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A Method for Proving the Non-existence of Limit Cycles

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A general criterion is established for showing the non-existence of periodic solutions and closed phase polygons of differential equations. Several known criteria of this type are obtained as special cases. An extension is obtained for functional differential equations and several applications are given for specific special cases including homogeneous systems and population dynamics equations.

1. INTRODUCTION

The negative criterion of Bendixson and its generalization by Dulac are widely used to eliminate the possibility of limit cycles for differential equations in the plane. We present a natural generalization of these criteria which applies to situations not covered by the Bendixson–Dulac criterion, and which includes as direct special cases several known results which had been proved by other means. We illustrate the use of this criterion by obtaining computable necessary conditions that exclude limit cycles and closed phase polygons for certain classes of dynamical systems. In particular, we treat homogeneous systems and some population dynamics equations. Finally, we define the notion of a closed loop solution for retarded functional differential equations and derive criteria that rule out their existence.

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We start by giving our first result and its simple proof in sufficient generality to allow us to cover the application we have mentioned above.

THEOREM 1.1. *Let X be a real Banach space and $f\colon X \to X$ be Lipschitz continuous. Let $\gamma(t)$ be a closed, piecewise smooth curve which is the boundary of a smooth orientable surface $S \subset \mathbf{R}^3$ with unit normal ν. Set $\gamma(t) = \gamma_i(t)$, $t \in [t_{i-1}, t_i]$, $i = 1, ..., n$, and assume that $\gamma_i(t) = P_i x_i(t)$, where $P_i\colon X \to \mathbf{R}^3$ are continuous linear operators and $\gamma_i \in C^1([t_{i-1}, t_i], \mathbf{R}^3)$. Suppose that $g\colon \mathbf{R}^3 \to \mathbf{R}^3$ is defined and piecewise smooth in a neighborhood of S, and that it satisfies*

$$\int_{\gamma_i} g(\gamma_i(t)) \cdot P_i f(x_i(t))\, dt \leqslant 0 \qquad (or < 0), \tag{1.1}$$

and

$$\iint_S (\operatorname{curl} g) \cdot \nu\, dA > 0 \qquad (or \geqslant 0). \tag{1.2}$$

Then $x_i(t)$, $t \in [t_{i-1}, t_i]$ are not trajectories of

$$x' = f(x) \tag{1.3}$$

such that $\gamma_i(t) = P_i x_i(t)$ are traversed in the positive sense relative to the orientation defined by ν.

Proof. Suppose that $x_i' = f(x_i)$ and note that by Stokes' theorem

$$\int_\gamma g(\gamma(t)) \cdot \gamma'(t)\, dt = \iint_S (\operatorname{curl} g) \cdot \nu\, dA.$$

Hence, by (1.2)

$$\begin{aligned} 0 < \int_\gamma g(\gamma(t)) \cdot \gamma'(t)\, dt &= \sum_{i=1}^n \int_{\gamma_i} g(\gamma_i(t)) \cdot P_i x_i'(t)\, dt \\ &= \sum_{i=1}^n \int_{\gamma_i} g(\gamma_i(t)) \cdot P_i f(x_i(t))\, dt, \end{aligned}$$

since $x_i' = f(x_i)$ for $t \in [t_{i-1}, t_i]$. By (1.1) the last expression in this inequality is non-positive, hence, we have a contradiction, and the theorem is proved. The alternate case of the theorem follows in the same manner. ∎

The following result is an immediate corollary of the above theorem

COROLLARY 1.2. *Suppose that in Theorem* 1.1, $P: X \to \mathbf{R}^3$ *is a continuous linear operator, and conditions* (1.1) *and* (1.2) *are replaced by*

$$g(Px) \cdot Pf(x) \leqslant 0 \qquad (or < 0) \qquad a.e. \text{ on } S, \tag{1.1'}$$

and

$$(\operatorname{curl} g) \cdot v > 0 \qquad (or \geqslant 0) \qquad a.e. \text{ on } S. \tag{1.2'}$$

Then there does not exist any closed, piecewise smooth curve $\gamma(t)$ *on* S *such that* $\gamma(t) = Px(t)$, *where* x *is a solution of* (1.3) *and* $\gamma(t)$ *is traversed in the positive sense relative to the orientation defined by* v.

Remark 1. The simplicity of the proof of these results points to the fact that much of the work in applying them lies in the construction of the function g. In both aspects, simplicity of proof and the labor necessary for application, this mirrors the situation of the Bendixson–Dulac criterion (see Hahn [9, p. 67] or Lefschetz [14]). We show that the present generalization allows the treatment of many cases not covered by the Bendixson–Dulac criterion. Our aim here is to illustrate the breath of these applications. The Bendixson–Dulac criterion is also related to the theorem of Liouville which connects the divergence of f to the evolution of the volume in state space under the flow induced by (1.1) (see Arnold [1, p. 198] and Hartmann [12, Chap. 14]). Recent extensions of this latter type of result with applications to the question of existence of periodic solutions are given by Butler, Schmid, and Waltman [6] and by Muldowney [15].

