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Positive solutions to logistic type equations with harvesting

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ABSTRACT

We use comparison principles, variational arguments and a truncation method to obtain positive solutions to logistic type equations with harvesting both in $\mathbb{R}^N$ and in a bounded domain $\Omega \subset \mathbb{R}^N$, with $N \geqslant 3$, when the carrying capacity of the environment is not constant. By relaxing the growth assumption on the coefficients of the differential equation we derive a new equation which is easily solved. The solution of this new equation is then used to produce a positive solution of our original problem.

1. Introduction

In this paper we mainly study the existence of *positive* solutions to the problem

$$\begin{cases} -\Delta u = \lambda a u - b g(u) - \mu h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

when $\Omega = \mathbb{R}^N$, in which case the boundary condition is understood as $\lim_{|x|\to\infty} u(x) = 0$, as well as when $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain. Here $N \geqslant 3$, and both the functions a, b, h, and the

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parameters λ, μ are nonnegative. Problem (1) can be thought of as the steady state of the reaction–diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + \lambda a u - b g(u) - \mu h, & x \in \Omega, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,\infty). \end{cases}$$

We interpret this as the evolution equation arising from the population biology of one species. As such the function u represents the population density of the species. Throughout we assume that

$$\lim_{s \to 0} \frac{g(s)}{s} = 0 \quad \text{and} \quad \lim_{s \to \infty} \frac{g(s)}{s} = \infty, \tag{2}$$

so that the nonlinearity $\lambda a u - b g(u)$ represents a logistic type growth. Furthermore note that both coefficients a and b depend on the spatial variable, indicating variable linear growth and competition rates in the environment. The function h is interpreted as the harvesting distribution and μh is the harvesting rate. Hence, such equations have been used, for example, to model fishery or hunting management problems. We refer to [9] for further historical background and references. Intuitively, one expects the survival of the species, i.e. the existence of a positive solution to (1), only for small values of μ.

Mathematically, the presence of the harvesting term introduces a number of challenging issues in the study of existence of positive solutions. Indeed the harvesting term makes the right-hand side of the equation negative at $u = 0$, and therefore our problem belongs to the class of so-called semi-positone problems (see [2]). This prevents the direct application of the maximum principle.

The main inspiration for our study was the recent work [3]. There the authors consider problem (1) in $\mathbb{R}^N$ with the positive and bounded function $a \in L^{N/2}(\mathbb{R}^N)$, the natural setting for the eigenvalue problem

$$-\Delta u = \lambda a u, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N),$$

where $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^1(\mathbb{R}^N)$ with respect to the norm $(\int |\nabla u|^2)^{1/2}$. In addition, they assume that $\frac{g(u)}{u}$ is monotone, $g(u)$ behaves like u^p, $p > 1$, at infinity and most significantly $b = a$. These assumptions play a crucial role in the variational approach presented in [3], where, using some delicate integral inequalities, the authors prove, for a certain range of λ, the existence of a positive solution bounded below by $1/|x|^{N-2}$ at infinity, for μ sufficiently small. On the other hand, problem (1) was also considered by Du and Ma in [4] and [5] for $g(u) = u^p$ in the absence of the harvesting term. The existence of a positive solution was then proved with *no restriction* on the growth of the nonnegative function b.

Our first motivation for this work was to study the existence of a positive solution in $\mathbb{R}^N$ in the presence of harvesting under minimal restriction on the growth of b. The novelty of our approach is that it not only enables us to relax the hypotheses on the nonlinear term $g(u)$ to the more natural conditions (2), so that it does not require the usual monotonicity and power-like behavior, but also, more importantly, that it allows for consideration of a broad class of functions b. In particular we will be able to handle some functions b satisfying $b(x) \to +\infty$ as $|x| \to \infty$, reflecting the assumption that the life conditions are less and less favorable as one moves to infinity.

In our approach we are naturally led to consider equations of the form

$$-\Delta u = \lambda a u \left[1 - k\left(\frac{u}{d}\right)\right] - \mu h, \tag{3}$$

where k is increasing and d is a given function. We note that this reduces to the classical logistic model if $k(u) = u$ and d is a constant. Therefore in line with the classical terminology, letting

$\varsigma = \max k^{-1}(1)$, one may call ςd the carrying capacity of the environment because without harvesting or diffusion the growth rate of the population, $\lambda au[1 - k(\frac{u}{d})]$, is negative for $u > \varsigma d$.

As it turns out, for suitable choices of the function d Eq. (3) is relatively simple to solve. In fact, using variational arguments, the maximum principle and comparison principles, we first prove the existence of a positive solution to (3). Afterwards this solution is used to obtain a solution of the original problem decaying at infinity not faster than d. Our method is not only simpler than that in [3] but also provides more general results under less restrictive hypotheses on the coefficients.

In Section 7 we apply the ideas developed to deal with the case of whole space $\mathbb{R}^N$ to the bounded domain case. This in particular allows us to consider the situation where b blows up at the boundary of Ω, which to our knowledge has not been considered before. Indeed since the boundary of Ω is hostile to the population, it is natural to assume that the carrying capacity of the environment should go to zero at $\partial\Omega$. The blow up of b at the boundary of the domain can then be interpreted as a consequence of the vanishing of the carrying capacity of the environment at the boundary of the domain. Our analysis will show that in some sense it is natural to consider a carrying capacity for the environment that is proportional to the distance to $\partial\Omega$. Our results in this chapter complement and extend known results in the bounded domain case (see [9]).

The organization of the paper is as follows. In Section 2 we state our hypotheses and make some preliminary observations. We set up problem (1) in $\mathbb{R}^N$ when b does not grow "too fast." In Section 3 we consider Eq. (3) and obtain a solution for this equation. The existence of a *positive* solution for (3) is then proved in Section 4. In Section 5 we use this solution to get a positive solution to (1) when the function b grows not faster than a certain power of the distance to the origin. In Section 6 we discuss the case when the function b does not satisfy the growth requirements of the previous section. Section 7 deals with the case of a bounded domain. In Section 8 we generalize to the case where the function g also depends on the spatial variable. Finally, in Appendix A we prove some auxiliary results.

Throughout we denote by $\mathcal{H} := \mathcal{D}^{1,2}(\mathbb{R}^N)$, $N \geqslant 3$, and $\|u\| = \|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = (\int |\nabla u|^2)^{1/2}$ the norm on $\mathcal{H}$. When the region of integration is omitted it is understood to be $\mathbb{R}^N$.

2. The setup in $\mathbb{R}^N$

We wish to prove the existence of a positive weak solution to the equation

$$-\Delta u = \lambda au - bg(u) - \mu h, \quad u \in \mathcal{H}. \tag{4}$$

We define a weak solution to be a function $u \in \mathcal{H}$ satisfying

$$\int \nabla u \cdot \nabla v = \lambda \int auv - \int bg(u)v - \mu \int hv \tag{5}$$

for all $v \in \mathcal{D}(\mathbb{R}^N)$. We state our assumptions.

(Ha) The function $a : \mathbb{R}^N \to \mathbb{R}$ is positive and belongs to $L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

We call

$$\lambda_1 = \inf_{u \in \mathcal{H} \setminus \{0\}} \frac{\|u\|^2}{\int au^2}.$$

(Hg) The function $g : \mathbb{R} \to \mathbb{R}_0^+$ is continuous, with $g(s) = 0$ for $s \leqslant 0$. Furthermore, it satisfies

$$\limsup_{s \to 0} \frac{g(s)}{s^{1+\beta}} < \infty, \tag{6}$$

where $\beta > 0$ is a fixed constant, and

$$\lim_{s\to+\infty} \frac{g(s)}{s} = +\infty. \tag{7}$$

(H$\mathscr{b}$) The measurable function $\mathscr{b} : \mathbb{R}^N \to \mathbb{R}$ is nonnegative, not identically equal to zero, and satisfies

$$\mathscr{b} \leqslant C_1 a \mathscr{d}^{-\beta} \tag{8}$$

for some $C_1 > 0$, where $\mathscr{d} : \mathbb{R}^N \to \mathbb{R}$ is the Aubin–Talenti instanton defined by

$$\mathscr{d}(x) = \left(1 + |x|^2\right)^{-(N-2)/2}. \tag{9}$$

Let $B_0 = \{x \in \mathbb{R}^N\colon\ \mathscr{b}(x) = 0\}$. We assume either B_0 has measure zero, or $B_0 = \overline{\operatorname{int} B_0}$ with ∂B_0 Lipschitz.

In the former case we set $\lambda_* = +\infty$ and in the latter case

$$\lambda_* = \inf_{u\in\mathcal{D}^{1,2}(\operatorname{int} B_0)\setminus\{0\}} \frac{\int_{B_0} |\nabla u|^2}{\int_{B_0} au^2}.$$

By the unique continuation principle [10, p. 519] $\lambda_1 < \lambda_*$.

(Hλ) The value λ is such that $\lambda_1 < \lambda < \lambda_*$.
(H$\mathscr{h}$) The nonnegative and not identically equal to zero function $\mathscr{h}$ belongs to the space $\mathscr{h} \in L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)$, for some $q > \frac{N}{2}$ and some $s > N$, and there exists a constant $C_2 > 0$ such that

$$R^{N/r} |\mathscr{h}|_{L^q(\mathbb{R}^N\setminus B_R(0))} \leqslant C_2 \quad \text{for all } R \in \mathbb{R}^+ \tag{10}$$

with $\frac{1}{q} + \frac{1}{r} = 1$. Here $B_R(0)$ denotes the ball centered at zero with radius R.
(Hμ) The parameter μ is nonnegative.

Remark 2.1. Under the above hypotheses any positive weak solution u of (4) belongs to $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. Furthermore, $\lim_{|x|\to\infty} u(x) = 0$.

