PERGAMON Computers and Mathematics with Applications 39 (2000) 57–69

An International Journal
computers & mathematics with applications

www.elsevier.nl/locate/camwa

Applications of Fractional Calculus to Parabolic Starlike and Uniformly Convex Functions

H. M. SRIVASTAVA
Department of Mathematics and Statistics, University of Victoria
Victoria, British Columbia V8W 3P4, Canada
harimsri@math.uvic.ca

A. K. MISHRA
Department of Mathematics, Berhampur University
Berhampur 760 007, Orissa, India

(Received July 1999; accepted August 1999)

Abstract—Let $\mathcal{A}$ be the class of analytic functions in the open unit disk $\mathcal{U}$. Given $0 \leq \lambda < 1$, let Ω^λ be the operator defined on $\mathcal{A}$ by

$$\left(\Omega^\lambda f\right)(z) = \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z),$$

where $D_z^\lambda f$ is the fractional derivative of f of order λ. A function f in $\mathcal{A}$ is said to be in the class SP_λ if $\Omega^\lambda f$ is a parabolic starlike function. In this paper, several basic properties and characteristics of the class SP_λ are investigated. These include subordination, inclusion, and growth theorems, class-preserving operators, Fekete-Szegő problems, and sharp estimates for the first few coefficients of the inverse function.

Keywords—Uniformly convex functions, Parabolic starlike functions, Fractional derivative, Subordination, Fekete-Szegő problems, Inverse functions.

1. INTRODUCTION

Let $\mathcal{A}$ be the class of functions analytic in the *open* unit disk

$$\mathcal{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and let $\mathcal{A}_0$ be the family of functions f in $\mathcal{A}$ satisfying the normalization condition (cf. [1]):

$$f(0) = f'(0) - 1 = 0.$$

A function f in $\mathcal{A}_0$ is said to be *uniformly convex* in $\mathcal{U}$ if f is a *univalent convex* function along with the property that, for every circular arc γ contained in $\mathcal{U}$, with centre ζ also in $\mathcal{U}$, the image

The present investigation was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353.

0898-1221/00/$ - see front matter Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX
PII: S0898-1221(99)00333-8

curve $f(\gamma)$ is a convex arc. The class of uniformly convex functions is denoted by UCV (for details, see [2]). It is well known from [3,4] that

$$f \in \mathrm{UCV} \Longleftrightarrow \left|\frac{zf''(z)}{f'(z)}\right| < \mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}, \qquad (z \in \mathcal{U}). \tag{1.1}$$

Condition (1.1) implies that

$$1+\frac{zf''(z)}{f'(z)}$$

lies in the interior of the parabolic region

$$\mathcal{R} := \left\{w : w = u+iv \text{ and } v^2 < 2u-1\right\}$$

for every value of $z \in \mathcal{U}$. Let

$$P := \{p : p \in \mathcal{A},\, p(0)=1 \text{ and } \mathrm{Re}\,\{p(z)\} > 0,\ (z \in \mathcal{U})\}$$

and

$$\mathrm{PAR} := \{p : p \in P \text{ and } p(\mathcal{U}) \subset \mathcal{R}\}.$$

A function $f \in \mathcal{A}_0$ is said to be in the class of *parabolic starlike functions*, denoted by SP (cf. [4]), if

$$\frac{zf'(z)}{f(z)} \in \mathcal{R}, \qquad (z \in \mathcal{U}).$$

Let

$$\varphi(a,c;z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}, \qquad (z \in \mathcal{U};\ c \neq 0,-1,-2,\dots), \tag{1.2}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of Gamma functions, by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & (n=0), \\ \lambda(\lambda+1)\cdots(\lambda+n+1), & (n \in \mathbb{N} := \{1,2,3,\dots\}). \end{cases}$$

Further, let (cf. [5,6])

$$\mathcal{L}(a,c)\,f(z) = \varphi(a,c;z) * f(z), \qquad (f \in \mathcal{A}), \tag{1.3}$$

in terms of the Hadamard product (or convolution). Note that $\mathcal{L}(a,a)$ is the identity operator and

$$\mathcal{L}(a,c) = \mathcal{L}(a,b)\,\mathcal{L}(b,c), \qquad (b,c \neq 0,-1,-2,\dots).$$

It is well known that, if $c > a > 0$, then $\mathcal{L}$ maps $\mathcal{A}$ into itself.

We also need the following definitions of a *fractional derivative*.

DEFINITION 1. *(See [6,7]; see also [8,9].) Let the function $f(z)$ by analytic in a simply-connected region of the z-plane containing the origin. The fractional derivative of f of order λ is defined by*

$$D_z^\lambda f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda}\, d\zeta, \qquad (0 \le \lambda < 1),$$

where the multiplicity of $(z-\zeta)^\lambda$ is removed by requiring $\log(z-\zeta)$ *to be real when* $z-\zeta > 0$.

Using Definition 1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [6] introduced the operator $\Omega^\lambda : \mathcal{A} \to \mathcal{A}$ defined by

$$\Omega^\lambda f(z) := \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z), \qquad (\lambda \neq 2,3,4\dots). \tag{1.4}$$

DEFINITION 2. *Suppose that the functions f and F are in $\mathcal{A}$. We say that f is subordinate to F in $\mathcal{U}$, written as $f \prec F$, if F is univalent in $\mathcal{U}$,*

$$f(0) = F(0) \qquad \text{and} \qquad f(\mathcal{U}) \subseteq F(\mathcal{U}).$$

In a recent paper [10], the authors studied the class $\mathcal{S}_\lambda(\alpha)$ $(0 \le \alpha < 1;\ 0 \le \lambda < 1)$ of functions f such that $\Omega^\lambda f$ is a *univalent starlike function of order* α. In the present sequel to our earlier paper [10], we study a class of analytic functions, related to UCV and SP, using the operator Ω^λ defined by (1.4).

