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Identification of time-varying nonlinear systems using Chebyshev polynomials

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Abstract

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This paper is devoted to an identification of time-varying nonlinear systems using Chebyshev polynomials of the first kind. For systems being linear relatively unknown functional parameters, a method of approximate determination of these parameters has been worked out. As the identification problems are ill-posed to solve the obtained redefinite system of linear algebraic equations, a regularization method is used.

Keywords: Identification; nonlinear systems; Chebyshev polynomials; numerical methods; ill-posed problems

0. Introduction

The identification of dynamic systems is called finding and specifying of a mathematical model of this system by experiments results. We shall consider the identification problems in which a structure of the mathematical model is known, and it is needed to find approximately some functional parameters from the mathematical description of the system.

The identification methods for nonlinear systems using expansions into orthogonal polynomials have been developed in some papers [3–6]; however, methods stated in these papers are computationally unstable. We have extended an analogous method to a wider class of nonlinear systems and have worked out the computationally stable algorithms.

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1. Statement of the problem

We shall consider continuous time-varying nonlinear systems, which are linear relatively unknown functional parameters:

$$\dot{x}_i(t) = \sum_{j=1}^{k} a_{i,j}(t) f_{i,j}(t, x_1(t), \ldots, x_n(t), y_1(t), \ldots, y_p(t)), \tag{1}$$

$i = 1, 2, \ldots, n$, $t \in [t_0, t_f]$.

The functions $f_{i,j}(\cdot)$ are assumed to be the known and smooth enough on $[t_0, t_f]$. For example, one may demand that the functions $f_{i,j}$ and their partial derivatives of $x_1, \ldots, x_n$, $y_1, \ldots, y_p$ are continuous in the domain $[t_0, t_f] \times \mathbb{R}^{n+p}$. The unknown functions $a_{i,j}(t)$ are assumed to belong to some Holder space $\boldsymbol{H}^{r,\alpha}$, where $r + \alpha > 1$, $i = 1, \ldots, n$, $j = 1, \ldots, p$.

For approximate finding of the unknown functions, we have carried out a sufficiently large number of experiments. In every experiment with the number i, some p-dimensional vector

$$\boldsymbol{Y}^{(i)}(t) = \left[y_1^{(i)} \; y_2^{(i)} \; \ldots \; y_p^{(i)} \right]^{\mathrm{T}}$$

is applied to the input of the investigated system, and an n-dimensional vector

$$\boldsymbol{X}^{(i)}(t) = \left[x_1^{(i)} \; x_2^{(i)} \; \ldots \; x_n^{(i)} \right]^{\mathrm{T}}$$

is measured on the output in a sufficiently large number of points of $[t_0, t_f]$. We assume that for every $\boldsymbol{Y}^{(i)}(t)$, with corresponding initial conditions, system (1) has the only solution $\boldsymbol{X}^{(i)}(t)$ on $[t_0, t_f]$.

It is necessary to determine approximately the unknown coefficients $a_{i,j}(t)$, $i = 1, \ldots, n$, $j = 1, \ldots, k$.

2. Approximation of the system

The unknown functions can be approximately found with the help of some numerical parameters. In order to solve efficiently the identification problem, it is very important to select such a method of the approximation which allows to approximate functions with a given accuracy by as little numerical parameters as possible. If a function is smooth enough, one of the more effective methods of approximation is the method of expansion to Fourier series by such systems of orthogonal polynomials as, for example, systems of shifted Chebyshev polynomials of the first kind:

$$\boldsymbol{T}_i^*(t) = \cos\left(i \arccos\left(\frac{2(t - t_0)}{t_f - t_0} - 1 \right) \right), \quad i = 0, 1, 2, \ldots,$$

with the weight

$$\rho(t) = \frac{1}{\mathrm{sqrt}((t - t_0) \cdot (t_f - t))}.$$

When one uses Fourier–Chebyshev expansions to the identification of systems of the form (1), it is possible to propose two approaches. The first approach is based on the introduction of the functional $J(a_{1,1}(\cdot), \ldots, a_{n,p}(\cdot))$ which characterizes the quadratic residuals of all experiment results. This functional must be regularized, as the problem of its minimization is an ill-posed problem [7]. Then the regularized functional is approximated by the Ritz–Galerkin method using a basis from the m first Chebyshev polynomials.

We shall consider the second approach, based on the immediate transition from the differential form of the system description to the approximate description with the help of Fourier–Chebyshev coefficient sets.