Indeed, u satisfies

$$-\Delta u - \lambda a u \leqslant 0.$$

Therefore by [7, Theorem 8.17], for any $x \in \mathbb{R}^N$, we have

$$\sup_{B_1(x)} u \leqslant C|u|_{L^{2N/(N-2)}(B_2(x))} \leqslant C\|u\| \leqslant C.$$

So $u \in L^\infty(\mathbb{R}^N)$, and $\lim_{|x|\to\infty} u(x) = 0$. From elliptic regularity theory [7], it follows $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. We use the letter C to represent various positive constants.

The setting in which we make assumption (Hλ) is clarified in

Proposition 2.2. *Suppose $u \in \mathcal{H}$ is a positive weak solution to* (4)*.*

(i) *The value* λ *satisfies* $\lambda_1 \leqslant \lambda$. *This inequality is strict if* $\mu > 0$ *or if the restriction of* g *to* $\mathbb{R}^+$ *is positive.*

Suppose in addition $\operatorname{int} B_0 \neq \emptyset$.

(ii) *If* $\hbar = 0$ *on* B_0, *then* $\lambda < \lambda_*$.
(iii) *The inequality* $\lambda < \lambda_*$ *might not hold if* $h \not\equiv 0$ *on* B_0 *and* $\mu > 0$.

The proof is given in Appendix A so that we focus first on the more important part of the paper. In the sequel we will sometimes abbreviate weak solution to solution.

3. A related problem

From (6) there exist $0 < s_0 \leqslant 1$ and $C_4 > 1$ such that

$$\frac{g(s)}{s} \leqslant \lambda \frac{C_4}{C_1} s^\beta \quad \text{for } s \leqslant s_0.$$

We may assume $C_4 \geqslant \frac{1}{s_0^\beta}$. We take

$$l := \left(\frac{1}{C_4}\right)^{1/\beta}, \tag{11}$$

so

$$l \leqslant s_0. \tag{12}$$

Using (8),

$$\not{b}\, \frac{g(s)}{s} \leqslant \lambda a \left(\frac{s}{l\not{d}(x)}\right)^\beta \quad \text{for } s \leqslant s_0.$$

We define

$$k(s) = s^\beta \tag{13}$$

for $s > 0$, $k(s) = 0$ for $s \leqslant 0$. We have

$$\not{b}\, g(s) \leqslant \lambda a s k\left(\frac{s}{l\not{d}}\right) \quad \text{for } s \leqslant s_0. \tag{14}$$

We first consider the equation

$$-\Delta u = \lambda a u\left[1 - k\left(\frac{u}{l\not{d}}\right)\right] - \mu \hbar. \tag{15}$$

Although we are primarily interested in the case where k is as in (13), we more generally assume

(Hk) $k(s) = 0$ for $s \leqslant 0$, k is continuous, increasing (not necessarily strictly) and $k(\varsigma) = 1$ for some $\varsigma > 0$.

In this and the next sections instead of (Hλ) we assume

(Hλ)$'$ The value λ is such that $\lambda > \lambda_1$.

Theorem 3.1. *Under* (Ha), (Hk), (Hλ)$'$ *and* (H$\hbar$), *there exists $\mu_0 > 0$ such that for all $0 \leqslant \mu \leqslant \mu_0$ Eq.* (15) *has a positive weak solution $\underline{u}_\mu \in \mathcal{H} \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. Furthermore, there exists $C_3 > 0$ such that for all $0 \leqslant \mu \leqslant \mu_0$ this weak solution $\underline{u}_\mu$ satisfies*

$$\underline{u}_\mu(x) \geqslant \frac{C_3}{|x|^{N-2}} \quad \textit{for large } |x|. \tag{16}$$

In this section we prove existence of a solution to (21) below. This solution will be used in the next section to establish Theorem 3.1. We define $\hat{l}$ by

$$\hat{l} = \varsigma l. \tag{17}$$

Remark 3.2. The function $\hat{l}\mathscr{d}$ is a supersolution of (15).

Indeed, this follows from $-\Delta \mathscr{d} = N(N-2)\mathscr{d}^{2^*-1} > 0$, where $2^* = 2N/(N-2)$. Consider $\overline{G} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ with $\overline{G}(x,u) := \lambda a(x) \int_0^u sk(\frac{s}{l\mathscr{d}(x)})\,ds$ and the functional $I_\mu : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$I_\mu(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int a\big(u^+\big)^2 + \int \overline{G}(\cdot,u) + \mu\int \hbar u \tag{18}$$

if $\int \overline{G}(\cdot,u) < \infty$, and $I_\mu(u) = +\infty$ otherwise. We have used the standard notation $u^+ = \max\{0,u\}$. The function $\mathscr{d}$ belongs to $\mathcal{H}$. The function $\hbar$ belongs to the space $L^{2N/(N+2)}(\mathbb{R}^N)$ because $1 < 2N/(N+2) < N/2 < q$. So we have $I_\mu(\hat{l}\mathscr{d}) < \infty$ since $\int \overline{G}(\cdot,\hat{l}\mathscr{d}) < \infty$. Indeed, k increasing in $\mathbb{R}^+$ implies

$$\overline{G}(x,u) \leqslant \lambda a(x) u^2 k\left(\frac{u}{l\mathscr{d}(x)}\right). \tag{19}$$

Hence,

$$\begin{aligned}\int G(\cdot,\hat{l}\mathscr{d}) &\leqslant \lambda \hat{l}^2 \int a\mathscr{d}^2 \\ &< C\|a\|_{L^{N/2}(\mathbb{R}^N)}\|\mathscr{d}\|^2_{L^{2N/(N-2)}(\mathbb{R}^N)} \\ &\leqslant C\left(\int_0^\infty \frac{1}{(1+r^2)^{(N+1)/2}}\,dr\right)^{(N-2)/N} < \infty.\end{aligned}$$

We define the set

$$N = \big\{u \in \mathcal{H}\colon\ u \leqslant \hat{l}\mathscr{d} \text{ a.e. in } \mathbb{R}^N\big\}. \tag{20}$$

The set N is weakly closed.

Lemma 3.3. *Let $L \geqslant 0$. The functional I_μ is coercive on N, uniformly in μ with $0 \leqslant \mu \leqslant L$, i.e. for each $C > 0$, there exists $R > 0$ such that for all $0 \leqslant \mu \leqslant L$ and $u \in N$, if $\|u\| > R$ then $I_\mu(u) > C$.*

Proof. Suppose by contradiction there exists $u_n \in N$ with $\|u_n\| \to \infty$, and $\mu_n \in [0, L]$ such that $I_{\mu_n}(u_n) \leqslant C$. The sequence $v_n := u_n/\|u_n\|$ is bounded in $\mathcal{H}$ and so we may assume $v_n \rightharpoonup v$ in $\mathcal{H}$, $v_n \to v$ a.e. in $\mathbb{R}^N$. Since $u_n \leqslant \hat{l}d$ we have $v^+ \equiv 0$. Thus $\int a(v_n^+)^2 = o(1)$. Clearly,

$$I_{\mu_n}(u_n) \geqslant \|u_n\|^2\left(\frac{1}{2} + o(1) - C\frac{|h|_{L^{2N/(N+2)}(\mathbb{R}^N)}}{\|u_n\|}\right) \to \infty.$$

This contradiction proves the lemma. $\square$

Since the functional I_μ is weakly lower semi-continuous on $\mathcal{H}$, it admits a minimizer $\hat{u}_\mu$ on N for each $\mu \geqslant 0$. We note the derivative $I'_\mu(\hat{u}_\mu)\varphi$ is well defined for any $\varphi \in \mathcal{H} \cap L^\infty(\mathbb{R}^N)$ with compact support because $\sup \hat{u}_\mu$ is uniformly bounded (by $\hat{l}d$). In Lemma 5.4 we prove the differentiability of a related functional in a more general situation when we do not know a priori $\sup \hat{u}_\mu$ is uniformly bounded.

Lemma 3.4. *The function $\hat{u}_\mu$ is a solution to the equation*

$$-\Delta u = \lambda a u^+ - \lambda a u k\left(\frac{u}{ld}\right) - \mu h. \tag{21}$$

The argument of the proof is identical to the one in [11, Section I.2.3].

Lemma 3.5. *There exist $\mu_1, C_5 > 0$ such that for $0 \leqslant \mu \leqslant \mu_1$, we have $\inf_N I_\mu \leqslant -C_5 < 0$.*

Proof. From the definition of λ_1, there exists a sequence $u_n \in \mathcal{D}(\mathbb{R}^N) \setminus \{0\}$ satisfying

$$\frac{\|u_n\|^2}{\int a u_n^2} \to \lambda_1.$$

Since

$$\min\left\{\frac{\|u_n^+\|^2}{\int a(u_n^+)^2}, \frac{\|u_n^-\|^2}{\int a(u_n^-)^2}\right\} \leqslant \frac{\|u_n\|^2}{\int a u_n^2}$$

if u_n changes sign, we may assume each function u_n is nonnegative. Fix an n large enough so

$$\frac{\|u_n\|^2}{\int a u_n^2} < \lambda$$

and let K be the support of u_n. For small $t \in \mathbb{R}^+$, the energy of tu_n is

$$\begin{aligned} I_\mu(tu_n) &= \frac{t^2}{2}\|u_n\|^2 - \frac{\lambda t^2}{2}\int_K a u_n^2 + \int_K G(\cdot, tu_n) + \mu t \int_K h u_n \\ &\leqslant \frac{t^2}{2}\|u_n\|^2\left(1 - \lambda\frac{\int_K a u_n^2}{\|u_n\|^2}\right) + t^2 o(1) + \mu t \int_K h u_n. \end{aligned}$$

Here $o(1) \to 0$ as $t \to 0$. We have used (19), k is continuous at zero with $k(0) = 0$ and $u_n \in \mathcal{D}(\mathbb{R}^N)$. Note $d^{-1} \in L^\infty(K)$. We fix t small enough so $tu_n \in N$ and the sum of the first two terms is negative, say equal to $-C$, with $C > 0$. For μ sufficiently small, $0 \leqslant \mu \leqslant \mu_1$, the last term can be made smaller than $-C/2$. This shows $\inf_N I_\mu \leqslant -C/2 =: -C_5$. □

As in [3, Proposition 1.4], there exist $0 < r_0 < R_0$ such that

$$0 \leqslant \mu \leqslant \mu_1 \quad \Longrightarrow \quad r_0 \leqslant \|\hat{u}_\mu\| \leqslant R_0. \tag{22}$$

Indeed, the inequality

$$I_\mu(u) \geqslant -C\|u\|^2 + \int G(\,\cdot\,, u) - C\|u\| \geqslant -C\|u\|^2 - C\|u\|$$

implies

$$\liminf_{u \to 0} I_\mu(u) \geqslant 0.$$

Thus (22) follows from Lemmas 3.3 and 3.5.

