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On the number of linear forms in logarithms

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Abstract

Let n be a positive integer. In this paper we estimate the size of the set of linear forms $b_1 \log a_1 + b_2 \log a_2 + \cdots + b_n \log a_n$, where $|b_i| \leqslant B_i$ and $1 \leqslant a_i \leqslant A_i$ are integers, as $A_i, B_i \to \infty$.

1. Introduction

The theory of linear forms in logarithms, developed by A. Baker [1,2] in the 60's, is a powerful method in the transcendental number theory. It consists of finding lower bounds for $|b_1 \log a_1 + b_2 \log a_2 + \cdots + b_n \log a_n|$, where the b_i are integers and the a_i are algebraic numbers for which $\log a_i$ are linearly independent over $\mathbb{Q}$. We consider the simpler case where the $a_i > 0$ are integers, and we let $B_j = \max\{|b_j|, 1\}$, and $B = \max_{1 \leqslant j \leqslant n} B_j$.

Lang and Waldschmidt [4, Introduction to chapter X and XI, p. 212] conjectured the following

Conjecture. *Let $\epsilon > 0$. There exists $C(\epsilon) > 0$ depending only on ϵ, such that*

$$|b_1 \log a_1 + b_2 \log a_2 + \cdots + b_n \log a_n| > \frac{C(\epsilon)^n B}{(B_1 \cdots B_n a_1 \cdots a_n)^{1+\epsilon}}.$$

One part of the argument they used to motivate the Conjecture, is that the number of distinct linear forms $b_1 \log a_1 + b_2 \log a_2 + \cdots + b_n \log a_n$, where $|b_j| \leqslant B_j$ and $0 < a_j \leqslant A_j$, is $\asymp B_1 \cdots B_n A_1 \cdots A_n$, if the A_i and the B_i are sufficiently large.

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In this paper we estimate the number of these linear forms as $A_i, B_i \to \infty$.

An equivalent formulation of the problem is to estimate the size of the following set

$$R = R(A_1, \ldots, A_n, B_1, \ldots, B_n) := \left\{r \in \mathbb{Q}\colon\ r = a_1^{b_1} a_2^{b_2} \cdots a_n^{b_n},\ 1 \leqslant a_i \leqslant A_i,\ |b_i| \leqslant B_i\right\},$$

as $A_i, B_i \to \infty$.

For the easier case $A_i = A$ and $B_i = B$ for all i, a trivial upper bound on $|R|$ is $2^n A^n B^n / n! + o(A^n B^n)$, since permuting the numbers $a_i^{b_i}$ gives rise to the same number r.

We prove that this bound is attained asymptotically as $A, B \to \infty$. Also we deal with the general case, which is harder since not every permutation is allowed for all the ranges. Indeed the size of R depends on the ranges of the A_i and the B_i, as we shall see in Corollaries 1 and 2.

Let $E \subset \{(a_1, \ldots, a_n, b_1, \ldots, b_n),\ 1 \leqslant a_i \leqslant A_i,\ |b_i| \leqslant B_i\}$. We say that $r \in \mathbb{Q}$ has a representation in E, if $r = a_1^{b_1} a_2^{b_2} \cdots a_n^{b_n}$, for some $(a_1, \ldots, a_n, b_1, \ldots, b_n) \in E$.

For $r \in R$, if $\sigma \in S_n$ satisfies $1 \leqslant a_{\sigma(i)} \leqslant A_i$, and $|b_{\sigma(i)}| \leqslant B_i$ for all i, we say that σ permutes r, or σ is a possible permutation for the $a_i^{b_i}$. Finally we say that a permutation $\sigma \in S_n$ is permissible if

$$\left|\{r \in R\colon\ \sigma \text{ permutes } r\}\right| \gg A_1 \cdots A_n B_1 \cdots B_n.$$

The main result of this paper is the following

Theorem. *There exists a set $E \subset \{(a_1, \ldots, a_n, b_1, \ldots, b_n),\ 1 \leqslant a_i \leqslant A_i,\ |b_i| \leqslant B_i\}$ satisfying*

$$|E| \sim 2^n A_1 A_2 \cdots A_n B_1 B_2 \cdots B_n,$$

as $A_i, B_i \to \infty$, such that any rational number $r \in \{a_1^{b_1} a_2^{b_2} \cdots a_n^{b_n}\colon (a_1, \ldots, a_n, b_1, \ldots, b_n) \in E\}$ has a unique representation in E up to permissible permutations.

