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Poisson manifolds with compatible pseudo-metric and pseudo-Riemannian Lie algebras

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Abstract

In a previous paper (C. R. Acad. Sci. Paris Sér. I 333 (2001) 763–768), the author introduced a notion of compatibility between a Poisson structure and a pseudo-Riemannian metric. In this paper, we introduce a new class of Lie algebras called pseudo-Riemannian Lie algebras. The two notions are closely related: we prove that the dual of a Lie algebra endowed with its canonical linear Poisson structure carries a compatible pseudo-Riemannian metric if and only if the Lie algebra is a pseudo-Riemannian Lie algebra. Moreover, the Lie algebra obtained by linearizing at a point a Poisson manifold with a compatible pseudo-Riemannian metric is a pseudo-Riemannian Lie algebra. We also give some properties of the symplectic leaves of such manifolds, and we prove that every Poisson manifold with a compatible Riemannian metric is unimodular. Finally, we study Poisson Lie groups endowed with a compatible pseudo-Riemannian metric, and we give the classification of all pseudo-Riemannian Lie algebras of dimension 2 and 3.

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1. Introduction

The starting point of this work was the search of a notion of compatibility between a Poisson structure and a pseudo-Riemannian metric. Given a Poisson manifold endowed with a pseudo-Riemannian metric (P, π, g), it is natural to look for a notion of compatibility between the Poisson tensor π and the metric g. The first idea is to assume that $\nabla\pi = 0$, where ∇ is the Levi-Civita connection associated with g. Since

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the parallel transport preserves the rank, the Poisson tensor must have a constant rank. However, the most interesting Poisson structures are degenerate, and this condition is too strong. That is why we introduced in [1] another notion of compatibility which will be studied in this paper. It is based on the notion of contravariant connection. This notion has been introduced by Vaisman (see [7, p. 55]) as contravariant derivative. Recently, a geometric approach of this notion was given by Fernandes in [2].

Let us summarize the contents of this paper. Let P be a Poisson manifold with Poisson tensor π. A pseudo-Riemannian metric of signature (p,q) on T^*P is a smooth symmetric contravariant 2-form $\langle\,,\rangle$ on P such that, at each point $x \in P$, $\langle\,,\rangle_x$ is non-degenerate on T^*_xP with signature (p,q). For each pseudo-Riemannian metric $\langle\,,\rangle$ on T^*P, we consider the contravariant connection D defined in [1] by

$$\begin{aligned}2\langle D_\alpha\beta,\gamma\rangle &= \pi(\alpha).\langle\beta,\gamma\rangle + \pi(\beta).\langle\alpha,\gamma\rangle - \pi(\gamma).\langle\alpha,\beta\rangle\\ &\quad + \big\langle[\alpha,\beta]_\pi,\gamma\big\rangle + \big\langle[\gamma,\alpha]_\pi,\beta\big\rangle + \big\langle[\gamma,\beta]_\pi,\alpha\big\rangle,\end{aligned} \tag{1}$$

where $\alpha,\beta,\gamma \in \Omega^1(P)$ and the Lie bracket $[\,,\,]_\pi$ is given by

$$[\alpha,\beta]_\pi = L_{\pi(\alpha)}\beta - L_{\pi(\beta)}\alpha - d\big(\pi(\alpha,\beta)\big);$$

here, $\pi : T^*P \to TP$ denotes the bundle map given by

$$\beta\big[\pi(\alpha)\big] = \pi(\alpha,\beta).$$

The connection D is the contravariant analogue of the usual Levi-Civita connection. We call it the Levi-Civita contravariant connection associated with the couple $(\pi,\langle\,,\rangle)$. The connection D has vanishing torsion, i.e.,

$$D_\alpha\beta - D_\beta\alpha = [\alpha,\beta]_\pi.$$

Moreover, it is compatible with the pseudo-Riemannian metric $\langle\,,\rangle$, i.e.,

$$\pi(\alpha).\langle\beta,\gamma\rangle = \langle D_\alpha\beta,\gamma\rangle + \langle\beta, D_\alpha\gamma\rangle.$$

As usual, we denote by $X_f = \pi(df)$ the hamiltonian vector field associated with the function $f \in C^\infty(P)$.

The following definition was given in [1] with a different terminology.

Definition 1.1. With the notations above, the triple $(P,\pi,\langle\,,\rangle)$ is called a pseudo-Riemannian Poisson manifold if, for any $\alpha,\beta,\gamma \in \Omega^1(P)$,

$$D\pi(\alpha,\beta,\gamma) := \pi(\alpha).\pi(\beta,\gamma) - \pi(D_\alpha\beta,\gamma) - \pi(\beta,D_\alpha\gamma) = 0. \tag{2}$$

When $\langle\,,\rangle$ is positive definite we call the triple a Riemannian Poisson manifold.

Now, we give the definition of a pseudo-Riemannian Lie algebra. Let $(\mathcal{G},[\,,\,])$ be a Lie algebra and let a be a bilinear, symmetric and non-degenerate form on $\mathcal{G}$. We define a bilinear map $A : \mathcal{G}\times\mathcal{G} \to \mathcal{G}$ by

$$2a(A_uv,w) = a\big([u,v],w\big) + a\big([w,u],v\big) + a\big([w,v],u\big) \tag{3}$$

for any $u,v,w \in \mathcal{G}$. We call A the infinitesimal Levi-Civita connection associated with a. Indeed, if G is a connected Lie group whose Lie algebra is $\mathcal{G}$, a defines a left invariant pseudo-Riemannian metric on G. The Levi-Civita connection ∇ associated with this metric is given by

$$\nabla_{u^l}v^l = (A_uv)^l, \quad u,v \in \mathcal{G},$$

where u^l denotes the left invariant vector field associated with u.

