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The Engel Structure of Linear Groups

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1. DISCUSSION OF THE RESULTS

1.1. By a linear group over a commutative ring R we shall understand here a group of automorphisms of some finitely generated module over R. Our object is to establish Theorem 0 below. But the proof of this theorem involves two results that are useful in other contexts. The first of these (Theorem 2) states that if $\mathfrak{X}$ is a class of groups that is subject to two mild restrictions and such that all finite $\mathfrak{X}$-groups are soluble, then every linear $\mathfrak{X}$-group over a noetherian ring is also soluble. The second result (Theorem 4), which merely codifies a well-known method of proof, asserts that if $\mathfrak{Y}$ is a class of groups satisfying a certain list of conditions, then every linear group over a noetherian ring is a $\mathfrak{Y}$-group. Our arguments depend on a simple but rather useful reduction principle (Proposition 1). This may be paraphrased as stating that if a theorem of a certain type can be established for linear groups over R when R is a field, then the same theorem is automatically true when R is a noetherian ring.

In order to state Theorem 0 we must introduce some notation and terminology. Let G be a group. We denote the *Hirsch-Plotkin radical* of G (i.e., the unique maximal locally nilpotent normal subgroup) by $\eta(G)$ and the *Fitting subgroup* of G (i.e., the product of all nilpotent normal subgroups) by $\eta_1(G)$. The upper central series of G is the ascending series $(\alpha_\lambda(G); \lambda \leqslant \tau)$, where $\alpha_0(G) = 1$, for each $\lambda < \tau$ the center of $G/\alpha_\lambda(G)$ is nontrivial and equals $\alpha_{\lambda+1}(G)/\alpha_\lambda(G)$, but $G/\alpha_\tau(G)$ has trivial center. The order type τ of the series will be called the *central height* of G and $\alpha_\tau(G)$ is the *hypercenter* of G. When we wish to leave the central height of G unspecified we shall write the hypercenter as $\alpha(G)$. A group which coincides with its hypercenter will be called a *hypercentral group*. (This would seem to be a more natural and informative term than the frequently used "ZA-group.") We also adopt the notation

$$[x, \underbrace{y, ..., y}_{r}] = [x, {}_r y].$$

THEOREM 0. *Let R be a noetherian commutative ring, A a finitely generated R-module and G a group of R-automorphisms of A. Then G has the following properties:*

(1) *$\eta(G)$ is hypercentral and $a \in \eta(G)$ if, and only if, $[x, {}_r a] = 1$ for all x in G and some r, possibly depending on x;*

(2) *$\eta_1(G)$ is nilpotent and $a \in \eta_1(G)$ if, and only if, $[x, {}_r a] = 1$ for all x in G and some r independent of x;*

(3) *G has central height at most $\omega + k$, for some finite k, and $a \in \alpha(G)$ if, and only if, $[a, {}_r x] = 1$ for all x in G and some r, possible depending on x;*

(4) *$a \in \alpha_\omega(G)$ if, and only if, $[a, {}_r x] = 1$ for all x in G and some r independent of x.*

To the best of my knowledge, this theorem includes all the results on this topic that have been obtained so far. These all refer to the case when R is a field. Then G may be thought of as a group of matrices over R. Garaščuk [5] (see also [14], Proposition 2.3) proved that $\eta(G)$ is hypercentral; Garaščuk and Suprunenko [6] showed that if $[x, {}_r y] = 1$ for all x, y in G with r possibly depending on x and y, then $G = \eta(G)$; and finally, the same authors [7] established that if $[x, {}_r y] = 1$ for all x and y in G and some fixed r, then G is nilpotent. These facts are all contained in parts (1) and (2) of Theorem 0. The first mentioned result of Garaščuk and Suprunenko reappears below in a considerably generalized form as the Corollary to Proposition 4 (Section 2.1).

It is easy to see that if G is a linear group, then $\eta(G)$, $\eta_1(G)$, $\alpha(G)$ and $\alpha_\omega(G)$ (the four subgroups of Theorem 0) may all be distinct from each other. For let ω_i be a primitive complex 2^ith root of 1 ($i = 1, 2, \cdots$) and let s_i, t_i be the automorphisms of the two-dimensional complex vector space $\mathbf{C}^2$ given, respectively, by $(x, y) \rightarrow (\omega_i x, y)$ and $(x, y) \rightarrow (x, \omega_i y)$. If g is $(x, y) \rightarrow (y, x)$, and A is the subgroup generated by g and s_i, t_i, $i = 1, 2, \cdots$, then A is hypercentral of central height $\omega + 1$. Let B denote the symmetric group on three letters in its faithful permutational representation on $\mathbf{C}^3$. Then $A \times B$ is a subgroup of the automorphism group of $\mathbf{C}^5$ and in $A \times B$ the functions η, η_1, α, α_ω determine four distinct subgroups.

1.2. Theorem 0 is really a join of two results: first, it asserts that the various types of Engel elements constitute subgroups and secondly, it gives information on the structure of these subgroups.