From this result we can deduce that $|R|$ is asymptotic to the cardinality of the set of $2n$-tuples $\{(a_1, \ldots, a_n, b_1, \ldots, b_n),\ 1 \leqslant a_i \leqslant A_i,\ |b_i| \leqslant B_i\}$ modulo permissible permutations.

In the case $A_i = A$, $B_i = B$, every permutation is permissible and we deduce the following corollary.

Corollary 1. *As $A, B \to \infty$, we have*

$$\left|\left\{r \in \mathbb{Q}\colon\ r = a_1^{b_1} a_2^{b_2} \cdots a_n^{b_n},\ 1 \leqslant a_i \leqslant A,\ |b_i| \leqslant B\right\}\right| = \frac{2^n A^n B^n}{n!} + o\big(A^n B^n\big).$$

Now suppose that $A_i = o(A_{i+1})$ for all $1 \leqslant i \leqslant n-1$, or $B_i = o(B_{i+1})$ for all $1 \leqslant i \leqslant n-1$. For a non-identity permutation $\sigma \in S_n$, there exists j for which $\sigma(j) \neq j$. Therefore if σ permutes $r = a_1^{b_1} a_2^{b_2} \cdots a_n^{b_n}$, we must have $1 \leqslant a_j, a_{\sigma(j)} \leqslant \min(A_j, A_{\sigma(j)})$ and $-\min(B_j, B_{\sigma(j)}) \leqslant b_j, b_{\sigma(j)} \leqslant \min(B_j, B_{\sigma(j)})$. And so we deduce that

$$\begin{aligned}
&\left|\left\{r \in \mathbb{Q}\colon\ r = a_1^{b_1} a_2^{b_2} \cdots a_n^{b_n},\ 1 \leqslant a_i \leqslant A_i,\ |b_i| \leqslant B_i\colon \sigma \text{ permutes } r\right\}\right| \\
&\quad \leqslant 2^n A_1 \cdots A_n B_1 \cdots B_n \left(\frac{\min(A_j, A_{\sigma(j)}) \min(B_j, B_{\sigma(j)})}{\max(A_j, A_{\sigma(j)}) \max(B_j, B_{\sigma(j)})}\right) = o(A_1 \cdots A_n B_1 \cdots B_n),
\end{aligned}$$

by our assumption on the A_i and B_i. Thus in this case no permutation $\sigma \neq 1$ is permissible. Therefore we have

Corollary 2. *If $A_i = o(A_{i+1})$ for all $1 \leqslant i \leqslant n-1$, or $B_i = o(B_{i+1})$ for all $1 \leqslant i \leqslant n-1$, then*

$$\left|\left\{r \in \mathbb{Q}\colon\ r = a_1^{b_1} a_2^{b_2} \cdots a_n^{b_n},\ 1 \leqslant a_i \leqslant A_i,\ |b_i| \leqslant B_i\right\}\right| \sim 2^n A_1 \cdots A_n B_1 \cdots B_n,$$

as $A_i, B_i \to \infty$.

We can observe that Corollaries 1 and 2 correspond to extreme cases: in Corollary 1 all permutations are permissible, while none is permissible in Corollary 2. Indeed we can prove

Corollary 3. *As $A_i, B_i \to \infty$, we have*

$$\frac{2^n}{n!} A_1 \cdots A_n B_1 \cdots B_n \lesssim |R| \lesssim 2^n A_1 \cdots A_n B_1 \cdots B_n.$$

Moreover two bounds are optimal.