4. A positive solution for the related problem

In this section we use the minimizers $\hat{u}_\mu$ of I_μ on N obtained above, Lemmas 3.3 and 3.5, and (22) to complete the

Proof of Theorem 3.1. By the Riesz Representation Theorem there exists $w \in \mathcal{H}$ satisfying

$$\int \nabla w \cdot \nabla \phi = \int \hbar \phi \tag{23}$$

for all $\phi \in \mathcal{H}$, as $\hbar \in L^{2N/(N+2)}$. Since also $\hbar \in L^s$ for some $s > N$, by elliptic regularity theory w belongs to the space $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\alpha > 0$. We can rewrite (21) as

$$-\Delta(\hat{u}_\mu + \mu w) = \lambda a \hat{u}_\mu^+ \left[1 - k\left(\frac{\hat{u}_\mu}{l\mathcal{d}}\right)\right].$$

The right-hand side satisfies $0 \leqslant \lambda a \hat{u}_\mu^+[1 - k(\frac{\hat{u}_\mu}{l\mathcal{d}})] \leqslant \lambda a \hat{u}_\mu^+$, since $\hat{u}_\mu \leqslant \hat{l}\mathcal{d}$ and k is increasing in $\mathbb{R}^+$. As $\hat{u}_\mu^+ \in L^\infty(\mathbb{R}^N)$ and $a \in L^\infty(\mathbb{R}^N)$, by elliptic regularity theory $\hat{u}_\mu \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$.

There exists $0 < \mu_2 \leqslant \mu_1$ such that for all $0 \leqslant \mu \leqslant \mu_2$ one can choose $x_0(\mu)$ where $\hat{u}_\mu(x_0(\mu)) > 0$. Otherwise $\hat{u}_\mu \leqslant 0$ and

$$\begin{aligned} I_\mu(\hat{u}_\mu) &= \frac{1}{2}\|\hat{u}_\mu\|^2 + \mu \int \hbar \hat{u}_\mu \\ &\geqslant \frac{1}{2}\|\hat{u}_\mu\|^2 - \mu |\hbar|_{L^{2N/(N+2)}} C \|\hat{u}_\mu\| \geqslant 0 \end{aligned}$$

for small μ because $r_0 \leqslant \|\hat{u}_\mu\| \leqslant R_0$ (see (22)). This contradicts Lemma 3.5.

Because the function $\hat{u}_{\mu_2}$ is a solution of (21) for $\mu = \mu_2$, the function $\hat{u}_{\mu_2}$ is a subsolution of (21) for $0 \leqslant \mu \leqslant \mu_2$. Using Lemma 3.3, we minimize the functional I_μ over the set

$$M = \left\{u \in \mathcal{H}\colon\ \hat{u}_{\mu_2} \leqslant u \leqslant \hat{l}\mathcal{d} \ \text{ a.e. in } \mathbb{R}^N\right\}. \tag{24}$$

Thus, for $0 \leqslant \mu \leqslant \mu_2$, obtain new solutions $\underline{u}_\mu$ of (21), which means

$$\int \nabla \underline{u}_\mu \cdot \nabla v = \lambda \int a\underline{u}_\mu^+ v - \lambda \int a\underline{u}_\mu k\left(\frac{\underline{u}_\mu}{l\mathcal{d}}\right) v - \mu \int \hbar v \tag{25}$$

for all $v \in \mathcal{D}(\mathbb{R}^N)$.

For later reference, we note that using Lemma 3.5, inequality (22) and observing that

$$I_\mu(\underline{u}_\mu) \leqslant I_{\mu_2}(\hat{u}_{\mu_2}) + C|\mu - \mu_2| R_0,$$

we may assume, by decreasing μ_2 if necessary, that

$$I_\mu(\underline{u}_\mu) = \inf_M I_\mu \leqslant -\frac{C_5}{2} < 0, \quad 0 \leqslant \mu \leqslant \mu_2. \tag{26}$$

Here the constant C_5 is as in Lemma 3.5.

We fix $x_0 = x_0(\mu_2)$. There exists $\rho > 0$ such that

$$\inf_{B_\rho(x_0)} \hat{u}_{\mu_2} > 0.$$

Choose ε sufficiently small satisfying

$$\frac{\varepsilon}{|x - x_0|^{N-2}} < \hat{u}_{\mu_2}(x) = \underline{u}_{\mu_2}(x) \quad \text{if } x \in \partial B_\rho(x_0).$$

All the $\underline{u}_\mu$ lie above $\underline{u}_{\mu_2}$ and w is positive so

$$\inf_{B_\rho(x_0)} \underline{u}_\mu \geqslant \inf_{B_\rho(x_0)} \underline{u}_{\mu_2} > 0 \tag{27}$$

and

$$\frac{\varepsilon}{|x - x_0|^{N-2}} < (\underline{u}_\mu + \mu w)(x) \quad \text{if } x \in \partial B_\rho(x_0)$$

for all $0 \leqslant \mu \leqslant \mu_2$. Let

$$S_\mu = \left\{ x \in B_\rho(x_0)^C \colon \ \frac{\varepsilon}{|x - x_0|^{N-2}} > (\underline{u}_\mu + \mu w)(x) \right\}.$$

Note $0 \leqslant \lambda a\underline{u}_\mu k(\frac{\underline{u}_\mu}{l\mathcal{d}}) \leqslant \lambda a\underline{u}_\mu^+$. Let v be an arbitrary function in $\mathcal{H}$ and $v_n \in \mathcal{D}(\mathbb{R}^N)$, $v_n \to v$ in $\mathcal{H}$. Using equality (25) with v replaced by v_n and passing to the limit, we see (25) is valid for v in $\mathcal{H}$. Hence, using (23),

$$\int \nabla(\underline{u}_\mu + \mu w) \cdot \nabla \phi = \int \lambda a\hat{u}_\mu^+ \left[1 - k\left(\frac{\hat{u}_\mu}{l\mathcal{d}}\right)\right] \phi \quad \text{for all } \phi \in \mathcal{H}. \tag{28}$$

Also

$$\int \nabla\left(\frac{1}{|x - x_0|^{N-2}}\right) \cdot \nabla \phi = 0 \tag{29}$$

for all $\phi \in \mathcal{H}$ satisfying $\phi(x) = 0$ for $x \in B_\rho(x_0)$. Subtracting (29) from (28),

$$\int \nabla\left(\underline{u}_\mu + \mu w - \frac{\varepsilon}{|x - x_0|^{N-2}}\right) \cdot \nabla\phi = \int \lambda a \hat{u}_\mu^+ \left[1 - k\left(\frac{\hat{u}_\mu}{l\mathscr{d}}\right)\right]\phi$$

for all $\phi \in \mathcal{H}$ satisfying $\phi(x) = 0$ for $x \in B_\rho(x_0)$. The function $\phi := (\underline{u}_\mu + \mu w - \frac{\varepsilon}{|x-x_0|^{N-2}})\chi_{S_\mu}$ belongs to $\mathcal{H}$, is less than or equal to zero and has support in $B_\rho(x_0)^C$. Thus

$$\int_{S_\mu} \left|\nabla\left(\underline{u}_\mu + \mu w - \frac{\varepsilon}{|x - x_0|^{N-2}}\right)\right|^2 \leqslant 0.$$

Therefore S_μ is empty which means

$$\frac{\varepsilon}{|x - x_0|^{N-2}} \leqslant (\underline{u}_\mu + \mu w)(x) \quad \text{for all } x \in B_\rho(x_0)^C. \tag{30}$$

We now recall the following lemma due to Allegretto and Odiobala.

Lemma 4.1. *(See [1, Lemma 4].) Let $\hbar \in L^1(\mathbb{R}^N)$ and suppose* (10) *holds. Then there exists a constant C such that*

$$w(x) \leqslant \frac{C}{|x|^{N-2}} \quad \textit{for all } x \in \mathbb{R}^N \setminus \{0\}.$$

Combining the estimates (27) and (30) with Lemma 4.1, we conclude there exists $0 < \mu_0 \leqslant \mu_2$ such that for all $0 \leqslant \mu \leqslant \mu_0$ the function $\underline{u}_\mu$ is positive and $\underline{u}_\mu(x) \geqslant \frac{C_3}{|x|^{N-2}}$ for $x \in B_\rho(x_0)^C$. This completes the proof of Theorem 3.1. □

5. A positive solution in $\mathbb{R}^N$

We now turn to Eq. (4).