Remark 2. A special case of the above results is proved by Hall and Busenberg [11] where S is the surface of a sphere and $X = \mathbf{R}^3$. Busenberg and van den Driessche [4] prove the above result for $X = \mathbf{R}^3$ and $P_i =$ identity on $\mathbf{R}^3$, in order to resolve a question in population disease dynamics involving a homogeous dynamical system. A further application of the result in [4] to homogeneous systems is given in Hadeler [8].

Remark 3. It is clear that the conclusions of the theorem remain valid if the inequalities in (1.1) and (1.2) and the orientation of $\gamma(t)$ are all reserved. Also, if (1.1) is replaced by $g(\gamma_i(t)) \cdot P_i f(x_i(t)) = 0$ a.e. on $[t_{i-1}, t_i]$ then the result holds with either the strict inequality in (1.2), or with this inequality in (1.2) reserved.

In [4] we have shown that the Bendixson–Dulac criterion is a simple consequence of Theorem 1.1. We next show that the following result of Chen [7] also follows from Theorem 1.1.

COROLLARY 1.3. *Let $G \subset \mathbf{R}^2$ be an open region and let $P, Q: G \to \mathbf{R}^2$ be of class C^1. Suppose that there exist functions $B, M, N \in C^1(G, \mathbf{R}^2)$ with $M \geqslant 0$, $N \geqslant 0$, $M + N > 0$ on G, and*

$$\iint_G \left[\frac{\partial}{\partial x}(BP + NQ) + \frac{\partial}{\partial y}(BQ - MP)\right] dx\, dy > 0 \tag{1.4}$$

Then the two dimensional system

$$x' = P(x, y), \qquad y' = Q(x, y), \tag{1.5}$$

has no limit cycles or closed phase polygons in G which are traversed clockwise. If the inequality in (1.4) is reserved, there are no limit cycles or closed phase polygons of (1.5) in G which are traversed counterclockwise.

Proof. If γ is a limit cycle of (1.5) in G then it is also a limit cycle of the system $x' = P$, $y' = Q$, $z' = 0$, in $\mathbf{R}^3$. In Theorem 1.1, choose $X = \mathbf{R}^3$, $g = (BQ - MP, -BP - NQ, 0)$, and note that the vector $(P, Q, 0)$ is tangent to γ. However,

$$g(\gamma) \cdot (P, Q, 0) = -MP^2 - NQ^2 \leqslant 0,$$

and condition (1.1) of Theorem 1.1 holds. Now, if γ is traversed clockwise in the $x - y$ plane, the outward normal to the planar surface $S \subset G$ that it bounds in $\mathbf{R}^3$ is $v = (0, 0, -1)$. Thus

$$(\text{curl}\, g) \cdot v = \frac{\partial}{\partial x}(BP + NQ) + \frac{\partial}{\partial y}(BQ - MP),$$

and by hypothesis (1.4) condition (1.2) of Theorem 1.1 also holds. Consequently, γ cannot be a limit cycle of (1.5), contradicting our initial hypothesis. Thus γ cannot be traversed clockwise. The proof for the case of cycles which are traversed counterclockwise is entirely analogous. The extension to phase polygons is obtained by partitioning the polygon into n trajectories γ_i and applying Theorem 1.1 as above. ∎

A special case of Corollary 1.3, which parallels Corollary 1.2, is obtained by imposing the condition

$$\frac{\partial}{\partial x}(BP + NQ) + \frac{\partial}{\partial y}(BQ - MP) > 0 \qquad (\text{or } < 0) \qquad \text{a.e. on } G, \tag{1.4'}$$

and concluding that there are no periodic solutions of (1.5) in G which are traversed clockwise (or counterclockwise).

The hypotheses of Corollary 1.2 can be easily strengthened in order to eliminate periodic solutions regardless of the sense in which they are traversed.

COROLLARY 1.4. *Suppose that in Theorem* 1.1, $P: X \to \mathbf{R}^3$ *is a continuous linear operator, and conditions* (1.1) *and* (1.2) *are replaced by either*

$$g(Px) \cdot Pf(x) < 0 \qquad (\text{or } > 0) \qquad \text{a.e. on } S, \tag{1.6}$$

and

$$(\operatorname{curl} g) \cdot v = 0 \qquad \text{a.e. on } S; \tag{1.7}$$

or

$$g(Px) \cdot Pf(x) = 0 \qquad \text{a.e. on } S, \tag{1.8}$$

and

$$(\operatorname{curl} g) \cdot v < 0 \qquad (\text{or } > 0) \qquad \text{a.e. on } S. \tag{1.9}$$

Then there does not exist any closed, piecewise smooth curve $\gamma(t)$ *on* S *such that* $\gamma(t) = Px(t)$, *where* x *is a solution of* (1.3).