We propose to employ the same notation as in our earlier paper [10], but we recapitulate the salient features. If A and B are subsets of an arbitrary group G, then $A \mathrel{e} B$ means $[a, {}_r b] = 1$ for all a in A, b in B and some $r = r(a, b)$. We write $A \mathrel{|e} B$, or $A \mathrel{e|} B$, if r can be chosen independently of a in A, or of b in B, respectively. Then $L(G)$, $\bar{L}(G)$, $R(G)$, $\bar{R}(G)$ denote the sets of all g in G satisfying $G \mathrel{e} g, G \mathrel{|e} g, g \mathrel{e} G, g \mathrel{e|} G$, respectively (called the left, bounded left, right and bounded right Engel elements of G). (Cf. [*10*], Section 1.3, p. 439.) It is still an unsolved problem whether any of these subsets can ever fail to be a subgroup. (In this connexion, confer the Corollary to Proposition 4, below.)

The Engel elements are closely connected with four characteristic subgroups of G. Let H be a subgroup and suppose there exists an ascending series from H to G. We write $H \infty\triangleleft G$ and we shall call H an *ascendant subgroup* of G. (This term, due to P. Hall, is preferable to the term "serial subgroup" introduced in [*9*], p. 152.) If the series is finite we shall say H is *subnormal* in G (instead of finitely-serial) and write $H \triangleleft\triangleleft G$. In particular, $H\ k\triangleleft G$ shall mean that the series has length at most k (so that $H\ 1\triangleleft G$ or $H \triangleleft G$ means that H is normal in G). The four characteristic subgroups referred to above may now be defined as follows: (1) $\sigma(G)$ is the union of all cyclic ascendant subgroups of G; (2) $\bar{\sigma}(G)$ is the union of all cyclic subnormal subgroups of G; (3) $\rho(G)$ is the set of all a so that for every x in G we have $Gp\{x\} \infty\triangleleft Gp\{x, a^G\}$; and (4) $\bar{\rho}(G)$ is the set of all a so that for every x in G we have $Gp\{x\}\ k\triangleleft Gp\{x, a^G\}$ with some k independent of x. (Here a^G denotes the conjugacy class of a.) All these characteristic subgroups lie in the Hirsch-Plotkin radical $\eta(G)$. The Fitting subgroup $\eta_1(G)$ is contained in $\bar{\sigma}(G)$.

We let $\mathfrak{E}$ denote the class of all groups G in which $\sigma(G) = L(G)$, $\bar{\sigma}(G) = \bar{L}(G)$, $\rho(G) = R(G)$ and $\bar{\rho}(G) = \bar{R}(G)$. The first of these equalities implies $\sigma(G) = \eta(G)$. A result of L. G. Kovács and B. H. Neumann (unpublished) shows that this equality is not universally true. Hence $\mathfrak{E}$ is certainly not the class of all groups. It is now also known that the inequality $\eta(G) \leqslant L(G)$ may be strict: this follows from the recent important work of E. S. Golod and I. R. Šafarevič [*8*]. We remark that Golod's finitely generated non-nilpotent Engel group cannot be isomorphic to a linear group over any commutative ring: this is a consequence of the Corollary to Proposition 4 (Section 2.1).

The results of [*10*] (particularly Theorems 1.2 and 1.3, p. 438) suggest a criterion for judging whether the above four characteristic subgroups are well-behaved in a given group or not. Let $\mathfrak{D}$ denote the class of all groups G such that $\eta(G)$ is hypercentral (and therefore $\sigma(G) = \eta(G)$), $\bar{\sigma}(G)$ is nilpotent (and therefore $\eta_1(G) = \bar{\sigma}(G)$), $\rho(G) = \alpha(G)$ and $\bar{\rho}(G) = \alpha_\omega(G)$. The class $\mathfrak{D} \cap \mathfrak{E}$ may be thought of as the class of groups with a good Engel structure.

Let R be a commutative ring and $\mathfrak{L}(R)$ the class of all groups that are isomorphic to a group of automorphisms of some finitely generated R-module. Then Theorem 0 may be restated as

THEOREM 1. *If R is noetherian, then*

(i) $\mathfrak{L}(R) \leqslant \mathfrak{E}$;

(ii) $\mathfrak{L}(R) \leqslant \mathfrak{D}$; *and*

(iii) *all groups in* $\mathfrak{L}(R)$ *have central height* $< \omega 2$.

1.3. In order to prove Theorem 1 (i) it is enough to show that if $G \in \mathfrak{L}(R)$ and $G = Gp\{L(G) \cup R(G)\}$, then G is soluble. This follows from Theorem 1.5 of [*10*]. (The hypothesis of that theorem concerning local nilpotence is redundant in view of Theorem 4 of [*9*].)

We let $\mathfrak{A}$, $\mathfrak{F}$, $\mathfrak{G}$ and $\mathfrak{S}$ denote, respectively, the class of all abelian, finite, finitely generated, and soluble groups and we shall employ the usual product notation for extensions (cf. [*10*], Section 1.2, p. 347). In addition we use Hall's calculus of closure operations (cf. [*12*], Section 1.3, p. 533). We shall need S (subgroup closure), Q (quotient group closure), L (local closure) and R_0 which we define as follows: $\mathfrak{X} = \mathsf{R}_0\mathfrak{X}$ means that whenever $G/K_1 \in \mathfrak{X}$ and $G/K_2 \in \mathfrak{X}$ then $G/K_1 \cap K_2 \in \mathfrak{X}$.