Theorem 5.1. *Under* (Ha), (Hg), (H$\mathscr{b}$), (Hλ) *and* (H$\hbar$), *there exists $\mu_0 > 0$ such that for all $0 \leqslant \mu \leqslant \mu_0$ Eq.* (4) *has a positive weak solution $u_\mu \in \mathcal{H} \cap C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$. Furthermore, there exists $C_3 > 0$ such that for all $0 \leqslant \mu \leqslant \mu_0$ this weak solution u_μ satisfies*

$$u_\mu(x) \geqslant \frac{C_3}{|x|^{N-2}} \quad \textit{for large } |x|. \tag{31}$$

Proof. We take the function k as in (13) and apply Theorem 3.1 to obtain a positive solution $\underline{u}_\mu$ of (15) for $0 \leqslant \mu \leqslant \mu_0$. Using (14) and

$$\underline{u}_\mu \leqslant \hat{l}\mathscr{d} = \varsigma l\mathscr{d} = l\mathscr{d} \leqslant l \leqslant s_0 \tag{32}$$

(see (24), (17), (Hk) and (12)), the function $\underline{u}_\mu$ satisfies

$$-\Delta\underline{u}_\mu \leqslant \lambda a\underline{u}_\mu - \mathscr{b}g(\underline{u}_\mu) - \mu\hbar,$$

and so is a subsolution of our problem.

Fix any $1 < p \leqslant (N+2)/(N-2)$. For all integers m with $m \geqslant 1$ we define $j_m : \mathbb{R} \to \mathbb{R}$ by

$$j_m(s) = \begin{cases} g(s) & \text{for } s \leqslant m, \\ g(m) - m^p + s^p & \text{for } s > m. \end{cases} \tag{33}$$

We also define $j : \mathbb{R} \to \mathbb{R}$ by

$$j(s) = \inf_{m \geqslant 1} j_m(s).$$

The function j is measurable and in $L^1_{\mathrm{loc}}(\mathbb{R})$.

Lemma 5.2. *The function j satisfies*

$$\lim_{s \to +\infty} \frac{j(s)}{s} = +\infty. \tag{34}$$

Proof. By contradiction, suppose there exists a constant $C > 0$ and a sequence $s_n \to +\infty$ such that $\frac{j(s_n)}{s_n} \leqslant C$. Then there also exists a sequence (m_n) with $m_n \geqslant 1$ and

$$\frac{j_{m_n}(s_n)}{s_n} \leqslant C + 1.$$

From the definition of j_{m_n} and using $\frac{g(s_n)}{s_n} \to +\infty$, it follows $s_n > m_n$ for large n. So for large n

$$\frac{j_{m_n}(s_n)}{s_n} = \frac{g(m_n) - m_n^p + s_n^p}{s_n} = \frac{g(m_n) - m_n^p}{s_n} + s_n^{p-1} \leqslant C + 1.$$

The last inequality implies $g(m_n) < m_n^p$ for large n and $m_n \to +\infty$. Thus

$$C + 1 \geqslant \frac{j_{m_n}(s_n)}{s_n} \geqslant \frac{g(m_n) - m_n^p}{m_n} + s_n^{p-1} = \frac{g(m_n)}{m_n} - m_n^{p-1} + s_n^{p-1} \geqslant \frac{g(m_n)}{m_n}$$

for large n. From assumption (7), $\lim_{n\to\infty} \frac{g(m_n)}{m_n} = +\infty$. We have reached a contradiction. This proves (34). $\square$

For $0 \leqslant \mu \leqslant \mu_0$ the function $\underline{u}_\mu$ satisfies $0 < \underline{u}_\mu \leqslant \hat{l}d \leqslant \hat{l} = l \leqslant 1 \leqslant m$ (see (11) and (17)). Since every j_m coincides with g up to m, we have $\underline{u}_\mu$ satisfies

$$-\Delta \underline{u}_\mu \leqslant \lambda a \underline{u}_\mu - b j_m(\underline{u}_\mu) - \mu h.$$

For each $0 \leqslant \mu \leqslant \mu_0$, we define the set

$$M_\mu = \{ u \in \mathcal{H}\colon\ \underline{u}_\mu \leqslant u \ \text{ a.e. in } \mathbb{R}^N \}.$$

The set M_μ is weakly closed. Let $J_m(s) = \int_0^s j_m(t)\,dt$ and $J(s) = \int_0^s j(t)\,dt$. The function J is continuous. For $m \geqslant 1$ we also define $I_\mu^m : M_\mu \to \mathbb{R} \cup \{+\infty\}$ by

$$I_\mu^m(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2} \int a u^2 + \int b J_m(u) + \mu \int h u$$

if $\int b J_m(u) < \infty$, and $I_\mu^m(u) = +\infty$ otherwise. Similarly, we define I_μ^0 with J in the place of J_m.

Lemma 5.3. *The functionals I_μ^m are coercive on M_μ, uniformly in m and μ with $m \geqslant 1$ and $0 \leqslant \mu \leqslant \mu_0$, i.e. for each $L > 0$, there exists $R > 0$ such that for all $m \geqslant 1$, $0 \leqslant \mu \leqslant \mu_0$ and $u \in M_\mu$, if $\|u\| > R$ then $I_\mu^m(u) > L$.*

Proof. The argument is similar to the one in [5, proof of Theorem 6]. Suppose by contradiction there exists $\mu_n \in [0, \mu_0]$, $m_n \geqslant 1$ and $u_n \in M_{\mu_n}$ with $\|u_n\| \to \infty$, such that $I_{\mu_n}^{m_n}(u_n) \leqslant C$. From the definition of j we also have $I_{\mu_n}^0(u_n) \leqslant C$. Clearly

$$c_n^2 := \int a u_n^2 \to +\infty$$

since J is nonnegative, and $\int hu \geqslant 0$ for all $u \in M_\mu$. We define a sequence of functions, (v_n), with $v_n = \frac{u_n}{c_n}$, so that $\int a v_n^2 = 1$ and

$$\frac{1}{2}\|v_n\|^2 - \frac{\lambda}{2} + \frac{1}{c_n^2}\int b J(c_n v_n) + \frac{\mu_n}{c_n}\int h v_n \leqslant \frac{C}{c_n^2}. \tag{35}$$

Inequality (35) implies $\|v_n\|$ is uniformly bounded in n. Up to a subsequence, $v_n \rightharpoonup v$ in $\mathcal{H}$ and $v_n \to v$ a.e. in $\mathbb{R}^N$. The function v is nonnegative. Inequality (34) implies $\lim_{s\to+\infty} J(s)/s^2 = +\infty$. Taking the limit inferior on both sides of (35), and using Fatou's lemma,

$$\frac{1}{2}\|v\|^2 - \frac{\lambda}{2} + \int_{\{x\in\mathbb{R}^N:\ v(x)>0\}} b \times (+\infty) v^2 \leqslant 0.$$

The function v must be zero almost everywhere on the set where the function b is positive, i.e. (aside from a set of measure zero) v must have support in B_0. We also obtain $\|v\|^2 \leqslant \lambda$. On the other hand, since $\int a v_n^2 = 1$ and $\int a v_n^2 \to \int a v^2$, the function $v \not\equiv 0$ and $\int a v^2 = 1$. If B_0 has measure zero, then we are done. Otherwise, (Hb) implies $v \in \mathcal{D}^{1,2}(\operatorname{int} B_0)$ and

$$\lambda_* \leqslant \frac{\|v\|^2}{\int a v^2} \leqslant \lambda.$$

This contradicts $\lambda < \lambda_*$. The lemma is proved. □

For $0 \leqslant \mu \leqslant \mu_0$ and $m \geqslant 1$, the functional I_μ^m has a minimizer u_μ^m on M_μ, which of course is positive.

Lemma 5.4. *Suppose $v \in \mathcal{H}(\mathbb{R}^N)$ with compact support. For $u \in \mathcal{H}$ with $\int b J_m(u) < \infty$, the functional I_μ^m is differentiable in the direction v and*

$$\frac{d}{dt}\int b J_m(u + tv)\Big|_{t=0} = \int b j_m(u) v.$$

Proof. Our assumption on p and $b \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ imply $\int b J_m(u + tv) < \infty$. Suppose $0 < |t| \leqslant 1$.

$$\begin{aligned}\frac{\int b[J_m(u+tv) - J_m(u)]}{t} &= \int_{\{x\in\mathbb{R}^N:\ v(x)\neq 0\}} b\left(\frac{1}{tv}\int_u^{u+tv} j_m(s)\,ds\right) v\,dx \\ &= \int_{\{x\in\mathbb{R}^N:\ v(x)\neq 0\}} b(\overline{j_m})_t v\,dx,\end{aligned}$$

where $(\overline{j_m})_t : \{x \in \mathbb{R}^N\colon\ v(x) \neq 0\} \to \mathbb{R}$ is defined by

$$(\overline{j_m})_t(x) := \frac{1}{tv(x)} \int_{u(x)}^{u(x)+tv(x)} j_m(s)\,ds.$$

We have

$$\left|(\overline{j_m})_t\right| \leqslant \varepsilon\left(u^+ + v^+\right) + C_\varepsilon\left(\left(u^+\right)^p + \left(v^+\right)^p\right).$$

The function $\mathscr{b}[\varepsilon(u^+ + v^+) + C_\varepsilon((u^+)^p + (v^+)^p)]v$ is integrable. So the assertion of the lemma follows from Lebesgue's Dominated Convergence Theorem. □

