$. For $3 \leqslant n \leqslant 18$, Lübeck has produced tables [17, Tables A.6–21] which record the degrees and highest weights of all p-restricted irreducible SL_n-modules whose degree is at most some number $f(n)$, where $f(n) \geqslant f(3) = 400$ for all $3 \leqslant n \leqslant 18$. Using these results, together with the fact that $M(\lambda)^* \cong M(-w_o(\lambda))$, where w_o is the longest element of the Weyl group of G (see [8, 3.1.6]), the proposition follows immediately. □

Remark 2.4. It is not difficult to see that the irreducible representation in (ii) embeds SL_4 in SO_6, while in (iii) we have SL_6 embedded in Sp_{20} if $p \neq 2$, and SO_{20} if $p = 2$.

In a similar spirit, we will need the following result.

Proposition 2.5. *Let $G = SL_d$, with d odd. Then the only irreducible self-dual n-dimensional representation of G such that $n < 4d^2 - 4$ is the adjoint representation.*

In order to prove the proposition, we state two further results.

Lemma 2.6 [7, 10.3B]. *Let $\lambda = a_1\lambda_1 + \cdots + a_r\lambda_r$ be a dominant weight of G, where r denotes the rank. Then the stabiliser of λ in the Weyl group W of G is the parabolic subgroup generated by the reflections along the simple roots α_i for which $a_i \neq 0$.*

Lemma 2.7 [18, Premet]. *If the root system of G has different root lengths, we assume that $p \neq 2$, and if G is of type G_2, we also assume that $p \neq 3$. Let λ be a p-restricted dominant weight. Then the set of weights of the irreducible G-module $M(\lambda)$ is the union of the W-orbits of dominant weights ω with $\omega \leqslant \lambda$.*

Proof of Proposition 2.5. Let $M(\lambda)$ be an irreducible self-dual G-module of highest weight $\lambda = a_1\lambda_1 + \cdots + a_{d-1}\lambda_{d-1}$, and suppose that $\dim M(\lambda) < 4d^2 - 4$. Now, if $5 \leqslant d \leqslant 17$, then it is immediate from Lübeck's tables [17] that the only irreducible representation of G satisfying the hypotheses is the adjoint representation. Suppose now that $d \geqslant 19$. Recall that the Weyl group of G acts on the set of weights of $M(\lambda)$, and thus $\dim M(\lambda) \geqslant |W \cdot \lambda|$. Recall also that the Weyl group of SL_n is the symmetric group S_n. Now since $M(\lambda)^* \cong M(-w_o(\lambda))$, self-duality implies that $a_i = a_{d-i}$ for each i. Suppose that $a_3 = a_{d-3} \neq 0$. From Lemma 2.6, the W-stabiliser of λ is contained in a parabolic subgroup of type $A_2 \times A_{d-7} \times A_2$, so

$$\dim M(\lambda) \geqslant |W \cdot \lambda| = |W : W_\lambda| \geqslant \frac{d!}{3!3!(d-6)!} > 4d^2 - 4$$

when $d \geqslant 19$. Hence we must have $a_3 = a_{d-3} = 0$. Similarly, we also have $a_2 = a_{d-2} = a_4 = a_{d-4} = \cdots = a_{d/2-1/2} = a_{d/2+1/2} = 0$, i.e. $\lambda = a\lambda_1 + a\lambda_{d-1}$. If $a \geqslant 2$ then

$$\lambda - \alpha_1 - \alpha_{d-1} = (a-2)\lambda_1 + \lambda_2 + \lambda_{d-2} + (a-2)\lambda_{d-1}$$

is dominant. From Lemma 2.7, it follows that $\omega = \lambda - \alpha_1 - \alpha_{d-1}$ is a weight of $M(\lambda)$. As before, W_ω is contained in a parabolic subgroup of type $A_1 \times A_{d-5} \times A_1$, and it follows that $\dim M(\lambda) \geqslant |W : W_\omega| > 4d^2 - 4$ when $d \geqslant 19$. Hence $a = 1$ and $\lambda = \lambda_1 + \lambda_{d-1}$, which is the highest weight of the adjoint representation. □

For use in Section 4.2, we require the following result on self-dual minimal degree irreducible G-modules, where G is exceptional.

Proposition 2.8. *Let G be a simple algebraic group of exceptional type. The following table records the dimension n of the minimal degree non-trivial self-dual irreducible G-module*:

G	p	n
E_6	$\neq 2$	324
	2	351
E_7	arbitrary	56
E_8	arbitrary	248
F_4	$\neq 3$	26
	3	25
G_2	$\neq 2$	7
	2	6

Proof. Since $M(\lambda)^* \cong M(-w_o(\lambda))$, it follows that if G is one of E_7, E_8, F_4 or G_2 then every irreducible G-module is self-dual. If $G = E_6$ then $w_o = -\tau$, where τ is a graph automorphism of G induced from the order two symmetry of the Dynkin diagram for G. The data in the table now follows immediately from [17, Tables A.49–53]. □

Next we record a number of results concerning involutions in a simple algebraic group G.

Proposition 2.9. *Let G be a classical group, and suppose that $p \neq 2$. We have the following table of involution class representatives in G*:

G	t	$C_G(t)$	$\dim t^G$
A_l	$[-I_{2k}, I_{l+1-2k}]$	$T_1 A_{2k-1} A_{l-2k}$	$4k(l+1-2k)$
	$[-jI_{2k+1}, jI_{l-2k}]$	$T_1 A_{2k} A_{l-2k-1}$	$(4k+2)(l-2k)$
B_l	$[-I_{2k}, I_{2l+1-2k}]$	$D_k B_{l-k}$	$2k(2l+1-2k)$
C_l	$[-I_{2k}, I_{2(l-k)}]$	$C_k C_{l-k}$	$4k(l-k)$
	$[-iI_l, iI_l]^{\diamond}$	$T_1 A_{l-1}$	$l(l+1)$
D_l	$[-I_{2k}, I_{2(l-k)}]$	$D_k D_{l-k}$	$4k(l-k)$
	$[-iI_l, iI_l]^{\diamond}$	$T_1 A_{l-1}$	$l(l-1)$

Suppose that $G = Sp_{2m}$ or SO_{2m}, and $p = 2$. The Jordan canonical form of a unipotent involution $t \in G$ has the form $[J_2^l, I_{2(m-l)}]$, for some $1 \leqslant l \leqslant m$. We call such an element an l-*involution*. As described in [1], if l is even (which must be the case if $t \in SO_{2m}$) then there are precisely two distinct conjugacy classes of l-involutions in G, with representatives denoted by a_l and c_l. If $(\ ,\ )$ is the associated symmetric bilinear form on the natural G-module V_{2m}, then an l-involution $t \in G$ is said to be in a_l^G if and only if

$$\bigl(t(v), v\bigr) = 0 \quad \text{for all } v \in V.$$

Otherwise, $t \in c_l^G$. Therefore with no ambiguity we can take $c_l = [J_2^l, I_{2(m-l)}]$, where the basis is ordered in the usual way. We will use $[J_2^l, I_{2(m-l)}]_a$ to denote a member of the conjugacy class a_l^G. If l is odd there is a unique class of l-involutions in Sp_{2m}, and following [1], we denote this class by b_l^G.

Proposition 2.10. *Let G be a classical group over an algebraically closed field of characteristic $p \geqslant 0$, and let u be a non-identity unipotent element in G. Suppose for each i, the Jordan canonical form for u has n_i Jordan blocks of size i.*

(i) *If $G = SL_n$, then*

$$\dim C_G(u) = 2\sum_{i<j} i n_i n_j + \sum_i i n_i^2 - 1.$$

(ii) *Let $G = Sp_{2m}$ and $p = 2$. Then*

$$\dim C_G(a_{m-k}) = m^2 + m + k^2,$$
$$\dim C_G(b_{m-k}) = \dim C_G(c_{m-k}) = m^2 + k^2 + k.$$

(iii) *If $G = SO_{2m}$ and $p = 2$, we have*

$$\dim C_G(a_{m-k}) = m^2 + k^2 - k, \qquad \dim C_G(c_{m-k}) = m^2 - m + k^2.$$

Proof. Part (i) follows from [22, pp. 34–39], and (ii) and (iii) follow from [1, Sections 7 and 8]. □

Using Propositions 2.9 and 2.10, we have the following result.

Proposition 2.11. *Let G be a simple algebraic group of classical type. The following is a table of representatives of involution classes of maximal dimension in G, where n denotes the dimension of the natural G-module*:

G	p	n	t	$\dim t^G$
A_l	$\neq 2$	$l+1=2m$	$[-iI_m, iI_m]$	$(l+1)^2/2$
		$l=2m$	$\pm[-I_m, I_{m+1}]$	$l^2/2+l$
	$=2$	$l+1=2m$	$[J_2^m]$	$(l+1)^2/2$
		$l=2m$	$[J_2^m, 1]$	$l^2/2+l$
B_l	$\neq 2$	$2l+1$	$\pm[-I_l, I_{l+1}]$	l^2+l
C_l	$\neq 2$	$2l$	$[-iI_l, iI_l]^\diamond$	l^2+l
	$=2$	$2l$	$[J_2^l]$	l^2+l
D_l	$\neq 2$	$2l \equiv 0 \pmod 4$	$[-I_l, I_l]$	l^2
		$2l \equiv 2 \pmod 4$	$[-I_{l-1}, I_{l+1}]$	l^2-1
	$=2$	$2l \equiv 0 \pmod 4$	$[J_2^l]$	l^2
		$2l \equiv 2 \pmod 4$	$[J_2^{l-1}, I_2]$	l^2-1

Next we state a well-known result concerning involutions in exceptional groups.