Proof. The proof follows the lines of that of Theorem 1.1 with the contradiction occurring because one of the equivalent integrals in that proof is zero by condition (1.7) or (1.8), while the other integral must be either strictly positive or strictly negative by conditions (1.6) or (1.9), regardless of the orientation of $\gamma(t)$. ∎

Another special result which includes the Bendixson–Dulac criterion as a special case is given in the following corollary.

COROLLARY 1.5. *Let* $G \subset \mathbf{R}^2$ *be an open region and let* $P, Q: G \to \mathbf{R}^2$ *be of class* C^1. *Suppose that there exists a function* $B \in C^1(\mathbf{R}^3, \mathbf{R})$ *such that*

$$-(P+Q)\frac{\partial B}{\partial z} + \frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} \neq 0, \qquad \forall (x, y) \in G, \tag{1.10}$$

then (1.5) *has no limit cycles or closed phase polygons in* G.

Proof. Consider the system

$$x' = P(x, y), \qquad y' = Q(x, y), \qquad z' = -P(x, y) - Q(x, y), \tag{1.11}$$

and note that the plane $S = \{(x, y, x): x + y + z = 1\}$ with normal

$v=(1, 1, 1)$ is invariant under the flow induced by (1.11). Letting $g=(-BQ, BP, 0)$, we have $g \cdot (P, Q, -P-Q)=0$, and

$$(\operatorname{curl} g) \cdot v = -(P+Q)\frac{\partial B}{\partial z} + \frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} \neq 0.$$

The proof now follows from a direct application of Corollary 1.4. ▮

Note that by choosing $B(x, y, z)=\rho(x, y)$ condition (1.10) reduces to the Bendixson–Dulac criterion. If we choose $B(x, y, z)=\rho(x, y)\, e^{kz}$, (1.6) becomes

$$k\rho(P+Q) + \frac{\partial(\rho P)}{\partial x} + \frac{\partial(\rho Q)}{\partial y} \neq 0, \tag{1.12}$$

which again yields the Bendixson–Dulac criterion when $k=0$.

The proofs of Corollaries 1.3 and 1.5 illustrate the ease with which Theorem 1.1 can be used to obtain results concerning planar systems. In the remainder of this paper we present a number of other applications of this theorem. In Section 2 we treat a general class of systems which are motivated by population dynamics and epidemic modelling problems and derive results for the non-existence of limit cycles, periodic solutions, or closed phase polygons for such systems. These results are natural extensions of our previous work [4]. In Section 3 we treat homogeneous systems and derive general computable conditions for the non-existence of limit cycles. In Section 4 we consider certain classed of retarded functional differential equations. We introduce the notion of closed loop solutions, and give necessary conditions for the non-existence of such solutions. In each of these sections we discuss the applicability of the criteria we derive.

2. A Class of Population Dynamics Problems

A number of population dynamics and epidemiological models lead to dynamical systems of the form

$$x' = Ax + f(x), \qquad x \in \mathbf{R}^n_+, \tag{2.1}$$

with $A=(a_{ij})$ an essentially non-negative $n \times n$ matrix, that is,

$$a_{ij} \geqslant 0, \qquad \text{if} \quad i \neq j, \tag{2.2}$$

and f continuous and homogeneous of degree 1, that is, $f(\alpha x)=\alpha f(x)$, for all $\alpha > 0$. The components x_i of x are restricted to be non-negative, and we assume that the components f_i of f satisfy $f_i|_{x_i=0} \geqslant 0$, hence, the flow induced by (2.1) leaves $\mathbf{R}^n_+$ invariant.

Introducing the variable

$$y = \frac{x}{\sum_{i=1}^n x_i}, \tag{2.3}$$

we note that $\sum y_i = 1$, and that y satisfies

$$y' = Ay - \left(\sum_{i,j=1}^n a_{ij} y_j\right) y - \left(\sum_{i=1}^n f_i(y)\right) y + f(y). \tag{2.4}$$

Clearly, the simplex

$$S = \left\{ y : y_i \geqslant 0, \sum_{i=1}^n y_i = 1 \right\} \tag{2.5}$$

is invariant under the flow induced by (2.4) since on S, $(\sum y_i)' = 0$. We are interested in obtaining general conditions which rule out the existence of periodic solutions of (2.4) on S. These results have some direct applications, and we refer the reader to our previous work [4, 5] for a detailed discussion of a class of epidemic models which are special cases of (2.1). Here we consider systems of the form (2.1) which occur in ecological models. The case of mutualistic (or competitive) systems has been widely studied and there are several basic results that limit the complexity of their dynamic behavior, see, for example, Hirsch [13], Smith [17], and Butler, Schmid, and Waltman [6]. In such situations the homogeneous non-linearity often takes the form

$$f(x) = (\text{Diag } x)\, \phi(x) = (x_1 \phi_1(x), ..., x_n \phi_n(x)), \tag{2.6}$$

where Diag x denotes the diagonal matrix with x_i as the ith diagonal entry, and ϕ is continuous and homogeneous of degree zero:

$$\phi(\alpha x) = \phi(x), \quad \forall \alpha > 0. \tag{2.7}$$

In this case we have the following theorem.