Using Lemma 5.4, I_μ^m is differentiable at u_μ^m in the direction of functions φ of compact support. As in Lemma 3.4 one can prove u_μ^m is a solution of

$$-\Delta u = \lambda a u - \mathscr{b} j_m(u) - \mu \mathscr{h}, \tag{36}$$

by showing $(I_\mu^m)'(u_\mu^m)\varphi = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$. The functions u_μ^m satisfy

$$-\Delta u_\mu^m - \lambda a u_\mu^m \leqslant 0.$$

By [7, Theorem 8.17] we have

$$\sup_{\mathbb{R}^N} u_\mu^m \leqslant C_6 \left\|u_\mu^m\right\|, \tag{37}$$

where the constant C_6 depends only on N, λ and the norm $|a|_{L^\infty(\mathbb{R}^N)}$. Furthermore, from (14), (32), $s_0 \leqslant 1 \leqslant m$ and (26), we have

$$I_\mu^m\left(u_\mu^m\right) \leqslant I_\mu(\underline{u}_\mu) < 0.$$

So using Lemma 5.3 there exists an $R > 0$ such that $\|u_\mu^m\| \leqslant R$. It follows $\sup_{\mathbb{R}^N} u_\mu^m \leqslant C_6 R =: C_7$. If we take any constant $m \geqslant C_7$, the function u_μ^m is a solution of (4). Since the right-hand side of (4) belongs to $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ and $s > N$ by elliptic regularity theory $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\alpha > 0$. Estimate (31) is immediate from (16). The proof of Theorem 5.1 is complete. □

Suppose $\tilde{\mathscr{d}}$ is another function satisfying the properties that we used concerning the function $\mathscr{d}$, i.e. suppose $\tilde{\mathscr{d}} \in \mathcal{H}$ is continuous, $\tilde{\mathscr{d}} \not\equiv 0$ and $-\Delta \tilde{\mathscr{d}} \geqslant 0$. Multiplying the last inequality by $\tilde{\mathscr{d}}^-$ and integrating, $\tilde{\mathscr{d}}^- \equiv 0$. From [7, Theorem 8.19], there exists C such that $\inf_{x \in \overline{B_1(0)}} \tilde{\mathscr{d}}(x) = C > 0$. Hence,

$$\tilde{\mathscr{d}}(x) \geqslant \frac{C}{|x|^{N-2}} \tag{38}$$

for $x \in \partial B_1(0)$. As $x \mapsto \frac{C}{|x|^{N-2}}$ is harmonic in $B_1(0)^C$, by the maximum principle inequality (38) also holds for $x \in B_1(0)^C$. So $\tilde{\mathscr{d}} \geqslant C\mathscr{d}$. If $\mathscr{b} \leqslant \tilde{C}_1 a \tilde{\mathscr{d}}^{-\beta}$ for some constant $\tilde{C}_1 > 0$, then $\mathscr{b} \leqslant C_1 a \mathscr{d}^{-\beta}$ for some constant $C_1 > 0$. So we cannot apply the proof above if $\mathscr{b}$ grows faster than in (8). In addition, inequality (31) shows the bound $\underline{u}_\mu \leqslant \hat{l}\mathscr{d}$ is sharp.

6. The case where $\mathcal{b}$ grows fast

Eq. (4) may have positive solutions for $\mathcal{b}$ growing faster than in (8), or in other words for $\mathcal{d}$ going faster to zero than $1/|x|^{N-2}$ as $|x| \to \infty$. We now prove a theorem regarding such a situation. We will relax the growth condition on $\mathcal{b}$ at infinity and the condition on g at zero, at the expense of assuming a more restrictive hypothesis for $\mathcal{h}$.

Instead of (Hg), (H$\mathcal{b}$) and (H$\mathcal{h}$) we now assume

(Hg)′ The function $g:\mathbb{R}\to\mathbb{R}_0^+$ is continuous, with $g(s)=0$ for $s\leqslant 0$. Furthermore,

$$\lim_{s\to 0}\frac{g(s)}{s}=0$$

and (7) holds.

(H$\mathcal{b}$)′ The measurable function $\mathcal{b}:\mathbb{R}^N\to\mathbb{R}$ is nonnegative, not identically equal to zero, and satisfies $\mathcal{b}=\lambda a\Upsilon$ with $\Upsilon\in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$. Let $B_0=\{x\in\mathbb{R}^N\colon\ \Upsilon(x)=0\}$. We assume either B_0 has measure zero, or $B_0=\overline{\operatorname{int}B_0}\neq\mathbb{R}^N$ with $\operatorname{int}B_0\neq\emptyset$ and ∂B_0 Lipschitz.

(H$\mathcal{h}$)′ The measurable, nonnegative and not identically equal to zero function $\mathcal{h}$ has compact support and there exists a constant C_8 such that $\mathcal{h}\leqslant C_8 a$.

Theorem 6.1. *Under* (Ha), (Hg)′, (H$\mathcal{b}$)′, (Hλ) *and* (H$\mathcal{h}$)′, *there exists* $\mu_3>0$ *such that for all* $0\leqslant\mu\leqslant\mu_3$ *Eq.* (4) *has a positive weak solution* $u_\mu\in\mathcal{H}\cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. *Furthermore, there exists a constant* $C>0$ *such that, for all* $0\leqslant\mu\leqslant\mu_3$, $\|u_\mu\|_{L^\infty(\mathbb{R}^N)}\leqslant C$.

Proof. To solve Eq. (4), we first consider

$$-\Delta u=\lambda au-2\tilde{\mathcal{b}}\tilde{g}(u), \tag{39}$$

where $\tilde{\mathcal{b}}=\lambda a\tilde{\Upsilon}$, with $\tilde{\Upsilon}=\max\{\Upsilon,1\}$, and $\tilde{g}(u)=g(u)+(u^+)^2$. Obviously, zero is a solution to this equation. We define the set

$$M=\bigl\{u\in\mathcal{H}\colon\ u\geqslant 0 \text{ a.e. in } \mathbb{R}^N\bigr\}. \tag{40}$$

For all integers $m\geqslant 1$, we define $I^m:M\to\mathbb{R}\cup\{+\infty\}$ by

$$I^m(u)=\frac{1}{2}\|u\|^2-\frac{\lambda}{2}\int au^2+2\int\tilde{\mathcal{b}}J_m(u)$$

if $\int\tilde{\mathcal{b}}J_m(u)<\infty$, and $I^m(u)=+\infty$ otherwise. Here J_m is as in Section 5 with g replaced by $\tilde{g}$. As in Lemma 5.3, the functionals I^m are coercive on M, uniformly in m. Indeed, $\{x\in\mathbb{R}^N\colon\ \tilde{\mathcal{b}}(x)=0\}=\emptyset$. For $m\geqslant 1$, the functional I^m has a minimizer $\underline{u}^m$ on M. As a consequence of the analogue of Lemma 3.5, $I^m(\underline{u}^m)<0$. Lemma 5.4 applies as well as the subsequent discussion. Eq. (39) has a nonnegative solution $\underline{u}\in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. We observe that $\underline{u}\not\equiv 0$ since it has negative energy. We prove that $\underline{u}$ is positive. We may rewrite (39) as

$$-\Delta u=\lambda au\bigl(1-2\tilde{\Upsilon}k(u)\bigr),$$

with $k(s)=\tilde{g}(s)/s$ for $s\neq 0$ and $k(0)=0$. Suppose by contradiction $\underline{u}$ vanishes at some point x_0. Because $\underline{u}$ and k are continuous, $k(\underline{u}(x_0))=0$ and $\tilde{\Upsilon}\in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, there exists $r>0$ such that $1-2\tilde{\Upsilon}(x)k(\underline{u}(x))>0$ for $x\in B_r(x_0)$. Thus $-\Delta\underline{u}(x)\geqslant 0$ in the sense of distributions for $x\in B_r(x_0)$.

From [7, Theorem 8.19], it follows $\underline{u} \equiv 0$ in $B_r(x_0)$. By the unique continuation principle [10, p. 519] $\underline{u} \equiv 0$ in $\mathbb{R}^N$. We have reached a contradiction so $\underline{u}$ is positive.

There exists a constant $c > 0$ such that $\underline{u}(x) \geqslant c$ for x in the support of h. Then $\tilde{g}(\underline{u}(x)) \geqslant c^2$ for x in the support of h. Let $0 \leqslant \mu \leqslant \mu_3 := \frac{\lambda c^2}{C_8}$. Taking into account (H$b$)′ and (H$h$)′, $\tilde{b} \geqslant \lambda a$ and $h \leqslant C_8 a \leqslant \frac{C_8}{\lambda} \tilde{b}$. Then in the support of h, we have

$$\mu h \leqslant \frac{\lambda c^2}{C_8} h \leqslant c^2 \tilde{b} \leqslant \tilde{b} \tilde{g}(\underline{u});$$

thus $\mu h \leqslant \tilde{b}\tilde{g}(\underline{u})$ everywhere on $\mathbb{R}^N$. So $\underline{u}$ satisfies

$$-\Delta \underline{u} \leqslant \lambda a \underline{u} - \tilde{b}\tilde{g}(\underline{u}) - \mu h \leqslant \lambda a \underline{u} - b g(\underline{u}) - \mu h.$$

We also have

$$\begin{aligned} \tilde{I}_\mu(\underline{u}) &:= \frac{1}{2} \int |\nabla \underline{u}|^2 - \frac{\lambda}{2} \int a \underline{u}^2 + \int b G(\underline{u}) + \mu \int h \underline{u} \\ &\leqslant \frac{1}{2} \int |\nabla \underline{u}|^2 - \frac{\lambda}{2} \int a \underline{u}^2 + \int \tilde{b} \tilde{G}(\underline{u}) + \mu \int h \underline{u} \leqslant C < \infty \end{aligned}$$

because $I^m(\underline{u}) < 0$, and h has compact support and belongs to the space $L^\infty(\mathbb{R}^N)$. (We could even take C to be zero if we restricted $0 \leqslant \mu \leqslant \frac{\lambda c^2}{3C_8}$ because this would imply $\mu \int h \underline{u} \leqslant \int \tilde{b} \tilde{G}(\underline{u})$). Repeating the arguments in Section 5 we obtain a positive solution u_μ of (4) with $\tilde{I}_\mu(u_\mu) \leqslant \tilde{I}_\mu(\underline{u})$. The uniform bound on the $L^\infty(\mathbb{R}^N)$ norm on u_μ follows from the uniform coercivity in Lemma 5.3 and (37). □

We mention it is possible to construct examples where Eq. (4) has a positive solution for a b growing faster than in (8) and an h without compact support.