Proposition 2.12 [11, 1.2], [15, 4.3]. *Let G be adjoint and of exceptional type. When $p \neq 2$, the centralisers in G of involutions are as follows*:

G	involution centralisers	c
E_8	A_1E_7, D_8	128
E_7	$A_1D_6, (A_7).2, (T_1E_6).2$	70
E_6	A_1A_5, D_5T_1	40
F_4	A_1C_3, B_4	28
G_2	A_1^2	8

For each G, we also record c, the maximal dimension of a conjugacy class of involutions in G. This upper bound is also realised when $p = 2$.

The following result regarding long root elements in a simple algebraic group G is also well known.

Proposition 2.13 [11, 1.12]. *If U_α denotes a long root subgroup of G, and $1 \neq t \in U_\alpha$, then* $\dim t^G$ *is given in the following table*:

G	A_l	B_l	C_l	D_l	G_2	F_4	E_6	E_7	E_8
$\dim t^G$	$2l$	$4l-4$	$2l$	$4l-6$	6	16	22	34	58

Recall that if α is a long root of a simple algebraic group G, with corresponding root subgroup U_α, then $\langle U_\alpha, U_{-\alpha}\rangle \cong SL_2(K)$, unless of course $G = PSL_2(K)$ and $p \neq 2$. Suppose $p \neq 2$. A *fundamental involution* in G (relative to some long root α) is defined to be the unique involution $t \in \langle U_\alpha, U_{-\alpha}\rangle$. This implies that $C_G(t)$ must have an A_1 factor, and in view of Proposition 2.12, this completely determines the conjugacy class of t in G when G is exceptional. For use in Section 5, Table 2.3 is a table of fundamental involutions, with corresponding centralisers.

Table 2.3
Fundamental involutions

G	t	$C_G(t)$
A_l	$[-I_2, I_{l-1}]$	$T_1A_1A_{l-2}$
B_l	$[-I_4, I_{2l-3}]$	$A_1^2B_{l-2}$
C_l	$[-I_2, I_{2l-2}]$	A_1C_{l-1}
D_l	$[-I_4, I_{2l-4}]$	$A_1^2D_{l-2}$
E_8	–	A_1E_7
E_7	–	A_1D_6
E_6	–	A_1A_5
F_4	–	A_1C_3
G_2	–	A_1^2

The next result provides us with a method for calculating $\dim C_{\Omega}(t)$. It reduces the problem to a calculation of conjugacy class dimensions.

Proposition 2.14 [11, 1.14]. *Let G be an algebraic group, and let H be a closed subgroup. If Ω denotes the coset variety G/H, then for $x \in H$,*

$$\dim C_{\Omega}(x) = \dim \Omega - \dim x^G + \dim\bigl(x^G \cap H\bigr).$$

All of our calculations rely on this important result. In practise however, it is often difficult to calculate $\dim(t^G \cap H)$ directly and so we use the fact that $\dim(t^G \cap H) \geqslant \dim t^{H^\circ}$ to obtain a lower bound for $\dim C_{\Omega}(t)$.

We are now in a position to prove Theorem 1 in the case where H is a finite subgroup of G.

Proposition 2.15. *Let G be a simple algebraic group over an algebraically closed field K. If H is finite, then the conclusion of Theorem* 1 *is true.*

Proof. We can assume that $|H|$ is even. Let $t \in H$ be an involution. Since $\dim \Omega = \dim G$ and $\dim(t^G \cap H) = 0$, it follows from Proposition 2.14 that $\dim C_{\Omega}(t) = \dim G - \dim t^G$. Using the upper bounds provided by Propositions 2.11 and 2.12, we obtain the results in Table 2.4, which are independent of characteristic, and from which Theorem 1 follows immediately. There n denotes the dimension of the natural G-module for G classical. □

Remark 2.16. As we shall see in Section 4, there are examples where equality can hold. This illustrates the fact that the bound in Theorem 1(i) is close to best possible.

Table 2.4
H finite

G	h	n	$f_{\Omega}(t) \geqslant$
A_l	$l+1$	$l+1$ even	$\frac{1}{2} - \frac{1}{2l(h+1)}$
		$l+1$ odd	$\frac{1}{2}$
B_l	$2l$	$2l+1$	$\frac{1}{2} - \frac{1}{2(h+1)}$
C_l	$2l$	$2l$	$\frac{1}{2} - \frac{1}{2(h+1)}$
D_l	$2l-2$	$2l \equiv 0 \pmod 4$	$\frac{1}{2} - \frac{1}{2(h+1)}$
		$2l \equiv 2 \pmod 4$	$\frac{1}{2} - \frac{l-2}{2l(h+1)}$
E_8	30	–	$\frac{1}{2} - \frac{1}{2(h+1)}$
E_7	18	–	$\frac{1}{2} - \frac{1}{2(h+1)}$
E_6	12	–	$\frac{1}{2} - \frac{1}{6(h+1)}$
F_4	12	–	$\frac{1}{2} - \frac{1}{2(h+1)}$
G_2	6	–	$\frac{1}{2} - \frac{1}{2(h+1)}$

In order to calculate $\dim\Omega$ when $\Omega = G/H$ and $H = P_i$ is a maximal parabolic subgroup, we make use of the following well-known result.

Proposition 2.17. *Let P_i be a parabolic subgroup of a simple algebraic group G, and let $P_i = Q_i L_i$ be a Levi decomposition, where $Q_i = R_u(P_i)$. Then,*

$$\dim G - \dim P_i = \dim Q_i = \left|\Sigma^+(G)\right| - \left|\Sigma^+\left(L_i'\right)\right|,$$

where $|\Sigma^+(X)|$ denotes the number of positive roots in the associated root system of the semisimple group X.

We finish this preliminary section with three technical propositions which will be needed to deal with the case where H is a maximal parabolic subgroup of G.

Proposition 2.18 [20, p. 54]. *If u is a unipotent element of the simple algebraic group G, and B is a Borel subgroup of G, then*

$$\dim\left(u^G \cap B\right) = \frac{1}{2}\dim u^G.$$

Proposition 2.19. *Let G be a simple algebraic group of exceptional type, and let s be a non-identity semisimple element of G lying in the maximal parabolic subgroup $P_i = Q_i L_i$. If $\Omega_i = G/P_i$ and $D = C_G(s)$, then*

(i) *$D \cap P_i$ is a parabolic subgroup of D*;
(ii) $\dim C_{\Omega_i}(s) \geqslant \dim R_u(D \cap P_i) = |\Sigma^+(D)| - |\Sigma^+(C_{L_i}(s))|$.

Proof. For (i), see [11, 3.1]. Since $\dim\Omega_i = \dim Q_i$ (2.17) and $\dim(s^G \cap P_i) \geqslant \dim s^{P_i}$, it follows from Proposition 2.14 that $\dim C_{\Omega_i}(s) \geqslant \dim R_u(D \cap P_i)$. The last part follows from Proposition 2.17 since $C_{L_i}(s)$ is a Levi factor of $D \cap P_i$. □

Proposition 2.20 [11, 2.1]. *Let $u_\alpha = u$ be a long root element of the simple algebraic group G, and let $P_i = Q_i L_i$ be a maximal parabolic subgroup, where $Q_i = R_u(P_i)$ and L_i a Levi subgroup. If $u \in L_i$, then*

$$\dim u^G - \dim\left(u^G \cap P_i\right) = \frac{1}{2}\left(\dim u^G - \dim u^{L_i}\right).$$

3. Proof of Theorem 1, Part I: G classical, $H \in \mathcal{C}(G)$

In this section we deal with the case where G is classical and H is a member of one of the classes $\mathcal{C}_i$ (see Theorem 2.1). Treating each collection $\mathcal{C}_i$ in turn, we seek to find best possible lower bounds for $f_\Omega(t)$ and obtain Theorem 1 as a corollary of this work. Throughout this section we repeatedly apply Propositions 2.9, 2.10, and 2.14.

Lemma 3.1. *If $H \in \mathcal{C}_1$ then the conclusion of Theorem 1 is true.*

Table 3.1
$H = P_i$

G	$\dim \Omega$	p	t	$\dim t^H$	$\dim t^G$	$f_\Omega(t) \geqslant$
A_l	$i(l+1-i)$	$\neq 2$	$[-j, jI_l]$	$l+i-1$	$2l$	$1-\frac{2}{l+1}$
			$[jI_l, -j]$	$2l-i$	$2l$	
		2	$[J_2, I_{l-1}]$	$l+i-1$	$2l$	$1-\frac{2}{l+1}$
			$[I_{l-1}, J_2]$	$2l-i$	$2l$	
B_l	$2li-3i^2/2+i/2$	$\neq 2$	$[-I_{2l}, 1]$	$2l-i$	$2l$	$1-\frac{2}{l+1}$
C_l	$2li-3i^2/2+i/2$	$\neq 2$	$[-I_2, I_{2l-2}]$	$2l+i-3$	$4l-4$	$1-\frac{2}{l+1}$
			$[I_{2l-2}, -I_2]$	$3l-3$, if $i=l$; $4l-2i-4$, otherwise	$4l-4$	
D_l	$l^2/2-l/2$, if $i=l-1$; $2li-3i^2/2-i/2$, otherwise	$\neq 2$	$[-I_2, I_{2l-2}]$	$3l-3$, if $i=l-1$; $2l+i-3$, otherwise	$4l-4$	$1-\frac{2}{l}$
			$[I_{2l-2}, -I_2]$	$3l-3$, if $i=l$; $4l-2i-4$, otherwise	$4l-4$	

Proof. The maximal parabolic subgroups of $G = Cl(V)$ are the stabilisers of totally singular subspaces of V. We adopt the standard notation P_i, $1 \leqslant i \leqslant \operatorname{rank} G$. Following [14], N_i will denote the stabiliser in G of an i-dimensional non-degenerate subspace of V. Beginning with the maximal parabolic subgroups, we have Table 3.1. We now justify the information in this table. The stated values for $\dim t^G$ follow from Propositions 2.9 and 2.10, and we use Proposition 2.17 to calculate $\dim \Omega$. Thus in view of Proposition 2.14, we only need to justify the stated values for $\dim t^H$. Once this is achieved, one can readily check via Proposition 2.14 that in each case (G, P_i, p), the lower bound for $f_\Omega(t)$ in the last column is realised for at least one of the listed involutions t.