THEOREM 2.1. *Let A be a 3×3 matrix whose entries satisfy condition* (2.2), *let $\phi \in C^1(\mathbf{R}^3_+, \mathbf{R}^3_+)$ satisfy* (2.7), *and f be given by* (2.6). *If*

$$\frac{\partial \phi_i}{\partial x_i} + \frac{\partial \phi_j}{\partial x_j} - \left(\frac{\partial \phi_i}{\partial x_j} + \frac{\partial \phi_j}{\partial x_i}\right) \leqslant 0, \qquad i \neq j, \qquad i = 1, 2, 3, \tag{2.8}$$

then the system (2.1), *leaves the simplex $S = \{y_i \geqslant 0, \sum y_i = 1\}$ invariant and has no periodic solutions, limit cycles, or closed phase polygons on S.*

Proof. Starting with (2.4) and using (2.6) we obtain

$$y' = Ay - \left(\sum_{i,j} a_{ij} y_j\right) y + (\text{Diag } y)\,\phi(y) - \left(\sum_i y_i \phi_i(y)\right) y. \tag{2.9}$$

Using the fact that $\sum y_i = 1$, it is easily seen that the vector field defined by the right-hand side of (2.9) is orthogonal to the vector (1, 1, 1), hence, that it is tangent to S. From this follows the invariance of S under the flow induced by (2.9).

Next, for simplicity, denote the right-hand side of (2.9) by $h(y)$, with components $h_i(y)$, $i = 1, 2, 3$, and note that for $y \in S$, we can rewrite $h_i(y)$ in the equivalent forms

$$\begin{aligned}
H_{12}(y_1, y_2) &\equiv h_1(y_1, y_2, 1 - y_1 - y_2) = h_1(y_1, 1 - y_1 - y_3, y_3) \\
&\equiv H_{13}(y_1, y_3) \\
H_{21}(y_1, y_2) &\equiv h_2(y_1, y_2, 1 - y_1 - y_2) = h_2(1 - y_2 - y_3, y_2, y_3) \\
&\equiv H_{23}(y_2, y_3) \\
H_{31}(y_1, y_3) &\equiv h_3(y_1, 1 - y_1 - y_3, y_3) = h_3(1 - y_2 - y_3, y_2, y_3) \\
&\equiv H_{32}(y_2, y_3).
\end{aligned}$$

Now, let $g: \mathbf{R}^3_+ \to \mathbf{R}^3_+$ be defined by

$$g(y) = \frac{1}{y_1 y_2 y_3} \begin{pmatrix} y_2 H_{31}(y_1, y_3) - y_3 H_{21}(y_1, y_2) \\ y_3 H_{12}(y_1, y_2) - y_1 H_{32}(y_2, y_3) \\ y_1 H_{23}(y_2, y_3) - y_2 H_{13}(y_1, y_3) \end{pmatrix}$$

and compute curl g to obtain, after some lengthy but straightforward calculations,

$$\begin{aligned}
(\text{curl } g) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = &-\frac{1}{y_1 y_2 y_3} \sum_{i,j,\, i \neq j} \frac{y_j}{y_i} a_{ij} \\
&+ \frac{1}{y_3}\left[\frac{\partial \phi_1}{\partial y_1} + \frac{\partial \phi_2}{\partial y_2} - \left(\frac{\partial \phi_1}{\partial y_2} + \frac{\partial \phi_2}{\partial y_1}\right)\right] \\
&+ \frac{1}{y_2}\left[\frac{\partial \phi_1}{\partial y_1} + \frac{\partial \phi_3}{\partial y_3} - \left(\frac{\partial \phi_1}{\partial y_3} + \frac{\partial \phi_3}{\partial y_1}\right)\right] \\
&+ \frac{1}{y_1}\left[\frac{\partial \phi_2}{\partial y_2} + \frac{\partial \phi_3}{\partial y_3} - \left(\frac{\partial \phi_2}{\partial y_3} + \frac{\partial \phi_3}{\partial y_2}\right)\right] < 0,
\end{aligned} \tag{2.10}$$

the last inequality following by the hypotheses (2.2) and (2.8) of the theorem. Now, by the definition of g, we see that

$$g|_S = \frac{y}{y_1 y_2 y_3} \times h,$$

hence, $g \cdot h = 0$ on S, and applying Corollary 1.4, the theorem is proved. ∎

Remark 4. The conclusions of Theorem 2.1 hold if the inequalities in (2.2) and (2.8) are both reversed. Also, a somewhat more general condition than the requirement that (2.2) and (2.8) hold, and which still leaves the theorem valid, is that the inequality in (2.10) holds. This, of course, is a more complicated condition to check.