DEFINITION 3. *Let SP_λ $(0 \le \lambda \le 1)$ be the class of functions $f \in \mathcal{A}_0$ satisfying the inequality:*

$$\left| \frac{z\left(\Omega^\lambda f(z)\right)'}{\Omega^\lambda f(z)} - 1 \right| < \operatorname{Re}\left\{ \frac{z\left(\Omega^\lambda f(z)\right)'}{\Omega^\lambda f(z)} \right\}, \qquad (z \in \mathcal{U}). \tag{1.5}$$

It follows that

$$\mathrm{SP}_1 \equiv \mathrm{UCV} \quad \text{and} \quad \mathrm{SP}_0 \equiv \mathrm{SP}. \tag{1.6}$$

We investigate here several basic properties and characteristics of the general class SP_λ. These include inclusion, subordination, and growth theorems, class-preserving operators (like the Hadamard product and various integral transforms), Fekete-Szegő problems, and sharp estimates for the first few coefficients of the inverse function.

2. BASIC PROPERTIES OF THE CLASS SP_λ

We need the following results in our investigation of the class SP_λ.

LEMMA 1. *(See [11].) Let F and G be univalent convex functions in $\mathcal{U}$. Then the Hadamard product $F * G$ is also univalent convex in $\mathcal{U}$.*

LEMMA 2. *(See [12].) Let the functions F and G be univalent convex in $\mathcal{U}$. Also let $f \prec F$ and $g \prec G$. Then $f * g \prec F * G$.*

LEMMA 3. *(See [11].) Let each of the functions f and g be univalent starlike of order 1/2. Then, for every function $F \in \mathcal{A}$,*

$$\frac{f(z) * g(z)F(z)}{f(z) * g(z)} \in \overline{\mathrm{CH}}\{F(\mathcal{U})\}, \qquad (z \in \mathcal{U}),$$

where $\overline{\mathrm{CH}}$ denotes the closed convex hull.

THEOREM 1. *If $0 \le \mu < \lambda < 1$, then*

$$\mathrm{SP}_\lambda \subset \mathrm{SP}_\mu.$$

PROOF. Let $f \in \mathrm{SP}_\lambda$. Then

$$\begin{aligned}\Omega^\mu f &= \mathcal{L}(2, 2-\mu)f = \mathcal{L}(2-\lambda, 2-\mu)\Omega^\mu f \\ &= \varphi(2-\lambda, 2-\mu; z) * \Omega^\lambda f\end{aligned}$$

and

$$\begin{aligned}z\left(\Omega^\mu f\right)' &= \mathcal{L}(2,1)\,\mathcal{L}(2-\lambda, 2-\mu)\Omega^\lambda f \\ &= \varphi(2-\lambda, 2-\mu; z) * \left\{ z\left(\Omega^\lambda f\right)' \right\}.\end{aligned}$$

Also it is known that (cf. [13])

$$\varphi(2-\lambda, 2-\mu; z) \in \mathcal{S}^*\left(\frac{1}{2}\right).$$

Since $\mathcal{R}$ is a convex region, using Lemma 3, we get

$$\frac{z\left(\Omega^\mu f(z)\right)'}{\Omega^\mu f(z)} = \frac{\varphi(2-\lambda, 2-\mu; z) * \left(\left(z\left(\Omega^\lambda f(z)\right)'\right) \Big/ \left(\Omega^\lambda f(z)\right)\right)\Omega^\lambda f(z)}{\varphi(2-\lambda, 2-\mu; z) * \Omega^\lambda f(z)} \in \mathcal{R}.$$

Thus $f \in \mathrm{SP}_\mu$. This completes the proof of Theorem 1.

COROLLARY 1. *If* $0 \le \lambda < 1$, *then the functions in* SP_λ *are parabolic starlike and* $\mathrm{UCV} \subset \mathrm{SP}$.

PROOF. Clearly, we have (cf. equation (1.6))

$$\mathrm{SP}_\lambda \subset \mathrm{SP}_0 \equiv \mathrm{SP} \qquad \text{and} \qquad \mathrm{UCV} \equiv \mathrm{SP}_1 \subset \mathrm{SP}_0 \equiv \mathrm{SP}.$$

It can be verified that the Riemann map q of $\mathcal{U}$ onto the region $\mathcal{R}$, satisfying $q(0) = 1$ and $q'(0) > 0$, is given by

$$q(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 \tag{2.1}$$

$$= 1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{2k+1}\right)z^n$$

$$= \sum_{n=0}^{\infty} B_n z^n = 1 + \frac{8}{\pi^2}\left(z + \frac{2}{3}z^2 + \frac{23}{45}z^3 + \frac{44}{105}z^4 + \cdots\right), \qquad (z \in \mathcal{U}). \tag{2.2}$$

We define the function G by

$$G(z) := \frac{1}{z}\left\{\mathcal{L}(2-\lambda, 2)z \exp\left(\int_0^z \frac{q(s)-1}{s}\,ds\right)\right\}, \qquad (z \in \mathcal{U}). \tag{2.3}$$

THEOREM 2. *Let* $0 \le \lambda < 1$ *and let* $G(z)$ *be defined by (2.3). Then* $G(z)$ *is a convex univalent function. Furthermore, if* $f \in \mathrm{SP}_\lambda$, *then*

$$\frac{f(z)}{z} \prec G(z).$$

PROOF. We first note that

$$G(z) = \frac{\varphi(2-\lambda, 2; z)}{z} * \exp\left(\int_0^z \frac{q(s)-1}{s}\,ds\right), \qquad (z \in \mathcal{U}), \tag{2.4}$$

where each member of the Hadamard product in (2.4) is known to be a convex univalent function (cf. [3,13]). Therefore, by Lemma 1, $G(z)$ is a univalent convex function.

Next, if $f \in \mathrm{SP}_\lambda$, then (cf. Definition 2)

$$\frac{z\left(\Omega^\lambda f(z)\right)'}{\Omega^\lambda f(z)} \prec q(z).$$

Thus there exists a function ψ satisfying the Schwarz Lemma such that

$$\frac{\Omega^\lambda f(z)}{z} = \exp\left(\int_0^z \frac{q(\psi(s))-1}{s}\,ds\right), \qquad (z \in \mathcal{U}).$$

Since $q(z) - 1$ is a univalent convex function, a result of [14]; (see also [15, p. 50]) yields

$$\frac{\Omega^\lambda f(z)}{z} \prec \exp\left(\int_0^z \frac{q(s)-1}{s}\,ds\right).$$

It now follows from a known result of [13, p. 508, Theorem 2] that

$$\frac{f(z)}{z} \prec G(z).$$

The proof Theorem 2 is evidently completed.

