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Separating subsets and stable values

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Abstract

Let X be a metric continuum. In this paper we prove that if there exist pariwise disjoint terminal subcontinua $A_1, \ldots, A_n$ of X such that $X - (A_1 \cup \cdots \cup A_n)$ is disconnected, then each onto map $f : Y \to X$ has a stable value.

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A *continuum* is a compact connected metric space with more than one point. A *map* is a continuous function. A *stable value of* a map between continua $f : Y \to X$ is a point $p \in X$ for which there exists $\varepsilon > 0$ such that p is in the image of all maps within ε of f in the supremum metric. A subcontinuum A of X is said to be *terminal* provided that any subcontinuum of X that intersects both A and $X - A$ contains A. A *regular curve* is a continuum such that each point has arbitrarily small neighborhoods whose boundaries are finite [1, Definition 10.14, p. 171]. A continuum is said to be in Class(S) ($X \in$ Class(S)) provided that every map of any continuum onto X has a stable value.

The concept of Class(S) was recently introduced by Nadler, Jr. in [2] where, among other results, he proved that if a Peano continuum X is separated by some finite set, then every map of any Peano continuum onto X has a stable value [2, Corollary 4.2]. In this paper we generalize Nadler's result by proving the following theorem.

Theorem 1. *Let X be a continuum. Suppose that there exist pairwise disjoint, terminal subcontinua $A_1, \ldots, A_n$ of X such that $X - (A_1 \cup \cdots \cup A_n)$ is disconnected. Then $X \in$ Class(S).*

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Proof. Let $f : Y \to X$ be an onto map. Suppose that $X - (A_1 \cup \cdots \cup A_n) = U \cup V$, where U and V are nonempty open disjoint subsets of X.

Let $W_1, \ldots, W_n$ be open subsets of X such that $A_i \subset W_i$ for each $i = 1, \ldots, n$, $\mathrm{cl}_X(W_1) \cup \cdots \cup \mathrm{cl}_X(W_n)$ neither contains U nor V and $\mathrm{cl}_X(W_1), \ldots, \mathrm{cl}_X(W_n)$ are pairwise disjoint. Let $H = f^{-1}(U \cap (\mathrm{bd}_X(W_1) \cup \cdots \cup \mathrm{bd}_X(W_n)))$, $K = f^{-1}(V \cap (\mathrm{bd}_X(W_1) \cup \cdots \cup \mathrm{bd}_X(W_n)))$ and $L = f^{-1}(\mathrm{cl}_X(W_1) \cup \cdots \cup \mathrm{cl}_X(W_n))$. Since $H = f^{-1}((X - V) \cap (\mathrm{bd}_X(W_1) \cup \cdots \cup \mathrm{bd}_X(W_n)))$, we obtain that H is closed in Y. Similarly, K is closed in Y. Next, we prove the following claim:

Claim. *There exists a subcontinuum A of Y such that $A \subset L$, $A \cap H \neq \emptyset$ and $A \cap K \neq \emptyset$.*

In order to prove this claim, suppose to the contrary, that no connected subset A of L intersects both H and K. Applying the Cut Wire Theorem [1, Theorem 5.2] to the space L and the closed subsets H and K of L, we obtain that there exist disjoint compact subsets M and N of L such that $H \subset M$, $K \subset N$ and $L = M \cup N$.

Let $R = f^{-1}(X - (V \cup W_1 \cup \cdots \cup W_n)) \cup M$ and $S = f^{-1}(X - (U \cup W_1 \cup \cdots \cup W_n)) \cup N$. Clearly, R and S are nonempty closed subsets of Y and $Y = R \cup S$. Suppose that there exists a point $y \in f^{-1}(X - (V \cup W_1 \cup \cdots \cup W_n)) \cap N$. Then $f(y) \in U \cap (\mathrm{cl}_X(W_1) \cup \cdots \cup \mathrm{cl}_X(W_n)) - (W_1 \cup \cdots \cup W_n)$. Thus $y \in H \cap N$, a contradiction. Therefore, $f^{-1}(X - (V \cup W_1 \cup \cdots \cup W_n)) \cap N = \emptyset$. Similarly, $f^{-1}(X - (U \cup W_1 \cup \cdots \cup W_n)) \cap M = \emptyset$. This implies that R and S are disjoint. We have obtained a separation of Y. This is absurd, so the claim holds.

Since $f(A)$ is a connected subset of $\mathrm{cl}_X(W_1) \cup \cdots \cup \mathrm{cl}_X(W_n)$, we may assume that $f(A) \subset \mathrm{cl}_X(W_1)$. We claim that the points in A_1 are stable points of f. Fix points $p \in A \cap H$ and $q \in A \cap K$. Choose $\varepsilon > 0$ with the following properties. The ε-neighborhood around $f(p)$ (respectively, $f(q)$) is contained in U (respectively, V), and the ε-neighborhood around $\mathrm{cl}_X(W_1)$ does not intersect $\mathrm{cl}_X(W_2) \cup \cdots \cup \mathrm{cl}_X(W_n)$. Choose $\delta > 0$ satisfying the definition of uniform continuity of f for the number ε.

Let $g : Y \to X$ be a map such that g is δ-close of f. By the choice of ε and δ, $g(p) \in U$, $g(q) \in V$ and $g(A) \cap (\mathrm{cl}_X(W_2) \cup \cdots \cup \mathrm{cl}_X(W_n)) = \emptyset$. Then $g(A)$ is a subcontinuum of X which intersects U and V and does not intersect $A_2 \cup \cdots \cup A_n$. This implies that $g(A)$ intersects A_1 and $g(A)$ is not contained in A_1. Thus $A_1 \subset g(A)$.

Therefore, the points in A_1 are stable values of f. This completes the proof of the theorem. □

Corollary 2. *If a continuum X is separated by some finite set, then $X \in$* Class(S).

Corollary 3 (Compare with [2, Corollary 4.3]). *Every regular curve X is in* Class(S).

References

[1] S.B. Nadler Jr, Continuum Theory, An Introduction, in: Monographs Textbooks Pure Appl. Math., Vol. 158, Marcel Dekker, New York, 1992.
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