Let us fix some experiment with the number l. First of all, from system (1) with some initial conditions $x_i(t_0)=x_i^{(l)}$, $i=1,\dots,n$, we pass on the Volterra–Hammerstein integral equations

$$x_i^{(l)}(t)=\sum_{j=1}^{k}\int_{t_0}^{t}a_{i,j}(s)f_{i,j}\big(s,\ x_1^{(l)}(s),\dots,x_n^{(l)}(s),\ y_1^{(l)}(s),\dots,y_p^{(l)}(s)\big)\,\mathrm{d}s+x_i^{(l)},$$
$$i=1,2,\dots,n,\quad t\in[t_0,\ t_f].\tag{2}$$

From the integral equations (2) let us pass on in its turn to Galerkin approximate equations of the kind

$$\tilde{x}_i^{(l)}(t)=\sum_{j=1}^{k}\sigma_m\int_{t_0}^{t}\tilde{a}_{i,j}(s)\tilde{f}_{i,j}\big(s,\ x_1^{(l)}(s),\dots,x_n^{(l)}(s),\ y_1^{(l)}(s),\dots,y_p^{(l)}(s)\big)\,\mathrm{d}s+x_i^{(l)},$$
$$i=1,2,\dots,n,\quad t\in[t_0,\ t_f],\tag{3}$$

where σ_m is the operator of transition from a function to its Fourier–Chebyshev partial sum of the order m, and the symbol $\sim$ over a function means that it is replaced by the Fourier–Chebyshev partial sum of the order m.

This method of approximation is correct in the following sense. Using [1, Theorem 19.1], one can prove that with the imposed conditions on the system (1) there are solutions of the approximate system (3) for large m and they converge to the exact solution of the Cauchy problem for system (1) as $m\to\infty$.

3. Transition to algebraic problems

Let us denote the columns of the first m Fourier–Chebyshev coefficients of the functions $x_i^{(l)}(t)$, $a_{i,j}(t)$ and $f_{i,j}(t,\ x_1^{(l)}(t),\dots,x_n^{(l)}(t),\ y_1^{(l)}(t),\dots,y_p^{(l)}(t))$ by $\boldsymbol{X}_i^{(l)}$, $\boldsymbol{A}_{i,j}$ and $\boldsymbol{F}_{i,j}^{(l)}$, respectively ($i=1,\dots,n$, $j=1,\dots,k$). Further we introduce the matrix operator of integration $\boldsymbol{H}_m$ by

$$\boldsymbol{H}_m=(t_f-t_0)\begin{pmatrix}\frac{1}{2} & -\frac{1}{8} & \cdots & \dfrac{(-1)^{m-2}}{2(m-1)(m-3)} & \dfrac{(-1)^{m-1}}{2m(m-2)}\\ \frac{1}{2} & 0 & \cdots & 0 & 0\\ 0 & \frac{1}{8} & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & \dfrac{-1}{4(m-2)}\\ 0 & 0 & \cdots & \dfrac{1}{4(m-1)} & 0\end{pmatrix}$$

and the matrix operator of multiplication $P_m(G)$ by $\tilde{g}(t)=\sum_{i=0}^{m-1} g_i T_i^*(t)$:

$$P_m(G)=\begin{pmatrix} g_0 & \frac{1}{2}g_1 & \frac{1}{2}g_2 & \cdots & \frac{1}{2}g_{m-2} & \frac{1}{2}g_{m-1} \\ g_1 & g_0+\frac{1}{2}g_2 & \frac{1}{2}(g_1+g_3) & \cdots & \frac{1}{2}(g_{m-3}+g_{m-1}) & \frac{1}{2}g_{m-2} \\ g_2 & \frac{1}{2}(g_1+g_3) & g_0+\frac{1}{2}g_4 & \cdots & \frac{1}{2}(g_{m-4}) & \frac{1}{2}g_{m-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ g_{m-1} & \frac{1}{2}g_{m-2} & \frac{1}{2}g_{m-3} & \cdots & \frac{1}{2}g_1 & g_0 \end{pmatrix}.$$

We note that these matrices are slightly different from the analogous matrices used, for example, in [6]. Using these matrix operators, it is easy to establish the following result: if X is an m-dimensional column of Fourier–Chebyshev coefficients of a function $x(t)$, then the Fourier–Chebyshev coefficients of functions $\sigma_m \int_{t_0}^t \tilde{x}(s)\,\mathrm{d}s$ and $\sigma_m(\tilde{g}(t)\tilde{x}(t))$ can be noted in the form $H_m X$ and $P_m(G)X$.