Definition 1.2. With the notations above, the triple $(\mathcal{G}, [\ ,\ ], a)$ is called a pseudo-Riemannian Lie algebra if

$$[A_u v, w] + [u, A_w v] = 0 \tag{4}$$

for all $u, v, w \in \mathcal{G}$. When a is positive definite we call the triple $(\mathcal{G}, [\ ,\ ], a)$ a Riemannian Lie algebra.

Let us state the main results of the paper. In Section 2, we give some basic properties of a pseudo-Riemannian Poisson manifold and we prove the following theorems:

Theorem 1.1. *Let $(P, \pi, \langle\ ,\ \rangle)$ be a pseudo-Riemannian Poisson manifold and $x \in P$ a point where the restriction of $\langle\ ,\ \rangle$ to $\operatorname{Ker}\pi(x)$ is non-degenerate. Then the isotropy Lie algebra $\mathcal{G}_x$ at x is a pseudo-Riemannian Lie algebra.*

(The definition of the isotropy Lie algebra is given in Section 2.2.)

Theorem 1.2. *Let $\mathcal{G}$ be a Lie algebra. The dual $(\mathcal{G}^*, \pi)$ of $\mathcal{G}$ endowed with its canonical linear Poisson structure carries a pseudo-Riemannian metric $\langle\ ,\ \rangle$ for which the triple $(\mathcal{G}^*, \pi, \langle\ ,\ \rangle)$ is a pseudo-Riemannian Poisson manifold if and only if $\mathcal{G}$ is a pseudo-Riemannian Lie algebra.*

Remark. The assumption on x in Theorem 1.1 is a property of the symplectic leaf through x. This means the following: if the restriction of $\langle\ ,\ \rangle$ to $\mathcal{G}_x = \operatorname{Ker}\pi(x)$ is non-degenerate, then it is non-degenerate at every point of the symplectic leaf through x, since the parallel transport determined by D (see Fernandes [2]) preserves both $\operatorname{Ker}\pi$ and $\langle\ ,\ \rangle$.

In Section 3, we prove the following theorems:

Theorem 1.3. *Let $(P, \pi, \langle\ ,\ \rangle)$ be a Riemannian Poisson manifold and let S be a symplectic leaf of P. Then S is a Kähler manifold.*

Theorem 1.4. *Let $(P, \pi, \langle\ ,\ \rangle)$ be a Riemannian Poisson manifold and let S be a regular symplectic leaf of P. Then the holonomy group of S is finite.*

In Section 4, we recall the definition of an unimodular Poisson manifold and we prove the following theorem:

Theorem 1.5. *Every Riemannian Poisson manifold is unimodular. In particular, every Riemannian Lie algebra is unimodular.*

In Section 5, we give a necessary and sufficient condition for a left invariant pseudo-Riemannian metric on a Poisson Lie group G to be compatible with the Poisson structure of G. We also give a proof of the following theorem which is the key for finding interesting examples of compact Riemannian Poisson manifolds.

Theorem 1.6. *Let G be a connected Lie group and $\mathcal{G}$ its Lie algebra. Let $\langle\ ,\ \rangle$ be a bi-invariant pseudo-Riemannian metric on T^*G. Let $r \in \mathcal{G} \wedge \mathcal{G}$ be a solution of the generalized Yang–Baxter equation and let*

π be the Lie Poisson structure on G given by r. Then $(G, \pi, \langle\, ,\, \rangle)$ is a pseudo-Riemannian Poisson Lie group if and only if

$$[\mathrm{ad}^*_{r(\alpha)}\gamma, \beta]_r + [\alpha, \mathrm{ad}^*_{r(\beta)}\gamma]_r = 0, \quad \forall \alpha, \beta, \gamma \in \mathcal{G}^*,$$

where the bracket $[\, ,\,]_r$ is given by

$$[\alpha, \beta]_r = \mathrm{ad}^*_{r(\beta)}\alpha - \mathrm{ad}^*_{r(\alpha)}\beta$$

and where r also denotes the linear map from $\mathcal{G}^*$ to $\mathcal{G}$ induced by r.

Remark. If G is connected compact or connected semi-simple, any Poisson Lie group structure on G is given by a solution of the generalized Yang–Baxter equation and G carries a bi-invariant pseudo-Riemannian metric. Consequently, to get a structure of Riemannian Poisson Lie group on G, it suffices to find a solution of the generalized Yang–Baxter equation which is also a solution of the equation in Theorem 1.6.

The proof of the following theorem is a very long calculation. We will omit it here.

Theorem 1.7. (1) *The 2-dimensional abelian Lie algebra is the unique 2-dimensional pseudo-Riemannian Lie algebra.*

(2) *The 3-dimensional Lie algebras which have a pseudo-Riemannian Lie algebra structure are*:

(a) *The Heisenberg Lie algebra given by*

$$[e_1, e_2] = e_3, \quad [e_3, e_1] = [e_3, e_2] = 0.$$

(b) *A family of Lie algebras given by*

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = \gamma e_2 - \alpha e_3, \quad [e_2, e_3] = 0,$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $\alpha^2 + \beta\gamma \neq 0$.