THEOREM 2. *If* $\mathfrak{X}$ *is a class of groups satisfying* (i) $\mathsf{Q}\mathfrak{X} = \mathfrak{X}$, (ii) $\mathfrak{X} \leqslant \mathsf{L}(\mathfrak{G} \cap \mathfrak{X})$, (iii) $\mathfrak{X} \cap \mathfrak{F} \leqslant \mathfrak{S}$, *and* R *is a noetherian commutative ring, then* $\mathfrak{X} \cap \mathfrak{L}(R) \leqslant \mathfrak{S}$.

Condition (ii) says that if G is in $\mathfrak{X}$, then every finitely generated subgroup lies in a finitely generated $\mathfrak{X}$-subgroup. Of course, this is automatically true if $\mathfrak{X}$ is S-closed. But we wish to apply the theorem when $\mathfrak{X}$ is all groups G such that $G = Gp\{L(G) \cup R(G)\}$. There is no obvious reason why this class need be S-closed. But it clearly satisfies (ii). Also (i) is obvious and (iii) is a theorem of Baer [*1*]. Hence linear groups in the class are soluble and therefore, by our earlier remark, Theorem 1 (i) follows.

Other applications of Theorem 2 come to mind. Thus $\mathfrak{X}$ could be the class of groups in which every element has finite order divisible by at most two fixed primes: condition (iii) is the *pq*-theorem of Burnside. Again, $\mathfrak{X}$ could be the class of all groups in which every two-generator subgroup is soluble: here (iii) is an unpublished result of John Thompson.

1.4. Turning now to part (ii) of Theorem 1, we shall deduce this from the following result.

THEOREM 3. *Let* R *be a noetherian commutative ring and* $G \in \mathfrak{L}(R)$. *If* H *is a normal subgroup of* G, *then*

(i) $H \mathfrak{e} G$ *implies* $H \leqslant \alpha(G)$; *while*

(ii) $H \,|\mathfrak{e}\, G$ *implies* $H \leqslant \alpha_t(G)$ *for some finite* t.

Proof of Theorem 1 (ii). Let $G \in \mathfrak{L}(R)$. Then $\rho(G) \mathfrak{e} G$ implies $\rho(G) \leqslant \alpha(G)$, by Theorem 3 (i), and thus $\rho(G) = \alpha(G)$. In particular, $\rho(\eta(G)) = \alpha(\eta(G))$. But $\rho(\eta(G)) = \eta(G)$, by Theorem 1 (i), and therefore $\eta(G)$ is hypercentral.

If $a \in \bar{\rho}(G)$ and $A = Gp\{a^G\}$, then $A \mid \mathfrak{e}\, G$. Thus $A \leqslant \alpha_t(G)$, by Theorem 3 (ii), and so $\bar{\rho}(G) = \alpha_\omega(G)$. Finally, $G \mid \mathfrak{e}\, \bar{\sigma}(G)$ implies $\bar{\sigma}(G) \mid \mathfrak{e}\, \bar{\sigma}(G)$ and therefore, again by Theorem 3 (ii), $\bar{\sigma}(G)$ is nilpotent.

Theorem 3 will follow, in turn, from

THEOREM 4. *Let R be a noetherian commutative ring and suppose $\mathfrak{Y}$ is a class of groups subject to the following conditions:*

(i) $\mathfrak{L}(R) \cap \mathfrak{Y}\mathfrak{F} \leqslant \mathfrak{Y}$;

(ii) *if G is a group of automorphisms of a finitely generated R-module A and Z is the group of all automorphic scalar multiplications (i.e., all automorphisms $x1_A$, where x is an invertible element of R), then $GZ \in \mathfrak{Y}$ if, and only if, $G \in \mathfrak{Y}$;*

(iii) *if $G \in \mathfrak{L}(R)$ and $\eta(G) \in \mathfrak{F}$ then $G \in \mathfrak{Y}$;*

(iv) *if G is a group of automorphisms of a finitely generated R-module A and B is a G-invariant submodule such that the automorphism groups induced by G on A/B and B both lie in $\mathfrak{Y}$ then G is in $\mathfrak{Y}$.*

Under these circumstances, $\mathfrak{L}(R) \leqslant \mathfrak{Y}$.

Let $\mathfrak{Y}_1$ be the class of all groups such that for every locally nilpotent normal subgroup H satisfying $H\, \mathfrak{e}\, G$ we have $H \leqslant \alpha(G)$. Further, let $\mathfrak{Y}_2$ be the class of all groups with the following property: if H is a locally nilpotent normal subgroup of G such that $H \mid \mathfrak{e}\, G$, then $H \leqslant \alpha_t(G)$ for some finite t. Suppose we can verify the listed conditions for both $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$. Then Theorem 4 asserts $\mathfrak{L}(R) \leqslant \mathfrak{Y}_1 \cap \mathfrak{Y}_2$. If now $G \in \mathfrak{L}(R)$, $H \lhd G$ and $H\, \mathfrak{e}\, G$, then $H \leqslant \eta(G)$ by Theorem 1 (i). Since $G \in \mathfrak{Y}_1 \cap \mathfrak{Y}_2$, Theorem 3 is proved.