Proof. From the Theorem we have that

$$|R| \sim \sum_{\substack{1 \leqslant a_1 \leqslant A_1 \\ |b_1| \leqslant B_1}} \cdots \sum_{\substack{1 \leqslant a_n \leqslant A_n \\ |b_n| \leqslant B_n}} \frac{1}{|\{\sigma \in S_n\colon\ \sigma \text{ is possible for the } a_i^{b_i}\}|}.$$

The result follows from the fact that $1 \leqslant |\{\sigma \in S_n\colon\ \sigma \text{ is possible for the } a_i^{b_i}\}| \leqslant n!$. □

For the simple case $n = 2$, there is only one non-trivial permutation $\sigma = (12)$. This permutation is possible only if $1 \leqslant a_1, a_2 \leqslant \min(A_1, A_2)$ and $|b_1|, |b_2| \leqslant \min(B_1, B_2)$. Then by the Theorem, and after a simple calculation we deduce that

$$\begin{aligned}
&\left|\left\{r \in \mathbb{Q}\colon\ r = a_1^{b_1} a_2^{b_2},\ 1 \leqslant a_1 \leqslant A_1,\ 1 \leqslant a_2 \leqslant A_2,\ |b_1| \leqslant B_1,\ |b_2| \leqslant B_2\right\}\right| \\
&\quad \sim 4 A_1 A_2 B_1 B_2 - 2 \min(A_1, A_2)^2 \min(B_1, B_2)^2.
\end{aligned}$$

In general the size of $|R|$ is asymptotic to a homogeneous polynomial of degree $2n$ in the variables $A_1, \ldots, A_n, B_1, \ldots, B_n$. Moreover it is also necessary to order the A_i's and B_i's, so without loss of generality we assume that $A_1 \leqslant A_2 \leqslant \cdots \leqslant A_n$ and $B_{\pi(1)} \leqslant B_{\pi(2)} \leqslant \cdots \leqslant B_{\pi(n)}$, where $\pi \in S_n$ is a permutation.

We prove the following

Proposition. *Suppose that $A_1 \leqslant A_2 \leqslant \cdots \leqslant A_n$ and $B_{\pi(1)} \leqslant B_{\pi(2)} \leqslant \cdots \leqslant B_{\pi(n)}$, where $\pi \in S_n$ is a permutation. Also let $A_0 = B_{\pi(0)} = 1$.*

Then $|R|$ is asymptotic to

$$2^n \sum_{\substack{i_1 = 1 \\ 1 \leqslant j_1 \leqslant \pi^{-1}(1)}} \sum_{\substack{1 \leqslant i_2 \leqslant 2 \\ 1 \leqslant j_2 \leqslant \pi^{-1}(2)}} \cdots \sum_{\substack{1 \leqslant i_n \leqslant n \\ 1 \leqslant j_n \leqslant \pi^{-1}(n)}} \frac{\prod_{k=1}^{n} (A_{i_k} - A_{i_k - 1})(B_{\pi(j_k)} - B_{\pi(j_k - 1)})}{|\{\sigma \in S_n\colon\ i_{\sigma(l)} \leqslant l,\ j_{\sigma(l)} \leqslant \pi^{-1}(l),\ \forall 1 \leqslant l \leqslant n\}|},$$

as $A_i, B_i \to \infty$.

2. Preliminary lemmas

Let C be a positive real number. We say that the n-tuple $(a_1, \dots, a_n)$ satisfies condition (1_C), if there exists a prime p, such that $p^k \mid a_1 a_2 \cdots a_n$ where $k \geqslant 2$, and $p^k \geqslant C$.

Lemma 1. *We have*

$$\left|\left\{(a_1, \dots, a_n),\ 1 \leqslant a_i \leqslant A_i\colon \text{ which satisfy } (1_C)\right\}\right| \ll_n \frac{A_1 \cdots A_n (\log C)^n}{\sqrt{C}}.$$

Proof. First we have

$$\begin{aligned}
&\left|\left\{(a_1, \dots, a_n), 1 \leqslant a_i \leqslant A_i\colon \text{ which satisfy } (1_C)\right\}\right| \\
&\quad \leqslant \sum_p \left|\left\{(a_1, \dots, a_n), 1 \leqslant a_i \leqslant A_i\colon \exists k \geqslant 2,\ p^k \geqslant C, \text{ and } p^k \mid a_1 a_2 \cdots a_n\right\}\right|. \qquad (1)
\end{aligned}$$

Case 1. $p \leqslant \sqrt{C}$.