Remark 5. The conditions (2.2) and (2.8) can be viewed from the perspective of the population model in several ways depending on particular added hypotheses. For example, in a model of symbiotic mutualism, but where crowding causes self regulation of each of the species, then (2.2) and (2.8) are automatically satisfied, and Theorem 2.1 rules out the possibility of periodic solutions. On the other hand, when crowding can result in both interspecific and intraspecific competition, requiring $\partial \phi_i / \partial x_j \leqslant 0$, then (2.8) states that the net effect of interspecific competition between any pair of the three species does not exceed the total effect of intraspecific competition for the same two species. In that situation, the theorem again rules out periodic states. Other similar interpretations that apply to different biological situation can clearly be obtained from either (2.2) and (2.8), or from (2.10); or else from situations where the signs are reversed in (2.2) and (2.8) or in (2.10). A parallel result to Theorem 2.1 which imposes conditions that are specific to epidemic models is given in [5].

3. Homogeneous Systems

Let $f: \mathbf{R}^3 \to \mathbf{R}^3$ be continuous and positive homogeneous of degree δ. That is,

$$f(\alpha x) = \alpha^\delta f(x), \qquad x \in \mathbf{R}^3, \qquad \text{and} \qquad \alpha > 0.$$

We consider the homogeneous, autonomous system

$$x' = f(x), \tag{3.1}$$

and use Corollary 1.2 to obtain conditions which exclude certain types of oscillatory solutions. We then combine these results with a condition characterizing the stability of homogeneous systems that is due to Busenberg and Jaderberg [3] to obtain computable conditions for the asymptotic stability of the origin of such systems.

First, let $\phi: \mathbf{R}^3 \to \mathbf{R}_+$ be of class C^1, homogeneous of degree one, and positive definite. Define

$$y = \frac{x}{\phi(x)}, \qquad S = \{x : \phi(x) = 1\}, \tag{3.2}$$

and note that $\phi(y) = \phi'(y) \cdot y = 1$, the second equality following from the Euler equation for homogeneous functions, and where the gradient of ϕ is denoted by ϕ'. Now, use the homogeneity of f and ϕ to get

$$y' = \phi^{\delta-1}(x)[f(y) - \phi'(y) \cdot f(y)\, y].$$

For any solution $x(t)$ which does not vanish for any $t > 0$, introduce the new time variable

$$\tau = \int_0^t \phi^{\delta-1}(x(s))\, ds$$

to obtain the equivalent autonomous system

$$\frac{dy}{d\tau} = f(y) - \phi'(y) \cdot f(y)\, y, \tag{3.3}$$

which describes the projection of the flow of (3.1) onto the surface S. Noting that (3.3) is the same for all nonvanishing solutions $x(t)$ of (3.1), we have the following lemma.

LEMMA 3.1. *Let $f: \mathbf{R}^3 \to \mathbf{R}^3$ be continuous and homogeneous of degree δ, and let $\phi \in C^1(\mathbf{R}^3, \mathbf{R}_+)$ be homogeneous of degree one and positive definite. Then, the projection $y(t)$ of any non-zero solution $x(t)$ of* (3.1) *on the level surface S coincides with an orbit of* (3.3).

Applying Corollary 1.4, we note that if $g: \mathbf{R}^3 \to \mathbf{R}^3$ is orthogonal to the right hand side of (3.3) on S (whose unit normal is $v(y) = \phi'(y)/[\phi'(y) \cdot \phi'(y)]$), that is, if

$$g(y) \cdot [f(y) - \phi'(y) \cdot f(y)\, y] = 0, \tag{3.4}$$

then the condition

$$(\operatorname{curl} g(y)) \cdot \phi'(y) > 0 \qquad (\text{or } < 0), \qquad y \in S \tag{3.5}$$

rules out the existence of periodic solutions to (3.1) on S. Note that, when $g(y)=\rho(y)\,\phi'(y)$, $\rho: \mathbf{R}^3 \to \mathbf{R}$, then (3.4) is automatically satisfied, but $(\operatorname{curl} g(y)) \cdot \phi'(y)=0$, hence, (3.5) cannot hold. Thus we need to use vector fields g which satisfy (3.4) and are orthogonal to $\phi'(y)$ for $y \in S$. One such class of fields is

$$g(y)=\rho(y)\,\phi'(y) \times [f(y)-\phi'(y)\cdot f(y)\,y]. \tag{3.6}$$

Different choices of ρ yield different sufficient criteria for the non-existence of limit cycles. For example, choosing

$$\rho(y)=\frac{1}{\phi'(y)\cdot\phi'(y)},$$

substituting (3.6) in (3.5), and collecting terms, we obtain

$$\begin{aligned}\phi'(y)\cdot(\operatorname{curl} g(y)) &= \operatorname{trace} f'(y)-3\phi'(y)\cdot f(y)\\ &>0 \qquad (\text{or } <0) \qquad \text{on } S\end{aligned} \tag{3.7}$$

as a condition for the non-existence of periodic solutions, limit cycles, or closed phase polygons of (3.3) on S. The condition (3.7) was obtained for the case $\delta=1$ by Hadeler [8] who gave a proof based on the special case of Theorem 1.1 that we gave in [4].