Condition (ii) obviously holds for both $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$, while the verification of (i) and (iii) is straightforward (and given in Section 3.2). Condition (iv) is more interesting. If G is a group as in (iv) and K is the subgroup of all elements that act trivially on both A/B and B, then $G/K \in \mathrm{R}_0\mathfrak{Y}$. It is obvious that $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$ are R_0-closed. Hence if $\mathfrak{Y} = \mathfrak{Y}_i$ $(i = 1, 2)$, $G/K \in \mathfrak{Y}_i$. To deduce that G itself is in $\mathfrak{Y}_i$ we shall need

THEOREM 5. *If R is a noetherian commutative ring, A is a finitely generated R-module and Λ is a R-algebra of endomorphisms of A, generated as R-module by locally nilpotent endomorphisms, then Λ is nilpotent.*

We mention in passing the following immediate corollary.

COROLLARY. *If R is a noetherian commutative ring and Λ is a R-algebra that is generated as R-module by a finite number of nilpotent elements, then Λ is nilpotent.*

This corollary improves Proposition 2.1 of [*10*] as well as two theorems of Barnes [*2*].

1.5. In order to prove Theorems 2, 4, and 1 (iii) we shall make use of the following result.

PROPOSITION 1. *Let $\mathfrak{X}$ be a Q-closed class of groups and $\mathfrak{Y}$ a class satisfying condition (iv) of Theorem 4. If $\mathfrak{X} \cap \mathfrak{L}(R) \leqslant \mathfrak{Y}$ for all fields R, then $\mathfrak{X} \cap \mathfrak{L}(R) \leqslant \mathfrak{Y}$ for all noetherian commutative rings R.*

Let $\mathfrak{Z}$ denote the class of all groups with central height $< \omega 2$. We shall find that this class satisfies condition (iv) of Theorem 4 and thus, by Proposition 1 (with $\mathfrak{X}$ the class of all groups), we only need to prove Theorem 1 (iii) when R is a field. Then we may even assume R to be an algebraically closed field. To deal with this case we must know that $\alpha(G)$ contains a triangulable subgroup, say M, of finite index in $\alpha(G)$ and *normal in* G. Then $\alpha(G)/M$ is a finite normal subgroup of G/M. Therefore there exists an integer t such that $[\alpha(G), {}_tG] = T$ is contained in M and hence T is triangulable. The proof of Theorem 1 (iii) will then be completed by establishing

PROPOSITION 2. *If F is a field, $G \in \mathfrak{L}(F)$ and T is a triangulable normal subgroup of G contained in the hypercenter, then $T \leqslant \alpha_\omega(G)$.*

The proof of the existence of the subgroup M, mentioned above, involves the well-known theorem of Kolchin-Mal'cev that every soluble group of matrices over an algebraically closed field contains a triangulable normal subgroup of finite index. It may be worth remarking (although we shall make no use of it here) that Proposition 1 enables one to generalize this theorem in the following form:

if R is a noetherian commutative ring and G is a soluble group of automorphisms of the finitely generated R-module A, then G contains a subgroup of finite index whose commutator group stabilizes a finite series of A.

The remainder of the paper is divided into three sections. The first of these contains the proofs of Theorems 2 and 4; the second is devoted to the proof of Theorem 5 and the verification of conditions (i), (iii), and (iv) of Theorem 4 for the classes $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$ (thereby completing the proof of Theorem 3); and the last section contains the proof of Theorem 1 (iii).

2. PROOFS OF THEOREM 2 AND THEOREM 4

2.1. If A is a finitely generated R-module, we shall denote the group of all R-automorphisms of A by $\mathrm{Aut}_R\, A$. If G is a subgroup of $\mathrm{Aut}_R\, A$ and B is a G-invariant submodule of A, we shall write $G_{A/B}$ for the image of G under the homomorphism induced by $A \to A/B$ and G_B for the image of G under the restriction $G \to \mathrm{Aut}_R\, B$.

LEMMA 1. *Suppose $\mathfrak{X}$ is a class of groups satisfying the conditions* (i) *and* (iii) *of Theorem 2 and that R is a finitely generated ring. Let A be a free R-module of rank n and $\mathfrak{r}^t = 0$, where $\mathfrak{r}$ is the nilradical of R. If G is an $\mathfrak{X}$-subgroup of* $\mathrm{Aut}_R A$, *then $G \in \mathfrak{A}^k$, where k depends only on n and t.*

Proof. If $\mathfrak{m}$ is a maximal ideal of R then $R/\mathfrak{m}$ is finite and $\mathfrak{r}$ is the intersection of all such $\mathfrak{m}$ (cf. Corollary 1, p. 68 [*4*]).[1] For each $\mathfrak{m}$, $G_{A/A\mathfrak{m}}$ is finite and hence soluble (by the hypothesis on $\mathfrak{X}$). Since $A/A\mathfrak{m}$ is a vector space over the field $R/\mathfrak{m}$ of dimension n, by a theorem of Zassenhaus [*15*], there exists an integer n', depending only on n, so that $G_{A/A\mathfrak{m}}$ has derived length at most n'. Hence $G/K \in \mathfrak{A}^{n'}$ where K is the intersection of the kernels of all homomorphisms $G \to G_{A/A\mathfrak{m}}$. Since A is a direct sum of copies of R, $\cap A\mathfrak{m} = A\mathfrak{r}$ and thus $A(K - 1) \leqslant A\mathfrak{r}$.