In this case pick k to be the smallest integer such that $p^k \geqslant C$, i.e. $k = [\log C / \log p] + 1$. Then the number of $(a_1, \dots, a_n)$ such that $p^k \mid a_1 a_2 \cdots a_n$ is equal to

$$\sum_{d_1 d_2 \cdots d_n = p^k} \prod_{i=1}^{n} \sum_{\substack{1 \leqslant a_i \leqslant A_i \\ d_i \mid a_i}} 1 \leqslant d_n\left(p^k\right) \frac{A_1 \cdots A_n}{p^k} \leqslant d_n\left(p^k\right) \frac{A_1 \cdots A_n}{C}.$$

Now $d_n(p^k) = \binom{n+k-1}{k}$, and by Stirling's formula, for k large enough we have

$$\begin{aligned}
\log d_n\left(p^k\right) &= \left(n + k - 1 + \frac{1}{2}\right) \log(n + k - 1) - \left(k + \frac{1}{2}\right) \log k \\
&\quad - \left(n - 1 + \frac{1}{2}\right) \log(n - 1) + O(1) \\
&\leqslant \left(k + \frac{1}{2}\right) \log\left(1 + \frac{n-1}{k}\right) + \left(n - \frac{1}{2}\right) \log\left(\frac{n - 1 + k}{n - 1}\right) \\
&\leqslant n \log k.
\end{aligned}$$

Then summing over these primes gives

$$\sum_{p \leqslant \sqrt{C}} \left|\left\{(a_1, \dots, a_n),\ 1 \leqslant a_i \leqslant A_i\colon\ p^k \mid a_1 a_2 \cdots a_n\right\}\right| = O_n\left(\frac{A_1 \cdots A_n (\log C)^n}{\sqrt{C}}\right). \qquad (2)$$

Case 2. $p > \sqrt{C}$.

In this case pick $k = 2$. Then the number of $(a_1, \dots, a_n)$ such that $p^2 \mid a_1 a_2 \cdots a_n$ is $O(A_1 \cdots A_n / p^2)$, where the constant involved in the O depends only on n. Therefore summing over these primes gives

$$\sum_{p>\sqrt{C}} \left|\left\{(a_1,\dots,a_n),\ 1\leqslant a_i\leqslant A_i\colon\ p^2 \mid a_1a_2\cdots a_n\right\}\right| = O_n\left(\frac{A_1\cdots A_n}{\sqrt{C}}\right). \tag{3}$$

Thus combining (1)–(3) gives the result. □

We say that $(a_1,\dots,a_n)$ satisfies condition (2_C) if at least one of the a_i is C-smooth: that is has all its prime factors lying below C.

Lemma 2. *Write $C^{u_i}=A_i$ for all $1\leqslant i\leqslant n$. Then uniformly for $\min_{1\leqslant i\leqslant n} A_i\geqslant C\geqslant 2$, we have*

$$\left|\left\{(a_1,\dots,a_n),\ 1\leqslant a_i\leqslant A_i\colon \textit{ which satisfy } (2_C)\right\}\right| \ll_n A_1A_2\cdots A_n\left(\sum_{i=1}^n e^{-u_i/2}\right).$$

Proof. We have that

$$\left|\left\{(a_1,\dots,a_n),\ 1\leqslant a_i\leqslant A_i\colon \text{ which satisfy } (2_C)\right\}\right| \ll_n A_1A_2\cdots A_n\sum_{i=1}^n \frac{\Psi(A_i,C)}{A_i},$$

where $\Psi(x,y)$ is the number of y-smooth positive integers below x. The result follows by the following Theorem of de Bruijn [3]

$$\Psi(A_i,C)\ll A_i e^{-u_i/2},$$

uniformly for $A_i\geqslant C\geqslant 2$. □

We say that $(b_1,b_2,\dots,b_n)$ satisfy condition (3_C), if there exists an n-tuple of integers $|c_i|\leqslant 2\log C$ not all zero, such that $c_1b_1+c_2b_2+\cdots+c_nb_n=0$.

Lemma 3. *We have that*

$$\left|\left\{(b_1,\dots,b_n),\ |b_i|\leqslant B_i\colon \textit{ which satisfy condition } (3_C)\right\}\right| \leqslant B_1B_2\cdots B_n\sum_{i=1}^n\left(\frac{(9\log C)^n}{B_i}\right).$$

