Topology Vol. 32, No. 4, pp. 757–772, 1993.
Printed in Great Britain

0040–9383/93 $6.00 + 0.00

DEFORMATIONS OF SEXTIC SURFACES

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(*Received* 23 *October* 1992)

§1. INTRODUCTION

IN THIS paper we shall determine all possible structures on surfaces which are deformations of a non-singular surface of degree 6 in $\mathbf{P}^3$ defined over $\mathbf{C}$. If a surface S is a deformation of a sextic surface, then it has the following numerical characters

$$p_g = 10, \quad q = 0, \quad \text{and} \quad K^2 = 24,$$

where p_g, q, and K denote the geometric genus, irregularity, and canonical bundle of S, respectively. Moreover S has an even intersection form on $H^2(S, \mathbf{Z})$.

As the first main theorem, we determine the structures on the surfaces with these properties. Since S is even, the canonical bundle K is divisible by 2 as $K = 2L$ with a line bundle L.

THEOREM 1. *The possible structures on S can be classified into the following six types in terms of the rational map Φ_L associated to L:*

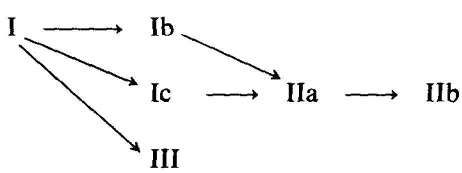
(Ia) *S is birationally equivalent to a sextic surface in $\mathbf{P}^3$ with at most rational double points.*
(Ib) *Φ_L is a generically* 2*-sheeted map onto a cubic surface in $\mathbf{P}^3$.*
(Ic) *Φ_L is a generically* 3*-sheeted map onto a quadratic surface in $\mathbf{P}^3$.*
(IIa) *Φ_L is a generically* 2*-sheeted map onto a smooth quadratic surface in $\mathbf{P}^3$.*
(IIb) *Φ_L is a generically* 2*-sheeted map onto a singular quadratic surface in $\mathbf{P}^3$.*
(III) *Φ_L is composed of a pencil of curves of genus* 3 *of non-hyperelliptic type.*

In the course of the proof, the construction of these surfaces will become clear.

As the second main theorem, we shall prove the following.

THEOREM 2. *All the surfaces in Theorem* 1 *are specializations of non-singular sextic surfaces. In other words, the moduli space of even surfaces with the above numerical characters is irreducible.*

Our proof is very constructive, and we shall give, for each type of the above six, defining equation of the desired family. We obtain the following specialization diagram:



Since S is automatically minimal, the above list exhausts all possible complex structures on the underlying differentiable manifold.

An outline of the results was stated in [7, §4].

§2. SURFACES OF TYPE I

Let S be a minimal non-singular algebraic surface with $p_g = 10$, $q = 0$ and $K^2 = 24$. We further assume that the canonical bundle K is divisible by 2, and let $K = 2L$, where L is a line bundle on S. We use the abbreviated symbol $h^i(L)$ to denote $\dim H^i(S, \mathcal{O}(L))$ etc.

LEMMA 2.1. *We have* $h^0(L) = 4$ *and* $h^1(L) = 0$.

Proof. We have $10 = h^0(2L) \geq 2h^0(L) - 1$, and the Riemann-Roch theorem yields

$$2h^0(L) - h^1(L) = -\frac{1}{2}L^2 + 11 = 8.$$

It follows that $h^0(L) = 4$ or 5.

Suppose $h^0(L) = 5$ and consider the rational map Φ_L associated to L. We claim that Φ_L is composed of a pencil. If this is not so, then the image $W = \Phi_L(S)$ is a surface in $\mathbf{P}^4$ which is contained in 5 linearly independent quadrics. In particular $\deg W \leq 4$. Combined with the inequality $\deg \Phi_L \cdot \deg W \leq 6$, it follows that $\deg W = 3$. Hence, as is well known, W is either the Hirzebruch surface Σ_1, i.e., the $\mathbf{P}^1$-bundle $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(1))$ over $\mathbf{P}^1$, or a cone over a cubic rational curve in $\mathbf{P}^3$. In either case, W is contained only in 3 quadrics. Now, L can be written as $|L| = |4L_0| + F$, with some irreducible pencil L_0 and the fixed part F. It easily follows that $LL_0 = 1$, $LF = 2$. Then the first equality implies that $L_0^2 = 0$, $L_0F = 1$. Therefore, L_0 is a pencil of curves of genus 2 without base points. From the list of six types (0)–(V) of the singular fibres (see [5]), we see that any singular fibre other than those of type (0) contains a divisor with self-intersection number -1. Since S is even, only fibres of type (0) can occur. But this implies that $K^2 = 2p_g - 4$, which is absurd.

LEMMA 2.2. *If* $|L|$ *has no base point nor fixed component, then* Φ_L *is one of the following*:

(Ia) Φ_L *induces a birational map of* S *onto a sextic surface with at most rational double points.*
(Ib) Φ_L *is a generically 2-sheeted map onto a cubic surface.*
(Ic) Φ_L *is a generically 3-sheeted map onto a quadratic surface.*

Although the proof is obvious, we state here the existence of surfaces of types Ib and Ic. Suppose first that $\deg \Phi_L(S) = 3$. Let $\{x_0, x_1, x_2, x_3\}$ be a basis of $H^0(S, \mathcal{O}(L))$. Then, by assumption, there is one cubic relation $g(x_0, x_1, x_2, x_3) = 0$. Since $h^0(3L) = h^0(K + L) = 20$, by the Riemann-Roch theorem and the vanishing theorem, we have a new element $w \in H^0(S, \mathcal{O}(3L))$ which is independent of the cubic monomials in x_i. Let C be a smooth member of the linear system $|L|$ associated to the line bundle L. Then $K + L$ induces the canonical bundle K_C. If C is hyperelliptic then the restriction $L|_C$ is of the form $[4P_0 + P_1 + P_2]$, where P_0 is a Weierstrass point and P_1 and P_2 are fixed points, because it is a subsystem of $|K_C|$ with $h^0(L|_C) = 3$ (see Lemma 3.2 below). This would imply that $|L|$ has base points. Therefore C is not hyperelliptic. Furthermore, comparing the dimensions, we see that w and the cubic monomials of the x_i generate the space $H^0(C, \mathcal{O}(K_C))$. It also follows that there is no relation of the form

$$h(x_0, x_1, x_2, x_3)w + f(x_0, x_1, x_2, x_3) = 0, \; h \nmid g.$$

This implies that the monomials in w and x_i, linear in w, form a basis of $H^0(S, \mathcal{O}(6L))$. Therefore, there is a relation of the form

$$w^2 + h(x_0, x_1, x_2, x_3)w + f(x_0, x_1, x_2, x_3) = 0\,,$$

where $\deg h = 3$, $\deg f = 6$. By linear change, we may assume $h = 0$.

Thus, we have shown that Φ_L is lifted to a map of S into a weighted projective space $V = \mathbf{P}(3, 1, 1, 1, 1)$ with coordinates (w, x_0, x_1, x_2, x_3) of weights (3, 1, 1, 1, 1) whose image S' is defined by

$$g = 0, \quad w^2 + f = 0.$$

Since S' coincides with the canonical model of S, it has at most rational double points. Conversely, if we choose general g and f, then the above equations determine a smooth S'. It remains to show that such S', or its minimal resolution is an even surface with the desired numerical characters. This follows from a standard calculation. But here we prove the following proposition.