Furthermore, there is no Riemannian Lie algebra structure on the Heisenberg Lie algebra, and a Lie algebra among the above family has a structure of Riemannian Lie algebra if and only if $\alpha^2 + \beta\gamma < 0$ and $\gamma > \beta$.

Finally, we remark that the most general setup for contravariant connections appears in the context of Lie algebroids (see [3]). It will be interesting to study the possible extension of the results discussed here in this general context.

2. From Riemannian Poisson manifolds to Riemannian Lie algebras

2.1. Basic material

In this subsection, we collect the basic material which will be used throughout this paper. For more background material we refer to Vaisman's monograph [7].

Let P be a Poisson manifold with Poisson tensor π. Let $\langle\, ,\, \rangle$ be a pseudo-Riemannian metric on T^*P and D the Levi-Civita contravariant connection associated with the couple $(\pi, \langle\, ,\, \rangle)$.

We give now another condition of compatibility between the Poisson tensor π and the pseudo-Riemannian metric $\langle\,,\rangle$ which will be useful later. Since D has vanishing torsion and since the contravariant exterior differential d_π associated with the bracket $[\,,\,]_\pi$ is given by $d_\pi = -[\pi, .]_S$, we can obviously deduce that, for any $\alpha, \beta, \gamma \in \Omega^1(P)$,

$$0 = -[\pi,\pi]_S(\alpha,\beta,\gamma) = D\pi(\alpha,\beta,\gamma) + D\pi(\beta,\gamma,\alpha) + D\pi(\gamma,\alpha,\beta). \tag{5}$$

From this formula and a straightforward calculation, we obtain, for any $\alpha, \beta, \gamma \in \Omega^1(P)$,

$$D\pi(\gamma,\alpha,\beta) = -d\gamma\big(\pi(\alpha),\pi(\beta)\big) - \pi(D_\alpha\gamma,\beta) - \pi(\alpha, D_\beta\gamma). \tag{6}$$

Proposition 2.1. *Let $(P,\pi,\langle\,,\rangle)$ be a Poisson manifold with a pseudo-Riemannian metric on T^*P and let D be the Levi-Civita contravariant connection associated with the couple $(\pi,\langle\,,\rangle)$. The following assertions are equivalent*:

(1) *The triple $(P,\pi,\langle\,,\rangle)$ is a pseudo-Riemannian Poisson manifold.*

(2) *For any $\alpha,\beta \in \Omega^1(P)$ and any $f \in C^\infty(P)$,*

$$\pi(D_\alpha df,\beta) + \pi(\alpha, D_\beta df) = 0. \tag{7}$$

(3) *For any $\alpha,\beta,\gamma \in \Omega^1(P)$,*

$$d\gamma\big(\pi(\alpha),\pi(\beta)\big) + \pi(D_\alpha\gamma,\beta) + \pi(\alpha, D_\beta\gamma) = 0. \tag{8}$$

2.2. Proof of Theorem 1.1

Let us recall the definition of the isotropy Lie algebra at a point in a Poisson manifold (P,π) (see [8]). If $x \in P$, we set $\mathcal{G}_x = \operatorname{Ker}\pi(x)$ and, for any $\alpha,\beta \in \mathcal{G}_x$, we define the bracket of α and β by

$$[\alpha,\beta]_x = d_x\big(\pi(\tilde\alpha,\tilde\beta)\big),$$

where $\tilde\alpha, \tilde\beta \in \Omega^1(P)$, $\tilde\alpha_x = \alpha$ and $\tilde\beta_x = \beta$. The definition of $[\,,\,]_x$ is independent of the extensions and $(\mathcal{G}_x, [\,,\,]_x)$ is a Lie algebra called the isotropy Lie algebra at x.

Let $(P,\pi,\langle\,,\rangle)$ be a pseudo-Riemannian Poisson manifold. Fix a point $x \in P$ such that the restriction of $\langle\,,\rangle$ to $\operatorname{Ker}\pi(x)$ is non-degenerate. We denote by a the restriction of $\langle\,,\rangle$ to $\mathcal{G}_x$ and by A be the infinitesimal Levi-Civita connection associated with a.

For every $\alpha \in \mathcal{G}_x$, we denote by $\tilde\alpha$ any 1-form in $\Omega^1(P)$ such that $\tilde\alpha_x = \alpha$.

For any $\alpha,\beta \in \mathcal{G}_x$, we claim that

$$(D_{\tilde\alpha}\tilde\beta)_x = A_\alpha\beta.$$

In fact, for any $\gamma \in \mathcal{G}_x$, we deduce from (1) and (3)

$$2\big\langle (D_{\tilde\alpha}\tilde\beta)_x, \gamma\big\rangle = 2a(A_\alpha\beta,\gamma).$$

It remains to show that $(D_{\tilde\alpha}\tilde\beta)_x \in \mathcal{G}_x$. Indeed, for every $\mu \in \Omega^1(P)$, we have

$$\mu\big(\pi(D_{\tilde\alpha}\tilde\beta)\big) = \pi(D_{\tilde\alpha}\tilde\beta,\mu) = \pi(\tilde\alpha).\pi(\tilde\beta,\mu) - \pi(\tilde\beta, D_{\tilde\alpha}\mu) = \pi(\tilde\alpha).\pi(\tilde\beta,\mu) - D_{\tilde\alpha}\mu\big(\pi(\tilde\beta)\big).$$

Since $\pi(\tilde\alpha)_x = \pi(\tilde\beta)_x = 0$, we deduce that $(D_{\tilde\alpha}\tilde\beta)_x \in \mathcal{G}_x$.