REMARK 1. Taking $\lambda = 1$ in Theorem 2, we immediately obtain a subordination result due to [3, p. 169, Theorem 3].

THEOREM 3. *Let* $0 \le \lambda < 1$. *If* $f \in \mathrm{SP}_\lambda$, *then*

$$G(-r) \le \left|\frac{f(z)}{z}\right| \le G(r), \qquad (|z| = r) \tag{2.5}$$

and

$$\left|\arg\left(\frac{f(z)}{z}\right)\right| \le \max_{\theta\in[0,2\pi]} \left\{\arg\left(G\left(re^{i\theta}\right)\right)\right\}, \qquad \left(z = re^{i\theta}\right), \tag{2.6}$$

where $G(z)$ *is defined by (2.3). Equality holds true in (2.5) and (2.6) for some* $z \ne 0$ *if and only if* f *is a rotation of* $zG(z)$.

PROOF. Let $f \in \mathrm{SP}_\lambda$. Then, by Theorem 2 and Lindelöf's principle of subordination, we get

$$\begin{aligned} \inf_{|z|=r} \operatorname{Re}\{G(z)\} &\le \inf_{|z|\le r} \operatorname{Re}\left\{\frac{f(z)}{z}\right\} \le \sup_{|z|\le r} \operatorname{Re}\left\{\frac{f(z)}{z}\right\} \\ &\le \sup_{|z|\le r}\left|\frac{f(z)}{z}\right| \le \sup_{|z|\le r} \operatorname{Re}\{G(z)\}. \end{aligned} \tag{2.7}$$

Since $G(z)$ is a univalent convex function and has real coefficients, $G(\mathcal{U})$ is a convex region symmetric with respect to real axis. Hence,

$$\inf_{|z|\le r} \operatorname{Re}\{G(z)\} = \inf_{-r\le x\le r} G(x) = G(-r) \tag{2.8}$$

and

$$\sup_{|z|\le r} \operatorname{Re}\{G(z)\} = \sup_{-r\le x\le r} G(x) = G(r). \tag{2.9}$$

Thus, (2.7) gives the assertion (2.5) of Theorem 3. Also we readily have the assertion (2.6) of Theorem 3.

The sharpness in inequalities (2.5) and (2.6) is also a consequence of the principle of subordination. This completes the proof of Theorem 3.

COROLLARY 2. *Let* $f \in \mathrm{SP}_\lambda$. *Then*

$$\{w : |w| \le G(-1)\} \subseteq f(\mathcal{U}).$$

The result is sharp.

REMARK 2. Taking $\lambda = 1$ in Theorem 3, we get a result of [3, p. 170, Corollary 3].

3. CLASS-PRESERVING OPERATORS AND TRANSFORMS

THEOREM 4. *If* $f \in \mathcal{S}_\lambda(1/2)$ *and* $g \in \mathrm{SP}_\mu$ $(\lambda \le \mu)$, *then*

$$\Omega^\lambda f * \Omega^\mu g \in \mathrm{SP}_\mu.$$

In particular, if $f \in \mathcal{S}_\lambda(1/2)$ *and* $g \in \mathrm{SP}_\lambda$, *then*

$$\Omega^\lambda f * \Omega^\lambda g \in \mathrm{SP}_\lambda.$$

PROOF. Let $f \in \mathcal{S}_\lambda(1/2)$ and $g \in \mathrm{SP}_\mu$ $(\lambda \le \mu)$. By definition,

$$f \in \mathcal{S}^*\left(\frac{1}{2}\right) \qquad \text{and} \qquad \Omega^\mu g \in \mathrm{SP} \subset \mathcal{S}^*\left(\frac{1}{2}\right).$$

The commutative and associative properties of the Hadamard product yield

$$\begin{aligned} z\left(\Omega^\lambda f * \Omega^\mu g\right)' &= \mathcal{L}(2,1)\left(\Omega^\lambda f * \Omega^\mu g\right) \\ &= \Omega^\lambda f * \mathcal{L}(2,1)\,\Omega^\mu g \\ &= \Omega^\lambda f * \left\{z\left(\Omega^\mu g\right)'\right\}. \end{aligned} \tag{3.1}$$

Therefore, using Lemma 3, we get

$$\frac{z\left(\Omega^\lambda f * \Omega^\mu g\right)'}{\Omega^\lambda f * \Omega^\mu g} = \frac{\Omega^\lambda f * \left(\left(z\left(\Omega^\mu g\right)'\right) / \left(\Omega^\mu g\right)\right)\Omega^\mu g}{\Omega^\lambda f * \Omega^\mu g} \in \mathcal{R}, \qquad (z \in \mathcal{U}). \tag{3.2}$$

This completes the proof of Theorem 4.

COROLLARY 3. *(See [4].) If $f \in S^*(1/2)$ and $g \in \mathrm{SP}$, then $f * g \in \mathrm{SP}$. In particular, if $f \in \mathrm{SP}$ and $g \in \mathrm{SP}$, then $f * g \in \mathrm{SP}$.*

THEOREM 5. *Let $f \in \mathrm{SP}$ and $g \in \mathrm{SP}_\mu$ $(0 \le \mu \le 1)$. Then $f * g \in \mathrm{SP}_\mu$. In particular, if $f \in \mathrm{UCV}$ and $g \in \mathrm{UCV}$, then $f * g \in \mathrm{UCV}$.*

PROOF. The proof of Theorem 5 is similar to that of Theorem 4. Let $f \in \mathrm{SP}$ and $g \in \mathrm{SP}_\mu$ $(0 \le \mu \le 1)$. We first note that

$$z\left(\Omega^\mu (f * g)(z)\right)' = \Omega^\mu g(z) * z f'(z) \qquad \text{and} \qquad \Omega^\mu (f * g)(z) = \Omega^\mu g(z) * f(z). \tag{3.3}$$

Therefore, using Lemma 3, we get

$$\frac{z\left(\Omega^\mu (f * g)(z)\right)'}{\Omega^\mu (f * g)(z)} = \frac{\Omega^\mu g(z) * \left(\left(z f'(z)\right) / \left(f(z)\right)\right) f(z)}{\Omega^\mu g(z) * f(z)} \in \mathcal{R}, \qquad (z \in \mathcal{U}). \tag{3.4}$$

Thus $f * g \in \mathrm{SP}_\mu$. Next, by Corollary 1, $\mathrm{UCV} \subset \mathrm{SP}$. Thus, by taking $\mu = 1$, we see that the Hadamard product of two uniformly convex functions is a uniformly convex function. This completes the proof of Theorem 5.