PROPOSITION 2.1. *Surfaces of type* Ib *and type* Ic *are deformations of smooth sextic surfaces.*

Proof. Actually we shall show that a surface S' defined above is a deformation of sextic surfaces. For this, let t be a parameter varying in a neighborhood of 0 in $\mathbf{C}$, and consider the subvarieties of V defined by

$$tw - g = 0, \; w^2 + f = 0.$$

If S_t' is a surface with parameter t, then S_t' has at most rational double points, because S_0' is so. For $t \neq 0$, S_t' is nothing but a sextic surface defined by $g^2 + t^2 f = 0$. Therefore the minimal resolution of S_t' is a deformation of the minimal resolution of S_0' (see below for surfaces of type Ic). q.e.d.

By a similar argument we can show that a surface of type Ic is defined in the weighted projective space $\mathbf{P}(2, 1, 1, 1, 1)$ by the equations

$$u^3 + A_2 u^2 + A_4 u + A_6 = 0, \; g = 0\,,$$

where $\deg u = 2$ and A_{2j} and g are homogeneous polynomials in the variables (x_0, x_1, x_2, x_3) of degree $2j$ and 2, respectively. Surfaces of type Ic can be deformed to sextic surfaces by replacing the equation $g = 0$ by $tu - g = 0$.

§3. SURFACES OF TYPE II

PROPOSITION 3.1. *Suppose Φ_L is not holomorphic nor composed of a pencil. Then Φ_L induces a generically 2-sheeted map onto a quadratic surface in $\mathbf{P}^3$, i.e., S is of type* IIa *or* IIb. *Moreover, $|L|$ is written $|H| + F$, where $|H|$ has no base point and $HF = 2$, $F^2 = -2$, $LF = 0$.*

Let $\pi: \tilde{S} \to S$ be a composition of blowings up such that $\Phi_L \circ \pi$ is holomorphic. We write $|\pi^* L| = |H| + F$, where $|H|$ is the variable and F is the fixed part. Let $\tilde{K}$ be the canonical bundle on $\tilde{S}$ and let $\tilde{K} = \pi^* K + [E]$, where E is a sum of the exceptional curves. Since we only need to blow up base points, we may assume $F \geq E$. From

$$L^2 = H^2 + HF + \pi^* L \cdot F, \; HF \geq 0, \; \pi^* L \cdot F \geq 0,$$

we obtain $H^2 \leq L^2 = 6$. If $H^2 = 6$, then $|L|$ would have no base point. If $H^2 = 5$, then S would be birationally equivalent to a quintic surface. Hence we get $H^2 = 4$, because $H^2 \leq 3$ is absurd. This implies that Φ_L is generically 2-sheeted onto a quadratic surface and it also follows

$$HF + \pi^* L \cdot F = 2.$$

From the inequality $HE \leq HF = 2$, we have $HE = 0$, 1, or 2. Let $|L_0| + F_0$ be the decomposition of $|L|$ into the variable and fixed part (on S). Then

$$L_0^2 \leq L^2 = 6,$$
$$H^2 = L_0^2 - \sum m_i^2,$$
$$HE = \sum m_i,$$

where the m_i are the multiplicities of the base points appearing in π.

If $HE = 1$, then π is a single blowing up and $L_0^2 = 5$. This contradicts that S is even. If $HF = 0$ then $F^2 \leq 0$ by Hodge's index theorem. This contradicts $L^2 \geq H^2$. Therefore, we have 3 possibilities:

(1) $HE = 2$,
(2) $HE = 0$, $HF = 2$, $F^2 = -2$,
(3) $HE = 0$, $HF = 1$, $F^2 = 0$.

LEMMA 3.1. *$HE = 2$ does not happen.*

Proof. If $HE = 2$, then, by the above consideration, π is a composite of two blowings up, and $L^2 = L_0^2$, which in turn implies that $|L|$ has no fixed component. In particular, any general member $C \in |L|$ is a non-singular curve, which is hyperelliptic. We recall the following lemma.

LEMMA 3.2. *Suppose that C is a hyperelliptic curve and let $|\Lambda|$ be a complete linear system such that $|K_C - \Lambda| \neq \emptyset$. Then $|\Lambda|$ is of the form $|\nu\vartheta|$ +* (fixed points), *where ϑ is a hyperelliptic divisor* (i.e., $\deg \vartheta = h^0(\vartheta) = 2$), *and $\nu = \dim |\Lambda|$.*

Proof. Let $\varphi: C \to \mathbf{P}^n$ be the map associated to the canonical system $|K_C|$. By assumption, the map Φ_Λ is dominated by φ, so that the variable part of $|\Lambda|$ is induced by a line bundle on $\varphi(C) = \mathbf{P}^1$. q.e.d.

Returning to the proof of Lemma 3.1, we can write the restriction $|L_C|$ in the form $|2\vartheta| + P_1 + P_2$, because $h^0(L_C) = 3$. Since $3L$ induces the canonical bundle $K_C = [9\vartheta]$, it follows that $3P_1 + 3P_2 \in |3\vartheta|$. If P_1' is the point conjugate to P_1, i.e., $P_1 + P_1' \in |\vartheta|$, then $3P_2$ is linearly equivalent to $3P_1'$. Since P_2 cannot coincide with P_1', this contradicts Lemma 3.2. This proves Lemma 3.1.

It remains to prove the following lemma.

LEMMA 3.3. *The case* (3) *does not occur.*

Proof. The equality $HE = 0$ implies that $|L_0|$ has no base point. So we take $\pi = \mathrm{id}$. Since $H^2 = 4$, Φ_H defines a generically 2-sheeted holomorphic map onto a quadric W in $\mathbf{P}^3$. We first suppose that the image is non-singular. Then, since $W = \mathbf{P}^1 \times \mathbf{P}^1$, $|L|$ is of the form $|D + D_1| + F$, where $|D|$ and $|D_1|$ are linear pencils with $DD_1 = 2$. By interchanging $|D|$ and $|D_1|$, if necessary, we may assume $FD = 0$, $FD_1 = 1$. Then F is contained in a fibre of the holomorphic map Φ_D. From $F^2 = 0$, it follows that F is a rational multiple of a total fibre. Combining with $2HF = HD$, we get $2F \in |D|$. This implies that $2L$ is linearly equivalent to $3D + 2D_1$ and we get $h^0(2L) \geq 12$, which contradicts the equality $p_g = 10$.

Next suppose that the image $\Phi_L(S)$ is singular. In this case, $|L|$ is of the form $|2D + G|$, where $|D|$ is a linear pencil possibly with base point and G is effective (see [4, p. 46]). Since $|H|$ has no base point, we have $HG = 0$, which in turn implies that $HD = 2$ and $2D^2 + DG = 2$. Since D^2 is even, we get $D^2 = 0$, $DG = 2$. Therefore D has no base point and $G \neq 0$. It follows that Φ_D is holomorphic and that Φ_H can be lifted to a holomorphic map $f: S \to \Sigma_2$, of S onto the Hirzebruch surface of degree 2.

We set $DF = k \geq 0$. If $k = 0$, then we have $2F \in |D|$ and get a contradiction as above. If $k > 0$, then $FG = 1 - 2k$ and $f_*F = k\Delta_0 + \Gamma$, where Δ_0 is a section of $\Sigma_2 \to \mathbf{P}^1$ with $\Delta_0^2 = -2$ and Γ is a fibre. From $H(D + G - 2F) = 0$, we have

$$0 \geq (D + G - 2F)^2 = -4(1 - k).$$

Hence we conclude that $k = 1$ and that $D + G$ is numerically equivalent to $2F$.

To see that $f^*f_*F - 2F$ is effective, we calculate the ramification divisor R of f, which turns out to be linearly equivalent to $8D + 4G + 2F$. Hence the branch locus B is in the linear system $|10\Delta_0 + 18\Gamma|$ on Σ_2. Since $f^*B - 2R \in |2(D + G - 2F)|$ is effective [4, Lemma 3], it follows that $f^*f_*F - 2F$ is effective. Combining with the preceding fact, we conclude that $2F$ is linearly equivalent to $D + G$. Therefore, we get a contradiction $h^0(2L) \geq h^0(5D + 3G) \geq 12$, again. This completes the proof of Proposition 3.1.