If $G = A_l$, we can calculate $\dim t^{P_i}$ directly. For example, if $t = [-j, jI_l]$, then $C_{P_i}(t) \cong (GL_1 \times GL_l) \cap P_i$, so $\dim C_{P_i}(t) = 1 + (i-1)^2 + l(l+1-i) - 1$. To calculate $\dim t^{P_i}$ for the other types of G when p is odd, we interpret $C_{P_i}(t)$ in terms of smaller parabolics. For example, if $t = [-I_2, I_{2l-2}] \in Sp_{2l}$, then

$$C_{P_i}(t) \cong (P_1 \text{ in } Sp_2) \times (P_{i-1} \text{ in } Sp_{2l-2}).$$

Now suppose that $G = C_l$ or D_l, and $p = 2$. We claim that if $t \in P_i$ is an involution, then

$$f_\Omega(t) \geqslant \begin{cases} 1 - \frac{2}{l+1}, & \text{if } G = C_l, \\ 1 - \frac{4}{l+5}, & \text{if } G = D_l. \end{cases}$$

Table 3.1.1

(G, H)	p	t	$f_\Omega(t) \geqslant$
(A_2, P_1)	$\neq 2$	$[-I_2, 1]$	1/2
	2	$[1, J_2]$	1/2
(A_2, P_2)	$\neq 2$	$[1, -I_2]$	1/2
	2	$[J_2, 1]$	1/2
(C_2, P_1)	2	$[I_2, J_2]$	2/3
(C_2, P_2)	$\neq 2$	$[-I_2, I_2]$	2/3

Let $G = C_l$. If $1 \leqslant i \leqslant l-1$ then $t = [J_2, I_{2l-2}]$ is a long root element contained in the simple factor C_{l-i} of a Levi subgroup L_i of P_i. Thus, using Proposition 2.13, we have $\dim t^G = 2l$ and $\dim t^{L_i} \geqslant 2l - 2i$, and applying Proposition 2.20, we have $f_\Omega(t) \geqslant 1 - 2/(l+4)$. When $i = l$, we take the same involution and use Spaltenstein's result (2.18) to establish a lower bound of $1 - 2/(l+1)$. Hence, in either case, a bound of $1 - 2/(l+1)$ holds. We treat the case $G = D_l$ in a similar way; for $1 \leqslant i \leqslant l-3$, let t be a long root element lying in the D_{l-i} simple factor of a Levi subgroup of P_i. Now t is G-conjugate to $[J_2^2, I_{2l-4}]_a$ and, using Proposition 2.20 in conjunction with Proposition 2.13, we have $f_\Omega(t) \geqslant 1 - 4/(l+8)$. Similarly, if $i = l-2$, choosing a long root element t in the A_{l-3}, factor of the Levi subgroup gives $\dim t^G - \dim t^{L_i} = 2l$, and finally, if $i = l-1$ or l, we have $\dim t^G - \dim t^{L_i} = 2l - 4$ when t is a long root element in the Levi factor A_{l-1}. Thus in all cases, a lower bound of $1 - 4/(l+5)$ holds.

As it stands, in some low-rank cases the above work is not sufficient to establish the conclusion of Theorem 1 when H is a maximal parabolic subgroup. However, as detailed in Table 3.1.1, it is possible to derive better lower bounds. For $(G, H, p) = (C_2, P_1, \neq 2)$ and $(C_2, P_2, 2)$, the best lower bound is $1/3 < 7/18 = 1/2 - 1/(2h+1)$. These exceptions are recorded in Table 1.1.

We now consider the stabilisers $H = N_i$ of non-degenerate subspaces of the natural G-module. Note that if $G = B_l$ then $N_{2i+1} \cong N_{2(l-i)}$, so we need only consider even-dimensional non-degenerate subspaces in this case. We have Table 3.2.

We now justify the information in this table. If U is a non-degenerate subspace of the natural G-module V then $H = G_U \cong Cl(U) \times Cl(U^\perp)$ and $\dim \Omega$ follows immediately. By exploiting this isomorphism, we can easily calculate $\dim C_H(t)$ for a given involution t. For example, if $G = C_l$, $H = N_{2i}$, $p = 2$ and $t = [J_2, I_{2l-2}]$ then

$$C_H(t) \cong C_{Sp_{2i}}(s) \times Sp_{2l-2i},$$

where $s = [J_2, I_{2i-2}] \in Sp_{2i}$. We calculate $\dim C_{Sp_{2i}}(s)$ via Proposition 2.10.

This leaves us to deal with one remaining case, namely $(G, H, p) = (D_l, N_1, 2)$, where N_1 is the stabiliser in G of a 1-dimensional non-singular subspace U of the natural module V. Relative to a standard basis, we take $U = \langle e_1 + f_1 \rangle$ and the corresponding data in the table follows from the well-known fact that $N_1 \cong Sp_{2l-2}$ (see [9, 4.1.7]).

As with Table 3.1, it is straightforward to check the validity of the stated lower bounds in the last column using Proposition 2.14. To establish Theorem 1, we need to make alternative choices for t in some small rank cases. These are given in Table 3.2.1.

Table 3.2
$H = N_i$

G	H	$\dim \Omega$	p	t	$\dim t^H$	$\dim t^G$	$f_\Omega(t) \geqslant$
B_l	N_{2i}	$4li + 2i - 4i^2$	$\neq 2$	$[-I_{2l}, 1]$	$2l - 2i$	$2l$	$1 - \frac{2}{l+1}$
				$[-I_2, I_{2l-1}]$	$4i - 4$	$4l - 2$	
C_l	N_{2i}	$4li - 4i^2$	$\neq 2$	$[-I_2, I_{2l-2}]$	$4i - 4$	$4l - 4$	$1 - \frac{2}{l}$
				$[I_{2l-2}, -I_2]$	$4l - 4i - 4$	$4l - 4$	
			2	$[J_2, I_{2l-2}]$	$2i$	$2l$	$1 - \frac{2}{l+1}$
				$[I_{2l-2}, J_2]$	$2l - 2i$	$2l$	
D_l	N_{2i}	$4li - 4i^2$	$\neq 2$	$[-I_2, I_{2l-2}]$	$4i - 4$	$4l - 4$	$1 - \frac{2}{l}$
				$[I_{2l-2}, -I_2]$	$4l - 4i - 4$	$4l - 4$	
			2	$[J_2^2, I_{2l-4}]$	$4i - 4$	$4l - 4$	$1 - \frac{2}{l}$
				$[I_{2l-4}, J_2^2]$	$4l - 4i - 4$	$4l - 4$	
	N_{2i-1}	$4li - 2l - 4i^2$ $+4i - 1$	$\neq 2$	$[I_{2l-2}, -I_2]$	$4l - 4i - 2$	$4l - 4$	$1 - \frac{2}{l+1}$
	N_1	$2l - 1$	2	$[J_2^2, I_{2l-4}]$	$4l - 6$	$4l - 4$	$1 - \frac{2}{2l-1}$

Table 3.2.1

(G, H)	p	t	$f_\Omega(t) \geqslant$
(B_2, N_2)	$\neq 2$	$[-I_4, 1]$	2/3
(B_2, N_4)	$\neq 2$	$[-I_2, I_3]$	1/2
(C_2, N_2)	$\neq 2$	$[-iI_2, iI_2]^\diamond$	1/2
	2	$[J_2, I_2]$	1/2
(C_3, N_2)	$\neq 2$	$[I_4, -I_2]$	1/2
(D_3, N_2)	$\neq 2$	$[I_4, -I_2]$	1/2
	2	$[I_2, J_2^2]$	1/2

With the exception of the case (B_1, N_2), the data in Tables 3.2 and 3.2.1 is sufficient to establish the fact that Theorem 1 holds when $H = N_i$. To deal with (B_1, N_2), one observes that $H = N_2$ corresponds to the subgroup of monomial matrices in $PSL_2 \cong SO_3$. We shall see in Lemma 3.2 that in this case, a lower bound of $1/2$ holds.

This completes the proof of Theorem 1 when G is classical and $H \in \mathcal{C}_1$. □

Lemma 3.2. *If $H \in \mathcal{C}_2$ then the conclusion of Theorem 1 is true.*

Proof. We have Table 3.3, where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Note that the subgroup $H^\circ = (O_1)^a \cap G$ in both $G = B_l$ and D_l is finite and so in view of Proposition 2.15, Theorem 1 holds in these cases.

The stated values of $\dim t^{H^\circ}$ are easy to verify; given $t \in H^\circ = (Cl_m)^a \cap G$, let s denote the restriction of t to V_1, where the natural G-module V admits the orthogonal decomposition $V = \bigoplus_{i=1}^a V_i$. It is clear in each case that $\dim t^{H^\circ} = \dim s^{Cl_m}$, which we can calculate in the usual manner.