Now, for each $\gamma \in \mathcal{G}_x$ and for each $f \in C^\infty(P)$ such that $d_x f = \gamma$, we have from (7),

$$\pi(D_{\tilde\alpha} df, \tilde\beta) + \pi(\tilde\alpha, D_{\tilde\beta} df) = 0.$$

Differentiating this relation at x, we obtain

$$[A_\alpha \gamma, \beta]_x + [\alpha, A_\beta \gamma]_x = 0$$

and the theorem follows.

2.3. Proof of Theorem 1.2

Let a be a bilinear, symmetric and non-degenerate form on $\mathcal{G}$ such that the triple $(\mathcal{G}, [\ ,\ ], a)$ is a pseudo-Riemannian Lie algebra. We define on $T^*\mathcal{G}^* = \mathcal{G}^* \times \mathcal{G}$ a pseudo-Riemannian metric $\langle\ ,\ \rangle$ by

$$\big\langle (\mu, u), (\mu, v) \big\rangle = a(u, v), \qquad \mu \in \mathcal{G}^*,\ (u, v) \in \mathcal{G}^2.$$

Each vector $v \in \mathcal{G}$ defines a linear form on $\mathcal{G}^*$ which will be denoted also by v. For any $u, v \in \mathcal{G}$ and any $\mu \in \mathcal{G}^*$, we have

$$\pi(dv, du)(\mu) = \mu\big([u, v]\big), \quad [du, dv]_\pi = d[u, v], \quad \text{and} \quad D_{du} dv = d(A_u v).$$

It is easy now to show that (4) and (7) are equivalent and hence, $(\mathcal{G}^*, \pi, \langle\ ,\ \rangle)$ is a pseudo-Riemannian Poisson manifold.

Conversely, if $(\mathcal{G}^*, \pi)$ carries a pseudo-Riemannian Poisson structure, the isotropy Lie algebra at the origin of $\mathcal{G}^*$ is $\mathcal{G}$ and the theorem follows by Theorem 1.1.

3. Symplectic leaves of a Riemannian Poisson manifold

This section is devoted to the proofs of Theorems 1.3 and 1.4. Before we give these proofs, we need some lemmas.

Lemma 3.1. *Let $(P, \pi, \langle\ ,\ \rangle)$ be a pseudo-Riemannian Poisson manifold, let $S \subset P$ be a symplectic leaf an let $U \subset P$ be an open subset. If $\alpha, \beta \in \Omega^1(P)$ are* 1*-forms such that $\pi(\alpha)|_{S\cap U} = 0$ or $\pi(\beta)|_{S\cap U} = 0$ then*

$$\pi(D_\alpha \beta)|_{S\cap U} = 0,$$

where D is the Levi-Civita contravariant connection associated with $(\pi, \langle\ ,\ \rangle)$.

Proof. Since the torsion of D vanishes, we have

$$\pi(D_\alpha \beta) - \pi(D_\beta \alpha) = \big[\pi(\alpha), \pi(\beta)\big].$$

Hence, if $\pi(\alpha)|_{S\cap U} = 0$ or $\pi(\beta)|_{S\cap U} = 0$, then $\pi(D_\alpha\beta)|_{S\cap U} = \pi(D_\beta\alpha)|_{S\cap U}$.

Suppose that $\pi(\beta)|_{S\cap U} = 0$. For any $\gamma \in \Omega^1(P)$,

$$\gamma\big(\pi(D_\alpha\beta)\big) = \pi(D_\alpha\beta, \gamma) = \pi(\alpha).\pi(\beta, \gamma) - \pi(\beta, D_\alpha\gamma) = 0$$

since $\pi(\alpha)$ is tangent to $S \cap U$. □

Lemma 3.2. *Let $(P, \pi, \langle\,,\,\rangle)$ be a Riemannian Poisson manifold. Let O be the regular open set where the rank of the Poisson tensor is locally constant. Then,*

(1) *D is a $\mathcal{F}$-connection on O in the sense of Fernandes* [2]; *this means that, for any $x \in O$ and every $\alpha \in T_x^*P$, we have*

$$\pi(\alpha) = 0 \quad \Rightarrow \quad D_\alpha = 0;$$

(2) *D is a basic connection on O in the sense of Fernandes* [2]; *this means that for any symplectic leaf $S \subset O$ and for any $\alpha, \beta \in \Omega^1(O)$ such that $\pi(\beta)|_S = 0$, we have*

$$(D_\alpha \beta)|_S = [\alpha, \beta]_\pi|_S.$$

Proof. Let U be an open set in O where the rank of the Poisson tensor is constant. On U, the cotangent bundle splits

$$T^*P = \operatorname{Ker}\pi \oplus (\operatorname{Ker}\pi)^\perp,$$

where $(\operatorname{Ker}\pi)^\perp$ is the $\langle\,,\,\rangle$-orthogonal of $\operatorname{Ker}\pi$.