Hereafter we assume that $|L| = |H| + F$, where $|H|$ has no base points with $H^2 = 4$, $HF = 2$ and $F^2 = -2$. A general member $C \in |H|$ is a non-singular hyperelliptic curves of genus 9, and the restriction $|L_C|$ is of the form $|2\vartheta| + P_1 + P_2$, where ϑ is a hyperelliptic divisor. Since $2L + H$ induces the canonical bundle $K_C = [8\vartheta]$, it follows that $2P_1 + 2P_2 \in |2\vartheta|$. If P_1' is conjugate to P_1, then $2P_2$ is linearly equivalent to $2P_1'$. Since $P_1 + P_2$ is the fixed part of $|L_C|$, P_2 cannot coincide with P_1'. Hence P_1 and P_2 are the Weierstrass points on C, which are distinct.

Let $\zeta \in H^0(S, \mathcal{O}([F]))$ be a non-zero section and let $\{z_0, z_1, z_2, z_3\}$ be a basis of $H^0(S, \mathcal{O}(H))$. We assume that C is defined by $z_3 = 0$ on S. There is one quadratic relation $q(z_0, z_1, z_2, z_3) = 0$, and the monomials $z_i z_j \zeta^2$ generate 9-dimensional subspace of $H^0(S, \mathcal{O}(2L))$. Hence there is an element $u \in H^0(S, \mathcal{O}(2L))$ which is linearly independent from these monomials.

Since $H|_C$ is of the form $[2\vartheta]$, we may assume that $z_0 = st$, $z_1 = s^2$, $z_2 = t^2$ on C. Moreover we may assume that P_1 and P_2 are the points $s = 0$ and $t = 0$, respectively, so that $\zeta^2|_C$ is a constant multiple of z_0. The restriction $u|_C$ is a sextic form in (s, t), and hence can be written as a cubic form in the z_i. That is

$$u = h_0(z_0, z_1, z_2) \quad \text{on } C.$$

Multiplying $z_0\zeta$, we obtain

$$z_0\zeta u = h_0(z_0\zeta, z_1\zeta, z_2\zeta) \quad \text{on } C.$$

This holds not only on C, but also on $C + F$, i.e., the difference is divisible by $z_3\zeta$. Introducing $x_i = z_i\zeta$, $0 \leq i \leq 3$, we obtain

$$x_0 u \equiv h_0(x_0, x_1, x_2) \mod x_3.$$

From the exact sequence

$$0 \to \mathcal{O}(2L) \to \mathcal{O}(3L) \to \mathcal{O}_{C+F}(3L|_{C+F}) \to 0,$$

it follows that the above relation is lifted to

$$l(x_0, x_3)u = h(x_0, x_1, x_2, x_3),$$

where $\deg l = 1$, $\deg h = 3$. We may assume $l = x_0$.

In $H^0(S, \mathcal{O}(3L))$ we have the elements

$$x_i x_j x_k, \ x_i u.$$

In the first group, there are 16 ($= 20 - 4$) linearly independent elements, and three more in the second group. Since $h^0(3L) = 20$, there is a new element $w \in H^0(S, \mathcal{O}(3L))$. The restriction $w|_C$ is a section of $[8\vartheta + P_1 + P_2] = [K_C + P_1 + P_2]$.

Recall that $KF = 0$ and $F^2 = -2$. Hence F is a fundamental cycle arizing from a resolution of a rational double point. It follows that $h^0(\mathcal{O}_F) = 1$ (see [1]). Therefore, we have $h^1(K + F) = h^1(-F) = 0$. Hence the restriction map $H^0(S, \mathcal{O}(3L)) \to H^0(C, \mathcal{O}(3L_C))$ is surjective. Let $C \to \mathbf{P}^1$ be the double covering associated to $|K_C|$. Then its ramification divisor R_C is linearly equivalent to 10ϑ. Therefore, $R_C - P_1 - P_2 \sim 8\vartheta + P_1 + P_2 \sim 3L_C$, where $\sim$ denotes linear equivalence on C. Combining these facts, we may assume that $(w|_C) = R_C - P_1 - P_2$.

To get the equation for w, we consider the space $H^0(S, \mathcal{O}(6L)) \cong \mathbf{C}^{83}$. Here we find the following elements:

sextics in the x_i $\mod q$	: 49 elements,
$u\cdot$(quartics in the x_i) $\mod(q, x_0)$	: 9 elements,
$u^2\cdot$(quadrics in the x_i) $\mod(q, x_0)$	: 5 elements,
u^3	: 1 element,
$w\cdot$(cubics in the x_i) $\mod q$	: 16 elements,
wux_i $(i \neq 0)$	: 3 elements,
w^2	: 1 element,

Since these are 84 in total, there is one non-trivial relation. The coefficient of w^2 in this relation is not 0, for otherwise w cannot separate the two sheets on C. Thus we obtain

$$w^2 = A_0u^3 + A_2u^2 + A_4u + A_6,$$

where the A_{2j} are forms of degree $2j$ in the x_i. This implies

$$x_0(x_0w)^2 = A_0h^3 + A_2x_0h^2 + A_4x_0^2h + A_6x_0^3. \tag{3.1}$$

This may be interpreted as follows. Suppose that a quadratic surface W: $q = 0$ is given in $\mathbf{P}^3$. Then $l = 0$ defines a divisor Δ on W, and $h = 0$ cuts 9 points P_i, $1 \leq i \leq 9$ on Δ (distinct provided l and h are chosen to be general). The above equation means that x_0^2w is the square root of

$$x_0(A_0h^3 + A_2x_0h^2 + A_4x_0^2h + A_6x_0^3).$$

This determines a double covering S' of W which is birationally equivalent to the original S. It is easy to see that, if A_{2j} are general, then the minimal resolution $\hat{S}$ of S' satisfies $p_g = 10$, $q = 0$ and $K^2 = 24$. Furthermore, since Δ is contained in the branch locus, the inverse image of Δ on $\hat{S}$ decomposes as

$$2\tilde{\Delta} + (\text{curves contracted to a point on } W),$$

From this, we infer that the canonical bundle of $\hat{S}$ is induced by $2\tilde{H} + 2\tilde{\Delta}$, where $\tilde{H}$ is the pull-back of the hyperplane bundle on $\mathbf{P}^3$. This proves that $\hat{S}$ is even.

We can put the result more formally as follows:

PROPOSITION 3.2. *Let S be a surface of type* IIa *or* IIb. *Then its canonical model is in the weighted projective space* $V = \mathbf{P}(1, 1, 1, 1, 2, 3)$ *defined by*

$$\begin{cases} q = 0, \\ x_0u = h, \\ w^2 = u^3 + A_2u^2 + A_4u + A_6, \end{cases}$$

where $(x_0, x_1, x_2, x_3, u, w)$ *is a system of coordinates on* V *with* $\deg x_i = 1$, $\deg u = 2$, $\deg w = 3$ *and* q, h, A_{2j} *are homogeneous forms in the* x_i *of degree* 2, 3, $2j$, *respectively.*

Remarks. (1) If A_0 should be 0, then x_0 could be factored out in (3.1), and S would have smaller value of p_g.

(2) In case of type IIb, W is a quadratic cone. In this case, the map $S \to W$ can be lifted to a holomorphic map $S \to \Sigma_2$ (cf. the proof of Lemma 3.3).

(3) In both cases of type IIa and IIb, S admits a pencil of hyperelliptic curves of genus 4. To make S even, we needed a special component Δ in the branch locus. Consideration similar to (2) shows that any surface of type Ic admits a pencil of curves of genus 4 which is of non-hyperelliptic type. (Note that general curves of genus 4 are trigonal.)