Table 3.3
$H \in \mathcal{C}_2$

G	H°	$\dim \Omega$	p	t	$\dim t^{H^\circ}$	$\dim t^G$	$f_\Omega(t) \geqslant$
A_l	$(GL_m)^a \cap G$	$l^2 - l(m-2)$	$\neq 2$	$[-j, jI_l]$	$2m-2$	$2l$	$1 - \frac{2}{l+1}$
	$m > 1$	$-m+1$	2	$[J_2, I_{l-1}]$	$2m-2$	$2l$	$1 - \frac{2}{l+1}$
	$(GL_1)^{l+1} \cap G$	$l^2 + l$	$\neq 2$	$[-j, jI_l]$	0	$2l$	$1 - \frac{2}{l+1}$
			2	$[A, I_{l-1}]$	1	$2l$	$1 - \frac{2l-1}{l(l+1)}$
B_l	$(O_m)^a \cap G$	$2l^2 + 2l - lm$	$\neq 2$	$[-I_2, I_{2l-1}]$	$2m-4$	$4l-2$	$1 - \frac{4}{2l+1}$
	$m > 1$	$+1/2 - m/2$					
C_l	$(Sp_{2m})^a$	$2l^2 - 2lm$	$\neq 2$	$[-I_2, I_{2l-2}]$	$4m-4$	$4l-4$	$1 - \frac{2}{l}$
			2	$[J_2, I_{2l-2}]$	$2m$	$2l$	$1 - \frac{1}{l}$
D_l	$(O_m)^a \cap G$	$2l^2 - lm$	$\neq 2$	$[-I_2, I_{2l-2}]$	$2m-4$	$4l-4$	$1 - \frac{2}{l}$
	$m > 1$		2	$[J_2^2, I_{2l-4}]$	$2m-4$	$4l-4$	$1 - \frac{2}{l}$

Table 3.3.1

G	H°	p	t	$f_\Omega(t) \geqslant$
A_1	$(GL_1)^2 \cap G$	$\neq 2$	B	$1/2$
A_2	$(GL_1)^3 \cap G$	$\neq 2$	$[A, -1]$	$1/2$
C_2	$(Sp_2)^2$	$\neq 2$	$[-iI_2, iI_2]^\diamond$	$1/2$
C_3	$(Sp_2)^3$	$\neq 2$	$[-iI_3, iI_3]^\diamond$	$1/2$
D_3	$(O_3)^2 \cap G$	$\neq 2$	$[-I_2, 1, -I_2, 1]$	$5/9$

As in the previous lemma, we need to make alternative choices in some small rank cases in order to deduce that the conclusion of Theorem 1 is true when $H \in \mathcal{C}_2$. This is detailed in Table 3.3.1, where B denotes the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. This just leaves the case $G = D_3$, $H = (O_2 \wr S_3) \cap G$. This is dealt with by interpreting H as the subgroup of monomial matrices in $PSL_4 \cong SO_6$, where according to Table 3.3, a lower bound of $1/2$ can be established. $\square$

Lemma 3.3. *If $H \in \mathcal{C}_3$ then the conclusion of Theorem 1 is true.*

Proof. Here we have $G = C_l$ or D_l. With respect to the natural G-module bases introduced earlier, let $W = \langle e_1, \ldots, e_l \rangle$ and $W' = \langle f_1, \ldots, f_l \rangle$. It is clear that if $H = G_{\{W,W'\}}$, then $H^\circ = G_{W,W'} \cong GL_l$, via the isomorphism

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} \in G,$$

where the matrices in G are written with respect to the basis ordering $(e_1, \ldots, e_l, f_1, \ldots, f_l)$. If $t' \in GL_l$, let $t \in H^\circ$ be the image under this isomorphism. The lemma now follows

Table 3.4
$H \in \mathcal{C}_3$

G	H°	$\dim \Omega$	p	t'	$\dim t'^{H^\circ}$	$\dim t^G$	$f_\Omega(t)$
C_l	GL_l	l^2+l	$\neq 2$	$[-1, I_{l-1}]$	$2l-2$	$4l-4$	$1-\frac{2(l-1)}{l(l+1)}$
			2	$[J_2, I_{l-2}]$	$2l-2$	$4l-4$	$1-\frac{2(l-1)}{l(l+1)}$
D_l	GL_l	l^2-l	$\neq 2$	$[-1, I_{l-1}]$	$2l-2$	$4l-4$	$1-\frac{2}{l}$
			2	$[J_2, I_{l-2}]$	$2l-2$	$4l-6$	$1-\frac{2l-4}{l(l-1)}$

from Table 3.4. Note that when p is even, in both cases above the listed involution t is G-conjugate to a_2.

Observe that apart from the case $G = D_l,\ p \neq 2$, we always have $f_\Omega(t) \geqslant 2/3$ for each involution t in the table. However, as stated in the definition of the subgroup collection $\mathcal{C}_3$, if $G = D_l$ and l is odd, then the corresponding subgroup H is not maximal in G. Thus, we can ignore the case $G = D_3$, and conclude that a lower bound of at least $1/2$ holds when $G = D_l$ and $p \neq 2$. □

Lemma 3.4. *If $H \in \mathcal{C}_4$ then the conclusion of Theorem* 1 *is true.*

Proof. We begin with some preliminary remarks on notation. If $G = Cl(V)$ and $V = \bigotimes_{i=1}^k V_i$ then we use $\bigotimes^k Cl(V_i)$ to denote the central product of classical groups, $Cl(V_1) \circ \cdots \circ Cl(V_k)$, acting naturally on the tensor product. Similarly, if $V = V_1 \otimes V_2 \otimes \cdots \otimes V_2$ (k factors), we adopt the notation $t = t_1 \otimes^{k-1} t_2$ to represent the element of G which acts naturally on the tensor product as t_1 on V_1 and t_2 on each subspace V_2.

We have Tables 3.5–3.8, where as in Lemma 3.2, A denotes the 2×2 matrix interchanging the standard basis vectors.

In each case, the central product acts naturally on the tensor product, so the action of a given involution t on the natural G-module V is easy to calculate, from which $\dim t^G$ follows in the usual way. One should note that if $G = D_l,\ p = 2$, and $H^\circ =$

Table 3.5
$H \in \mathcal{C}_4,\ G = A_{n-1}$

H°	$\dim \Omega$	p	t	$\dim t^{H^\circ}$	$\dim t^G$	$f_\Omega(t) \geqslant$
$SL_a \otimes SL_b$	n^2-a^2	$\neq 2$	$[-j, jI_{a-1}] \otimes I_b$	$2a-2$	$2nb-2b^2$	$1-\frac{2}{a+1}$
$a \geqslant b,\ a > 2$	$-b^2+1$	2	$[J_2, I_{a-2}] \otimes I_b$	$2a-2$	$2nb-2b^2$	$1-\frac{2}{a+1}$
$SL_2 \otimes SL_2$	9	$\neq 2$	$[-i, i] \otimes [-i, i]$	4	8	$\frac{5}{9}$
		2	$A \otimes A$	4	8	$\frac{5}{9}$
$\bigotimes^k SL_a$	n^2-1	$\neq 2$	$[-j, jI_{a-1}] \bigotimes^{k-1} I_a$	$2a-2$	$2n^2/a - 2n^2/a^2$	$\frac{1}{2}$
$a > 2$	$-ka^2+k$	2	$[J_2, I_{a-2}] \bigotimes^{k-1} I_a$	$2a-2$	$2n^2/a - 2n^2/a^2$	$\frac{1}{2}$
$\bigotimes^k SL_2$	n^2-1	$\neq 2$	$\bigotimes^k [-i, i]$	$2k$	$n^2/2$	$\frac{1}{2}$
	$-3k$	2	$\bigotimes^k A$	$2k$	$n^2/2$	$\frac{1}{2}$

Table 3.6
$H \in \mathcal{C}_4,\ G = B_l$

H°	$\dim \Omega$	p	t	$\dim t^{H^\circ}$	$\dim t^G$	$f_\Omega(t) \geqslant$
$SO_a \otimes SO_b$	$2l^2 + l$	$\neq 2$	$[-I_2, I_{a-2}] \otimes I_b$	$2a - 4$	$4lb + 2b$	$\frac{1}{2} - \frac{1}{2(h+1)}$
$a \geqslant b$	$-a^2/2 + a/2$				$-4b^2$	
	$-b^2/2 + b/2$					
$\bigotimes^k SO_a$	$2l^2 + l$	$\neq 2$	$[-I_2, I_{a-2}] \bigotimes^{k-1} I_a$	$2a - 4$	$2(2l+1)^2/a$	$\frac{1}{2}$
	$-ka^2/2 + ka/2$				$-4(2l+1)^2/a^2$	

Table 3.7
$H \in \mathcal{C}_4,\ G = C_l$

H°	$\dim \Omega$	p	t	$\dim t^{H^\circ}$	$\dim t^G$	$f_\Omega(t) \geqslant$
$Sp_{2a} \otimes SO_b$	$2l^2 + l$	$\neq 2$	$[-I_2, I_{2a-2}] \otimes I_b$	$4a - 4$	$4lb - 4b^2$	$\frac{1}{2}$
$b \leqslant 2a$	$-2a^2 - a$					
$a > 1$	$-b^2/2 + b/2$					
$Sp_2 \otimes SO_2$	6	$\neq 2$	$[-i, i] \otimes I_2$	2	4	$\frac{2}{3}$
$Sp_{2a} \otimes SO_b$	$2l^2 + l$	$\neq 2$	$I_{2a} \otimes [-I_2, I_{b-2}]$	$4b - 4$	$8la - 16a^2$	$\frac{1}{2}$
$b > 2a$	$-2a^2 - a$					
	$-b^2/2 + b/2$					
$\bigotimes^k Sp_{2a}$	$2l^2 + l$	$\neq 2$	$[-I_2, I_{2a-2}] \bigotimes^{k-1} I_{2a}$	$4a - 4$	$4l^2/a - 4l^2/a^2$	$\frac{1}{2}$
$a > 1$	$-2ka^2 - ka$					
$\bigotimes^k Sp_2$	$2l^2 + l$	$\neq 2$	$\bigotimes^k [-i, i]$	$2k$	l^2	$\frac{1}{2}$
	$-3k$					

$Sp_{2a} \otimes Sp_{2b}$ then $t = [J_2, I_{2a-2}] \otimes I_{2b}$ is G-conjugate to a_{2b}. Similarly, the element $[J_2, I_{2a-2}] \otimes^{k-1} I_{2a} \in \bigotimes^k Sp_{2a} < D_l$ is G-conjugate to $a_{l/a}$.