Let $x \in U$ and $\alpha \in T_x^*P$ such that $\pi(\alpha) = 0$. Choose a 1-form $\tilde{\alpha}$ on U such that $\pi(\tilde{\alpha}) = 0$ and $\tilde{\alpha}_x = \alpha$. For any $\beta \in \Omega^1(U)$ we have, by Lemma 3.1, $\pi(D_{\tilde{\alpha}}\beta) = 0$. We claim that $D_{\tilde{\alpha}}\beta \in (\operatorname{Ker}\pi)^\perp$. Indeed, for any $\gamma \in \operatorname{Ker}\pi$, we have

$$\langle D_{\tilde{\alpha}}\beta, \gamma\rangle = \pi(\tilde{\alpha}).\langle \beta, \gamma\rangle - \langle \beta, D_{\tilde{\alpha}}\gamma\rangle = -\langle \beta, D_{\tilde{\alpha}}\gamma\rangle.$$

Now, using the splitting of T^*P, we can write $\beta = \beta_1 + \beta_1^\perp$ and so $\langle \beta, D_{\tilde{\alpha}}\gamma\rangle = \langle \beta_1, D_{\tilde{\alpha}}\gamma\rangle$. This quantity vanishes from the definition of D, so D is a $\mathcal{F}$-connection. It is also a basic connection because its torsion vanishes. □

Lemma 3.3. *Let $(P, \pi, \langle\,,\,\rangle)$ be a pseudo-Riemannian Poisson manifold. Let O be the regular open set where the rank of the Poisson tensor is locally constant. Then,*

(1) *For any $x \in O$, for any $\alpha, \beta \in \operatorname{Ker}\pi(x)$ and for any $f \in C^\infty(O)$, we have*

$$L_{X_f}\big(\langle\,,\,\rangle\big)(\alpha, \beta) = 0;$$

(2) *For any Casimir functions f, g, the scalar product $\langle df, dg\rangle$ is also a Casimir function.*

Proof. (1) For any $f, g, h \in C^\infty(P)$, we have

$$\begin{aligned} L_{X_f}\big(\langle\,,\,\rangle\big)(dg, dh) &= \pi(df).\langle dg, dh\rangle - \langle L_{X_f}dg, dh\rangle - \langle dg, L_{X_f}dh\rangle \\ &= \langle D_{df}dg, dh\rangle + \langle dg, D_{df}dh\rangle - \big\langle [df, dg]_\pi, dh\big\rangle - \big\langle dg, [df, dh]_\pi\big\rangle \\ &= \langle D_{dg}df, dh\rangle + \langle dg, D_{dh}df\rangle. \end{aligned}$$

This implies that

$$L_{X_f}\big(\langle\,,\,\rangle\big)(\alpha, \beta) = \langle D_\alpha df, \beta\rangle + \langle \alpha, D_\beta df\rangle$$

and the property follows from Lemma 3.2.

(2) Let (f, g) be a couple of Casimir functions on P. For every $h \in C^\infty(P)$, we have

$$\begin{aligned} \big\{h, \langle df, dg\rangle\big\} &= X_h.\langle df, dg\rangle = \langle D_{dh}df, dg\rangle + \langle df, D_{dh}dg\rangle \\ &= \big\langle D_{df}dh + [dh, df]_\pi, dg\big\rangle + \big\langle df, D_{dg}dh + [dh, dg]_\pi\big\rangle. \end{aligned}$$

Now $[dh, df]_\pi = d\{h, f\} = 0$ and $D_{df}dh$ vanishes on O by Lemma 3.2. So $\{h, \langle df, dg\rangle\}$ is zero on P because O is dense. □

3.1. Proof of Theorem 1.3

Let $(P, \pi, \langle\,,\rangle)$ be a Riemannian Poisson manifold and let $S \subset P$ be a symplectic leaf. We denote by ω_S the symplectic form of S.

For any vectors fields X, Y tangent to S, we set

$$\nabla^S_X Y = \pi(D_\alpha \beta)|_S \tag{9}$$

where $\pi(\alpha)|_S = X$ and $\pi(\beta)|_S = Y$. It follows from Lemma 3.1 that ∇^S defines a torsionless covariant connection on S. Obviously, we have

$$\nabla^S \omega_S = 0.$$

For any $x \in S$, we have

$$T^*_x P = \operatorname{Ker}\pi(x) \oplus \big(\operatorname{Ker}\pi(x)\big)^\perp$$

and the linear map $\pi(x) : (\operatorname{Ker}\pi(x))^\perp \to T_x S$ is an isomorphism. For any $u, v \in T_x S$, we set

$$g_S(u, v) = \big\langle \pi(x)^{-1}(u), \pi(x)^{-1}(v)\big\rangle. \tag{10}$$

(S, g_s) is a Riemannian manifold and $\nabla^S g_S = 0$. This implies that ∇^S is the Levi-Civita connection of g_S.

Now, it is classical (see [7, p. 39]) that TS has a ∇^S-parallel complex structure $J = A(-A^2)^{-1/2}$, where A is given by

$$\omega_S(u, v) = g_S(Au, v).$$

So S is a Kähler manifold.

3.2. Proof of Theorem 1.4

The linear Poisson holonomy of a Poisson manifold was introduced by Ginzburg and Golubev in [4]. A deeper study of this notion was done by Fernandes in [2]. In his approach, basic connections play a fundamental role. Fernandes introduces also the notion of (non-linear) Poisson holonomy, and for a regular leaf he shows that the (linear) Poisson holonomy coincides with the standard (linear) holonomy.

Let $(P, \pi, \langle\,,\rangle)$ be a Riemannian Poisson manifold and let S be a regular symplectic leaf. We prove that the holonomy group of S is finite.

First step: The linear holonomy group of S is finite.