Proof. We note that

$$\begin{aligned}
&\left|\left\{(b_1,\dots,b_n),\ |b_i|\leqslant B_i\colon \text{ which satisfy condition } (3_C)\right\}\right|\\
&\quad\leqslant \sum_{\substack{|c_i|\leqslant 2\log C\\ (c_1,\dots,c_n)\neq(0,\dots,0)}} \left|\left\{(b_1,\dots,b_n),\ |b_i|\leqslant B_i\colon\ c_1b_1+c_2b_2+\cdots+c_nb_n=0\right\}\right|\\
&\quad\leqslant \sum_{\substack{|c_i|\leqslant 2\log C\\ (c_1,\dots,c_n)\neq(0,\dots,0)}} (2B_1+1)\cdots(2B_n+1)\sum_{i=1}^n\left(\frac{1}{2B_i+1}\right)\\
&\quad\leqslant B_1B_2\cdots B_n\sum_{i=1}^n\left(\frac{(9\log C)^n}{B_i}\right). \qquad\square
\end{aligned}$$

3. Proof of the results

Proof of the Theorem. We begin by choosing $C := \min(B_1, \ldots, B_n, \log A_1, \ldots, \log A_n)$. We consider the following set

$$E := \big\{(a_1, \ldots, a_n, b_1, \ldots, b_n),\ 1 \leqslant a_i \leqslant A_i,\ |b_i| \leqslant B_i\colon \\ (a_i) \text{ do not satisfy any of } (1_C), (2_C), (b_i) \text{ do not satisfy } (3_C)\big\}.$$

Then by our choice of C, if we combine Lemmas 1–3, we observe that $|E| = 2^n A_1 \cdots A_n B_1 \cdots B_n(1 + o(1))$.

Therefore it remains to prove that any representation of a rational number r as $a_1^{b_1} a_2^{b_2} \cdots a_n^{b_n}$ where $(a_1, \ldots, a_n, b_1, \ldots, b_n)$ belongs to E, is unique up to possible permutations of the $a_i^{b_i}$, and finally we can consider only permissible permutations (since the number of $r \in R$ which can be permuted by a non-permissible permutation is negligible).

We begin by considering the following equation

$$a_1^{b_1} a_2^{b_2} \cdots a_n^{b_n} = e_1^{f_1} e_2^{f_2} \cdots e_n^{f_n}, \tag{4}$$

where $(a_1, \ldots, a_n, b_1, \ldots, b_n)$ and $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ are in E. If for some i, a_i contains a prime factor p such that $p^2 \nmid a_1 a_2 \cdots a_n$ and $p^2 \nmid e_1 e_2 \cdots e_n$, then $b_i \in \{f_1, f_2, \ldots, f_n\}$. Now suppose that there exists $1 \leqslant j \leqslant n$ such that $b_j \notin \{f_1, f_2, \ldots, f_n\}$, then for the all primes p that divide a_j, there exists $k \geqslant 2$ for which $p^k \mid a_1 a_2 \cdots a_n$ or $p^k \mid e_1 e_2 \cdots e_n$, but the (a_i) and the (e_i) do not satisfy condition (1_C) and so we must have $p^k \leqslant C$, which implies that a_j is C-smooth; however this contradicts the fact that the (a_i) do not satisfy condition (2_C). Therefore we deduce that

$$\{b_1, b_2, \ldots, b_n\} = \{f_1, f_2, \ldots, f_n\}.$$

Then up to permutations, we have that $b_i = f_i$, and so Eq. (4) becomes

$$a_1^{b_1} a_2^{b_2} \cdots a_n^{b_n} = e_1^{b_1} e_2^{b_2} \cdots e_n^{b_n}. \tag{5}$$

Let p be any prime dividing $a_1 a_2 \cdots a_n$, and let $\alpha_i \geqslant 0$ and $\beta_i \geqslant 0$ be the corresponding powers of p in a_i and e_i respectively, and let $c_i = \alpha_i - \beta_i$. Then Eq. (5) implies that

$$c_1 b_1 + c_2 b_2 + \cdots + c_n b_n = 0.$$

Now the (a_i) and the (e_i) do not satisfy condition (1_C), and so $0 \leqslant \alpha_i, \beta_i \leqslant \log C / \log 2 \leqslant 2 \log C$, which implies that $|c_i| \leqslant 2 \log C$. And since the (b_i) do not satisfy condition (3_C), we deduce that $c_i = 0$, and then $\alpha_i = \beta_i$ for all $1 \leqslant i \leqslant n$. Since this is true for every prime factor of $a_1 a_2 \cdots a_n$, we must have $a_i = e_i$ for all $1 \leqslant i \leqslant n$, and our Theorem is proved. □

Proof of the Proposition. We want to count the number of elements $r = (r_1, \ldots, r_n)$, where $r_i = (a_i, b_i) \in [1, A_i] \times [-B_i, B_i] \cap \mathbb{Z} \times \mathbb{Z}$, modulo possible permutations of the r_i's.