7. The case of a bounded domain

As we noted in the last paragraph of Section 5, the upper bound (8) we imposed on b was the weakest one under which our proof goes through. In this sense, the choice we made for d in (9) was the best one possible. To treat the case of a bounded domain Ω we start by constructing the best function d for this setting. This is done in the next lemma. We note that in part (i) we do not assume Ω is bounded (having in mind future extensions to the case of unbounded domains which are not the whole space $\mathbb{R}^N$). In fact, if one is just concerned with the case of a bounded domain, then a shorter proof of (i) can be given.

Lemma 7.1. *Let Ω be a smooth domain in $\mathbb{R}^N$, $r > 0$, $y_0 \in \Omega$ with* $\operatorname{dist}(y_0, \partial\Omega) > 3r$, *and G be Green's function of the first kind for Ω. In* (ii) *and* (iii) *assume Ω is bounded.*

(i) *There exists a function $d \in C^2(\overline{\Omega})$, superharmonic in Ω and harmonic in $\Omega \setminus B_r(y_0)$, satisfying*

$$cG(x, y_0) \leqslant d(x) \leqslant CG(x, y_0) \quad \text{for } x \in \overline{\Omega} \setminus B_{2r}(y_0) \tag{41}$$

for some constants $c, C > 0$.

(ii) *A function $b : \Omega \to \mathbb{R}_0^+$ satisfies*

$$b \leqslant \overline{C}_1 a\big[\operatorname{dist}(\cdot, \partial\Omega)\big]^{-\beta} \tag{42}$$

for some constant $\overline{C}_1 > 0$ *if and only if the function* $\mathcal{b}$ *satisfies*

$$\mathcal{b} \leqslant C_1 a \mathcal{d}^{-\beta} \tag{43}$$

for some constant $C_1 > 0$ *and the function* $\mathcal{d}$ *as in* (i).

(iii) *If* $\tilde{\mathcal{d}} \in \mathcal{D}^{1,2}(\Omega)$ *is continuous,* $\tilde{\mathcal{d}} \not\equiv 0$, $-\Delta \tilde{\mathcal{d}} \geqslant 0$ *and* $\mathcal{b} \leqslant \tilde{C}_1 a \tilde{\mathcal{d}}^{-\beta}$ *for some constant* $\tilde{C}_1 > 0$, *then* $\mathcal{b} \leqslant C_1 a \mathcal{d}^{-\beta}$ *for some constant* $C_1 > 0$.

Proof. (i) Let

$$\Gamma(x) = \frac{1}{N(N-2)\omega_N} \cdot \frac{1}{|x|^{N-2}},$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$. The function Γ is uniformly continuous in $\mathbb{R}^N \setminus B_r(0)$. This means for each $\varepsilon > 0$ there exists $0 < \delta < r$ such that $y_1, y_2 \in B_r(0)^C$ and $|y_1 - y_2| < 2\delta$ implies $|\Gamma(y_1) - \Gamma(y_2)| < \varepsilon$. If $y_1, y_2 \in B_\delta(y_0)$ and $|x - y_1| \geqslant r$, $|x - y_2| \geqslant r$ then $|\Gamma(x - y_1) - \Gamma(x - y_2)| < \varepsilon$. Hence,

$$y_1, y_2 \in B_\delta(y_0) \text{ and } x \in \overline{\Omega} \setminus B_{r+\delta}(y_0) \quad \Longrightarrow \quad \bigl|\Gamma(x - y_1) - \Gamma(x - y_2)\bigr| < \varepsilon.$$

Green's function of the first kind for Ω is

$$G(x, y) = \Gamma(x - y) + h_y(x),$$

where

$$\begin{cases} -\Delta h_y(x) = 0 & \text{for } x \in \Omega, \\ h_y(x) = -\Gamma(x - y) & \text{for } x \in \partial\Omega. \end{cases}$$

When Ω is unbounded, we further assume h_y satisfies $\lim_{x\to\infty} h_y(x) = 0$. Then the existence of such an h_y can be established by adapting Perron's method or applying standard variational arguments. For $y_1, y_2 \in B_\delta(y_0)$ and $x \in \partial\Omega$, we have $|h_{y_1}(x) - h_{y_2}(x)| < \varepsilon$, so by the maximum principle

$$y_1, y_2 \in B_\delta(y_0) \text{ and } x \in \overline{\Omega} \setminus B_{r+\delta}(y_0) \quad \Longrightarrow \quad \bigl|G(x, y_1) - G(x, y_2)\bigr| < 2\varepsilon.$$

One easily obtains $x \in \partial B_{r+\delta}(y_0)$ implies

$$G(x, y_0) \geqslant \frac{1}{N(N-2)\omega_N r^{N-2}} \left(\frac{1}{2^{N-2}} - \frac{1}{3^{N-2}} \right) =: c > 0.$$

The value c only depends on r and N. Let

$$C = \max_{x \in \partial B_{r+\delta}(y_0)} G(x, y_0).$$

Choose $\varepsilon = c/4$. We have

$$y \in B_\delta(y_0) \text{ and } x \in \partial B_{r+\delta}(y_0) \quad \Longrightarrow \quad \frac{c}{2} \leqslant G(x, y) \leqslant C + \frac{c}{2}.$$

So $y \in B_\delta(y_0)$ and $x \in \partial B_{r+\delta}(y_0)$ implies

$$\frac{c}{2C}G(x,y_0)\leqslant G(x,y)\leqslant\left(\frac{C}{c}+\frac{1}{2}\right)G(x,y_0). \tag{44}$$

By the maximum principle the two inequalities of the last previous line also hold for $x\in\overline{\Omega}\setminus B_{r+\delta}(y_0)$. Let $\eta\in\mathcal{D}(B_\delta(y_0))$, $\eta\geqslant 0$ and $\int\eta=\rho>0$ and consider the function $d\in\mathcal{D}(\overline{\Omega})$ defined by

$$d(x)=\int G(x,y)\eta(y)\,dy. \tag{45}$$

Multiplying (44) by $\eta(y)$ and integrating, for $x\in\overline{\Omega}\setminus B_{r+\delta}(y_0)$,

$$\rho\frac{c}{2C}G(x,y_0)\leqslant d(x)\leqslant\rho\left(\frac{C}{c}+\frac{1}{2}\right)G(x,y_0).$$

Obviously $-\Delta d=\eta$ in Ω and $d=0$ on $\partial\Omega$.

(ii) Let (N_σ,proj) (with $\mathrm{proj}:N_\sigma\to\partial\Omega$) be a tubular neighborhood of $\partial\Omega$ in $\overline{\Omega}$ (see [8, p. 35]) with the length of the segment $\mathrm{proj}^{-1}(x)$ equal to σ for each $x\in\partial\Omega$. There exist $0<\sigma<\operatorname{dist}(y_0,\partial\Omega)-2r$ and $c>0$ satisfying

$$x\in N_\sigma\quad\Longrightarrow\quad-\frac{\partial d}{\partial\nu_{\mathrm{proj}\,x}}(x)\geqslant c. \tag{46}$$

The vector $\nu_{\mathrm{proj}\,x}$ is the exterior outward unit normal to $\partial\Omega$ at the point $\mathrm{proj}\,x$. Indeed, suppose by contradiction there exist $\sigma_n\searrow 0$ and $x_n\in N_{\sigma_n}$ satisfying

$$-\frac{\partial d}{\partial\nu_{\mathrm{proj}\,x_n}}(x_n)\leqslant\frac{1}{n}.$$

Modulo a subsequence, $x_n\to x_0\in\partial\Omega$. It follows $\mathrm{proj}\,x_n\to\mathrm{proj}\,x_0=x_0$, $\nu_{\mathrm{proj}\,x_n}\to\nu_{\mathrm{proj}\,x_0}$ and $-\frac{\partial d}{\partial\nu_{x_0}}(x_0)\leqslant 0$. This contradicts Hopf's lemma. We have established (46). Since $d\in C^2(\overline{\Omega})$, there exists $C>0$ such that

$$x\in N_\sigma\quad\Longrightarrow\quad-\frac{\partial d}{\partial\nu_{\mathrm{proj}\,x}}(x)\leqslant C. \tag{47}$$

Given $x\in N_\sigma$, we integrate $\frac{\partial d}{\partial\nu_{\mathrm{proj}\,x}}$ along the part of the segment $\mathrm{proj}^{-1}(\mathrm{proj}\,x)$ between $\mathrm{proj}\,x$ and x. This part of $\mathrm{proj}^{-1}(\mathrm{proj}\,x)$ has length $\operatorname{dist}(x,\partial\Omega)$. Using (46) and (47),

$$x\in N_\sigma\quad\Longrightarrow\quad c\operatorname{dist}(x,\partial\Omega)\leqslant d(x)\leqslant C\operatorname{dist}(x,\partial\Omega). \tag{48}$$

Suppose (42) holds. Using (48), $x\in N_\sigma\Rightarrow b(x)\leqslant C\,a(x)[d(x)]^{-\beta}$. On the other hand, there exist constants $c,C>0$ such that

$$\overline{\Omega}\setminus N_\sigma\quad\Longrightarrow\quad c\frac{\mathrm{diameter}(\Omega)}{2}\leqslant d(x)\leqslant C\sigma.$$

As a consequence,

$$x\in\overline{\Omega}\setminus N_\sigma\quad\Longrightarrow\quad c\operatorname{dist}(x,\partial\Omega)\leqslant d(x)\leqslant C\operatorname{dist}(x,\partial\Omega). \tag{49}$$

Taking into account (48) and (49), we conclude (42) and (43) are equivalent.