In almost all cases, we choose involutions of the form $t = s \otimes^{k-1} I_a$, where s is an involution in $Cl(V_1)$. In such cases, we calculate $\dim t^{H^\circ}$ by observing that $C_{H^\circ}(t) \cong C_{Cl(V_1)}(s)$. If $G = A_{n-1}$ or C_l and $t = \bigotimes^k [-i, i]$, then it is not difficult to check that $C_{H^\circ}(t)$ is isomorphic to the torus T_k. Similarly, if $G = A_{n-1}$ or D_l and $t = \bigotimes^k A$, then $C_{H^\circ}(t) \cong T_k$. In both cases, these observations provide us with the tabulated values for $\dim t^{H^\circ}$.

Finally, we need to make some remarks on the lower bounds stated in the last column of each table. Using the calculated data and Proposition 2.14, we obtain a lower bound for $\dim C_\Omega(t)$. Dividing by $\dim \Omega$, one can check that for all possible values of n, l, a, b or k, the resulting expression is always at least the stated bound in the table. This is clearly sufficient to establish that Theorem 1 holds in this case. However, it should be noted that we can obtain much stronger bounds than those stated as the rank of G increases. For example, consider the case $G = SL_n$ and $H = N_G(SL_a \otimes \cdots \otimes SL_a)$ (k factors), so $n = a^k$. Assuming $a \geqslant 3$ and referring to the data in Table 3.5, we have

$$f_\Omega(t) \geqslant 1 - \frac{2n^2a - 2n^2 - 2a^3 + 2a^2}{n^2a^2 - a^2 - ka^4 + ka^2}.$$

Table 3.8
$H \in \mathcal{C}_4,\ G = D_l$

H°	$\dim \Omega$	p	t	$\dim t^{H^\circ}$	$\dim t^G$	$f_\Omega(t) \geqslant$
$Sp_{2a} \otimes Sp_{2b}$	$2l^2 - l - 2a^2$	$\neq 2$	$[-I_2, I_{2a}] \otimes I_{2b}$	$4a - 4$	$8lb - 16b^2$	$\frac{1}{2}$
$a \geqslant b$	$-a - 2b^2 - b$		$[-iI_a, iI_a]^{\diamondsuit} \otimes I_{2b}$	$a^2 + a$	$l^2 - l$	
		$= 2$	$[J_2, I_{2a-2}] \otimes I_{2b}$	$2a$	$4lb - 4b^2 - 2b$	$\frac{1}{2}$
$SO_a \otimes SO_b$	$2l^2 - l$	$\neq 2$	$[-I_2, I_{a-2}] \otimes I_b$	$2a - 4$	$4lb - 4b^2$	$\frac{1}{2} - \frac{1}{2(h+1)}$ †
$a \geqslant b$	$-a^2/2 + a/2$					
	$-b^2/2 + b/2$					
$\bigotimes^k Sp_{2a}$	$2l^2 - l$	$\neq 2$	$[-I_2, I_{2a-2}] \bigotimes^{k-1} I_{2a}$	$4a - 4$	$4l^2/a - 4l^2/a^2$	$\frac{1}{2}$
$a > 2$	$-2a^2k - ak$					
$\bigotimes^k Sp_4$	$2l^2 - l - 10k$	$\neq 2$	$[-iI_2, iI_2]^{\diamondsuit} \bigotimes^{k-1} I_4$	6	$l^2 - l$	$\frac{1}{2} - \frac{1}{2(h+1)}$
$\bigotimes^k Sp_2$	$2l^2 - l - 3k$	$\neq 2$	$\bigotimes^k [-i, i]$	$2k$	l^2	$\frac{1}{2} - \frac{1}{2(h+1)}$
		$= 2$	$\bigotimes^k A$	$2k$	$\leqslant l^2$	$\frac{1}{2} - \frac{1}{2(h+1)}$
$\bigotimes^k Sp_{2a}$	$2l^2 - l$	$= 2$	$[J_2, I_{2a-2}] \bigotimes^{k-1} I_{2a}$	$2a$	$2l^2/a - l^2/a^2$	$\frac{1}{2}$
$a > 1$	$-2a^2k - ak$				$-l/a$	
$\bigotimes^k SO_a$	$2l^2 - l$	$\neq 2$	$[-I_2, I_{a-2}] \bigotimes^{k-1} I_a$	$2a - 4$	$8l^2/a - 16l^2/a^2$	$\frac{1}{2}$
$a \neq 2, 4$	$-ka^2/2 + ka/2$					

† For the given t, we can establish this bound except when $(a, b) = (4, 3)$ or $(4, 4)$. In these cases, let $t = [-iI_2, iI_2]^{\diamondsuit} \otimes I_b$ to obtain $f_\Omega(t) \geqslant 1/2$.

One can now check that this expression is always greater than $1/2$, and in fact, it is easy to check that for a fixed $a \geqslant 3$, this expression tends (from below) to $1 - (2a - 2)/a^2$, as the rank of G tends to infinity. □

Lemma 3.5. *If $H \in \mathcal{C}_5$ then the conclusion of Theorem* 1 *is true.*

Proof. This follows immediately from Proposition 2.15. □

Lemma 3.6. *If $H \in \mathcal{C}_6$ then the conclusion of Theorem* 1 *is true.*

Proof. Using Propositions 2.9 and 2.10, we have Table 3.9. Note that in all cases, the matrices are written with respect to the usual basis ordering corresponding to the appropriate non-degenerate form on the natural G-module.

To complete the proof of Theorem 1 when $H \in \mathcal{C}_6$, we need to look at the cases $(G, H^\circ) = (SL_4, Sp_4)$ when $p \neq 2$, and (SL_2, SO_2). For the latter case, one easily observes that $N_G(SO_2)$ is the subgroup of monomial matrices in SL_2, so this has been dealt with in Lemma 3.2. For the other case, let $t = [-iI_2, iI_2]^{\diamondsuit} \in Sp_4$ to obtain $f_\Omega(t) \geqslant 3/5$. □

This completes the proof of Theorem 1 in the case where G is classical and the maximal subgroup H lies in one of the classes $\mathcal{C}_i$. In the next section, we complete the classical case by considering the situation where H is maximal in G, but not a member of some $\mathcal{C}_i$.

Table 3.9
$H \in \mathcal{C}_6$

G	H°	$\dim \Omega$	p	t	$\dim t^{H^\circ}$	$\dim t^G$	$f_\Omega(t) \geqslant$
A_l	Sp_{l+1}	$l^2/2 + l/2 - 1$	$\neq 2$	$[-I_2, I_{l-1}]$	$2l-2$	$4l-4$	$1 - \frac{4}{l+2}$
			$= 2$	$[J_2, I_{l-1}]$	$l+1$	$2l$	$1 - \frac{2}{l+2}$
	SO_{l+1}	$l^2/2 + 3l/2$	$\neq 2$	$[-I_l, 1]$	l	$2l$	$1 - \frac{2}{l+3}$
	$l = 2m$						
	SO_{l+1}	$l^2/2 + 3l/2$	$\neq 2$	$[-I_2, I_{l-1}]$	$2l-2$	$4l-4$	$1 - \frac{4l-4}{l^2+3l}$
	$l = 2m+1$		$= 2$	$[J_2^2, I_{l-3}]$	$2l-2$	$4l-4$	$1 - \frac{4l-4}{l^2+3l}$
C_l	SO_{2l}	$2l$	$= 2$	$[J_2^2, I_{2l-4}]$	$4l-4$	$4l-2$	$1 - \frac{1}{l}$

4. Proof of Theorem 1, Part II: G classical, $H \notin \mathcal{C}(G)$

According to Theorem 2.1, if H is maximal in $G = Cl(V)$ but not a member of some $\mathcal{C}_i$, then $E(H)$ is simple and acts irreducibly on V. Of course, if $E(H)$ is finite then so is H and in this case Theorem 1 follows from Proposition 2.15. Hence we can assume that $E(H)$ is connected. If $E(H)$ is a classical group, say $E(H) \cong Cl(U)$, we shall adopt the following general strategy.

Using Proposition 2.11, we choose an involution $t \in Cl(U)$ so that $\dim t^{Cl(U)}$ is as large as possible. Now if $\varphi : Cl(U) \to G$ is an irreducible representation such that $\mathrm{Im}(\varphi) = E(H)$, then $\varphi(t)$ is also an involution and $\dim \varphi(t)^{E(H)} = \dim t^{Cl(U)}$. From Proposition 2.11 we obtain an upper bound for $\dim \varphi(t)^G$, and thus a lower bound for $\dim C_\Omega(t)$ in the usual manner. If $\dim U = d$ and $\dim V = n$ then from this lower bound we obtain a function $f(n,d)$ with the property that if $f(n,d) \geqslant 0$ then

$$f_\Omega(t) \geqslant \frac{1}{2} - \frac{1}{2(h+1)}.$$

Using Lübeck's results [17], we can show that in almost all cases, either $f(n,d) \geqslant 0$ is true, or otherwise, in those cases when the inequality fails to hold, n is small and we can explicitly calculate with the representation φ to establish a lower bound of $1/2 - 1/2(h+1)$. However, there are examples where we are forced to accept the slightly weaker bound of $1/2 - 1/(2h+1)$, with two further exceptional cases, namely $(G, E(H)) = (D_4, A_2)$ and (B_3, A_2).

If $E(H)$ is exceptional, then we shall choose t as in Proposition 2.12 to maximise $\dim t^{E(H)}$ and apply the same strategy, again utilising Lübeck's results. In this case, we are able to establish that the bound of $1/2 - 1/2(h+1)$ holds without exception. Clearly, this is sufficient to complete the proof of Theorem 1 when G is classical.