For each positive integer i, define K_i to be all elements g in G such that $A(g - 1) \leqslant A\mathfrak{r}^i$. Then K_i is the kernel of the homomorphism $G \to \mathrm{Aut}_R A/A\mathfrak{r}^i$, $K_1 = K$ and $K_t = 1$. If $x \in K_i$, $y \in K_j$, then clearly $A(x - 1)(y - 1) \leqslant A\mathfrak{r}^{i+j}$ and hence

$$A([x, y] - 1) = A(xy - yx) \leqslant A\mathfrak{r}^{i+j}.$$

Thus $[K_i, K_j] \leqslant K_{i+j}$ and so K is nilpotent of class $< t$.

PROPOSITION 3. *Let $\mathfrak{X}$ be a class of groups satisfying the conditions* (i), (ii), (iii) *of Theorem 2. Suppose R is a commutative ring with nilpotent nilradical and that A is a finitely generated free R-module. If G is an $\mathfrak{X}$-subgroup of* $\mathrm{Aut}_R A$, *then G is soluble.*

Proof. Let H be a finitely generated $\mathfrak{X}$-subgroup of G and suppose $H = Gp\{h_1, \cdots, h_m\}$, where each h_i^{-1} equals some h_j. If $A = a_1R \oplus \cdots \oplus a_nR$, then each a_jh_k can be written as a R-linear sum of $a_1, \cdots, a_n$, say

$$a_jh_k = \sum_{i=1}^{k} a_i r_{ij}^{(k)}, \qquad j = 1, \cdots, n; \qquad k = 1, \cdots, m.$$

Let S be the subring of R generated by

$$\{r_{ij}^{(k)}, i, j = 1, \cdots, n; k = 1, \cdots, m\}$$

and let $B = a_1S \oplus \cdots \oplus a_nS$. Then B is H-invariant and $H \leqslant \mathrm{Aut}_S B$. If $\mathfrak{r}^t = 0$, where $\mathfrak{r}$ is the nilradical of R, then $(S \cap \mathfrak{r})^t = 0$, where $S \cap \mathfrak{r}$ is the nilradical of S. It follows from Lemma 1 that $H \in \mathfrak{A}^k$, where $k = k(n, t)$.

[1] I wish to thank Hyman Bass for drawing my attention to the use of Hilbert's zero theorem in this connexion. Apart from this, the present proof is essentially a repeat of the argument in [*6*].

We now conclude, using condition (ii) on $\mathfrak{X}$, that $G \in \mathsf{L}\mathfrak{A}^k$ and thus $G \in \mathfrak{A}^k$.

Proof of Proposition 1. Let R be a noetherian ring, A a finitely generated R-module and G an $\mathfrak{X}$-subgroup of $\mathrm{Aut}_R A$. If $G \notin \mathfrak{Y}$, then there exists a G-invariant submodule C such that $G_{A/C} \notin \mathfrak{Y}$ but $G_{A/B} \in \mathfrak{Y}$ for all G-invariant submodules $B > C$. We shall obtain a contradiction by proving that $G_{A/C} \in \mathfrak{Y}$. Without loss of generality we may assume that $C = 0$.

Let $\mathfrak{p}$ be a maximal element of Ass (A) ([3], Corollary 1, p. 132) and denote by B the set of all a in A such that $a\mathfrak{p} = 0$. Then B is a fully invariant submodule and hence it admits G. Suppose a is a nonzero element in B and x an element in R such that $ax = 0$. Then the annihilator $\mathfrak{a}$ of a contains $\mathfrak{p}$ and x. But $\mathfrak{a} \leqslant \mathfrak{p}'$ for some $\mathfrak{p}' \in$ Ass (A) ([3], Proposition 8, p. 137) and hence $\mathfrak{p} = \mathfrak{p}'$ by the maximality of $\mathfrak{p}$. Therefore $x \in \mathfrak{p}$ and consequently B is torsion-free as a module over the integral domain $S = R/\mathfrak{p}$.

Let F be the quotient field of S. Then $b \to b \otimes 1$ is a S-monomorphism of B into $B_{(F)} = B \otimes_S F$ and this induces a monomorphism ζ: $G_B \to \mathrm{Aut}_F B_{(F)}$. Since $\mathfrak{X} = \mathsf{Q}\mathfrak{X}$, $G_B \in \mathfrak{X}$ and then, by hypothesis, $G_B\zeta \in \mathfrak{Y}$. Hence also $G_B \in \mathfrak{Y}$. Now $G_{A/B} \in \mathfrak{Y}$ by construction, and therefore, by condition (iv) on $\mathfrak{Y}$, $G \in \mathfrak{Y}$, as required.

Theorem 2 is now an immediate consequence. For by Proposition 3, $\mathfrak{X} \cap \mathfrak{L}(R) \leqslant \mathfrak{S}$ for all fields R and thus Proposition 1 yields the result provided $\mathfrak{S}$ is seen to satisfy the condition (iv) of Theorem 4. But this is clear: in the notation of that condition, if K is the set of all elements in G that induce the identity automorphism on A/B and on B, then G/K is soluble and K is Abelian since $A(x-1)(y-1) = 0$ for all x, y in K.