(iii) Suppose $\tilde{d} \in \mathcal{D}^{1,2}(\Omega)$ is continuous, $\tilde{d} \not\equiv 0$ and $-\Delta\tilde{d} \geqslant 0$. Multiplying the last inequality by $\tilde{d}^-$ and integrating, $\tilde{d}^- \equiv 0$. From [7, Theorem 8.19], $\inf_{x\in B_\delta(y_0)} \tilde{d}(x) > 0$. Thus there exists $C > 0$ such that

$$\tilde{d}(x) \geqslant C d(x) \tag{50}$$

for $x \in \overline{B_\delta(y_0)}$. By the maximum principle, as d is harmonic in $\Omega \setminus \overline{B_\delta(y_0)}$, inequality (50) also holds for $x \in \Omega \setminus \overline{B_\delta(y_0)}$. So (50) holds for $x \in \Omega$. The assertion follows. □

In the remainder of this section we suppose Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geqslant 3$. We wish to prove the existence of a positive solution to Eq. (4) where now $\mathcal{H} = \mathcal{D}^{1,2}(\Omega)$. We introduce

(Ha)″ The function $a : \Omega \to \mathbb{R}$ is positive and belongs to $L^\infty(\Omega)$.

(Hb)″ The measurable function $b : \Omega \to \mathbb{R}$ is nonnegative, not identically equal to zero, and satisfies

$$b \leqslant \overline{C}_1 a\big[\operatorname{dist}(\,\cdot\,, \partial\Omega)\big]^{-\beta}. \tag{51}$$

Let $B_0 = \{x \in \Omega\colon\ b(x) = 0\}$. We assume either B_0 has measure zero, or $B_0 = \overline{\operatorname{int} B_0}$ (closure in B_0) with ∂B_0 Lipschitz.

(Hh)″ The nonnegative and not identically equal to zero function h belongs to the space $L^s(\mathbb{R}^N)$, for some $s > N$.

Remark 7.2. Proposition 2.2 generalizes to the case of a bounded domain.

The proof is given in Appendix A.

Theorem 7.3. *Under* (Ha)″, (Hg), (Hb)″, (Hλ) *and* (Hh)″, *there exists* $\mu_4 > 0$ *such that for all* $0 \leqslant \mu \leqslant \mu_4$ *Eq.* (4) *has a positive weak solution* $u_\mu \in \mathcal{H} \cap C^{1,\alpha}(\Omega)$.

Proof. We fix any $x_1 \in \Omega$ and $r_1 < \operatorname{dist}(x_1, \partial\Omega)/3$. Let d be as in (i) of Lemma 7.1 with $y_0 = x_1$ and $r = r_1$. By (ii) of the same lemma, the function b satisfies (43). We repeat the arguments in Section 3 but with this new function d. For any nonnegative μ we obtain a solution $\hat{u}_\mu \in C^{1,\alpha}(\overline{\Omega})$ to (21). As in Lemma 3.5 there exist $\mu_5, C_9 > 0$ such that for $0 \leqslant \mu \leqslant \mu_5$, we have $\inf_N I_\mu \leqslant -C_9 < 0$ (with N as in (20)). As in the beginning of Section 4, there exists $0 < \mu_6 \leqslant \mu_5$ such that for all $0 \leqslant \mu \leqslant \mu_6$ one can choose $x_0(\mu)$ where $\hat{u}_\mu(x_0(\mu)) > 0$. In addition, there exists $\rho > 0$ such that

$$\inf_{\overline{B_\rho(x_0(\mu_6))}} \hat{u}_{\mu_6} > 0.$$

Let $r_0 < \min\{\rho, \operatorname{dist}(x_0(\mu_6), \partial\Omega)/3\}$. We again use (i) of Lemma 7.1, but this time with $y_0 = x_0(\mu_6)$ and $r = r_0$, to construct a function $\hat{d} \in C^2(\overline{\Omega})$, superharmonic in Ω and harmonic in $\Omega \setminus B_{r_0}(x_0(\mu_6))$ satisfying (41). We fix $\varepsilon > 0$ sufficiently small such that

$$\varepsilon\hat{d}(x) \leqslant \hat{u}_{\mu_6}(x) \quad \text{for } x \in B_\rho\big(x_0(\mu_6)\big).$$

Clearly,

$$\varepsilon\hat{d}(x) \leqslant (\hat{u}_{\mu_6} + \mu_6 w)(x) \quad \text{for } x \in B_\rho\big(x_0(\mu_6)\big)$$

with w as in (23). The maximum principle implies

$$\varepsilon\hat{d}(x) \leqslant (\hat{u}_{\mu_6} + \mu_6 w)(x) \quad \text{for } x \in \Omega \setminus B_\rho\big(x_0(\mu_6)\big).$$

As in Section 4, we use $\hat{u}_{\mu_6}$ as a subsolution to (21) when $0 \leqslant \mu \leqslant \mu_6$. We minimize I_μ over the set

$$\big\{u \in \mathcal{H}\colon\ \hat{u}_{\mu_6} \leqslant u \leqslant \hat{l}d \text{ a.e. in } \mathbb{R}^N\big\},$$

where $\hat{l}$ is as in (17), to obtain new solutions $\underline{u}_\mu$ of (21) for $0 \leqslant \mu \leqslant \mu_6$ with $I_\mu(\underline{u}_\mu) < 0$. These solutions satisfy

$$\varepsilon\hat{d} \leqslant \underline{u}_\mu + \mu w. \tag{52}$$

Combining (48) and (49), there exist constants $c, C > 0$ such that

$$c\,\mathrm{dist}(\,\cdot\,, \partial\Omega) \leqslant \hat{d} \leqslant C\,\mathrm{dist}(\,\cdot\,, \partial\Omega). \tag{53}$$

On the other hand, since $h \in L^s(\Omega)$ with $s > N$, $w \in C^{1,\alpha}(\overline{\Omega})$. Thus from (52) and (53) there exists $0 < \mu_7 \leqslant \mu_6$ such that for all $0 \leqslant \mu \leqslant \mu_7$ the function $\underline{u}_\mu$ is positive in Ω. Now we argue as in Section 5 and use $\underline{u}_\mu$ as subsolutions to (4). For $0 \leqslant \mu \leqslant \mu_7$ and all integers $m \geqslant 1$, we obtain a positive solution u_μ^m of (36) with $I_\mu^m(u_\mu^m) \leqslant I_\mu(\underline{u}_\mu) < 0$. This time we use [7, Theorem 8.25] to conclude the u_μ^m are uniformly bounded. Choosing any sufficiently large m we obtain a positive solution to (4). □

8. Further extensions

The results of the previous sections may be generalized to prove the existence of a positive solution to the equation

$$-\Delta u = \lambda a\big[u - g(\,\cdot\,, u)\big] - \mu h, \quad u \in \mathcal{H}. \tag{54}$$

We give two results related to Theorems 5.1 and 6.1 whose proofs we leave to the reader. First we replace (Hg) and (Hb) by

$(\mathrm{Hg})_d$ The function $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}_0^+$ is Carathéodory, with $g(x, s) = 0$ for $x \in \mathbb{R}^N$ and $s \leqslant 0$. Let $B_0 = \{x \in \mathbb{R}^N\colon\ g(x, s) = 0 \text{ for } s \in \mathbb{R}\}$. We assume either B_0 has measure zero, or $B_0 = \overline{\operatorname{int} B_0}$ with ∂B_0 Lipschitz. Furthermore, $g \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R})$,

$$\limsup_{s\to 0} \frac{[d(x)]^\beta g(x, s)}{s^{1+\beta}} < \infty \quad \text{uniformly for } x \in \mathbb{R}^N, \tag{55}$$

where $\beta > 0$ is a fixed constant and d is defined in (9), and

$$\lim_{s\to+\infty} \frac{g(x, s)}{s} = +\infty \quad \text{for each } x \in B_0^C.$$

Theorem 8.1. *Under* (Ha), $(\mathrm{Hg})_d$, (Hλ) *and* (Hh), *there exists $\mu_0 > 0$ such that for all $0 \leqslant \mu \leqslant \mu_0$ Eq.* (54) *has a positive weak solution $u_\mu \in \mathcal{H} \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. Furthermore, there exists $C_3 > 0$ such that for all $0 \leqslant \mu \leqslant \mu_0$ this weak solution u_μ satisfies*

$$u_\mu(x) \geqslant \frac{C_3}{|x|^{N-2}} \quad \text{for large } |x|.$$

Now we replace (Hg), (H$\mathcal{b}$) and (H$\hbar$) as follows:

(Hg)$_\Upsilon$ The function $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}_0^+$ is continuous, with $g(x,s) = 0$ for $x \in \mathbb{R}^N$ and $s \leqslant 0$. Let $B_0 = \{x \in \mathbb{R}^N\colon\ g(x,s) = 0 \text{ for } s \in \mathbb{R}\}$. We assume either B_0 has measure zero, or $B_0 = \overline{\operatorname{int} B_0}$ with ∂B_0 Lipschitz. Furthermore, $g \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R})$,

$$\lim_{s\to 0} \frac{g(x,s)}{s} = 0 \quad \text{uniformly for } x \text{ in compact subsets of } \mathbb{R}^N,$$

and

$$\lim_{s\to +\infty} \frac{g(x,s)}{s} = +\infty \quad \text{for each } x \in B_0^C.$$

(H$\hbar$)$'''$ The measurable, nonnegative and not identically equal to zero function $\hbar$ has compact support and there exists a constant $C > 0$ such that $\hbar \leqslant C a$.

Theorem 8.2. *Under* (Ha), (Hg)$_\Upsilon$, (Hλ) *and* (H$\hbar$)$'''$, *there exists* $\mu_3 > 0$ *such that for all* $0 \leqslant \mu \leqslant \mu_3$ *Eq.* (54) *has a positive weak solution* $u_\mu \in \mathcal{H} \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$.