THEOREM 6. *Let $f_j \in \mathrm{SP}_\lambda$ $(j = 1, \dots, n)$. Also let*

$$\alpha_j > 0 \qquad \text{and} \qquad \sum_{j=1}^{n} \alpha_j = 1. \tag{3.5}$$

Define a function g by

$$\Omega^\alpha g = \prod_{j=1}^{n} \left(\Omega^\lambda f_j\right)^{\alpha_j}. \tag{3.6}$$

Then $g \in \mathrm{SP}_\lambda$.

PROOF. Let $f_j \in \mathrm{SP}_\lambda$ $(j = 1, \dots, n)$ and let g be defined by (3.6). Direct calculation gives

$$\begin{aligned} \left|\frac{z\left(\Omega^\lambda g\right)'}{\Omega^\lambda g} - 1\right| &= \left|\sum_{j=1}^{n} \alpha_j \frac{z\left(\Omega^\lambda f_j\right)'}{\Omega^\lambda f_j} - 1\right| \\ &< \sum_{j=1}^{n} \alpha_j \,\mathrm{Re}\left(\frac{z\left(\Omega^\lambda f_j\right)'}{\Omega^\lambda f_j}\right) \\ &= \mathrm{Re}\left(\frac{z\left(\Omega^\lambda g\right)'}{\Omega^\lambda g}\right). \end{aligned} \tag{3.7}$$

Thus, by Definition 2, $g \in \mathrm{SP}_\lambda$. This completes the proof of Theorem 6.

THEOREM 7. *Let $f \in \mathrm{SP}_\lambda$ $(0 \le \lambda \le 1)$. Then the function $F(z)$ defined by the integral transform*

$$F(z) := \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt, \qquad (z \in \mathcal{U};\ c > -1) \tag{3.8}$$

is also in the class SP_λ.

PROOF. We begin by noting that

$$F(z) = \mathcal{L}(c+1, c+2) f(z)$$

and

$$\begin{aligned} z\left(\Omega^\lambda f(z)\right)' &= \mathcal{L}(2,1)\, \mathcal{L}(2, 2-\lambda)\, \mathcal{L}(c+1, c+2) f(z) \\ &= \mathcal{L}(c+1, c+2) \left\{ z\left(\Omega^\lambda f(z)\right)' \right\} = \varphi(c+1, c+2; z) * \left\{ z\left(\Omega^\lambda f(z)\right)' \right\}. \end{aligned}$$

Using a result of Bernardi [16], it can be verified that

$$\varphi(c+1, c+2; z) \in \mathcal{S}^*\left(\frac{1}{2}\right).$$

Also, by hypothesis, $\Omega^\lambda f(z) \in \mathrm{SP} \subseteq \mathcal{S}^*(1/2)$. Thus, using Lemma 3, we get

$$\frac{z\left(\Omega^\lambda f(z)\right)'}{\Omega^\lambda f(z)} = \frac{\varphi(c+1, c+2; z) * \left(\left(z\left(\Omega^\lambda f(z)\right)' \right) \Big/ \left(\Omega^\lambda f(z)\right) \right) \Omega^\lambda f(z)}{\varphi(c+1, c+2; z) * \Omega^\lambda f(z)} \in \mathcal{R}, \tag{3.9}$$

which completes the proof of Theorem 7.

4. THE FEKETE-SZEGŐ PROBLEM FOR THE CLASS SP_λ

Let the function f, given by

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \qquad (z \in \mathcal{U}), \tag{4.1}$$

be in the class SP_λ. Then there exists a function $w \in \mathcal{A}$, satisfying

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1, \qquad (z \in \mathcal{U}),$$

such that

$$\frac{z\left(\Omega^\lambda f(z)\right)'}{\Omega^\lambda f(z)} = q\left(w(z)\right), \qquad (z \in \mathcal{U}). \tag{4.2}$$

Let the function p_1 in P be defined by

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots. \tag{4.3}$$

Then, by using (2.2), (4.2), and (4.3) in the form

$$w(z) = \frac{p_1(z) - 1}{p_1(z) + 1},$$

we find that

$$a_2 = \frac{4\Gamma(3-\lambda)}{\pi^2 \Gamma(3) \Gamma(2-\lambda)} c_1, \tag{4.4}$$

$$a_3 = \frac{2\Gamma(4-\lambda)}{\pi^2 \Gamma(4) \Gamma(2-\lambda)} \left[c_2 - \frac{1}{6}\left(1 - \frac{24}{\pi^2}\right) c_1^2 \right], \tag{4.5}$$

and

$$a_4 = \frac{4\Gamma(5-\lambda)}{3\pi^2\Gamma(2-\lambda)\Gamma(5)}\left[c_3 - \frac{1}{3}\left(1-\frac{18}{\pi^2}\right)c_1c_2 + \frac{2}{45}\left(1-\frac{45}{2\pi^2}+\frac{180}{\pi^4}\right)c_1^3\right]. \tag{4.6}$$

These expressions shall be used throughout the rest of the paper.

Define the functions $k(z,\tau,\nu)$ in SP_λ by

$$k(z,\tau,\nu) = \mathcal{L}(2-\lambda,2)z\exp\left(\int_0^z\left[q\left(\frac{e^{i\tau}\zeta(\zeta+\nu)}{1+v\zeta}\right)-1\right]\frac{d\zeta}{\zeta}\right), \quad (0\le\tau\le 2\pi; 0\le\nu\le 1). \tag{4.7}$$

Note that $k(z,0,1) = zG(z)$ defined by (2.3) and that $k(z,\tau,0)$ is an odd function. We also need the following lemma in our investigation.