The linear holonomy group of S coincides with the linear Poisson holonomy because the leaf is regular. Now, the linear Poisson holonomy can be determined by the parallel transport associated with a basic connection (see [2]). The Levi-Civita contravariant connection D is, by Lemma 3.2, a basic connection on a neighbourhood of S. D is also a $\mathcal{F}$-connection, so the linear holonomy group is discrete. Moreover, the parallel transport defined by D is given by isometries on $\operatorname{Ker}\pi$. This gives the claim.

Second step: The holonomy group of S is finite.

The holonomy group coincides with the Poisson holonomy group because the leaf is regular. The Poisson holonomy is given by Hamiltonian flows. Now, let $x \in S$ and $(q_1, \ldots, q_k, p_1, \ldots, p_k, y_1, \ldots, y_l)$

a Darboux coordinates defined on a neighbourhood U of x. We consider the submanifold N of U defined by $p = q = 0$. We define a metric g_N on T^*N by

$$g_N(dy_i, dy_j) = \langle dy_i, dy_j \rangle.$$

By considering (N, g_N), we have according to Lemma 3.3 a well-defined notion of "transverse Riemannian structure" along S and the holonomy preserves this transverse structure. Now, the elements of the holonomy group are isometries and their differentials at x are elements of the linear holonomy group which is finite. Consequently, the holonomy group is finite.

4. Riemannian Poisson manifolds are unimodular

This section is devoted to the proof of Theorem 1.5. Let us recall the definition of an unimodular Poisson manifold (for more details see [10]).

The modular class of a Poisson manifold (P, π) is the obstruction to the existence of a volume form on P which is invariant with respect to Hamiltonian flows. More explicitly, let μ be a volume form on P. As shown in [10], the operator $\phi_\mu : f \mapsto \operatorname{div}_\mu X_f$ is a derivation and hence a vector field called the modular vector field of (P, π) with respect to the volume form μ. Moreover, $L_{\phi_\mu}\pi = 0$ and $L_{\phi_\mu}\mu = 0$.

If we replace μ by $a\mu$, where a is a positive function, the modular vector fields becomes

$$\phi_{a\mu} = \phi_\mu + X_{\ln a}.$$

Thus the first Poisson cohomology class of ϕ_μ is independent of μ. We call it the modular class of (P, π). The Poisson manifold is unimodular if its modular class vanishes.

Now, we give a metric version of the modular vector field.

Let (P, π) be a Poisson manifold and g a Riemannian metric on TP. We denote by $\#_g : T^*P \to TP$ the musical isomorphism associated with g and by μ_g the volume form on P given by g. The modular vector field with respect to μ_g is given by

$$\phi_{\mu_g}(f) = \operatorname{div}_g(X_f) = -\sum_{i=1}^n g(\nabla_{e_i} X_f, e_i), \tag{11}$$

where ∇ is the Levi-Civita connection associated with g and $(e_1, \ldots, e_n)$ is a local orthonormal basis of vector fields.

If $h(u, v) = g(Ju, v)$ is another Riemannian metric, we have $\mu_h = \sqrt{\det J}\,\mu_g$ and

$$\phi_{\mu_h} = \phi_{\mu_g} - \frac{1}{2} X_{\ln(\det J)}. \tag{12}$$

Now, we define a Riemannian metric $\langle\, ,\rangle$ on T^*P by

$$\langle \alpha, \beta \rangle = g\big(\#_g(\alpha), \#_g(\beta)\big), \quad \alpha, \beta \in T^*P$$

(remark that any Riemannian metric on T^*P can be obtained in this way).

Let D be the Levi-Civita contravariant connection associated with $(\pi, \langle\, ,\rangle)$. We claim that

$$\phi_{\mu_g} = \frac{1}{2} \sum_{i=1}^n L_{X_f}\big(\langle\, ,\rangle\big)(\alpha_i, \alpha_i) = \sum_{i=1}^n \langle D_{\alpha_i} df, \alpha_i \rangle, \tag{13}$$

where $(\alpha_1,\dots,\alpha_n)$ is a local orthonormal basis of 1-forms.

Indeed, we have, for any $\alpha,\beta\in T^*P$,

$$\begin{aligned}L_{X_f}g\big(\#_g(\alpha),\#_g(\beta)\big) &= X_f.g\big(\#_g(\alpha),\#_g(\beta)\big)-g\big(\big[X_f,\#_g(\alpha)\big],\#_g(\beta)\big)-g\big(\big[X_f,\#_g(\beta)\big],\#_g(\alpha)\big)\\ &= g\big(\nabla_{\#_g(\alpha)}X_f,\#_g(\beta)\big)+g\big(\nabla_{\#_g(\beta)}X_f,\#_g(\alpha)\big).\end{aligned}$$

We have also, for any $h\in C^\infty(P)$,

$$\begin{aligned}\big[X_f,\#_g(\alpha)\big](h) &= X_f.\#_g(\alpha)(h)-\#_g(\alpha)\big(X_f(h)\big)\\ &= X_f.\langle\alpha,dh\rangle-\langle\alpha,L_{X_f}dh\rangle\\ &= L_{X_f}\big(\langle\,,\rangle\big)(\alpha,dh)+\langle L_{X_f}\alpha,dh\rangle.\end{aligned}$$

This implies that

$$L_{X_f}\big(\langle\,,\rangle\big)(\alpha,\beta)=-L_{X_f}g\big(\#_g(\alpha),\#_g(\beta)\big),\quad \alpha,\beta\in T^*P.$$

The claim is a consequence of the formula given in the proof of Lemma 3.3.