Other such computable sufficient conditions can be derived from Corollary 1.2. For example, if we choose g such that

$$\phi'(y)\cdot(\operatorname{curl} g(y))=0, \tag{3.8}$$

then the condition

$$g(y)\cdot[f(y)-\phi'(y)\cdot f(y)\,y]>0 \qquad (\text{or } <0) \qquad \text{a.e. on } S \tag{3.9}$$

again rules out periodic solutions. An example of a special case of (3.9) is obtained by choosing

$$g(y)=c\begin{pmatrix} y_1^{\alpha_1}\\ y_2^{\alpha_2}\\ y_3^{\alpha_3}\end{pmatrix},$$

where $c \neq 0$ is a constant. Then (3.8) holds, and the criterion becomes

$$\begin{aligned}&c\left[\sum_{i=1}^{3} y_i^{\alpha_i} f_i(y)-\phi'(y)\cdot f(y)\sum_{i=1}^{3} y_i^{\alpha_i+1}\right]\\ &\qquad >0 \qquad (\text{or } <0) \text{ on } S.\end{aligned} \tag{3.10}$$

In terms of the original variable x, when $\alpha_i = \alpha$, this condition becomes

$$c\left[\phi(x) \sum_{i=1}^{3} x_i^{\alpha} f_i(x) - \phi'(x) \cdot f(x) \sum_{i=1}^{3} x_i^{\alpha+1}\right] > 0 \qquad (\text{or } <0, \text{ for } x \neq 0). \tag{3.11}$$

A particularly simple version of (3.10) is obtained by choosing $\alpha_i = 0$ or $\alpha_i = -1$ for all $i = 1, 2, 3$, in which case (3.10) becomes, respectively,

$$\sum_{i=1}^{3} f_i(y) - \phi'(y) \cdot f(y) \sum_{i=1}^{3} y_i > 0 \qquad (\text{or } <0), \text{ on } S, \tag{3.12}$$

and

$$\sum_{i=1}^{3} \frac{f_i(y)}{y_i} - 3\phi'(y) \cdot f(y) > 0 \qquad (\text{or } <0), \text{ on } S. \tag{3.13}$$

We collect these special results in the following theorem.

THEOREM 3.2. *Let $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ be continuous and homogeneous of degree δ, and let $\phi\colon \mathbf{R}^3 \to \mathbf{R}$ be positive definite, homogeneous of degree one, and smooth. Suppose that any one of the conditions* (3.7), (3.9)–(3.13) *holds. Then the system* (3.3) *has no periodic solutions, limit cycles, or closed phase polygons on $S = \{y \in \mathbf{R}^3 : \phi(y) = 1\}$.*

We now use the above results in order to obtain computable criteria for the asymptotic stability of the origin for certain classes of homogeneous systems. In order to do this we recall the following result of Busenberg and Jaderberg [3].

THEOREM 3.3 *Let $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ be continuous and homogeneous of degree $\delta > 0$, and let $\phi\colon \mathbf{R}^3 \to \mathbf{R}_+$ be positive definite, homogeneous of degree one, and smooth. Then the origin is an asymptotically stable solution of* (3.1), *if and only if every solution $y(\tau)$ of* (3.3) *satisfies*

$$\limsup_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} \phi'(y(s)) \cdot f(y(s))\, ds < 0. \tag{3.14}$$

Remark 6. Theorem 3.3 gives only one of several equivalent conditions obtained in [3]. The proofs in [3] are actually given only for $\phi(x) = \sqrt{\sum x_i^2}$, however, they extend directly to the more general case that we are discussing here.

THEOREM 3.4. *Let f and $\phi \in C^2$ satisfy the conditions of Theorem 3.3. Suppose that the nonlinear eigenvalue problem $f(y) = \lambda y$ has only a finite number of solutions (λ_i, y_i) on $S = \{y \in \mathbf{R}^3 : \phi(y) = 1\}$, and that any one of the conditions* (3.7), (3.9)–(3.13) *hold. Then the origin is an asymptotically stable solution of* (3.1) *if and only if*

$$\phi'(y_i) \cdot f(y_i) < 0, \tag{3.15}$$

for every eigenvector y_i of f on S.