We remark that Theorem 2 also has the following corollary.

PROPOSITION 4. *Let $\mathfrak{X}$ be a class of groups satisfying conditions* (i), (ii), (iii) *of Theorem 2.* Then $\mathfrak{X} \cap \mathfrak{L}(R) \leqslant \mathsf{L}\mathfrak{S}$ *for every commutative ring R.*

Proof. If $G \in \mathfrak{X} \cap \mathfrak{L}(R)$, choose a finite subset and a finitely generated $\mathfrak{X}$-subgroup H containing this. As in the proof of Proposition 3, we find a finitely generated subring S of R such that the submodule B generated by a set of R-generators of A is H-invariant and $H \leqslant \mathrm{Aut}_S B$. (The fact that A was a direct sum in Proposition 3 was not relevant to that part of the proof.) Now S is noetherian and hence H is soluble by Theorem 2.

COROLLARY. *If $G \in \mathfrak{L}(R)$, for any commutative ring R, then $L(G) = \eta(G)$ and this contains $R(G)$ as a subgroup.*

Proof. If we take $\mathfrak{X}$ to be the class of groups generated by their Engel elements, then Proposition 4 asserts that $H = Gp\{L(G) \cup R(G)\}$ is locally soluble. Hence (e.g., by Theorem 4 of [9]), H is locally nilpotent and thus equals $\eta(G)$. Since now $R(G) \leqslant \eta(G)$, $R(G)$ is a subgroup, by Lemma 14 of [*9*].

2.2. *Proof of Theorem 4.* If we take $\mathfrak{X}$ to be the class of all groups in Proposition 1, then the proposition asserts that we only need to consider the case when R is a field. It is clear that we may even assume R to be algebraically closed. The proof now follows the usual pattern for results of this type (cf., in particular, the proof of Theorem 9.4 in [*11*].)

Let $G \leqslant \mathrm{Aut}_R V$, where V is a n-dimensional vector space over R. We shall prove $G \in \mathfrak{Y}$ by an induction on n. If W is a G-invariant subspace such that $0 < W < V$, then $G_W, G_{V/W} \in \mathfrak{Y}$ by induction and hence $G \in \mathfrak{Y}$ by condition (iv). So assume that V is a simple RG-module.

If $K \lhd G$, then V is semi-simple as RK-module and we may write $V = V_1 \oplus \cdots \oplus V_r$, where each V_i is a direct sum of isomorphic simple RK-modules and each simple summand of V_i is nonisomorphic to each in V_j for all $i \neq j$. Then G acts as a permutation group on $\{V_1, \cdots, V_r\}$ and the kernel H of this representation consists of all g in G that fix each V_i. If $r > 1$, then H_{V_1} and H_{V/V_1} lie in $\mathfrak{Y}$ by the induction hypothesis and hence $H \in \mathfrak{Y}$ by condition (iv). Thus $G \in \mathfrak{Y}\mathfrak{F}$ and hence $G \in \mathfrak{Y}$ by condition (i).

Henceforth we assume that for every $K \lhd G$, V is the direct sum of isomorphic simple RK-modules. If K is Abelian then every element of K is a scalar multiple of the identity mapping (because R is algebraically closed).

Let Z be the subgroup of $\mathrm{Aut}_R V$ consisting of all $x1_V$, for $x \neq 0$ in R, and form $G_1 = GZ$. Then also $G_1 = HZ$, where H is the set of all elements in G_1 of determinant 1. By condition (ii), $G \in \mathfrak{Y}$ if, and only if, $G_1 \in \mathfrak{Y}$ and this holds if, and only if, $H \in \mathfrak{Y}$. We may therefore confine attention to H.

Every Abelian normal subgroup of H is contained in Z. Hence every nilpotent normal subgroup N of H has its center in Z, and thus, since all elements are of determinant 1, N is finite (Lemma 9.3, [*11*]). Now $\eta(H)$ is locally nilpotent and hence is soluble (by the theorem of Zassenhaus referred to in the proof of Lemma 1). Thus $\eta(H)$ is finite by Lemma 2, below, and consequently $H \in \mathfrak{Y}$, by condition (iii) on $\mathfrak{Y}$.

The following lemma occurs as part of the argument on p. 40 of [*11*]. We wish to refer to it again in the next section and therefore reproduce it for the convenience of the reader.

LEMMA 2. ***If S is a soluble normal subgroup of a group G such that every nilpotent normal subgroup of G contained in S is finite, then S itself is finite.***

Proof. Let $S = S^{(0)} > S^{(1)} > \cdots > S^{(k)} > 1$ be the derived series of S. Suppose $S^{(r)}$ is finite and let C be the centralizer of $S^{(r)}$ in $S^{(r-1)}$. Then C is normal in G, $S^{(r-1)}/C$ is finite and $[C', C] = 1$ (because $C' \leqslant S^{(r)}$). Hence

C is finite, by hypothesis, so that $S^{(r-1)}$ is finite. Therefore S is finite, by an induction on k.

3. Proofs of Theorem 3 and Theorem 5

3.1. *Proof of Theorem 5.* Since A satisfies the maximal condition on R-submodules, we may assume (cf. the proof of Proposition 1) that the action of A on A/B is nilpotent for every A-invariant nonzero submodule B but that the action of A on A is not nilpotent. Our aim is to produce a contradiction.