PROPOSITION 3.3. *Let S be a surface of type* IIa *or type* IIb, *and let $t = (t_1, t_2)$ be a system of parameters ranging in a neighborhood of the origin in $\mathbf{C}^2$. Then the following system of the equations*

$$\begin{cases} q - t_1 u = 0, \\ x_0 u - h - t_2 w = 0, \\ w^2 = u^3 + A_2 u^2 + A_4 u + A_6 \end{cases}$$

determines a family $\{S_t\}$ of deformations of $S = S_0$ such that

(1) *for $t_1 t_2 \neq 0$, S_t is a sextic surface in $\mathbf{P}^3$,*
(2) *for $t_1 = 0$, $t_2 \neq 0$, S_t is of type* Ib,
(3) *for $t_1 \neq 0$, $t_2 = 0$, S_t is of type* Ic.

Proof. If both t_1 and t_2 are non-zero, then one can solve the first two equations in u and w. So the above system reduces to a single equation of degree 6 in the x_i. In view of the description in §2, (2) and (3) are proved analogously. q.e.d.

From the defining equations it is obvious that the pencil of hyperelliptic curves of genus 4 on S is deformed to non-hyperelliptic pencil on surfaces of type Ic.

§4. SURFACES OF TYPE III

In the remaining sections we shall study the case in which Φ_L is composed of a pencil $|D|$. Since this pencil is necessarily linear, we can write

$$|L| = |3D| + F,$$

where F is the fixed part. From $L^2 = 6$ follows that $LD = 1$ or 2. In either case, we obtain $D^2 = 0$. Therefore, if $LD = 1$, then $|D|$ is a pencil of curves of genus 2 on an even surface S. This implies that S has only singular fibres of type (0) (see [5]), and S necessarily satisfies $K^2 = 2p_g - 4$. Hence we conclude that $|D|$ is a pencil of curves of genus 3 without base point.

PROPOSITION 4.1. *The general members of $|D|$ are not hyperelliptic.*

Proof is postponed to §7.

We now study the structure of this non-hyperelliptic pencil. In general, for a minimal surface S with such pencil, one has $K^2 \geq 3p_g - 7$. Since we have $K^2 = 3p_g - 6$ in our case, S will have one degeneration into hyperelliptic curve (see [7, §3]). In the present case, this is some special kind, because S is even. Note that we have

$$LD = 2, \quad LF = 0, \quad \text{and} \quad F^2 = -6.$$

Also

$$h^0(mL) = 3m(m-2) + 11, \quad \text{for } m \geq 3.$$

To simplify the notation, for any divisor Z, we write $H^0(Z)$ instead of $H^0(S, \mathcal{O}([Z]))$. Let $\{u_0, u_1\}$ be a basis of $H^0(D)$, and let $\zeta \in H^0(F)$ be non-zero, so that $x_i = u_0^i u_1^{3-i}\zeta$, $0 \le i \le 3$ form a basis of $H^0(L)$. Since $h^0(6D + 2F) = p_g = 10$ and $6D + 2F$ induces a canonical divisor on D, it follows $h^0(5D + 2F) \ge 7$. Therefore, besides $\zeta^2 u_0^i u_1^{5-i}$ $(0 \le i \le 5)$, there is a new section $\xi \in H^0(5D + 2F)$, i.e., ξ is linearly independent of the preceding sections. We postulate the following lemma, which will be proved at the end of this section.

LEMMA 4.1. $h^0(5D + 2F) = 7$.

Note that this implies that there is a new section $\eta \in H^0(6D + 2F)$ and that ζ^2, ξ, η induce a basis of $H^0(D, \mathcal{O}(K_D))$. In $H^0(9D + 3F) = H^0(3L)$ we have 19 sections:

$$u_0^i u_1^{9-i}\zeta^3 \quad (0 \le i \le 9), \qquad u_0^i u_1^{4-i}\zeta\xi \quad (0 \le i \le 4), \qquad u_0^i u_1^{3-i}\zeta\eta \quad (0 \le i \le 3).$$

Since $h^0(3L) = 20$ we can find a new section $\psi \in H^0(9D + 3F)$.

In $H^0(10D + 4F)$, we have 22 elements as above plus ξ^2, $\zeta u_0 \psi$, $\zeta u_1 \psi$.

LEMMA 4.2. *We have* $h^0(10D + 4F) = 24$.

Proof. First note that $H^0(12D + 4F) \cong \mathbf{C}^{35}$ contains all the six quadratic forms $\zeta^2, \zeta\xi, \ldots, \eta^2$, it follows that $h^0(11D + 4F) = 29$. Similarly, $H^0(11D + 4F)$ contain five of these. So $h^0(10D + 4F) \le 29 - 5 = 24$. Since we have $h^0(10D + 3F) = 24$, this proves the lemma.

By this lemma there is a relation of the form

$$\zeta L(u_0, u_1)\psi = f(u_0, u_1, \zeta^2, \xi, \eta).$$

Since $L(u_0, u_1) \ne 0$, by a linear change, we may assume $L = u_0$:

$$\zeta u_0 \psi = f(u_0, u_1, \zeta^2, \xi, \eta). \tag{4.1}$$

Here we introduce a (bi)degree by assigning (k, l) to the elements of $H^0(kD + 2lF)$. Therefore, f is of degree $(10, 2)$.

Next we look at $H^0(18D + 6F) \cong \mathbf{C}^{83}$. Here we have the following 83 elements

$$\begin{array}{llll}
\zeta^6 u_0^i u_1^{18-i} & (0 \le i \le 18), & \zeta^4 \xi u_0^i u_1^{13-i} & (0 \le i \le 13), \\
\zeta^4 \eta u_0^i u_1^{12-i} & (0 \le i \le 12), & \zeta^2 \xi^2 u_0^i u_1^{8-i} & 0 \le i \le 8), \\
\zeta^2 \xi\eta u_0^i u_1^{7-i} & (0 \le i \le 7), & \zeta^2 \eta^2 u_0^i u_1^{6-i} & (0 \le i \le 6), \\
\xi^3 u_0^i u_1^{3-i} & (0 \le i \le 3), & \xi^2 \eta u_0^i u_1^{2-i} & (0 \le i \le 2), \\
\xi\eta^2 u_0^i u_1^{1-i} & (0 \le i \le 1), & \eta^3, & \\
\psi\zeta^3 u_1^9, & & \psi\zeta\xi u_1^4, & \\
\psi\zeta\eta u_1^3. & & &
\end{array}$$

The first 80 elements are obviously linearly independent. Therefore, if these are linearly dependent, then we have a relation of the form

$$\zeta L'(u_1, \zeta^2, \xi, \eta)\psi = f'(u_0, u_1, \zeta^2, \xi, \eta),$$

where L' is non-trivial of degree $(9, 1)$. It would follow that $u_0 f' - L'f = 0$, which is a cubic relation among ζ^2, ξ and η. Thus we have seen that the above list gives a basis of $H^0(18D + 6F)$. Hence ψ^2 is a linear combination of them, and, by a linear change, we may assume

$$\psi^2 = g(u_0, u_1, \zeta^2, \xi, \eta),$$

where g is of degree (18,3).