Proof. Since the surface $S \subset \mathbf{R}^3$ is two-dimensional and of class C^2, any solution of the system (3.3) on S either tends to an equilibrium, that is, to an eigenvector y_i of f on S, or to a limit cycle, or to an oriented phase polygon on S. This follows by the generalization due to Schwartz [16] of the Poincaré–Bendixson theorem on S. Now, by Theorem 3.2, any one of the conditions (3.7), (3.9)–(3.13) implies that such a solution must tend to an equilibrium y_i on S, that is, $y(t) \to y_i$ as $t \to \infty$. Thus, the limit in (3.14) exists and is equal to $\phi'(y_i) \cdot f(y_i)$. Applying Theorem 3.3, we see that the origin is asymptotically stable if, and only if, (3.15) holds for every eigenvector y_i of f on S. ∎

Special choices of ϕ can yield particularly simple criteria for the non-existence of limit cycles. For example, if

$$\phi(x) = \left(\sum_{i=1}^{3} x_i^2 \right)^{1/2},$$

then condition (3.12) becomes

$$\sum_{i=1}^{3} f_i(y) - \left(\sum_{i=1}^{3} y_i \right) \sum_{i=1}^{3} y_i f_i(y)$$
$$> 0 \qquad (\text{or } < 0) \text{ on } S = \{ y \in \mathbf{R}^3 : \phi(y) = 1 \}.$$

Remark 7. The above results have a natural generalization to the case where $f: K \to K$, where K is a cone with nonempty interior in $\mathbf{R}^3$. In that situation it suffices to require that the hypotheses of Theorem 3.2 hold on $S_K = \{x \in K : \phi(x) = 1\}$ only, in order to obtain the non-existence of limit cycles or phase polygons on S_K. Similarly, in this situation, if the conditions of Theorem 3.4 hold only on S_K, then the origin will attract all solutions of (3.1) whose initial point lies in K. A special case of such a cone K which occurs in many biological and physical applications is $\mathbf{R}^3_+$. In this case, we can take $\phi: \mathbf{R}^3_+ \to \mathbf{R}_+$, as for example, $\phi(x) = \sum x_i$.

4. Application to Delay Differential Equations

In this section we apply our results to the study of certain classes of solutions of delay differential equations. Using the standard notation in this field (see the basic books by Bellman and Cooke [2] and by Hale [10]), we let C be the Banach space of continuous functions $h: [-r, 0] \to \mathbf{R}^n$ with uniform norm. For any continuous function $x: [-r, \infty]$ we let $x_t \in C$ be the function giving the "past history" of x at $t \geqslant 0$ and defined by $x_t(s) = x(t+s)$, $s \in [-r, 0]$. We let $f: C \to \mathbf{R}^n$, and consider the delay differential equation

$$x' = f(x_t). \tag{4.1}$$

We give criteria for the non-existence of certain types of solutions for such equations.

Before proceeding, we note that the graphs of periodic solutions of (4.1) in $\mathbf{R}^n$ may have several self intersection points, and in fact, entire intervals where they self intersect. This is possible because the Cauchy problem for (4.1) has a unique solution in the function space C but not necessarily a non-intersecting trajectory in the state space $\mathbf{R}^n$. For example, the delay differential equation for $(x(t), y(t)) \in \mathbf{R}^2$

$$x'(t) = -x\left(t - \frac{\pi}{2}\right), \qquad y'(t) = -x\left(t - \frac{\pi}{2}\right), \tag{4.2}$$

has the 2π periodic solution $(x(t), y(t)) = (\sin t, \sin t)$ which traces the line (x, x), $x \in [-1, 1]$ in $\mathbf{R}^2$ twice over each period. Moreover, this periodic solution does not have a graph in $\mathbf{R}^2$ which is the boundary of any region. These considerations lead us to introduce the following definition.

Definition. A *simple loop solution* of (4.1) is any solution $x \in C$ of (4.1) with orbit $x(t) \in \mathbf{R}^n$, $n \geqslant 2$, $t \in \mathbf{R}$, containing a piecewise smooth, oriented, simple, closed curve in $\mathbf{R}^n$.

Note that a simple loop solution of (4.1) need not be a periodic or even bounded solution of (4.1) and, as shown in the above example, a periodic solution of a delay differential equation need not be a simple loop solution. This concept of simple loop solutions can be used with equations in $\mathbf{R}^1$ by introducing auxiliary variables. For example, if we define $y(t) = x(t+\tau)$, $\tau > 0$, and use the corresponding equation $y'(t) = f(x_{t+\tau})$, we can couple this equation with (4.1), to obtain a redundant system in $\mathbf{R}^2$ for which the above definition applies. For specific equations other auxiliary variables may be more natural. For instance, for the equation $x'(t) = f(x(t), x(t-1))$, the variable $y(t) = x(t-1)$ leads to the coupled system in $\mathbf{R}^2$, $x' = f(x, y)$, $y' = f(y, y_t)$.

We have the following result concerning simple loop solutions.