Remark 4.1. With reference to Remark 2.16, if $E(H)$ is finite then we can demonstrate that the bound in Theorem 1 is close to best possible. For example, suppose $E(H) \cong PSL_2(q)$, where q is a power of some odd prime. Then the irreducible Steinberg representation ψ embeds $PSL_2(q)$ in SO_q. Furthermore, ψ maps the unique class of involutions in $PSL_2(q)$

to the class of involutions in SO_q of largest dimension. Using Proposition 2.11, we deduce that $f_\Omega(t) = 1/2 - 1/2(h+1)$. When $E(H) = A_{2m}$ and $G = B_l$, C_l or D_l, we have been unable to establish a lower bound of $1/2 - 1/2(h+1)$. However, with two exceptions, $1/2 - 1/(2h+1)$ does hold, and it is in this sense that Theorem 1 can be described as being close to best possible.

4.1. $E(H)$ classical

Applying the strategy described above, we obtain the results in Table 4.1, where d and n denote the respective dimensions of the natural modules for $E(H)$ and G.

Table 4.1
$E(H)$ is classical

G	$E(H)$	$\dim \Omega$	$\dim \varphi(t)^{E(H)} \geqslant$	$\dim \varphi(t)^G \leqslant$	$f(n,d)$
A_l	A_r	$n^2 - d^2$	$d^2/2 - 1/2$	$n^2/2$	$n^2 - d^2$ $-n - 1$
	B_r	$n^2 - d^2/2$ $+d/2 - 1$	$d^2/4 - 1/4$	$n^2/2$	$2n^2 - 3n - d^2$ $+nd + 2d - 5$
	C_r	$n^2 - d^2/2$ $-d/2 - 1$	$d^2/4 + d/2$	$n^2/2$	$2n^2 + nd$ $-d^2 - 2n - 4$
	D_r	$n^2 - d^2/2$ $+d/2 - 1$	$d^2/4 - 1$	$n^2/2$	$2n^2 - 6n - d^2$ $+nd + 2d - 8$
B_l	A_r	$n^2/2 - n/2$ $-d^2 + 1$	$d^2/2 - 1/2$	$n^2/4 - 1/4$	$1 - d^2$
	B_r	$n^2/2 - n/2$ $-d^2/2 + d/2$	$d^2/4 - 1/4$	$n^2/4 - 1/4$	$nd - d^2$ $-n + d$
	C_r	$n^2/2 - n/2$ $-d^2/2 - d/2$	$d^2/4 + d/2$	$n^2/4 - 1/4$	$n - d - 1$
	D_r	$n^2/2 - n/2$ $-d^2/2 + d/2$	$d^2/4 - 1$	$n^2/4 - 1/4$	$nd - d^2$ $-4n + d$
C_l	A_r	$n^2/2 + n/2$ $-d^2 + 1$	$d^2/2 - 1/2$	$n^2/4 + n/2$	$1 - d^2$
	B_r	$n^2/2 + n/2$ $-d^2/2 + d/2$	$d^2/4 - 1/4$	$n^2/4 + n/2$	$nd - d^2$ $+2d - n - 1$
	C_r	$n^2/2 + n/2$ $-d^2/2 - d/2$	$d^2/4 + d/2$	$n^2/4 + n/2$	$n - d$
	D_r	$n^2/2 + n/2$ $-d^2/2 + d/2$	$d^2/4 - 1$	$n^2/4 + n/2$	$nd - d^2$ $+2d - 4n - 4$
D_l	A_r	$n^2/2 - n/2$ $-d^2 + 1$	$d^2/2 - 1/2$	$n^2/4$	$1 - d^2$
	B_r	$n^2/2 - n/2$ $-d^2/2 + d/2$	$d^2/4 - 1/4$	$n^2/4$	$nd - d^2$ $-n + 1$
	C_r	$n^2/2 - n/2$ $-d^2/2 - d/2$	$d^2/4 + d/2$	$n^2/4$	$n - d - 2$
	D_r	$n^2/2 - n/2$ $-d^2/2 + d/2$	$d^2/4 - 1$	$n^2/4$	$nd - d^2$ $-4n + 4$

Leaving the cases $(G, E(H)) = (B_l, A_r)$, (B_l, D_r), (C_l, A_r), (C_l, D_r), (D_l, A_r), and (D_l, D_r) for now, one easily checks that for the remaining cases we always have $f(n,d) \geqslant 0$. This follows immediately from the lower bounds on n which arise naturally from the dimensional constraints. For example, if $(G, E(H)) = (A_{n-1}, A_{d-1})$ then we must have $n \geqslant d+1$, which implies that $f(n,d) = n^2 - d^2 - n - 1 \geqslant 0$.

Now consider the case $G = C_l$ and $E(H) = A_r$. If d is even, then following Proposition 2.11 we can choose our involution t so that $\dim t^{E(H)} = d^2/2$ and thus $f(n,d) = n - d^2 + 2$. According to Proposition 2.3 and Remark 2.4, the only self-dual irreducible representation of SL_d with d even, embedding SL_d in Sp_n and satisfying $n < d^2 - 2$ is the 20-dimensional SL_6-module $V = \bigwedge^3 U$ when $p \neq 2$, where U is the natural SL_6-module. If $t = [-I_2, I_4] \in SL_6$, then the action of t on V is given by $[-I_{12}, I_8]$ and hence $f_\Omega(t) \geqslant 19/35$. When d is odd, $f(n,d) = 1 - d^2$ and we are forced to consider the slightly weaker lower bound of $1/2 - 1/(2h+1)$. For this to hold, we require $n \geqslant 4d^2 - 4$. In view of Proposition 2.5, this leaves the adjoint representation with which we can calculate explicitly. To be precise, let $t = [-I_{d-1}, 1] \in SL_d$ if $p \neq 2$, and $t = [J_2, I_{d-2}]$ if $p = 2$. Then $\mathrm{Ad}(t) = [-I_{2d-2}, I_{d^2-2d+1}]$ and $[J_2^{2d-2}, I_{d^2-4d+3}]$, respectively. In both cases, we have $\dim \mathrm{Ad}(t)^{E(H)} = 2d - 2$ and $\dim \mathrm{Ad}(t)^G = 2d^3 - 6d^2 + 6d - 2$, and from this one can easily deduce that $f_\Omega(\mathrm{Ad}(t)) \geqslant 1/2$.

If $G = C_l$ and $E(H) = D_r$, then $n \geqslant d(d-1)/2 - 2$ (see [17, Theorem 5.1]). When $d \geqslant 8$, this is sufficient to imply that $f(n,d) \geqslant 0$. Since $f(n,6) = 2n - 28$ and $SO_6 \cong PSL_4$, it follows from Proposition 2.3 that $f(n,6) \geqslant 0$, so a lower bound of $1/2 - 1/2(h+1)$ holds in this case. Similarly, if $G = D_l$ and $E(H) = D_r$, we use [17, Theorem 5.1] to show that $f(n,d) \geqslant 0$ when $d \geqslant 8$. Now, $f(n,6) = 2n - 32$, so Proposition 2.3 leaves us to deal with the adjoint representation. A direct calculation shows that a lower bound of $1/2 - 1/2(h+1)$ also holds in this case. Similar reasoning deals with the case $G = B_l$, $E(H) = D_r$.

Suppose now that $G = B_l$ and $E(H) = A_r$. As in the $G = C_l$ case, if d is even then we can use Proposition 2.3 to show that a lower bound of $1/2 - 1/2(h+1)$ holds, and when d is odd, we are forced to consider $1/2 - 1/(2h+1)$. For this to hold, we require $n \geqslant 4d^2 - 3$, and using Proposition 2.5, the only odd-dimensional self-dual irreducible representation for which this fails is the adjoint representation when p divides d. Calculating explicitly, we see that a bound of $1/2 - 1/2(h+1)$ holds when $d \geqslant 5$. However, if $d = 3$ and $p = 3$, then the adjoint representation embeds SL_3 in SO_7 and as stated in Table 1.1, a direct calculation shows that the best lower bound is $5/13 < 1/2 - 1/(2h+1) = 11/26$. Similarly, if $G = D_l$ and $E(H) = A_r$ then $1/2 - 1/2(h+1)$ holds when d is even, and when d is odd we use Propositions 2.3, 2.5, and a direct calculation with the adjoint representation to establish a lower bound of $1/2 - 1/(2h+1)$, with the exception of the case $E(H) = SL_3$ and $G = SO_8$, $p \neq 3$. Here the adjoint representation maps the unique class of involutions in SL_3 to the class in SO_8 of largest dimension, so $f_\Omega(t) = 2/5 < 1/2 - 1/(2h+1) = 11/26$ for any involution t in SL_3. This exceptional case is recorded in Table 1.1.

This completes the proof of Theorem 1 when H is not in $\mathcal{C}(G)$ and $E(H)$ is classical.

4.2. $E(H)$ exceptional

We now complete the proof of Theorem 1 when G is classical, employing the same strategy as in Section 4.1. Following Proposition 2.12, we choose $t \in E(H) = H^\circ$ to maximise $\dim t^{E(H)}$, and obtain a condition of the form $n \geqslant c$ which is sufficient to imply that $f_\Omega(t) \geqslant 1/2 - 1/2(h+1)$, where as before, n denotes the dimension of the natural G-module. If $G = A_l$ then it is clear that for this choice of t we have $f_\Omega(t) \geqslant 1/2$. For example, if $E(H) = E_7$, choose $t \in E(H)$ such that $\dim t^{E(H)} = 70$. Since $\dim t^G \leqslant (l+1)^2/2$ (see Proposition 2.11), we have $f_\Omega(t) \geqslant 1/2$.