Eliminating ψ we obtain

$$f^2 - u_0^2\zeta^2 g = 0.$$

These are geometrically interpreted as follows. The basis (u_0, u_1) determines a holomorphic map $S \to \mathbf{P}^1$. Let $V = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-5) \oplus \mathcal{O}(-6))$ be the $\mathbf{P}^2$-bundle over $\mathbf{P}^1$, and let Γ be a fibre of $V \to \mathbf{P}^1$. Then we can take homogeneous coordinates (Z_0, Z_1, Z_2) on the fibres such that

$$(Z_0) \sim (Z_1) - 5\Gamma \sim (Z_2) - 6\Gamma,$$

where $\sim$ denotes linear equivalence. The sections (ζ^2, ξ, η) determine a rational map h: $S \to V$ over $\mathbf{P}^1$ and its image S' is defined by the equation

$$f(u_0, u_1, Z_0, Z_1, Z_2)^2 - u_0^2 Z_0 g(u_0, u_1, Z_0, Z_1, Z_2) = 0, \tag{4.2}$$

where $\deg f = (10,2)$, $\deg g = (18,3)$ with $\deg u_i = (1,0)$ and $\deg Z_j = (0,1)$, $(5,1)$, $(6,1)$ $(j = 0, 1, 2)$. From this equation, it follows that S' has a double curve along $u_0 = f = 0$, i.e., along a conic Q in a fibre over $u_0 = 0$. Let $p: S'' \to S'$ be the blowing up of the conic Q, which is realized by introducing a new variable $w = f/u_0$ (there is no use of u_0/f). Therefore, by assigning $w = \zeta\psi$, h lifts to h'': $S \to S''$. Since the canonical bundle K_V is given by $[-3(Z_0) - 13\Gamma]$, the dualizing sheaf $\omega_{S'}$ is $\mathcal{O}([(Z_0) + 7\Gamma])$. Let $\mathscr{E} = \pi^{-1}(Q)$ be the exceptional divisor. Then the dualizing sheaf $\omega_{S''}$ is given by $p^*\omega_{S'} - \mathscr{E}$. Hence $H^0(S'', \omega_{S''})$ is the space of those $\varphi \in H^0(S', \mathcal{O}([(Z_0) + 7\Gamma]))$ which vanishes on Q. Since such φ must be divisible by u_0, we obtain the equality $h^0(\omega_{S''}) = h^0((Z_0) + 6\Gamma) = 10$, which in turn implies that S'' has at most rational double points.

It remains to show that (4.2) actually gives an even surface with prescribed numerical characters. We view the equation (4.2) as a system of surfaces which have double curve along the fixed conic Q: $u_0 = f(0, 1, Z_0, Z_1, Z_2) = 0$. Note that f is a combination of Z_0^2, Z_0Z_1, Z_0Z_2, Z_1^2 and does not involve Z_1Z_2 nor Z_2^2. For general choice of f and g, S' has no singularity outside Q. Since S'' is defined by

$$\begin{cases} u_0 w - f = 0, \\ w^2 - Z_0 g = 0, \end{cases} \tag{4.3}$$

we easily see that S'' is singular only at $u_0 = w = Z_0 = Z_1 = 0$, provided f and g are general. This singularity is locally given by

$$u_0 w - (aZ_1^2 + bw^2 + \cdots) = 0.$$

This yields an ordinary double point A_1. Hence, by blowing up this point, we obtain a non-singular model, which is minimal.

To study the canonical bundle of S, we look at the divisor defined by $Z_0 = 0$. Since f is not divisible by Z_0 (otherwise (4.2) yields a cubic equation), (4.2) reads

$$Z_1^4 = Z_0(g + \cdots).$$

This implies that, in S', $Z_0 = 0$ determines a curve Δ defined by $Z_0 = Z_1 = 0$, that $Z_1 = 0$ is a defining equation at its general point, and that Z_0 vanishes to the fourth order. Let $\tilde{\Delta}$ be the proper transform of Δ on S''. Then, (4.3) shows that w vanishes twice on $\tilde{\Delta}$. Let F_0 be the proper transform of $\tilde{\Delta}$ on the non-singular model S, and let F_1 be the rational curve obtained by resolving A_1-singularity on S''. As is well known, $(w) = 2F_0 + F_1$ on S. Hence we conclude $(Z_0) = 4F_0 + 2F_1$ and $K_S = 2(3D + 2F_0 + F_1)$.

The fact that S satisfies $K^2 = 24$ and $\chi(\mathcal{O}_S) = 11$ can be proved by standard calculations. (Or these facts follow from the existence of deformations to sextic surfaces.)

Proof of Lemma 4.1. We assume $h^0(5D + 2F) \geq 8$ and pursue similar construction, which will eventually lead to a contradiction.

Let $\{u_0, u_1\}$ be a basis of $H^0(D)$, and let $\zeta \in H^0(F)$ be a non-zero section. Since $h^0(6D + 2F) = 10$ and since

$$h^0((k+1)D + 2F) - h^0(kD + 2F)$$

is non-decreasig with k, we obtain $h^0(4D + 2F) \geq 6$. Hence we find a new section $\xi \in H^0(4D + 2F)$. By the equality $h^0(7D + 2F) = 13$, we can find another new section $\eta \in H^0(7D + 2F)$. Then the triple (ζ^2, ξ, η) determines a rational map of S into a $\mathbf{P}^2$-bundle $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-4) \oplus \mathcal{O}(-7))$ over $\mathbf{P}^1$. From $h^0(9D + 3F) = 20$, we infer that there exists a new element $\psi \in H^0(9D + 3F)$. Next we look at $H^0(9D + 4F)$. Here we find the following 21 elements:

$$\zeta^4 u_0^i u_1^{9-i}, \qquad \xi\zeta^2 u_0^i u_1^{5-i},$$

$$\eta\zeta^2 u_0^i u_1^{2-i}, \qquad \xi^2 u_0^i u_1^{1-i},$$

which are linearly independent. On the other hand, starting from the equality $h^0(12D + 4F) = 35$, we obtain $h^0(11D + 4F) \leq 30$, $h^0(10D + 4F) \leq 25$, and $h^0(9D + 4F) \leq 21$ (cf. Lemma 4.2). This implies that there exists a relation

$$\zeta\psi = f(u_0, u_1, \zeta^2, \xi, \eta), \tag{4.4}$$

where f is not divisible by ζ, i.e., f contains the term ξ^2.

Let D be a general member of $|D|$ and let $(\zeta|_D) = P_1 + P_2$. Then, we claim that ξ does not vanish at P_1 nor P_2. To see this, consider the space $H^0(10D + 3F) \cong \mathbf{C}^{24}$, which contains the following 24 elements:

$$\zeta^3 u_0^i u_1^{10-i}, \quad \zeta\xi u_0^i u_1^{6-i}, \quad \zeta\eta u_0^i u_1^{3-i}, \quad \psi u_0, \quad \psi u_1. \tag{4.5}$$

If these were linearly dependent, we would get a relation of the form

$$L(u_0, u_1)\psi = \zeta f'(u_0, u_1, \zeta^2, \xi, \eta).$$

It would follow that $\zeta^2 f' = Lf$. This contradicts that D is not hyperelliptic. Therefore (4.5) gives a basis of $H^0(10D + 3F)$. It follows that $\{\zeta^3, \zeta\xi, \zeta\eta, \psi\}$ induces a basis of $H^0(D, \mathcal{O}(3F_D))$. Since $|3F_D|$ has no base point, ψ does not vanish at P_1 nor P_2.

Since f involves ξ^2, the right hand side of (4.4) does not vanish at P_1 nor P_2. But this contradicts the appearance of ζ on the left. q.e.d.