THEOREM 4.1. *Suppose that* $f: C([-r, 0], \mathbf{R}^3) \to \mathbf{R}^3$ *is Lipschitz continuous and that* S *is an oriented smooth surface in* $\mathbf{R}^3$ *with unit normal* v. *Let the boundary of* S *consist of a smooth, simple, closed curve* $\gamma(t)$ *parametrized as a periodic function of* t. *If there exists a function* $g \in C^1(\mathbf{R}^3, \mathbf{R}^3)$ *such that*

$$g(\gamma(t)) \cdot f(\gamma_t) \leqslant 0 \tag{4.3}$$

and

$$(\operatorname{curl} g(x)) \cdot v \geqslant 0, \qquad \textit{on } S \textit{ and} \qquad >0 \textit{ at some } x_0 \in S, \tag{4.4}$$

then $\gamma(t)$ *is not the orbit of a simple loop solution of* (4.1) *which is traversed in the clockwise sense defined by* v. *If the inequality in* (4.3) *is replaced by equality, then* $\gamma(t)$ *is not the orbit of any simple loop solution of* (4.1)

Proof. The proof of this result follows the same lines as the proofs of Theorem 1.1 and Corollary 1.4. However, note that if $\gamma(t)$ is the orbit of a simple loop solution, then $\gamma'(t) = f(\gamma_t)$ since, by the definition of such solutions, this graph is traversed in one direction only. ▮

It is worth noting that the conclusion of this result cannot be strengthened to claim that $\gamma(t)$ is not an oriented subset of the graph of a simple loop solution, even when the period of $\gamma(t)$ is greater than or equal to the delay r, since the past history γ_t need not coincide with a point x_t of the solution of (4.1) on the space C.

An immediate corollary of this result is the following:

COROLLARY 4.2. *Suppose that* S, f, *and* g *satisfy the smoothness conditions of Theorem* 4.1 *and that* $g(\gamma(t) \cdot f(\gamma_t) = 0$ *holds for every smooth curve* $\gamma: [\tau - r, \tau] \to S$ *for some* $\tau \geqslant 0$. *Then if* (4.4) *holds, there is no simple loop solution of* (4.1) *whose orbit is in* S.

A simple application of the above result is obtained by considering the following delay differential equation system,

$$\begin{aligned} x'(t) &= F_1(x, y)\, h(x_t, y_t) + l_1(x, y) \\ y'(t) &= F_2(x, y)\, h(x_t, y_t) + l_2(x, y), \end{aligned} \tag{4.5}$$

with $l_2 F_1 = l_1 F_2$ and satisfying the smoothness conditions of Theorem 4.1. Suppose that there exists $\rho(x, y) \in C^1(A)$ for some open region $A \in \mathbf{R}^2$ with

$$\frac{\partial}{\partial x}(\rho F_1) + \frac{\partial}{\partial y}(\rho F_2) > 0 \qquad (\text{or } <0) \text{ on } A. \tag{4.6}$$

Then the system (4.5) cannot have any simple loop solutions with graph contained in A. In order to see this, note that we can augment (4.5) with an additional equation $z'(t)=0$, and choose $g=(-\rho F_2, \rho F_1, 0)$. Then, for any curve $\gamma(t)=(x(t), y(t))$ in A we have

$$g(\gamma(t))\cdot(F_1(\gamma(t))\,h(x_t, y_t)+l_1(\gamma(t)), F_2(\gamma(t))\,h(x_t, y_t)+l_2(\gamma(t)), 0)=0, \tag{4.7}$$

by the hypotheses on F_i and l_i. Now, when A is regarded as a surface S embedded in $\mathbf{R}^3$, it has the unit normal $\nu=(0,0,1)$, hence,

$$(\text{curl } g(x))\cdot\nu=\frac{\partial}{\partial x}(\rho F_1)+\frac{\partial}{\partial y}(\rho F_2),$$

which is either always positive or always negative. The conclusion now follows from an application of Corollary 4.2.

We note that for a delay differential equation with constant delays, the checking of condition (4.7) becomes simpler since h now depends on the values of the solution at a discrete set of points. For example, if $h(x_t, y_t)=H(x(t), x(t-r_1), y(t), y(t-r_2))$, then the condition $l_2F_1=l_1F_2$ can be replaced by the condition that for any two pairs of points $(x, y), (u, v)\in A$ we have

$$g(x,y)\cdot(F_1(x,y)\,h(u,v)+l_1(x,y), F_2(x,y)\,h(u,v)+l_2(x,y), 0)=0. \tag{4.8}$$

We note again that these conditions do not rule out the possibility of periodic solutions of the delay differential equations on the function space C. In fact, the system (4.2) satisfies the conditions of the above example with $l_i=0$, and $F_1=F_2=-1$. If one chooses $\rho(x,y)=x$, and the region A to be all of $\mathbf{R}^2$, we get

$$\frac{\partial}{\partial x}(\rho F_1)+\frac{\partial}{\partial y}(\rho F_2)=-1.$$

However, as we have seen above, this system has a period solution. This is not a simple loop solution since its graph in $\mathbf{R}^2$ is a line segment. In fact, our result shows that the system (4.2) does not have any simple loop solutions.

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