Since the number of r for which some b_i is 0, is $o(A_1 \cdots A_n B_1 \cdots B_n)$, we can suppose that all the b_i's are positive by symmetry.

Moreover let $R_i := [1, A_i] \times [1, B_i] \cap \mathbb{Z} \times \mathbb{Z}$, and define the following distinct discrete sets $R_{ij} := [A_{i-1}, A_i] \times [B_{\pi(j-1)}, B_{\pi(j)}] \cap \mathbb{Z} \times \mathbb{Z}$, for $1 \leqslant i, j \leqslant n$.

For every $1 \leqslant k \leqslant n$, we have

$$R_k = \bigsqcup_{\substack{1 \leqslant i_k \leqslant k \\ 1 \leqslant j_k \leqslant \pi^{-1}(k)}} R_{i_k j_k}. \tag{6}$$

This implies

$$R_1 \times R_2 \times \cdots \times R_n = \bigsqcup_{\substack{i_1 = 1 \\ 1 \leqslant j_1 \leqslant \pi^{-1}(1)}} \bigsqcup_{\substack{1 \leqslant i_2 \leqslant 2 \\ 1 \leqslant j_2 \leqslant \pi^{-1}(2)}} \cdots \bigsqcup_{\substack{1 \leqslant i_n \leqslant n \\ 1 \leqslant j_n \leqslant \pi^{-1}(n)}} R_{i_1 j_1} \times R_{i_2 j_2} \times \cdots \times R_{i_n j_n}.$$

Now consider the elements $r \in R_{i_1 j_1} \times R_{i_2 j_2} \times \cdots \times R_{i_n j_n}$, with $1 \leqslant i_k \leqslant k$ and $1 \leqslant j_k \leqslant \pi^{-1}(k)$ being fixed. If $\sigma \in S_n$ permutes r, then $r_{\sigma(k)} \in R_k$ for all $1 \leqslant k \leqslant n$, but $r_{\sigma(k)} \in R_{i_{\sigma(k)} j_{\sigma(k)}}$ also, which implies that $R_{i_{\sigma(k)} j_{\sigma(k)}} \bigcap R_k \neq \emptyset$. From (6) this is equivalent to $R_{i_{\sigma(k)} j_{\sigma(k)}} \subseteq R_k$, and thus to the fact that $i_{\sigma(k)} \leqslant k$ and $j_{\sigma(k)} \leqslant \pi^{-1}(k)$ for all $1 \leqslant k \leqslant n$.

Therefore for any $r \in R_{i_1 j_1} \times R_{i_2 j_2} \times \cdots \times R_{i_n j_n}$, the number of $\sigma \in S_n$ which permutes r is constant and equal to

$$\left|\left\{\sigma \in S_n \colon i_{\sigma(l)} \leqslant l, \ j_{\sigma(l)} \leqslant \pi^{-1}(l), \ \forall 1 \leqslant l \leqslant n\right\}\right|.$$

Thus the number of elements in $R_1 \times R_2 \times \cdots \times R_n$, modulo possible permutations is

$$\sum_{\substack{i_1 = 1 \\ 1 \leqslant j_1 \leqslant \pi^{-1}(1)}} \sum_{\substack{1 \leqslant i_2 \leqslant 2 \\ 1 \leqslant j_2 \leqslant \pi^{-1}(2)}} \cdots \sum_{\substack{1 \leqslant i_n \leqslant n \\ 1 \leqslant j_n \leqslant \pi^{-1}(n)}} \frac{\prod_{k=1}^{n} (A_{i_k} - A_{i_k - 1})(B_{\pi(j_k)} - B_{\pi(j_k - 1)})}{|\{\sigma \in S_n \colon i_{\sigma(l)} \leqslant l, \ j_{\sigma(l)} \leqslant \pi^{-1}(l), \ \forall 1 \leqslant l \leqslant n\}|},$$

which implies the result. □

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