Precisely as in the proof of Proposition 1, we construct a fully invariant nonzero submodule B such that $B\mathfrak{p} = 0$ where $\mathfrak{p}$ is a prime ideal of R and B is torsion-free as $R/\mathfrak{p}$-module. It follows from Lemma 2.2 of [*10*] (cf. the remark at the end of the proof there, on p. 447) that A has nilpotent action on B. Since the action on A/B is nilpotent by construction, the action on A is nilpotent and thus we have our desired contradiction.

Proposition 5. *Let A be a finitely generated module over the noetherian ring R and G a subgroup of* $\operatorname{Aut}_R A$. *Suppose B is a G-invariant submodule and H a normal subgroup of G such that*

$$A(h-1) \leqslant B \qquad \textit{and} \qquad B(h-1) = 0$$

for all h in H. Then $H \mathfrak{e} G$ implies $H \leqslant \alpha_t(G)$ for some finite t.

Proof. If $h \in H$, then $a + B \to a(h-1)$ is a well-defined mapping, call it h'. Clearly $h' \in \operatorname{Hom}_R(A/B, B)$ and $h \to h'$ is a monomorphism of H into the additive group $\operatorname{Hom}_R(A/B, B)$. Since R is noetherian and A is finitely generated, $\operatorname{Hom}_R(A/B, B)$ is finitely generated as an R-module. Therefore the R-submodule generated by H', call it M, is also finitely generated over R.

Now $\operatorname{Hom}_R(A/B, B)$ is a G-module in the standard way: if $\varphi \in \operatorname{Hom}_R(A/B, B)$ and $g \in G$, φg is the mapping

$$a + B \to ((ag^{-1} + B)\,\varphi)\,g.$$

On the other hand, the Abelian normal subgroup H is a G-module by conjugation. It is now easy to check that $h \to h'$ is a G-module homomorphism.

Hence $[h, g]' = h'(g-1)$ for h in H and g in G. Therefore $H \mathfrak{e} G$ implies that, for each g in G, $g - 1$ acts as a locally nilpotent endomorphism on M. Now by Theorem 5, there is an integer t such that

$$M(g_1 - 1) \cdots (g_t - 1) = 0$$

for all $g_1, \cdots, g_t$ in G. This implies $[H, g_1, \cdots, g_t] = 1$ for all $g_1, \cdots, g_t$ in G, as required.

3.2. We ask the reader to turn back to the two paragraphs immediately after the statement of Theorem 4 (in Section 1.4). He will then see that in order to complete the proof of Theorem 3 we need now only verify conditions (i), (iii), and (iv) of Theorem 4 for the classes $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$.

Verification of condition (iv). Let $G \leqslant \mathrm{Aut}_R A$ and suppose K is the intersection of the kernel of $G \to G_{A/B}$ with the kernel of $G \to G_B$. Let H be a locally nilpotent normal subgroup of G satisfying $H \mathfrak{e} G$. If $G_{A/B} \in \mathfrak{Y}_1$ and $G_B \in \mathfrak{Y}_1$, then HK/K is contained in $\alpha(G/K)$. Now $H \cap K \mathfrak{e} G$ and thus $H \cap K \leqslant \alpha_t(G)$ for some finite t, by Proposition 5. Hence $H \leqslant \alpha(G)$.

If $H \,|\mathfrak{e}\, G$, then we argue with $\mathfrak{Y}_2$ in exactly the same way and show $H \leqslant \alpha_{t+r}(G)$ for some finite $t + r$.

Verification of condition (iii). If $H \mathfrak{e} G$, where H is a normal subgroup of G contained in $\eta(G)$ and $\eta(G)$ is finite, then $H \leqslant \alpha_t(G)$ for some finite t, by Theorem 3.1 (i) of [*10*] (with N of that theorem equal to $\eta(G)$).

Verification of condition (i). Let N be a normal subgroup of finite index in G and H a locally nilpotent normal subgroup. We suppose either (a) that $H \mathfrak{e} G$ and $N \in \mathfrak{Y}_1$ or (b) that $H \,|\mathfrak{e}\, G$ and $N \in \mathfrak{Y}_2$. In either case, $K = [H, {}_rG] \leqslant N$ for some finite r (since $\mathfrak{F} \leqslant \mathfrak{Y}_1 \cap \mathfrak{Y}_2$). Now $K \leqslant \alpha(N)$ in case (a), while $K \leqslant \alpha_s(N)$ for some finite s, in case (b).

Let $K_\lambda = K \cap \alpha_\lambda(N)$ for each $\lambda \geqslant 0$. Then $K_{\lambda+1}/K_\lambda$ is a G/N-module and hence, by Proposition 2.2 of [*10*], $K_{\lambda+1}/K_\lambda \leqslant \alpha_\omega(G/K_\lambda)$ in case (a), while $K_{\lambda+1}/K_\lambda \leqslant \alpha_n(G/K_\lambda)$ for some finite n, in case (b). It follows that K, and therefore also H, is contained (a) in $\alpha(G)$ or (b) in $\alpha_t(G)$ for some finite t, as the case may be.

The proof of Theorem 3 is now complete.