LEMMA 4. *Let $g \in P$, where*

$$g(z) = 1 + c_1z + c_2z^2 + \cdots = 1 + G(z). \tag{4.8}$$

Then

$$|c_n| \le 2, \qquad (n\in\mathbb{N}) \tag{4.9}$$

and

$$\left|c_2 - \frac{1}{2}\mu c_1^2\right| \le 2 + \frac{1}{2}\left(|\mu-1|-1\right)|c_1|^2. \tag{4.10}$$

Furthermore, if we define the sequence $\{A_n\}_{n=1}^\infty$ by

$$\sum_{n=1}^\infty(-1)^{n-1}\gamma_{n-1}\{G(z)\}^n = \sum_{n=1}^\infty A_nz^n, \tag{4.11}$$

where

$$\gamma_0 = 1 \quad \text{and} \quad \gamma_n = \frac{1}{2^n}\left[1+\frac{1}{2}\sum_{j=1}^n\binom{n}{j}B_n\right], \tag{4.12}$$

and the sequence $\{B_n\}_{n=1}^\infty$ is given by

$$h(z) = 1 + \sum_{n=1}^\infty B_nz^n, \tag{4.13}$$

then

$$|A_n| \le 2, \qquad (n\in\mathbb{N}). \tag{4.14}$$

REMARK 3. Inequalities (4.9) and (4.10) are well known (cf. [1]) and estimate (4.14) can be found in [17]. For $n=2$ and $n=3$, (4.10) reduces to

$$|A_2| = \left|c_2 - \gamma_1c_1^2\right| \le 2 \tag{4.15}$$

and

$$|A_3| = \left|c_3 - 2\gamma_1c_1c_2 + \gamma_2c_1^3\right| \le 2, \tag{4.16}$$

respectively.

THEOREM 8. *Let the function f, given by (4.1), be in the class SP_λ. Then,*

$$\left|\mu a_2^2 - a_3\right| \le \begin{cases} \frac{4}{3\pi^2}(3-\lambda)(2-\lambda)\left(\frac{12(2-\lambda)\mu}{(3-\lambda)\pi^2}-\frac{4}{\pi^2}-\frac{1}{3}\right), & (\mu\ge\sigma_1), \\ \frac{2}{3\pi^2}(3-\lambda)(2-\lambda), & (\sigma_2\le\mu\le\sigma_1), \\ \frac{4}{3\pi^2}(3-\lambda)(2-\lambda)\left(\frac{1}{3}+\frac{4}{\pi^2}-\frac{12(2-\lambda)\mu}{\pi^2(3-\lambda)}\right), & (\mu\le\sigma_2), \end{cases} \tag{4.17}$$

where, for convenience,

$$\sigma_1 := \frac{3-\lambda}{2-\lambda}\left(\frac{1}{3}+\frac{5\pi^2}{72}\right) \quad \text{and} \quad \sigma_2 := \frac{3-\lambda}{2-\lambda}\left(\frac{1}{3}-\frac{\pi^2}{72}\right). \tag{4.18}$$

Each of the estimates in (4.17) is sharp.

PROOF. Using (4.4) and (4.5), we write

$$\left|\mu a_2^2 - a_3\right| = \frac{(2-\lambda)(3-\lambda)}{6\pi^2}\left|\left(\frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2}+\frac{1}{3}-\frac{8}{\pi^2}\right)c_1^2 - 2c_2\right| \tag{4.19}$$

$$\leq \frac{(2-\lambda)(3-\lambda)}{6\pi^2}\left(\left|\frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2}-\frac{5}{3}-\frac{8}{\pi^2}\right|\left|c_1\right|^2 + 2\left|c_1^2-c_2\right|\right). \tag{4.20}$$

If $\mu \geq \sigma_1$, then the expression inside the first modulus on the right-hand side of (4.20) is nonnegative. Thus, by applying Lemma 4, we get

$$\left|\mu a_2^2 - a_3\right| \leq \frac{4(2-\lambda)(3-\lambda)}{3\pi^2}\left(\frac{12(2-\lambda)\mu}{(3-\lambda)\pi^2}-\frac{4}{\pi^2}-\frac{1}{3}\right), \tag{4.21}$$

which is the first part of assertion (4.17). Equality in (4.21) (or, equivalently, in (4.20)) holds true if and only if $|c_1| = 2$. Thus the function f is $k(z,0,1)$ or one of its rotations for $\mu > \sigma_1$.

Next, if $\mu \leq \sigma_2$, we rewrite (4.19) as

$$\left|\mu a_2^2 - a_3\right| = \frac{(2-\lambda)(3-\lambda)}{6\pi^2}\left|2c_2 + \left(\frac{8}{\pi^2}-\frac{1}{3}-\frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2}\right)c_1^2\right|$$

$$\leq \frac{(2-\lambda)(3-\lambda)}{6\pi^2}\left\{2\left|c_2\right| + \left(\frac{8}{\pi^2}-\frac{1}{3}-\frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2}\right)\left|c_1\right|^2\right\}.$$

The estimates $|c_2| \leq 2$ and $|c_1| \leq 2$, after simplification, yield the second part of the assertion (4.17), in which equality holds true if and only if f is a rotation of $k(z,0,1)$ for $\mu < \sigma_2$.

If $\mu = \sigma_2$, then equality holds true if and only if $|c_2| = 2$. Equivalently, we have

$$p_1(z) = \frac{1+\nu}{2}\left(\frac{1+z}{1-z}\right) + \frac{1-\nu}{2}\left(\frac{1-z}{1+z}\right), \qquad (0<\nu<1;\ z \in \mathcal{U}).$$

Thus, the function f is $k(z,0,\nu)$ or one of its rotations.

If $\mu = \sigma_1$, then

$$\frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2}-\frac{5}{3}-\frac{8}{\pi^2}=0.$$

Therefore, equality holds true if and only if $|c_1^2 - c_2| = 2$. This happens if and only if

$$\frac{1}{p_1(z)} = \frac{1+\nu}{2}\left(\frac{1+z}{1-z}\right) + \frac{1-\nu}{2}\left(\frac{1-z}{1+z}\right), \qquad (0<\nu<1;\ z \in \mathcal{U}).$$

Thus the function f is $k(z,\pi,\nu)$ or one of its rotations.