For the other types of G, using Propositions 2.12 and 2.11, we derive the following values of c:

G	$E(H) = E_6$	E_7	E_8	F_4	G_2
B_l	39	19	31	13	7
C_l	38	18	30	12	6
D_l	40	20	32	14	8

Recalling that we only consider the case $G = B_l$ when $p \neq 2$, it follows immediately from Proposition 2.8 that we have $f_\Omega(t) \geqslant 1/2 - 1/2(h+1)$ when $G = B_l$ or C_l. If $G = D_l$ then from Proposition 2.8 and [17, Table A.49], we deduce that we need only consider the irreducible embedding $G_2 \hookrightarrow SO_6$ ($p = 2$). According to Proposition 2.11, if $t \in SO_6$ is an involution then $\dim t^{SO_6} \leqslant 8$. So if $t \in G_2$ is an involution such that $\dim t^{G_2} = 8$, then $\dim t^{SO_6} = 8$ and $f_\Omega(t) = 1$.

This completes the proof of Theorem 1 when G is a classical simple algebraic group. In the final section, we turn our attention to the exceptional groups.

5. Proof of Theorem 1, Part III: *G* exceptional

In this final section we consider the case where G is a simple algebraic group of exceptional type, and in doing so we complete the proof of Theorem 1. As in the classical case, the key result is the Liebeck/Seitz classification of the maximal subgroups of positive dimension (Theorem 2.2). In even characteristic, things are greatly simplified by the use of long root involutions. When $p \neq 2$, our initial strategy is to choose fundamental involutions where possible (see Table 2.3). For those cases for which this method fails, we will need to make alternative choices for t, and work harder to identify the centraliser $C_G(t)$, in some cases invoking results concerning the representation theory of the simple exceptional groups. We begin by dealing with the maximal parabolic subgroups. Throughout this section, G will always denote a simple algebraic group of exceptional type.

Lemma 5.1. *If H is a maximal parabolic subgroup, then the conclusion of Theorem* 1 *is true.*

Proof. Let $H = P_i = Q_i L_i$ be a maximal parabolic subgroup of G, and assume to begin with that $p = 2$. If $L_i' = X_1 X_2 \cdots X_s$, with each X_l simple, let t be a long root involution in the simple factor X_j, where the rank of X_j is maximal. Apart from the cases (F_4, P_2) and (G_2, P_1), this is always possible. Since $\dim t^{L_i} = \dim t^{X_j}$, we can easily calculate $\dim C_{\Omega_i}(t)$ via Propositions 2.13 and 2.20, where $\Omega_i = G/P_i$.

For example, suppose $G = E_7$ and $H = P_4$. Since $L_4 = T_1A_1A_2A_3$ is a Levi factor of P_4, it follows from Proposition 2.17 that $\dim \Omega_4 = 53$. Following the described method, let t be a long root involution in the A_3 factor of L_4, so from Proposition 2.13 we have $\dim t^G = 34$ and $\dim t^{L_4} = 6$. Proposition 2.20 now implies that $\dim C_{\Omega_4}(t) = 53 - 14 = 39$, and it is clear that Theorem 1 holds in this particular case.

In fact, with the exception of the two cases mentioned above, this method yields bounds which are always greater than 1/2. To deal with (F_4, P_2), we need to slightly adapt the method since $L_2' = A_1A_2$, and the fundamental roots of the highest rank A_2 factor are both short. However, by choosing a long root involution t in the A_1 factor, we have $\dim t^{L_2} = 2$ and $f_\Omega(t) \geqslant 13/20$. This leaves the case (G_2, P_1). Since P_1 contains a Borel subgroup it follows that $u^G \cap P_1 \neq \emptyset$ for any unipotent element $u \in G$. Hence if s is a long root involution in G then there exists some $g \in G$ such that $t = s^g \in P_1$. So $\dim t^G = 6$ and via Proposition 2.18, we have $\dim(t^G \cap P_1) \geqslant 3$ and hence $f_{\Omega_1}(t) \geqslant 2/5$. Since this is the best lower bound that we can obtain, and $2/5 < 11/26 = 1/2 - 1/(2h+1)$, this case is included in Table 1.1.

Now suppose that $p \neq 2$. If $L_i' = X_1X_2 \cdots X_s$, where each X_l is simple, let t be a fundamental involution in X_j, where the rank of X_j is maximal. Referring to Table 2.3, this gives us $C_{L_i}(t)$ and $D = C_G(t)$, and so via Proposition 2.19, we derive a lower bound for $\dim C_{\Omega_i}(t)$. For example, consider (E_8, P_6), where $L_6 = T_1D_5A_2$ and $\dim \Omega_6 = 97$. Let $t \in D_5$ be a fundamental involution, so from Table 2.3 we have $C_{D_5}(t) = A_1^2A_3$ and hence $C_{L_6}(t) = T_1A_1^2A_2A_3$ and $D = C_{E_8}(t) = A_1E_7$. Since $|\Sigma^+(A_1E_7)| = 64$ and $|\Sigma^+(C_{L_6}(t))| = 11$, it follows from Proposition 2.19 that $\dim C_{\Omega_6}(t) \geqslant 53$. It is readily checked that this method provides lower bounds in excess of $1/2 - 1/2(h+1)$ in almost all cases, the exceptions being (E_7, P_4), (E_6, P_i), $2 \leqslant i \leqslant 5$, (F_4, P_j), $1 \leqslant j \leqslant 3$, and when $G = G_2$. From Proposition 2.12 we know that G_2 contains a unique class of involutions with centraliser $D = A_1^2$, so $|\Sigma^+(D)| = 2$. Since $\dim \Omega_i = 5$ for $i = 1, 2$, it follows from Proposition 2.19 that if $t = t_1t_2 \in L_i = T_1A_1$, where $t_1 = t_2 = [-i, i]$, then $f_{\Omega_i}(t) \geqslant 2/5 < 11/26 = 1/2 - 1/(2h+1)$. This case is recorded in Table 1.1 of Theorem 1(iii).

We now deal with the other cases for which our initial method failed. Let $(G, H) = (E_6, P_3)$, so $\dim \Omega_3 = 25$ and $L_3 = T_1A_1A_4$. Let $t = [-I_4, 1] \in A_4 < L_3$; then $|\Sigma^+(C_{L_3}(t))| = 7$. Now A_4 naturally embeds in D_5. Viewing t as an element of D_5, we have $t = [-I_8, I_2]$, so $D_4 < C_{D_5}(t) < C_G(t)$. Since D_4 is not contained in A_1A_5, it follows from Proposition 2.12 that $D = C_G(t) = T_1D_5$, so $|\Sigma^+(D)| = 20$, and $\dim C_{\Omega_3}(t) \geqslant 13$. The case (E_6, P_5) is identical, and (E_6, P_2) is dealt with in a similar way. To be precise, if $t = [-I_4, I_2] \in A_5 < L_2$ then viewing t as an element of A_4 and arguing as before, we deduce that $C_G(t) = T_1D_5$, and hence $f_{\Omega_2}(t) \geqslant 13/21$.

For (E_6, P_4) we have $\dim \Omega_4 = 29$ and $L_4 = T_1A_1A_2^2$. Let $t = t_1t_2 \in A_2^2 < L_4$, where $t_1 = t_2 = [-I_2, 1] \in A_2$, so $|\Sigma^+(C_{L_4}(t))| = 3$. To show that $C_G(t) = T_1D_5$, we consider the restriction to A_2^3 of the 27-dimensional irreducible E_6-module $V_{27} = M(\lambda_1)$.

According to [13, Proposition 2.3],

$$V_{27} \downarrow A_2^3 = \big(M(\lambda_1) \otimes M(\lambda_2) \otimes 0\big) \oplus \big(M(\lambda_2) \otimes 0 \otimes M(\lambda_1)\big) \oplus \big(0 \otimes M(\lambda_1) \otimes M(\lambda_2)\big),$$

where λ_1 and λ_2 are fundamental dominant weights of A_2, and 0 denotes the trivial A_2-module. Let $t = t_1t_2t_3 \in A_2^3$, where $t_1 = I_3$ and $t_2 = t_3 = [-I_2, 1]$. Using the above decomposition, one easily shows that the action of t on V_{27} is given by $[-I_{16}, I_{11}]$. Now from [13, Proposition 2.3] we have

$$V_{27} \downarrow A_1A_5 = \big(M(\lambda_1) \otimes M(\lambda_1)\big) \oplus \big(0 \otimes M(\lambda_4)\big),$$

and from [13, Table 8.7],

$$V_{27} \downarrow D_5 = M(\lambda_1) \oplus M(\lambda_4) \oplus 0.$$

Hence if $t \in E_6$ is an involution and $C_{E_6}(t) = A_1A_5$, then (up to conjugacy), t acts on V_{27} as $[-I_{12}, I_{15}]$. On the other hand, if $C_{E_6}(t) = T_1D_5$, the action is given by $[-I_{16}, I_{11}]$. We conclude that $C_G(t) = T_1D_5$ and $f_\Omega(t) \geqslant 17/29$.

For (F_4, P_i), $i = 1, 2$, let $t \in P_i$ be the involution in $\langle U_{\alpha_3}, U_{-\alpha_3}\rangle \cong SL_2$. Viewing t in the B_4 subsystem subgroup of F_4, we deduce from Chevalley's commutator relations that t centralises a D_3 subgroup. Since $D_3 \not< A_1C_3$, it follows from Proposition 2.12 that $C_G(t) = B_4$. In the usual manner, we calculate that $f_{\Omega_1}(t) \geqslant 7/15$ and $f_{\Omega_2}(t) \geqslant 3/5$. To deal with (F_4, P_3), we let t be the involution in $\langle U_{\alpha_4}, U_{-\alpha_4}\rangle$. Then $t \in P_3$ and since α_3 and α_4 are of equal length, it follows from the above work that $C_G(t) = B_4$. From this we deduce that $f_{\Omega_3}(t) \geqslant 3/5$.