4. Proof of Theorem 1 (iii)

The class $\mathfrak{Z}$ of all groups of central height $< \omega 2$ satisfies condition (iv) of Theorem 4. For let $G_{A/B}$ and G_B be in $\mathfrak{Z}$ and suppose K is the meet of the kernel of $G \to G_{A/B}$ with the kernel of $G \to G_B$. If U is the inverse image in G of $\alpha_\omega(G_{A/B})$ and V the inverse image of $\alpha_\omega(G_B)$, then there exists a finite s such that

$$M = [\alpha(G), {}_sG] \leqslant U \cap V.$$

Hence $MK/K \leqslant \alpha_\omega(G/K)$, since clearly $(U \cap V)\, K/K \leqslant \alpha_\omega(G/K)$. But by Proposition 5, $\alpha(G) \cap K \leqslant \alpha_t(G)$ for some finite t. Therefore $M \leqslant \alpha_\omega(G)$, as required.

It now follows from Proposition 1 that we only have to prove Theorem 1 (iii) for linear groups over a field. Moreover, without loss of generality, we may suppose the field to be algebraically closed.

Let $G \in \mathfrak{L}(F)$, where F is an algebraically closed field, and denote by S the soluble radical of G (i.e., the unique maximal soluble normal subgroup of G). The existence of this radical was established by Zassenhaus [*15*] (cf. also [*13*], Theorem 8.12, p. 56). We wish to prove that S *contains a triangulable subgroup of finite index in S and normal in G.*

The best way to see this is to use the Zariski topology on G.[2] The closure, in this topology, of a soluble subgroup is itself soluble ([*13*], Corollary 2, p. 54) and therefore S is closed in G. It follows that the connected component of the identity in S, call it D, is closed in G and hence is normal in G. But D has finite index in S ([*13*], Lemma 4.5, p. 28) and D is triangulable by the theorem of Kolchin ([*13*], Theorem 4.11, p. 30).

Alternatively we may argue as follows. If $\mathfrak{Y}$ is the class of all groups in $\mathfrak{L}(F)$ which possess a triangulable normal subgroup and of finite index in their soluble radical, then it is easy to verify the four conditions listed in Theorem 4 for the class $\mathfrak{Y}$ (condition (iii) being an immediate consequence of Lemma 2). Hence a repeat of the argument used to establish Theorem 4 in the case when R was an algebraically closed field shows that $\mathfrak{L}(F) = \mathfrak{Y}$. (The reader should note however that $\mathfrak{Y}$ is here not an abstract class of groups.)

If D is the triangulable normal subgroup of G such that $(S : D)$ is finite, then $M = \alpha(G) \cap D$ is a triangulable normal subgroup of G and $(\alpha(G) : M)$ is finite (because $\alpha(G) \leqslant S$). Now the remarks immediately preceding the statement of Proposition 2, in Section 1.5, show that it only remains to establish that proposition.

Proof of Proposition 2. Let $G \leqslant \mathrm{Aut}_F V$ and use an induction on the dimension of V. Since T is triangulable, there exists a common eigenvector for the elements of T, say e. Let E be the FG-submodule generated by e and suppose $E \neq V$. Since eg, for every g in G, is an eigenvector for all the elements of T, therefore T_E is diagonable and $T_{V/E}$ is triangulable. Hence, by induction, $T_E \leqslant \alpha_\omega(G_E)$ and $T_{V/E} \leqslant \alpha_\omega(G_{V/E})$. If T_0 is the subgroup of T consisting of all elements that induce the identity on V/E and on E, then we have shown that $T/T_0 \leqslant \alpha_\omega(G/T_0)$. But $T_0 \leqslant \alpha_t(G)$ for some finite t, by Proposition 5. Hence $T \leqslant \alpha_\omega(G)$, as required.

Now assume $E = V$, i.e., T is diagonable. We may write $V = E_1 \oplus \cdots \oplus E_r$, where each E_i is a direct sum of isomorphic simple FT-modules and each simple summand of E_i is nonisomorphic to each in

[2] I am indebted to Mr. B. Wehrfritz for showing me this short proof.

E_j for all $i \neq j$. Thus if $t \in T$, the restriction of t to E_i is multiplication by some nonzero scalar, call it $i(t)$; and if $i \neq j$, there exists t in T such that $i(t) \neq j(t)$. Every element of G induces a permutation on $\{E_1, \cdots, E_r\}$. Let C be the kernel of this permutational representation of G. We assert C is the centralizer of T in G. For if g centralizes T, then for any e in E_i and t in T,

$$(eg)\, t = (et)\, g = i(t)\, (eg),$$

whence $eg \in E_i$. Thus $g \in C$. Conversely, if $g \in C$,

$$(eg)\, t = i(t)\, (eg) = (i(t)\, e)\, g = (et)\, g,$$

so that g centralizes T. It follows that T is a G/C-module, where G/C is a finite group. Now $T \triangleleft G$ implies $T \leqslant \alpha_\omega(G)$, by Proposition 2.2 (ii) of [*10*].

An interesting and still open question concerning Theorem 1 (iii) was raised by Dr. Otto Kegel. If G is a group of $n \times n$ matrices over a field, then we know that the central height of G is at most $\omega + k$, for some finite k. Kegel asks if k is bounded above by some integer depending only on n, and if so, what such a bound might be like.

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