Appendix A

Proof of Proposition 2.2. (i) We choose an $R > 0$ such that $B_R(0) \setminus B_0 \neq \emptyset$. If the restriction of g to $\mathbb{R}^+$ is positive, then $\mathcal{b} g(u)\chi_{B_R(0)} \not\equiv 0$. For all $v \in \mathcal{D}(\mathbb{R}^N)$ with $v \geqslant 0$

$$\int \nabla u \cdot \nabla v \leqslant \lambda \int a u v - \int \mathcal{b} g(u)\chi_{B_R(0)} v - \mu \int \hbar v. \tag{56}$$

So (56) holds for all $v \in \mathcal{H}$ with $v \geqslant 0$. Taking $v = u$ we obtain

$$\|u\|^2 \leqslant \lambda \int a u^2 - \int \mathcal{b} g(u) u \chi_{B_R(0)} - \mu \int \hbar u \leqslant \lambda \int a u^2$$

and the last inequality is strict if $\mu > 0$ or if the restriction of g to $\mathbb{R}^+$ is positive. The conclusion follows.

(ii) Suppose $\hbar = 0$ on B_0. We write $u = u_0 + u^\perp$ where $u_0|_{\operatorname{int} B_0}$ is the projection of u on $\mathcal{D}^{1,2}(\operatorname{int} B_0)$ and $u_0 = 0$ on $(\operatorname{int} B_0)^C$. This means $u_0|_{\operatorname{int} B_0} \in \mathcal{D}^{1,2}(\operatorname{int} B_0)$ and

$$\int \nabla u \cdot \nabla v = \int \nabla u_0 \cdot \nabla v \quad \text{for all } v \in \mathcal{D}^{1,2}(\operatorname{int} B_0).$$

The function $u^\perp := u - u_0$ so that $u = u^\perp$ on $(\operatorname{int} B_0)^C$. Note

$$\int \nabla u^\perp \cdot \nabla v = \int \nabla (u - u_0) \cdot \nabla v = 0 \quad \text{for all } v \in \mathcal{D}^{1,2}(\operatorname{int} B_0),$$

which means that $u^\perp$ is harmonic in $\operatorname{int} B_0$. Since u is superharmonic in $\operatorname{int} B_0$ and $u^\perp$ is harmonic in $\operatorname{int} B_0$, u_0 is superharmonic in $\operatorname{int} B_0$. Thus u_0 is nonnegative. The function u_0 cannot be identically zero. Otherwise in $\operatorname{int} B_0$ we would have $0 = -\Delta u^\perp = -\Delta u = \lambda a u^\perp$. This implies $u^\perp \equiv 0$ in $\operatorname{int} B_0$ and so $u \equiv 0$ in $\operatorname{int} B_0$, contradicting the fact that u is positive. The function u has a positive trace on ∂B_0. Also $u = u^\perp$ on ∂B_0. So from $u^\perp \in \mathcal{H}$, clearly $(u^\perp)^-|_{\operatorname{int} B_0} \in \mathcal{D}^{1,2}(\operatorname{int} B_0)$, and hence $(u^\perp)^-|_{\operatorname{int} B_0} \equiv 0$.

By the strong maximum principle $u^\perp > 0$ on B_0. Let

$$\begin{cases} -\Delta\phi_1^* = \lambda_* a\phi_1^* & \text{in } \operatorname{int} B_0, \\ \phi_1^* > 0 & \text{in } \operatorname{int} B_0, \\ \phi_1^* = 0 & \text{on } (\operatorname{int} B_0)^C. \end{cases} \tag{57}$$

One can easily see we may also take v such that $v|_{B_0} = \phi_1^*$ and $v|_{B_0^C} = 0$ in (5). Indeed, this follows from $\mathcal{b} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and $\phi_1^*|_{\operatorname{int} B_0} \in \mathcal{D}^{1,2}(\operatorname{int} B_0)$. We obtain

$$\int \nabla u_0 \cdot \nabla\phi_1^* + \int \nabla u^\perp \cdot \nabla\phi_1^* = \lambda \int a u_0\phi_1^* + \lambda \int a u^\perp\phi_1^*.$$

This yields

$$\lambda_* \int a u_0\phi_1^* = \lambda \int a u_0\phi_1^* + \lambda \int a u^\perp\phi_1^* > \lambda \int a u_0\phi_1^*,$$

and so $\lambda < \lambda_*$.

(iii) We give functions a, $\mathcal{b}$, g, $\mathcal{h}$ (with $\mathcal{h} \not\equiv 0$ on B_0), and a function $u \in \mathcal{H}$ which is a positive solution of (4) for $\lambda = \lambda_* + \mu$. Here $\mu > 0$ is the parameter in (4). Since all functions will be radially symmetric, we introduce the coordinate $r = |x|$ and write them in terms of r. We choose the set $B_0 = \{x \in \mathbb{R}^N\colon\ r \leqslant 1\}$. The functions a and g are

$$a(r) = \begin{cases} 1 & \text{for } r \leqslant 1, \\ \frac{1}{r^{(N-2)\beta}} & \text{for } r > 1, \end{cases}$$

$$g(u) = \begin{cases} 0 & \text{for } u \leqslant 0, \\ u^{1+\beta} & \text{for } u > 0, \end{cases}$$

with $\beta > 2$. We define u using (57),

$$u(r) = \begin{cases} \phi_1^* + \kappa & \text{for } r \leqslant 1, \\ \frac{\kappa}{r^{N-2}} & \text{for } r > 1, \end{cases}$$

with $\kappa = -\frac{1}{N-2}\frac{\partial\phi_1^*}{\partial r}|_{r=1}$ so that $u \in C^1(\mathbb{R}^N)$. This is possible because ϕ_1^* is spherically symmetric [6] and $\frac{\partial\phi_1^*}{\partial r}|_{r=1} < 0$ (by Hopf's lemma). The functions $\mathcal{b}$ and $\mathcal{h}$ are

$$\mathcal{b}(r) = \begin{cases} 0 & \text{for } r \leqslant 1, \\ \frac{\lambda}{\kappa^\beta} & \text{for } r > 1, \end{cases}$$

$$\mu\mathcal{h}(r) = \begin{cases} \mu\phi_1^*(r) + \lambda\kappa & \text{for } r \leqslant 1, \\ 0 & \text{for } r > 1. \end{cases}$$

Our assumptions are all satisfied except for (Hλ) of course. In particular, the function a is positive and belongs to $L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. The measurable function $\mathcal{b}$ is nonnegative, not identically equal to zero, and satisfies (8) for $C_1 = \frac{\lambda}{\kappa^\beta}$ as $a\mathcal{d}^{-\beta} > 1$. Note also $u \in \mathcal{H}$. The function u satisfies (4) in $B_1(0)$ and in $\overline{B_1(0)}^C$. In fact, for $r < 1$,

$$-\Delta\big(\phi_1^* + \kappa\big) = \lambda \cdot 1 \cdot \big(\phi_1^* + \kappa\big) - 0 - \big(\mu\phi_1^* + \lambda\kappa\big) = \lambda_*\phi_1^*.$$

For $r > 1$,

$$0 = \lambda\frac{1}{r^{(N-2)\beta}}\frac{\kappa}{r^{N-2}} - \frac{\lambda}{\kappa^\beta}\frac{\kappa^{1+\beta}}{r^{(N-2)(1+\beta)}} - 0.$$

Let $v \in \mathcal{D}(\mathbb{R}^N)$. We recall $u \in C^1(\mathbb{R}^N)$. Multiplying (4) by v and integrating over $B_1(0)$ we obtain

$$-\int_{\partial B_1(0)} \frac{\partial u}{\partial r} v + \int_{B_1(0)} \nabla u \cdot \nabla v = \lambda \int_{B_1(0)} auv - \int_{B_1(0)} \not{b} g(u) v - \mu \int_{B_1(0)} \hbar v. \tag{58}$$

Multiplying (4) by v and integrating over $\overline{B_1(0)}^C$ we obtain

$$\int_{\partial B_1(0)} \frac{\partial u}{\partial r} v + \int_{\overline{B_1(0)}^C} \nabla u \cdot \nabla v = \lambda \int_{\overline{B_1(0)}^C} auv - \int_{\overline{B_1(0)}^C} \not{b} g(u) v - \mu \int_{\overline{B_1(0)}^C} \hbar v. \tag{59}$$

Adding (58) and (59), the function u is a positive weak solution of (4). □

Proof of Remark 7.2. The proof of items (i) and (ii) is similar to the case of the space $\mathbb{R}^N$. To check item (iii) let $\Omega = B_2(0)$. We may take

$$a(r) = \begin{cases} 1 & \text{for } r \leqslant 1, \\ (\frac{1}{r^{N-2}} - \frac{1}{2^{N-2}})^\beta & \text{for } 1 < r < 2, \end{cases}$$

$$u(r) = \begin{cases} \phi_1^* + \kappa(1 - \frac{1}{2^{N-2}}) & \text{for } r \leqslant 1, \\ \kappa(\frac{1}{r^{N-2}} - \frac{1}{2^{N-2}}) & \text{for } 1 < r < 2, \end{cases}$$

$$\mu \hbar(r) = \begin{cases} \mu\phi_1^*(r) + \lambda\kappa(1 - \frac{1}{2^{N-2}}) & \text{for } r \leqslant 1, \\ 0 & \text{for } 1 < r < 2, \end{cases}$$

and all the parameters and other functions as in the proof of Proposition 2.2. There exists $\overline{C}_1 > 0$ such that (42) holds because

$$0 < \lim_{r \to 2} \left[\left(\frac{1}{r^{N-2}} - \frac{1}{2^{N-2}} \right) \frac{1}{2-r} \right]^\beta < \infty. \qquad \square$$

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