Finally, we see that

$$\left|\mu a_2^2 - a_3\right| = \frac{(2-\lambda)(3-\lambda)}{6\pi^2}\left|2\left(c_2-\frac{1}{2}c_1^2\right) + \left(\frac{8}{\pi^2}+\frac{2}{3}-\frac{24(2-\lambda)\mu}{3-\lambda}\right)c_1^2\right|$$

and

$$\max\left|\frac{8}{\pi^2}+\frac{2}{3}-\frac{24(2-\lambda)\mu}{3-\lambda}\right| \leq 1, \qquad (\sigma_2 \leq \mu \leq \sigma_1).$$

Therefore, using Lemma 4, we get

$$\left|\mu a_2^2 - a_3\right| \leq \frac{(2-\lambda)(3-\lambda)}{6\pi^2}\left\{2\left(2-\frac{1}{2}|c_1|^2\right)+|c_1|^2\right\}$$
$$= \frac{2}{3\pi^2}(2-\lambda)(3-\lambda), \qquad (\sigma_2 \leq \mu \leq \sigma_1).$$

If $\sigma_2 < \mu < \sigma_1$, then equality holds true if and only if $|c_1| = 0$ and $|c_2| = 2$. Equivalently, we have

$$p_1(z) = \frac{1+\nu z^2}{1-\nu z^2}, \qquad (0 \leq \nu \leq 1;\ z \in \mathcal{U}).$$

Thus the function f is $k(z,0,0)$ or one of its rotations. The proof of Theorem 8 is evidently completed.

REMARK 4. The second part of assertion (4.17) can be improved as follows:

$$\left|\mu a_2^2 - a_3\right| + \left\{\mu - \frac{3-\lambda}{2-\lambda}\left(\frac{1}{3}-\frac{\pi^2}{72}\right)\right\}|a_2|^2 \leq \frac{2}{3\pi^2}(3-\lambda)(2-\lambda), \qquad (\sigma_2 \leq \mu \leq \sigma_3) \tag{4.22}$$

and

$$\left|\mu a_2^2 - a_3\right| + \left\{\frac{3-\lambda}{2-\lambda}\left(\frac{1}{3}+\frac{5\pi^2}{72}\right)-\mu\right\}|a_2|^2 \leq \frac{2}{3\pi^2}(3-\lambda)(2-\lambda), \qquad (\sigma_3 \leq \mu \leq \sigma_1), \tag{4.23}$$

where σ_1 and σ_2 are given, as before, by (4.18), and

$$\sigma_3 := \frac{3-\lambda}{2-\lambda}\left(\frac{1}{3}+\frac{\pi^2}{36}\right). \tag{4.24}$$

PROOF. For the values of μ prescribed in (4.22), we have

$$\left|\mu a_2^2 - a_3\right| + \left\{\mu - \frac{3-\lambda}{2-\lambda}\left(\frac{1}{3}-\frac{\pi^2}{72}\right)\right\}|a_2|^2$$
$$= \frac{(2-\lambda)(3-\lambda)}{6\pi^2}\left\{\left|2\left(c_2-\frac{1}{2}c_1^2\right)+\left(\frac{8}{\pi^2}+\frac{2}{3}-\frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2}\right)c_1^2\right|\right.$$
$$\left. + \left(\frac{24(2-\lambda)\mu}{\pi^2(3-\lambda)}+\frac{1}{3}-\frac{8}{\pi^2}\right)|c_1|^2\right\}$$
$$\leq \frac{(2-\lambda)(3-\lambda)}{6\pi^2}\left\{4-|c_1|^2+\left(\frac{8}{\pi^2}+\frac{2}{3}-\frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2}\right)|c_1|^2\right.$$
$$\left. + \left(\frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2}+\frac{1}{3}-\frac{8}{\pi^2}\right)|c_1|^2\right\}$$
$$= \frac{2}{3\pi^2}(3-\lambda)(2-\lambda), \qquad (\sigma_2 \leq \mu \leq \sigma_3),$$

which establishes (4.22). Similarly, for the values of μ prescribed in (4.23), we have

$$\left|\mu a_2^2 - a_3\right| + \left\{\frac{3-\lambda}{2-\lambda}\left(\frac{1}{3}+\frac{5\pi^2}{72}\right)-\mu\right\}|a_2|^2$$
$$= \frac{(3-\lambda)(2-\lambda)}{6\pi^2}\left\{\left|2\left(c_2-\frac{1}{2}c_1^2\right)+\left(\frac{8}{\pi^2}+\frac{2}{3}-\frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2}\right)c_1^2\right|\right.$$
$$\left. + \left(\frac{8}{\pi^2}+\frac{5}{3}-\frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2}\right)|c_1|^2\right\}$$
$$\leq \frac{(3-\lambda)(2-\lambda)}{6\pi^2}\left\{4-|c_1|^2+\left(\frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2}-\frac{8}{\pi^2}-\frac{2}{3}\right)|c_1|^2\right.$$
$$\left. + \left(\frac{8}{\pi^2}+\frac{5}{3}-\frac{24(2-\lambda)\mu}{(3-\lambda)\pi^2}\right)|c_1|^2\right\}$$
$$= \frac{2}{3\pi^2}(3-\lambda)(2-\lambda), \qquad (\sigma_3 \leq \mu \leq \sigma_1),$$

which proves (4.24).

REMARK 5. A special case of Theorem 8 when $\lambda = 1$ would yield a result due to [18, p. 280, Theorem 2]. Furthermore, by setting $\lambda = 0$ in Theorem 8, we readily obtain the following corollary.