Finally, we consider the case (E_7, P_4). Let $t = t_1t_2 \in A_2A_3 < L_4 = T_1A_1A_2A_3$, where $t_1 = [-I_2, 1]$ and $t_2 = [-I_2, I_2]$. Clearly we can view t as an element of the subgroup $A_2^2 < A_2A_3$. Since $|Z(A_2^2)| = 3^2$, it follows that t lifts to an involution in $\widehat{E}_7$, the simply connected cover. From [11, 1.2], we know that an involution in E_7 which lifts to an involution in $\widehat{E}_7$ must have centraliser A_1D_6. Thus, $|\Sigma^+(C_{E_7}(t))| = 31$ and $f_{\Omega_4}(t) \geqslant 27/53$. □

Following Theorem 2.2(b), we now consider the case where $H = N_G(M)$ is a maximal reductive subgroup of maximal rank.

Lemma 5.2. *If $H = N_G(M)$, with M as in Table* 2.1, *then the conclusion of Theorem* 1 *is true.*

Proof. If $p = 2$, let $t \in X$ be a long root involution, where X is a simple factor of $M = H^\circ$ of largest possible rank. Using Proposition 2.13, we obtain $\dim t^G$ and $\dim t^X$, giving rise to a lower bound for $\dim C_\Omega(t)$. For example, suppose $G = E_8$ and $M = A_2E_6$, so $\dim \Omega = 162$. If $t \in E_6$ is a long root involution, then $\dim t^{E_6} = 22$ and $\dim t^G = 58$. Hence, $\dim C_\Omega(t) \geqslant 126$. This method yields lower bounds which are always greater than or equal to $1/2$, with the obvious exception of the cases $G = E_i$, $H = N_G(T_i)$, $6 \leqslant i \leqslant 8$, for which an alternative argument is required.

Let Φ denote the root system of $G = E_i$, and let $\alpha \in \Phi$. As previously remarked when discussing fundamental involutions, it is a basic fact that there exists an isomorphism

$$\psi : SL_2(K) \to \langle U_\alpha, U_{-\alpha} \rangle.$$

Following [3], let $n_\alpha = \psi\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. When $p = 2$, $n_\alpha \in N_G(T_i)$ is G-conjugate to a long root involution and we can use the results of Proposition 2.13. For example, if $G = E_8$ then $\dim \Omega = 240$, and choosing $t = n_\alpha$ for some root $\alpha \in \Phi$, we have $\dim t^G = 58$ and $\dim C_\Omega(t) \geqslant 182$.

When $p \neq 2$, we employ an analogous method to that used in the proof of Lemma 5.1. Choosing $t \in X$ to be a fundamental involution (where X is a simple factor of M of largest rank) yields lower bounds which are greater than or equal to $1/2$, with the exception of the following cases:

G	M
E_8	T_8
E_7	T_7, A_7, A_2A_5, A_1^7
E_6	T_6, A_1A_5, A_2^3
F_4	$A_1C_3, A_2\widetilde{A}_2$
G_2	$A_1\widetilde{A}_1, A_2, \widetilde{A}_2$ $(p = 3)$

If $(G, M) = (E_8, T_8)$, then $\dim \Omega = 240$ and since any semisimple element lies in a maximal torus, and all the maximal tori in G are conjugate, it is clear that we can choose an involution $t \in H$ such that $C_G(t) = A_1E_7$. So $\dim t^G = 112$ and $\dim C_\Omega(t) \geqslant 128$. Using the fact that each subgroup M is of maximal rank, in the same way we can deal with the cases (E_7, A_1^7), (E_7, T_7), (E_6, T_6) and $(F_4, A_2\widetilde{A}_2)$. To handle the remaining cases, we need to work harder.

Let $(G, M) = (E_7, A_7)$, and choose $t = [-I_4, I_4] \in A_7$, so $\dim \Omega = 70$ and $\dim t^H = 32$. It is clear that t lies in a subgroup of A_7 which is isomorphic to $A_6 = SL_7$. It now follows that t lifts to an involution in the simply connected group $\widehat{E}_7$, and as in Lemma 5.1, we deduce that $C_G(t) = A_1D_6$ and thus $\dim C_\Omega(t) = 38$. We handle the case (E_7, A_2A_5) in the same way. Let $t = t_1t_2 \in A_2^2 < A_2A_5$, where $t_1 = [-I_2, 1]$ and $t_2 = [-I_2, I_4]$. Since $|Z(A_2^2)| = 3^2$, it follows that the preimage of t in $\widehat{E}_7$ is also an involution and thus $f_\Omega(t) \geqslant 23/45$.

Suppose now that $(G, H) = (E_6, A_1A_5)$, and let $t = [-I_4, I_2] \in A_4 < A_5$. As in Lemma 5.1, A_4 embeds in D_5 and we have $D_4 < C_G(t)$, so $C_G(t) = T_1D_5$ and $f_\Omega(t) \geqslant 3/5$. If $t = t_1t_2t_3 \in A_2^3 < E_6$, where $t_1 = I_3$ and $t_2 = t_3 = [-I_2, 1]$, then from our work in Lemma 5.1 concerning (E_6, P_4), we know that $C_G(t) = T_1D_5$. This gives us a lower bound of 5/9 in this case. If $(G, H) = (F_4, A_1C_3)$, let $t = t_1t_2$, where $t_1 = [-i, i] \in A_1$ and $t_2 = [-iI_3, iI_3]^\diamond \in C_3$. Then $f_\Omega(t) \geqslant 1/2$. For $(G_2, A_1\widetilde{A}_1)$ we can establish a lower bound of 1/2 by choosing $t = t_1t_2$, where $t_1 = t_2 = [-i, i] \in A_1$.

Finally, we observe that for (G_2, A_2) and $(G_2, \widetilde{A}_2)$ there does not exist an involution $t \in M$ such that $f_\Omega(t) \geqslant 11/26 = 1/2 - 1/(2h + 1)$. Here we have $\dim \Omega = 6$ and $\dim t^G = 8$ for any involution $t \in G$. Since A_2 has a unique class of involutions, whose dimension is 4, it follows that 1/3 is the best possible lower bound in each case. This is recorded in Table 1.1. □

According to Theorem 2.2, when $p \neq 2$, $(2^2 \times D_4).S_3 < E_7$ is maximal; and $A_1 \times S_5 < E_8$ is maximal when $p \neq 2, 3, 5$. In the former case, $\dim \Omega = 105$ and if $t = [-I_4, I_4] \in D_4$ then $\dim t^{D_4} = 16$. Since $\dim t^G \leqslant 70$ we have $f_\Omega(t) \geqslant 17/35 > 35/74$. If $(G, H) = (E_8, A_1 \times S_5)$ then $\dim \Omega = 245$ and if $t = [-i, i] \in A_1 = PSL_2$, we have $\dim t^{H^\circ} = 2$ and thus $f_\Omega(t) \geqslant 119/245 > 59/122$ since $\dim t^G \leqslant 128$. Note that the A_1 factor here must be adjoint as otherwise H would be contained in a subgroup $C_G(t)$ for some involution t, contradicting the maximality of H.

To complete the proof of Theorem 1, we need to consider one final collection of maximal subgroups. As in Theorem 2.2(c), these are the subgroups $N_G(X)$, where X is as in Table 2.2.

Lemma 5.3. *If $H = N_G(X)$, with X as in Table 2.2, then the conclusion of Theorem 1 is true.*

Proof. Suppose $X = X_1 \cdots X_s$, where each X_i is simple. Since $N_G(X)$ is maximal, it is clear that for each i, $Z(X_i)$ cannot contain an involution t – if this were the case, we would have $N_G(X) < C_G(t)$, contradicting the maximality of $N_G(X)$. So in particular, if $p \neq 2$ and $X_i = A_1$ or C_l, then X_i must be adjoint. Let $t = t_1 \cdots t_s \in X$, where each $t_i \in X_i$ is an involution such that $\dim t_i^{X_i}$ is maximal. Using the upper bound for $\dim t^G$ from Proposition 2.12 we deduce that $f_\Omega(t) \geqslant 1/2 - 1/2(h+1)$, unless $G = E_7$, $p \geqslant 5$ and $X = A_2$. For example, suppose $G = E_8$ and $X = A_1 G_2^2$. Let $t = t_1 t_2 t_3$, where $t_1 = [-i, i]$ and $t_2 = t_3$ satisfies $\dim t_2^{G_2} = 8$. Hence $\dim t^X = 18$ and since $\dim \Omega = 217$ and $\dim t^G \leqslant 128$ we have $f_\Omega(t) \geqslant 107/217 > 15/31 = 1/2 - 1/2(h+1)$.

To deal with $(G, X) = (E_7, A_2)$, let $t = [-I_2, 1] \in X$. We claim that $C_G(t) = A_1 D_6$. To see this, consider the action of t on the 56-dimensional irreducible E_7-module $V_{56} = M(\lambda_7)$. According to [13, Table 8.6], if $p > 5$ then

$$V_{56} \downarrow A_2 = M(6\lambda_1) \oplus M(6\lambda_2),$$

and thus the action of t is given by $[-I_{24}, I_{32}]$. When $p = 5$, we see from [12, Table 10.2] that $V_{56} \downarrow A_2$ has the same composition factors as the A_2-module $S^6 V_3 \oplus (S^6 V_3)^*$ and hence the action of t is again given by $[-I_{24}, I_{32}]$. Now from [13, Proposition 2.3] we have

$$V_{56} \downarrow A_1 D_6 = \big(M(\lambda_1) \otimes M(\lambda_1)\big) \oplus \big(0 \otimes M(\lambda_5)\big),$$

so if $s \in E_7$ is an involution and $C_{E_7}(s) = A_1 D_6$, then (up to conjugacy), s acts on V_{56} as $[-I_{24}, I_{32}]$. We conclude that $C_G(t) = A_1 D_6$. Hence, $\dim t^G = 64$, and since $\dim t^X = 4$ and $\dim \Omega = 125$, we deduce that $f_\Omega(t) \geqslant 65/125 > 1/2$. □

This completes the proof of Theorem 1 in the case where G is a simple algebraic group of exceptional type. In view of the results of Sections 3 and 4, the proof of Theorem 1 is complete.

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