Now, we consider a Riemannian Poisson manifold $(P,\pi,\langle\,,\rangle)$ and we denote by $\phi_{\langle,\rangle}$ the modular vector field defined by (13). We will show that $\phi_{\langle,\rangle}$ is zero on the regular open O. This implies that it is zero on P, since O is dense in P.

Let $x\in O$ and S the symplectic leaf of x. We consider the symplectic form ω_S of S, the Riemannian metric g_S on S given by (10) and its Levi-Civita connection ∇^S.

We have, in a neighbourhood of x,

$$T^*P=\operatorname{Ker}\pi\oplus(\operatorname{Ker}\pi)^\perp.$$

We choose $(\alpha_1,\dots,\alpha_l)$ (respectively $(\beta_1,\dots,\beta_{n-l})$) a local orthonormal frame of $\operatorname{Ker}\pi$ (respectively $(\operatorname{Ker}\pi)^\perp$). According to Lemma 3.2, we have:

$$\phi_{\langle,\rangle}(f)(x)=\sum_{i=1}^{n-l}\langle D_{\beta_i}df,\beta_i\rangle=\sum_{i=1}^{n-l}g_S\big(\nabla^S_{\pi(\beta_i)}X_f,\pi(\beta_i)\big)=-\operatorname{div}_{g_S}(X_f)(x).$$

Now, according to Theorem 1.3 there is a Riemannian metric h on S such that (S,h,ω_S) is a Kähler manifold and such that the isomorphism J given by $h(u,v)=g_S(Ju,v)$ is ∇^S-parallel. It follows that $\det J$ is constant and $\operatorname{div}_{g_S}(X_f)=\operatorname{div}_h(X_f)$. To conclude, we recall the well-known fact that in a Kähler manifold the divergence with respect to the Kähler metric of any hamiltonian vector field is zero.

Now, let $\mathcal{G}$ be a Riemannian Lie algebra. According to Theorem 1.2, $\mathcal{G}^*$ inherits a structure of Riemannian Poisson manifold which is unimodular and the theorem follows.

5. Riemannian Poisson Lie groups

A Lie group G is called a Poisson Lie group if it is also a Poisson manifold such that the multiplication map $m:G\times G\to G$ is a Poisson map, where $G\times G$ is equipped with the product Poisson structure.

Let G be a Poisson Lie group with Lie algebra $\mathcal{G}$ and π the Poisson tensor on G. Pulling π back to the identity element e of G by left translations, we get a map $\pi_l:G\to\mathcal{G}\wedge\mathcal{G}$ defined by $\pi_l(g)=(L_{g^{-1}})_*\pi(g)$ where $(L_g)_*$ denotes the tangent map of the left translation of G by g. Let

$$d_e\pi:\mathcal{G}\to\mathcal{G}\times\mathcal{G}$$

be the intrinsic derivative of π at e given by

$$v \mapsto L_X\pi(e),$$

where X can be any vector field on G with $X(e) = v$.

The dual map of $d_e\pi$

$$[\ ,\ ]_e : \mathcal{G}^* \times \mathcal{G}^* \to \mathcal{G}^*$$

is exactly the Lie bracket on $\mathcal{G}^*$ obtained by linearizing the Poisson structure at e.

If there is on G a pseudo-Riemannian metric which is compatible with the Poisson structure, the Lie algebra $(\mathcal{G}^*, [\ ,\ ]_e)$ is a pseudo-Riemannian Lie algebra (see Theorem 1.1).

Now, let a be a bilinear, symmetric and non-degenerate form on $\mathcal{G}^*$ and $A : \mathcal{G}^* \times \mathcal{G}^* \to \mathcal{G}^*$ the infinitesimal Levi-Civita connection associated with $(a, [\ ,\ ]_e)$. We denote by $\langle\ ,\ \rangle$ the pseudo-Riemannian metric on T^*G given by $\langle\ ,\ \rangle_g = (L_g)_* a$.

Lemma 5.1. *With the notations above, $(G, \pi, \langle\ ,\ \rangle)$ is a pseudo-Riemannian Poisson Lie group if and only if*

$$\big[\mathrm{Ad}_g^*(A_\alpha\gamma + \mathrm{ad}^*_{\pi_l(g)(\alpha)}\gamma), \mathrm{Ad}_g^*(\beta)\big]_e + \big[\mathrm{Ad}_g^*(\alpha), \mathrm{Ad}_g^*(A_\beta\gamma + \mathrm{ad}^*_{\pi_l(g)(\beta)}\gamma)\big]_e = 0$$

for all $g \in G$ and for all $\alpha, \beta, \gamma \in \mathcal{G}^*$.

Proof. We denote by $\mathcal{G}_l^*$ the space of left invariant 1-forms on G and by $\mathcal{G}_r$ the Lie algebra of right invariant vector fields on G.