COROLLARY 4. *Let the function f, given by (4.1), be in the class* SP. *Then*

$$\left|\mu a_2^2 - a_3\right| \leq \begin{cases} \dfrac{8}{\pi^2}\left(\dfrac{4(2\mu-1)}{\pi^2} - \dfrac{1}{3}\right), & (\mu \geq \delta_1), \\ \dfrac{4}{\pi^2}, & (\delta_2 \leq \mu \leq \delta_1), \\ \dfrac{8}{\pi^2}\left(\dfrac{1}{3} - \dfrac{4(2\mu-1)}{\pi^2}\right), & (\mu \leq \delta_2), \end{cases} \tag{4.25}$$

where, for convenience,

$$\delta_1 := \frac{1}{2} + \frac{5\pi^2}{48} \quad \text{and} \quad \delta_2 := \frac{1}{2} - \frac{\pi^2}{48}. \tag{4.26}$$

5. COEFFICIENT BOUNDS FOR THE INVERSE FUNCTIONS OF SP_λ

We first state the following theorem.

THEOREM 9. *Let the function f, given by (4.1), be in the class SP_λ. Also let the function f^{-1}, defined by*

$$f^{-1}(f(z)) = z = f\left(f^{-1}(z)\right), \tag{5.1}$$

be the inverse of f. If

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n, \qquad \left(|w| < r_0;\, r_0 > \frac{1}{4}\right), \tag{5.2}$$

then

$$|d_2| \leq \frac{4(2-\lambda)}{\pi^2}, \qquad (0 \leq \lambda \leq 1), \tag{5.3}$$

$$|d_3| \leq \begin{cases} \dfrac{2}{3\pi^2}(3-\lambda)(2-\lambda), & (\rho_1 \leq \lambda \leq 1) \\ \dfrac{4}{3\pi^2}(3-\lambda)(2-\lambda), & (0 \leq \lambda \leq \rho_1), \end{cases} \tag{5.4}$$

and

$$|d_4| \leq \frac{(4-\lambda)(3-\lambda)(2-\lambda)}{9\pi^2}, \qquad (\rho_2 \leq \lambda \leq 1), \tag{5.5}$$

where, for convenience,

$$\rho_1 := \frac{216 - 15\pi^2}{120 - 5\pi^2} \cong 0.9618\cdots \quad \text{and} \quad \rho_2 := \frac{4\left(72 - 5\pi^2\right)}{162 - 5\pi^2} \cong 0.8443\cdots. \tag{5.6}$$

Each of the estimates in (5.3)–(5.5) is sharp.

PROOF. Relation (5.2) gives

$$d_2 = -a_2, \quad d_3 = 2a_2^2 - a_3, \quad \text{and} \quad d_4 = -a_4 + 5a_3a_2 - 5a_2^3. \tag{5.7}$$

Thus, making use of (4.4), we get

$$|d_2| = \left|-\frac{2(2-\lambda)}{\pi^2}c_1\right| \leq \frac{4(2-\lambda)}{\pi^2}, \qquad (0 \leq \lambda \leq 1),$$

in which equality holds true if and only if h is the inverse of $k(z,0,1)$ or one of its rotations.

An application of Theorem 8 (with $\mu = 2$) gives the estimate for $|d_3|$. Since the estimates in Theorem 8 are all sharp, the bounds on $|d_3|$ are also the best possible.

Next, using the expressions for a_2, a_3, and a_4 given by (4.4)–(4.6), respectively, we have

$$\begin{aligned}|d_4| &= \frac{\Gamma(5-\lambda)}{18\pi^2\Gamma(2-\lambda)}\left|c_3 - \frac{1}{3}\left\{1 - \frac{18}{\pi^2}\left(1 - \frac{10(2-\lambda)}{4-\lambda}\right)\right\}c_1c_2\right.\\ &\quad \left.+\left[\frac{2}{45} + \frac{1}{\pi^2}\left(\frac{10(2-\lambda)}{4-\lambda} - 1\right) + \frac{8}{\pi^4}\left\{1 + \frac{30(2-\lambda)}{4-\lambda}\left(\frac{3(2-\lambda)}{3-\lambda} - 1\right)\right\}\right]c_1^3\right| \\ &= \frac{\Gamma(5-\lambda)}{18\pi^2\Gamma(2-\lambda)}\left|c_3 - 2\gamma_1c_1c_2 + \gamma_2c_1^3\right|,\end{aligned} \tag{5.8}$$

where

$$\gamma_1 = \frac{1}{6}\left\{1 - \frac{18}{\pi^2}\left(1 - \frac{10(2-\lambda)}{4-\lambda}\right)\right\} = \frac{1}{2}\left(1 + \frac{1}{2}\beta_1\right) \tag{5.9}$$

and

$$\begin{aligned}\gamma_2 &= \frac{2}{45} - \frac{1}{\pi^2}\left(1 - \frac{10(2-\lambda)}{4-\lambda}\right) + \frac{8}{\pi^4}\left(1 - \frac{30(2-\lambda)}{4-\lambda} + \frac{90(2-\lambda)^2}{(4-\lambda)(3-\lambda)}\right)\\ &= \frac{1}{4}\left(1 + \beta_1 + \frac{1}{2}\beta_2\right).\end{aligned} \tag{5.10}$$

Thus,

$$\beta_1 = \frac{12}{\pi^2}\left(\frac{10(2-\lambda)}{4-\lambda} - 1\right) - \frac{4}{3} \tag{5.11}$$

and

$$\frac{1}{2}\beta_2 = \frac{23}{45} - \frac{8}{\pi^2}\left(\frac{10(2-\lambda)}{4-\lambda} - 1\right) + \frac{32}{\pi^4}\left(1 - \frac{30(2-\lambda)}{4-\lambda} + \frac{90(2-\lambda)^2}{(4-\lambda)(3-\lambda)}\right). \tag{5.12}$$

In order for the inequality

$$\left|c_3 - 2\gamma_1c_1c_2 + \gamma_2c_1^3\right| \leq 2$$

of Lemma 4 to be applicable here, we must have $|\beta_1| \leq 2$ and $|\beta_2| \leq 2$. The condition $|\beta_1| \leq 2$ simplifies to

$$\rho_2 \leq \lambda \leq 1, \tag{5.13}$$

where ρ_2 is given by (5.6). Using the estimates

$$\frac{2-\lambda}{3-\lambda} \geq \frac{1}{2} \quad \text{and} \quad \frac{2-\lambda}{4-\lambda} \leq \frac{1}{2}, \qquad (0 \leq \lambda \leq 1),$$

it can be verified that $\beta_2 > 0$. Thus, in order to show that $|\beta_2| \leq 2$, it is sufficient to establish that

$$-\frac{4}{\pi^2}\left(\frac{10(2-\lambda)}{4-\lambda} - 1\right) + \frac{16}{\pi^4}\left(1 - \frac{30(2-\lambda)}{4-\lambda} + \frac{90(2-\lambda)^2}{(4-\lambda)(3-\lambda)}\right) \leq \frac{11}{45}. \tag{5.14}$$