§5. SURFACES OF TYPE III—SEMI-CANONICAL RING

Let $\mathcal{R} = \oplus_{m \geq 0} H^0(mL)$ be the graded ring determined by the line bundle L. By comparing the dimensions we see that the following elements generate the ring $\mathcal{R}$:

$$H^0(L) \ni x_0 = \zeta u_0^3, \quad x_1 = \zeta u_0^2 u_1, \quad x_2 = \zeta u_0 u_1^2, \quad x_3 = \zeta u_1^3,$$

$$H^0(2L) \ni y_0 = \xi u_0, \quad y_1 = \xi u_1, \qquad z = \eta,$$

$$H^0(3L) \ni w = \psi.$$

The following relations $\Phi_i = 0$, $1 \le i \le 3$ and $\Psi_i = 0$, $1 \le i \le 3$ are obvious.

$$\Phi_1 = x_1^2 - x_0x_2,$$
$$\Phi_2 = x_0x_3 - x_1x_2,$$
$$\Phi_3 = x_2^2 - x_1x_3,$$
$$\Psi_1 = x_1y_1 - x_0y_2,$$
$$\Psi_2 = -x_2y_1 + x_1y_2,$$
$$\Psi_3 = x_3y_1 - x_2y_2.$$

Among these we have 8 syzygies $S_i = 0$ ($i = 1, 2$) and $T_j = 0$ ($1 \le j \le 6$):

$$S_1 = x_2\Phi_1 + x_1\Phi_2 + x_0\Phi_3,$$
$$S_2 = x_3\Phi_1 + x_2\Phi_2 + x_1\Phi_3,$$
$$T_1 = x_1\Psi_1 + x_0\Psi_2 - y_1\Phi_1,$$
$$T_2 = x_2\Psi_1 + x_1\Psi_2 - y_2\Phi_1,$$
$$T_3 = x_3\Psi_1 + x_2\Psi_2 + y_1\Phi_3 + y_2\Phi_2,$$
$$T_4 = x_1\Psi_2 + x_0\Psi_3 - y_1\Phi_2 - y_2\Phi_1,$$
$$T_5 = x_2\Psi_2 + x_1\Psi_3 + y_1\Phi_3,$$
$$T_6 = x_3\Psi_2 + x_2\Psi_3 + y_2\Phi_3.$$

In the relation (4.1) we may assume f does not depend on u_0. Then f can be written as

$$f = \xi^2 + a_4u_1^4\zeta\eta + a_5u_1^5\zeta\xi + a_{10}u_1^{10}\zeta^3,$$

where the a_j are constants. From this we obtain 3 relations $\Gamma_i = 0$ ($1 \le i \le 3$):

$$\Gamma_1 = x_0w - y_1^2 - x_2^2Q,$$
$$\Gamma_2 = x_1w - y_1y_2 - x_2x_3Q,$$
$$\Gamma_3 = x_2w - y_2^2 - x_3^2Q, \tag{5.1}$$

where $Q = a_4z + a_5y_2 + a_{10}x_3^2$.

We have 8 more syzygies $U_i = 0$ ($1 \le i \le 6$) and $V_j = 0$ ($j = 1, 2$):

$$U_1 = x_1\Gamma_1 - x_0\Gamma_2 + y_1\Psi_1 - x_2Q\Phi_2,$$
$$U_2 = x_2\Gamma_1 - x_1\Gamma_2 + w\Phi_1 - y_1\Psi_2 + x_2Q\Phi_3,$$
$$U_3 = x_3\Gamma_1 - x_2\Gamma_2 - w\Phi_2 + y_1\Psi_3,$$
$$U_4 = x_1\Gamma_2 - x_0\Gamma_3 - w\Phi_1 + y_2\Psi_1 - x_3Q\Phi_2,$$
$$U_5 = x_2\Gamma_2 - x_1\Gamma_3 - y_2\Psi_2 + x_3Q\Phi_3,$$
$$U_6 = x_3\Gamma_2 - x_2\Gamma_3 + w\Phi_3 - y_2\Psi_3,$$
$$V_1 = y_2\Gamma_1 - y_1\Gamma_2 + w\Psi_1 - x_2Q\Psi_3,$$
$$V_2 = y_2\Gamma_2 - y_1\Gamma_3 - w\Psi_2 - x_3Q\Psi_3.$$

Finally we have a relation Δ in $H^0(6L)$.

$$\begin{aligned} \Delta &= w^2 + Aw + B, \\ A &= (\alpha_1 x^3 + \alpha_2 y_2 + \alpha_3 z)x_3, \\ B &= B(x_0, x_1, x_2, x_3, y_1, y_2, z). \end{aligned} \tag{5.2}$$

Here we note that we do not normalize A to be 0 because of the previous normalization.

PROPOSITION 5.1. *The above $S_1, S_2, T_1, \ldots, T_6, U_1, \ldots, U_6, V_1, V_2$ generate all syzygies among the $\Phi_i, \Psi_i, \Gamma_i, \Delta$.*

Proof. Let $\tilde{V}$ be the blowing up of $V = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-5) \oplus \mathcal{O}(-6))$ along $u_0 = f = 0$. The equations $\Phi_i = \Psi_i = \Gamma_i = 0, 1 \le i \le 3$ determine a threefold which is a modification of $\tilde{V}$. Therefore Δ is independent from other equations.

LEMMA 5.1. *Consider the module of the solutions (A_0, A_1, A_2) of*

$$A_0 x_0 + A_1 x_1 + A_2 x_2 \equiv 0 \mod (\Phi_i, \Psi_i),$$

$$A_i \in \mathbf{C}[x_0, x_1, x_2, x_3, y_1, y_2, z].$$

Then this module is generated by

$$(x_{i+1}, -x_i, 0), (0, x_{i+1}, -x_i), \quad i = 0, 1, 2,$$

$$(y_2, -y_1, 0), (0, y_2, -y_1)$$

Proof. Considering the homogeneous components we may assume that the A_i do not depend on z and that they are homogeneous in y_1, y_2. Using the given relations, we subtract appropriate multiples of x_1, x_2, x_3, y_2 from A_0. Then A_0 is a multiple of $x_0^\alpha y_1^\beta = u_0^{\alpha+\beta}\zeta^\alpha\xi^\beta$. Since x_1 and x_2 are divisible by u_1, this proves $A_0 = 0$ We can similarly make $A_1 = 0$. q.e.d.

Consider the syzygy of the form

$$B_0\Gamma_1 + B_1\Gamma_2 + B_2\Gamma_3 = \sum C_i\Phi_i + \sum D_j\Psi_j.$$

If $B_i = A_i w^n +$ (lower terms) with some A_i being non-zero, then the left hand side is

$$(A_0 x_0 + A_1 x_1 + A_2 x_2)w^{n+1} + \cdots.$$

Hence, by virtue of Lemma 5.1, subtracting appropriate multiples of $U_1, \ldots, U_6, V_1, V_2$ we may assume that the B_i are of degree less than n. By induction, we may assume $B_i = 0$.

It remains to show that any syzygy of the form

$$\sum C_i\Phi_i = \sum D_j\Psi_j$$

is generated by $S_1, S_2, T_1, \ldots, T_6$ up to trivial syzygies. Considering each homogeneous part we may assume that C_i and D_j do not depend on z or w, and that they are homogeneous in y_1, y_2. Subtracting appropriate multiples of T_1, T_2, T_3, we may assume $D_1 = \sum cx_0^\alpha y_1^\beta y_2^\gamma$. Then the right hand side contains $\sum cx_0^{\alpha+1} y_1^\beta y_2^\gamma$. But this cannot appear on the left hand side. So $D_1 = 0$. Similarly, using T_4, T_5, T_6, we can make $D_2 = 0$.

Finally, there is no non-trivial syzygies among $\Psi_3, \Phi_1, \Phi_2, \Phi_3$ which involve Ψ_3. This can be seen by the fact that the equations $\Phi_1 = \Phi_2 = \Phi_3 = 0$ define a 4-dimensional cone in the weighted projective space $\mathbf{P}(1, 1, 1, 1, 2, 2, 3)$ over $\mathbf{P}^1$ and $\Psi_3 = 0$ cuts out a 3-dimensional subset of this cone. q.e.d.