Let D be the Levi-Civita contravariant connection associated with the couple $(\pi, \langle\ ,\ \rangle)$. Since $\langle\ ,\ \rangle$ is left invariant and since $\mathcal{G}_l^*$ is a Lie subalgebra of $(\Omega^1(G), [\ ,\ ]_\pi)$ (see [9]), we can deduce from the definition of D that $D_\alpha\beta$ is a left invariant 1-form whenever α and β are left invariant 1-forms. Now, we have from (8) that $(\pi, \langle\ ,\ \rangle)$ is compatible if and only if

$$Q := d\gamma\big(\pi(\alpha), \pi(\beta)\big) + \pi(D_\alpha\gamma, \beta) + \pi(\alpha, D_\beta\gamma) = 0, \quad \forall \alpha, \beta, \gamma \in \mathcal{G}_l^*. \tag{$*$}$$

Since $\pi(e) = 0$, we get $Q(e) = 0$ so $(*)$ is equivalent to

$$X.Q = 0, \quad \forall \alpha, \beta, \gamma \in \mathcal{G}_l^*,\ \forall X \in \mathcal{G}_r. \tag{$**$}$$

It is easy to see that $(**)$ is equivalent to

$$L_X d\gamma\big(\pi(\alpha), \pi(\beta)\big) + d\gamma\big(\big[X, \pi(\alpha)\big], \pi(\beta)\big) + d\gamma\big(\pi(\alpha), \big[X, \pi(\beta)\big]\big) + L_X\pi(D_\alpha\gamma, \beta)$$
$$+ L_X\pi(\alpha, D_\beta\gamma) + \pi\big(L_X(D_\alpha\gamma), \beta\big) + \pi(D_\alpha\gamma, L_X\beta) + \pi(L_X\alpha, D_\beta\gamma) + \pi\big(\alpha, L_X(D_\beta\gamma)\big) = 0$$

$\forall \alpha, \beta, \gamma \in \mathcal{G}_l^*$, $\forall X \in \mathcal{G}_r$. A straightforward calculation gives

$$\big[X, \pi(\alpha)\big] = L_X\pi(\alpha) + \pi(L_X\alpha).$$

Now, the Lie derivative of left invariant form by a right invariant vector field vanishes, so $(G, \pi, \langle\ ,\ \rangle)$ is a pseudo-Riemannian Poisson Lie group if and only if

$$L_X\pi(D_\alpha\gamma - i_{\pi(\alpha)}d\gamma, \beta) + L_X\pi(\alpha, D_\beta\gamma - i_{\pi(\beta)}d\gamma) = 0, \tag{14}$$

for any $\alpha, \beta, \gamma \in \mathcal{G}_l^*$ and for any $X \in \mathcal{G}_r$. $L_X\pi$ is right invariant (see [5]) so (14) is equivalent to

$$(L_X\pi)_e\big(T_e^*R_g[D_\alpha\gamma - i_{\pi(\alpha)}d\gamma], T_e^*R_g(\beta)\big) + (L_X\pi)_e\big(T_e^*R_g(\alpha), T_e^*R_g[D_\beta\gamma - i_{\pi(\beta)}d\gamma]\big) = 0,$$

$\forall\alpha,\beta,\gamma\in\mathcal{G}_l^*$, $\forall X\in\mathcal{G}_r$ and $\forall g\in G$. We can get easily the following formula

$$T_e^*R_g(i_{\pi(\alpha)}d\gamma)=-\operatorname{Ad}_g^*\circ\operatorname{ad}^*_{\pi_l(g)(\alpha_e)}\gamma_e.$$

To get the lemma, we use the fact that $(L_X\pi)_e=d_e\pi(X)$ and the fact that the bracket $[\,,\,]_e$ is the dual of $d_e\pi$. □

5.1. Proof of Theorem 1.6

See [6] and [5] for the properties of the Poisson Lie groups which arise from the classical r-matrices.

Let G be a Lie group and $\mathcal{G}$ its Lie algebra. Let $r\in\mathcal{G}\wedge\mathcal{G}$. Define a bivector π on G by

$$\pi(g)=(L_g)_*r-(R_g)_*r,\quad g\in G.$$

(G,π_r) is a Poisson Lie group if and only if the element $[r,r]\in\mathcal{G}\wedge\mathcal{G}\wedge\mathcal{G}$ defined by

$$[r,r](\alpha,\beta,\gamma)=\oint\alpha\big(\big[r(\beta),r(\gamma)\big]\big)$$

is ad-invariant. Such an r is called a solution of the generalized Yang–Baxter equation.

In this case, the bracket $[\,,\,]_e$ is given by

$$[\alpha,\beta]_e=\operatorname{ad}^*_{r(\beta)}\alpha-\operatorname{ad}^*_{r(\alpha)}\beta,\quad \alpha,\beta\in\mathcal{G}^* \tag{15}$$

and

$$\pi_l(g)=r-\operatorname{Ad}_g(r),\quad g\in G. \tag{16}$$

Now, if a is a bilinear, symmetric and non-degenerate form on $\mathcal{G}^*$ which is invariant with respect to the coadjoint representation of G on $\mathcal{G}^*$, we can get that the infinitesimal Levi-Civita connection $A:\mathcal{G}^*\times\mathcal{G}^*\to\mathcal{G}^*$ associated with $(a,[\,,\,]_e)$ is given by

$$A_\alpha\beta=-\operatorname{ad}^*_{r(\alpha)}\beta,\quad \alpha,\beta\in\mathcal{G}^*. \tag{17}$$

Theorem 1.6 follows from Lemma 5.1, (15), (16), (17) and the following formula

$$\operatorname{Ad}_g^*[\operatorname{ad}^*_{r(\alpha)}\beta]=\operatorname{ad}^*_{(\operatorname{Ad}_{g^{-1}}r)(\operatorname{Ad}_g^*\alpha)}(\operatorname{Ad}_g^*\beta),\quad \alpha,\beta\in\mathcal{G}^*,\ g\in G.$$

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