The left-hand side of inequality (5.14) can be rewritten as

$$\frac{4}{\pi^2}\left(\frac{10(2-\lambda)}{4-\lambda} - 1\right)\left[\frac{4}{\pi^2}\left(\frac{10(2-\lambda)}{4-\lambda} - 1\right) - 1\right] + \left(\frac{4}{\pi^2}\cdot\frac{10(2-\lambda)}{4-\lambda}\right)^2\left(\frac{\lambda(3-2\lambda)}{10(2-\lambda)(3-\lambda)}\right),$$

which, for the values of λ given by (5.12), is less than or equal to

$$\frac{10}{81} + \left(\frac{10}{9} + \frac{4}{\pi^2}\right)^2\frac{\lambda(3-2\lambda)}{10(2-\lambda)(3-\lambda)}.$$

Now it is sufficient to show that

$$\frac{10}{81}+\left(\frac{10}{9}+\frac{4}{\pi^2}\right)^2\frac{\lambda(3-2\lambda)}{10(2-\lambda)(3-\lambda)}\leq\frac{11}{45}.$$

This is equivalent to

$$\left[49\pi^4+4\left(5\pi^2+18\right)^2\right]\lambda^2-\left[245\pi^4+6\left(5\pi^4+18\right)^2\right]\lambda+294\pi^4\geq 0,$$

which is true for all real values of λ. Thus $|\beta_2|\leq 2$ for the values of λ given by (5.13).

Finally, by applying (4.17), we arrive at (5.5).

In order to establish that our estimates are sharp, we need to find a function f corresponding to (4.13) with β_1 and β_2 given as above. One such function is

$$f(z)=\left\{\frac{6}{\pi^2}\left(\frac{10(2-\lambda)}{4-\lambda}-1\right)-\frac{2}{3}\right\}\frac{1+z}{1-z}-\left\{\frac{6}{\pi^2}\left(\frac{10(2-\lambda)}{4-\lambda}-1\right)-\frac{5}{3}\right\}\frac{1+\kappa z^2}{1-\kappa z^2},\tag{5.15}$$

where

$$\kappa=\frac{\dfrac{53}{45}-\dfrac{14}{\pi^2}\left(\dfrac{10(2-\lambda)}{4-\lambda}-1\right)+\dfrac{32}{\pi^4}\left(1-\dfrac{30(2-\lambda)}{4-\lambda}+\dfrac{90(2-\lambda)^2}{(4-\lambda)(3-\lambda)}\right)}{\dfrac{5}{3}-\dfrac{6}{\pi^2}\left(\dfrac{10(2-\lambda)}{4-\lambda}-1\right)}.\tag{5.16}$$

This completes the proof of Theorem 9.

REMARK 6. The choice $\lambda=1$ in Theorem 9 gives a recent result of [18, p. 283, Theorem 3].

REFERENCES

1. P.L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Bd., Volume 259, Springer-Verlag, New York, (1983).
2. A.W. Goodman, On uniformly convex functions, *Ann. Polon. Math.* **56**, 87–92 (1991).
3. W. Ma and D. Minda, Uniformly convex functions, *Ann. Polon. Math.* **57**, 165–175 (1992).
4. E. Rønning, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.* **118**, 189–196 (1993).
5. B.C. Carlson and D.B. Shaffer, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.* **15**, 735–745 (1984).
6. S. Owa and H.M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.* **39**, 1057–1077 (1987).
7. S. Owa, On the distortion theorems I, *Kyungpook Math. J.* **18**, 53–59 (1978).
8. H.M. Srivastava and S. Owa, An application of the fractional derivative, *Math. Japon.* **29**, 383–389 (1984).
9. H.M. Srivastava and S. Owa, *Univalent Functions, Fractional Calculus, and Their Applications*, Halsted Press/John Wiley and Sons, Chichester/New York, (1989).
10. H.M. Srivastava, A.K. Mishra and M.K. Das, A nested class of analytic functions defined by fractional calculus, *Comm. Appl. Anal.* **2**, 321–332 (1998).
11. St. Ruscheweyh and T. Sheil-Small, Hadamard products of schlicht functions and the Pólya-Schoenberg conjecture, *Comment. Math. Helv.* **48**, 119–135 (1973).
12. St. Ruscheweyh and J. Stankiewicz, Subordination under convex univalent functions, *Bull. Polish Acad. Sci. Math.* **33**, 499–502 (1985).
13. Y. Ling and S. Ding, A class of analytic functions defined by fractional derivation, *J. Math. Anal. Appl.* **186**, 504–513 (1994).
14. G.M. Goluzin, On the majorization principle in function theory, (in Russian), *Dokl. Akad. Nauk SSSR* **42**, 647–650 (1935).
15. Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, (1975).
16. S.D. Bernardi, Convex and starlike univalent functions, *Trans. Amer. Math. Soc.* **135**, 429–446 (1969).
17. Z. Nehari and E. Netanyahu, On the coefficients of meromorphic schicht functions, *Proc. Amer. Math. Soc.* **8**, 15–23 (1957).
18. W. Ma and D. Minda, Uniformly convex functions II, *Ann. Polon. Math.* **58**, 275–285 (1993).
19. R.J. Libera and E.L. Zlotkiewicz, Early coefficients of the inverse functions of a regular convex function, *Proc. Amer. Math. Soc.* **85**, 225–230 (1982).
20. St. Ruscheweyh, *Convolutions in Geometric Function Theory*, Sem. Math. Sup., Volume 83, Presses Univ. de Montréal, (1982).
21. D.-G. Yang, The subclass of starlike functions of order λ, *Chinese Ann. Math. Ser. A* **8**, 687–692 (1987).