§6. SURFACES OF TYPE III—DEFORMATION TO SEXTIC SURFACES

Let S be a surface of type III as in the preceding two sections. In this section we construct a flat family of rings $\mathcal{R}_t$ parametrized by a parameter t such that $\mathcal{R}_0$ coincides with

$$\mathcal{R} = \bigoplus_{m \geq 0} H^0(S, \mathcal{O}(mL)),$$

for $t = 0$. This is modeled on [3].

The ring $\mathcal{R}_t$ is generated by the same variables $x_0, x_1, x_2, x_3, y_1, y_2, z$ as $\mathcal{R}$. They are to satisfy the relations $\tilde{\Phi}_i$, $\tilde{\Psi}_i$, $\tilde{\Gamma}_i$, $\tilde{\Delta}$, which are given below. To simplify the notation we suppress the tildes below.

$$\Phi_1 = x_1^2 - x_0x_2 + t^2Q + ct^m z,$$

$$\Phi_2 = x_3x_0 - x_1x_2 - ty_1,$$

$$\Phi_3 = x_2^2 - x_1x_3 + ty_2,$$

where $Q = a_4z + a_5y_2 + a_{10}x_3^2$ as is defined in (5.1), c is a constant and m is an integer ≥ 3 (c may be set 0 if $a_4 \neq 0$).

Next we set

$$\Psi_1 = x_1y_1 - x_0y_2 - t(Q + ct^{m-2}z)x_2,$$

$$\Psi_2 = -x_2y_1 - x_1y_2 + t(Q + ct^{m-2}z)x_3,$$

$$\Psi_3 = x_3y_1 - x_2y_2 - tw.$$

For simplicity, we set $\tilde{Q} = Q + ct^{m-2}z$. By direct calculation we obtain

$$x_2\Phi_1 + x_1\Phi_2 + x_0\Phi_3 + t\Psi_1 = 0,$$

$$x_3\Phi_1 + x_2\Phi_2 + x_1\Phi_3 - t\Psi_2 = 0,$$

$$x_1\Psi_1 + x_0\Psi_2 - y_1\Phi_1 - t\tilde{Q}\Phi_2 = 0,$$

$$x_2\Psi_1 + x_1\Psi_2 - y_2\Phi_1 + t\tilde{Q}\Phi_3 = 0,$$

$$x_3\Psi_1 + x_2\Psi_2 + y_1\Phi_3 + y_2\Phi_2 = 0.$$

These extend the syzygies S_1, S_2 and T_1, T_2, T_3.

We set

$$\Gamma_1 = x_0w - y_1^2 - x_2^2\tilde{Q},$$

$$\Gamma_2 = x_1w - y_1y_2 - x_2x_3\tilde{Q},$$

$$\Gamma_3 = x_2w - y_2^2 - x_3^2\tilde{Q}.$$

Then

$$x_1\Psi_2 + x_0\Psi_3 - y_1\Phi_2 - y_2\Phi_1 + t\Gamma_1 = 0,$$

$$x_2\Psi_2 + x_1\Psi_3 + y_1\Phi_3 + t\Gamma_2 = 0,$$

$$x_3\Psi_2 + x_2\Psi_3 + y_2\Phi_3 + t\Gamma_3 = 0.$$

These extend T_4, T_5, T_6. We also have

$$x_1\Gamma_1 - x_0\Gamma_2 + y_1\Psi_1 - x_2\tilde{Q}\Phi_2 = 0,$$
$$x_2\Gamma_1 - x_1\Gamma_2 + w\Phi_1 - y_1\Psi_2 - x_2\tilde{Q}\Phi_3 + t\tilde{Q}\Psi_3 = 0,$$
$$x_3\Gamma_1 - x_2\Gamma_2 - w\Phi_2 + y_1\Psi_3 = 0,$$
$$x_1\Gamma_2 - x_0\Gamma_3 - w\Phi_1 + y_2\Psi_1 - x_3\tilde{Q}\Phi_2 - t\tilde{Q}\Psi_3 = 0,$$
$$x_2\Gamma_2 - x_1\Gamma_3 - y_2\Psi_2 + x_3\tilde{Q}\Phi_3 = 0,$$
$$x_3\Gamma_2 - x_2\Gamma_3 + w\Phi_3 + y_2\Psi_3 = 0.$$

These extend $U_1, \ldots, U_6$. Also we obtain

$$y_2\Gamma_1 - y_1\Gamma_2 + w\Psi_1 - x_2\tilde{Q}\Psi_3 = 0,$$
$$y_2\Gamma_2 - y_1\Gamma_3 - w\Psi_2 - x_3\tilde{Q}\Psi_3 = 0,$$

which extend V_1, V_2.

Finally we set

$$\Delta = w^2 + Aw + B,$$

the same equation as (5.2). These relations determine the quotient ring

$$\mathbf{C}[x_0, x_1, x_2, x_3, y_1, y_2, z, w, t]/(\Phi_i, \Psi_i, \Gamma_i, \Delta).$$

Since all the syzygies are extended this ring is formally flat over $\mathbf{C}[t]$ ([2, Proposition 3.1]). For $t \neq 0$, we can solve $\Phi_1 = \Phi_2 = \Phi_3 = \Psi_3 = 0$ in terms of z, y_1, y_2 and w. Then, by the above syzygies Ψ_1, Ψ_2, Γ_1, Γ_2, Γ_3 automatically vanish. By substituting y_1, y_2, z, w in Δ by the polynomials in the x_i, we obtain an equation of degree 6 in the x_i. Since $S = \mathrm{Proj}(\mathscr{R})$ is smooth or with at most rational double points, it follows that $S_t = \mathrm{Proj}(\mathscr{R}_t)$ has the same property. This completes the proof.

§7. NON-EXISTENCE OF HYPERELLIPTIC PENCIL OF GENUS 3

In this section we prove Proposition 4.1. Suppose $|L|$ is composed of a pencil $|D|$ of genus 3, whose general member is hyperelliptic. We have

$$|L| = |3D| + F, \quad DL = DF = 2, \quad D^2 = 0, \quad \text{and} \quad F^2 = -6.$$

We take an irreducible component F_0 of F such that $DF_0 > 0$. Then, since $KF_0 = 0$, F_0 is a rational curve with $F_0^2 = -2$. From $DF = 2$ and $3DF_0 + FF_0 = 0$, it follows that $DF_0 = 1$ or 2 and $FF_0 = -3$ or -6. If $DF_0 = 2$, we must have $F \geq 3F_0$ to get $FF_0 = -6$. But this implies $DF \geq 3DF_0 = 6$, which contradicts $DF = 2$. So we obtain $DF_0 = 1$, $FF_0 = -3$. From the last equality and $DF = 2$, we infer that $F = 2F_0 + F'$ with $F_0 \not\subseteq F'$, $F_0F' = 1$. From $F^2 = -6$ we obtain $FF' = 0$ and $F'^2 = -2$. On the other hand, from $DF' = 0$, it follows that F' is disjoint from a general D. Therefore the canonical bundle K_D is induced by $4F_0$.

LEMMA 7.1. *The intersection $P = F_0 \cap D$ is a Weierstrass point on D.*

Proof. Let P' be the conjugate to P. Then $2(P + P')$ and $4P$ are both canonical divisor. Hence $|2P| = |2P'|$ has positive dimension. q.e.d.

Consider the exact sequence

$$0 \to \mathcal{O}(L) \to \mathcal{O}(4D + F) \to \mathcal{O}_D(F|_D) \to 0.$$

Since $h^1(L) = 0$ by Lemma 2.1, we obtain $h^0(4D + F) = 6$. Therefore, we can find a section η which does not vanish along F. Hence the linear system $|4D + F|$ defines a rational map Φ: $S \to \mathbf{P}^5$, whose image is a cone over a rational quartic in $\mathbf{P}^4$. Moreover, Φ is generically 2-sheeted.

Since $[4D + F] = [L + D]$ is numerically effective and $(4D + F)^2 = 10 = 8 + 2$, there are 3 possibilities (see §3):

(1) $|4D + F|$ has 2 base points (possibly infinitely near).
(2) $|4D + F|$ has only fixed part G with $G^2 = -2$.
(3) $|4D + F|$ has only fixed part G with $G^2 = 0$.

Assume that $|4D + F|$ has 2 isolated base points. Since $(4D + F)F' = 0$, η does not vanish anywhere on F'. Combining with $(4D + F)F_0 = 1$, we see that $(\eta) \cap F_0$ consists of one point Q, and η has a simple zero at Q. If we blow up Q, the proper transform $(\tilde{\eta})$ intersects the exceptional curve E_1 at a point Q_1, which remains to be a base point because $F = 2F_0 + F'$. Let π: $\tilde{S} \to S$ be the composition of the two blowings up, at Q and Q_1. Let E_2 be the second exceptional curve and let $\tilde{E}_1$ be the proper transform of E_1. Then

$$\pi^*(4D + F) = M + \tilde{E}_1 + 2E_2,$$

where $|M|$ has no base point, and the canonical bundle $\tilde{K}$ on $\tilde{S}$ is given by $\pi^*(6D + 2F) + \tilde{E}_1 + 2E_2$. It also follows that Φ lifts to a generically 2-sheeted holomorphic map f of $\tilde{S}$ onto the Hirzebruch surface Σ_4, such that

$$f^*(\Delta_0) = 2\tilde{F}_0 + F' + \tilde{E}_1,$$

$$f^*(\Delta_\infty) = \text{proper transform of } (\eta),$$

where Δ_0 denotes the 0-section with $\Delta_0^2 = -4$, Δ_∞ is another section with $\Delta_\infty^2 = 4$ and $\tilde{F}_0$ is the proper transform of F_0. Letting Γ denote a fibre of Σ_4, we see that the branch locus for f is linearly equivalent to $8\Delta_0 + 30\Gamma$. In particular, B contains Δ_0. The ramification divisor R is linearly equivalent to $\tilde{K} - f^*K_{\Sigma_4}$, or $12D + 8\tilde{F}_0 + 4F' + 7\tilde{E}_1 + 6E_2$. By [4, Lemma 3], $f^*B - 2R$ is effective. This implies that there exists $Z \in |D - \tilde{E}_1 - 2E_2|$ such that $3Z \in |f^*(B/2) - R|$. It follows that f is ramified along the fibre $f(E_2)$. Therefore, we can write $B = B_0 + \Delta_0 + f(E_2)$.

If the branch locus B were smooth, then the double covering would have $p_a = p_g - q = 18$ and $K^2 = 40$. It follows that we need to have

$$\sum \tfrac{1}{2}[m_i/2]([m_i/2] - 1) = 8.$$

So this sum is either 1×8, $3 + 1 \times 5$, $3 \times 2 + 1 + 1$, or $6 + 1 + 1$. in each case, K^2 decreases by

$$\sum 2([m_i/2] - 1)^2 = 16, 18, 20, \text{ or } 22.$$

Let $\hat{S}$ be the canonical resolution of the double covering with branch locus B (see [4, §2]). We claim that $\hat{S}$ does not coincide with $\tilde{S}$. On the canonical resolution $\hat{S}$, the canonical bundle $\hat{K}$ is given by

$$f^*(B/2) + f^*K_{\Sigma_4} - \sum([m_i/2] - 1)\mathscr{D}_i,$$

where the $\mathscr{D}_i$ denote the divisors corresponding to the blowings up of Σ_4 (cf. [4, §2]). On the other hand, the canonical bundle $\tilde{K}$ of $\tilde{S}$ is written as $R + f^*K_{\Sigma_4}$. Therefore $3Z$ coincides with $\sum([m_i/2] - 1)\mathscr{D}_i$. Although we cannot directly conclude that $[m_i/2] - 1$ must be 0 or 3, at least we see that, for the first blowing up, we have $[m_1/2] = 4$. But, from the above calculation, in order to get $\hat{K}^2 = 22$, we must have $\{[m_i/2]\} = \{3, 1, 1, 1, 1, 1\}$.

Since the canonical resolution $\hat{S}$ does not coincide with $\tilde{S}$, the branch locus B must have infinitely near 5-ple points or infinitely near triple points.

Here infinitely near triple points are not allowed. In fact, if E is an exceptional curve on $\hat{S}$ which is contracted to a point on $\tilde{S}$, it meets a divisor Y on $\hat{S}$ with $Y^2 = -2$. By contracting E, we obtain a divisor $\bar{Y}$ on S such that $\bar{Y}^2 \geq -1$. Since the arithmetic genus of $\bar{Y}$ is 1, we cannot have $\bar{Y}^2 = 0$. Therefore, we have $\bar{Y}^2 = -1$, which contradicts that S is even.

Hence we are left with the case where B has an infinitely near 5-ple points. Since this is on the fibre $f(E_2)$, it is only possible if B_0 has a quadruple point which has, after a blowing up, a quadruple point on the proper transform of $f(E_2)$. This implies $B_0 \cdot f(E_2) \geq 8$, which contradicts $B_0 \Gamma = 7$.

Next we study the case in which $|4D + F|$ has fixed part G with $G^2 = -2$. In this case, we have $(4D + F)G = 0$. Since the variable part $|M|$ satisfies $MD = 2$, we see that F_0 is not contained in G. In a manner similar to the preceding case, Φ_M defines a holomorphic map $S \to \mathbf{P}^5$, which induces a holomorphic map $f\colon S \to \Sigma_4$ such that $f^*\Delta_0 = 2F_0 + (F' - G)$. This implies that the ramification divisor R for f is linearly equivalent to $12D + 8F_0 + 4F' - 2G$, and that the branch locus B is linearly equivalent to $8\Delta_0 + 28\Gamma$. Hence B is a disjoint sum $B_0 + \Delta_0$. We also see that there is an effective divisor $Z \in |D - G|$.

If B were smooth, it would yield a surface with $p_a = 15$, $K^2 = 32$. Therefore, the singularity of B must contribute -5 to p_a. Since this contribution is a sum $-\sum \frac{1}{2}[m_i/2]([m_i/2] - 1)$, it decomposes as $5 = 1 + \cdots + 1$ or $5 = 3 + 1 + 1$. The corresponding contribution to K^2 is $2 + \cdots + 2$ or $8 + 2 + 2$, when K^2 is calculated on the canonical resolution. It follows that the canonical resolution $\hat{S}$ is not the minimal model. As in the previous case, this implies that B has infinitely near 5-ple points. Then we get a divisor $\bar{Y}$ on S with $\bar{Y}^2 \geq -1$ with arithmetic genus less than 3. Since S is even, we must have $\bar{Y}^2 = 0$. This implies that $\bar{Y}$ is a rational multiple of a whole fibre. But since $\bar{Y}$ is a component of Z, this fibre is $f^{-1}f(E_2)$. This is impossible, because $f^{-1}f(E_2)$ contains G.

Finally we suppose $|4D + F| = |M| + G$ with $MG = 1$, $G^2 = 0$. Then Φ_M defines a holomorphic map $S \to \mathbf{P}^5$, which lifts to $f\colon S \to \Sigma_4$ with $f^*\Delta_0 = 2F_0 + (F' - G)$. By a standard calculation, the ramification divisor R is linearly equivalent to $4F - 2G + 12D$. Hence, the branch locus on Σ_4 is linearly equivalent to $8\Delta_0 + 26\Gamma$. This implies that B contains Δ_0 as a double component, which is impossible. q.e.d.

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