Let $X_{i,0}^{(l)}=[x_i^{(l)}\ 0\ 0\ \ldots\ 0]^{\mathrm{T}}$ be the column of length m. Using the imposed matrix operators, one can easily obtain that system (3) is equivalent to the following system of algebraic equations:

$$X_i^{(l)}=H_m \sum_{j=1}^{k} P_m\big(F_{i,j}^{(l)}\big)A_{i,j}+X_{i,0}^{(l)}, \quad i=1,\ldots,n. \tag{4}$$

Let us introduce the column of the size $mkn\times 1$ from Fourier–Chebyshev coefficients of all unknown functions

$$Z_m=\left[A_{1,1}^{\mathrm{T}}\ \ldots\ A_{1,k}^{\mathrm{T}}\ A_{2,1}^{\mathrm{T}}\ \ldots\ A_{2,k}^{\mathrm{T}}\ \ldots\ A_{n,1}^{\mathrm{T}}\ \ldots\ A_{n,k}^{\mathrm{T}}\right]^{\mathrm{T}},$$

the matrix $W_m^{(l)}$ of size $mn\times mnk$:

$$W_m^{(l)}=\begin{pmatrix} H_m P_{1,1}^{(l)} & \cdots & H_m P_{1,k}^{(l)} & 0_m & \cdots & 0_m & \cdots & 0_m & \cdots & 0_m \\ 0_m & \cdots & 0_m & H_m P_{2,1}^{(l)} & \cdots & H_m P_{2,k}^{(l)} & & 0_m & \cdots & 0_m \\ \vdots & & \vdots & & & & & & & \\ 0_m & \cdots & 0_m & 0_m & \cdots & 0_m & & H_m P_{n,1}^{(l)} & \cdots & H_m P_{n,k}^{(l)} \end{pmatrix},$$

where 0_m is the zero matrix of size $m\times m$, $P_{i,j}^{(l)}=P(F_{i,j}^{(l)})$, $i=1,\ldots,n$, $j=1,\ldots,k$, and the column of size $mn\times 1$:

$$G_m^{(l)}=\left[\big(X_1^{(l)}-X_{1,0}^{(l)}\big)^{\mathrm{T}}\ \big(X_2^{(l)}-X_{2,0}^{(l)}\big)^{\mathrm{T}}\ \ldots\ \big(X_n^{(l)}-X_{n,0}^{(l)}\big)^{\mathrm{T}}\right]^{\mathrm{T}}.$$

Then the approximate equations (4) can be written in the form

$$W_m^{(l)}Z_m=G_m^{(l)}. \tag{5}$$

Let us join all equations of the form (5) for the fulfilled N experiments to the only system

$$W_m Z_m=G_m, \tag{6}$$

where

$$G_m=\left[G_m^{(1)\mathrm{T}}\ G_m^{(2)\mathrm{T}}\ \ldots\ G_m^{(N)\mathrm{T}}\right]^{\mathrm{T}},$$

$$W_m=\left[W_m^{(1)\mathrm{T}}\ W_m^{(2)\mathrm{T}}\ \ldots\ W_m^{(N)\mathrm{T}}\right]^{\mathrm{T}}.$$

4. Solving the algebraic system

If $N > k$, then system (6) is a redefinite system of linear algebraic equations. In many cases, system (6) is ill-posed and if it has been solved with the use of the least-squares method

$$Z_m = \left(W_m^{\mathrm{T}} W_m\right)^{-1} W_m^{\mathrm{T}} G_m,$$

one should obtain too much errors.

Let us introduce the regularized functional

$$\Phi_\alpha(Z_m) = (W_m Z_m - G_m)^{\mathrm{T}}(W_m Z_m - G_m) + \alpha Z_m^{\mathrm{T}} Z_m, \tag{7}$$

and assume that the stated identification problem has a solution or some solutions. If the degree of smoothness of the functions $x_i(t)$, $i = 1, 2, \ldots, n$, $y_j(t)$, $j = 1, 2, \ldots, p$, the maximal values of measurement errors, errors of the computation of Fourier–Chebyshev coefficients and elements of W_m and G_m agree, it is possible to indicate the rule of selection of the regularization parameter $\alpha(m) \to_{m\to\infty} 0$ with which the obtained solutions of the minimization problem for functional (7) will converge to a column of Fourier–Chebyshev coefficients of the exact solution by the functional

$$J(Z) = (W_m Z - G_m)^{\mathrm{T}}(W_m Z - G_m),$$

as the errors tend to zero. The conditions of this theorem are very cumbersome; therefore we do not formulate it exactly. For selection of the regularization parameter there are some known practical methods [7].

We used the next practical method based on the QR transformation [2]. The matrix $W_m^{\mathrm{T}} W_m$ can be decomposed into a product of two orthogonal matrices Q and R and a diagonal matrix V:

$$W_m^{\mathrm{T}} W_m = QVR.$$

If $v_{i,i} = 0$ for some i, we must either increase the number m or fulfil one more experiment and form again system (6). If all diagonal elements of V do not equal zero, we calculate the condition number

$$\kappa = \operatorname{cond}\left(W_m^{\mathrm{T}} W_m\right) = \frac{\max\{v_{i,i}\}}{\min\{v_{i,i}\}}.$$

When the value κ is not large, the least-squares method can be applied. If κ is large, we choose in the capacity of the regularization parameter the value $\alpha = \min\{v_{i,i}\}$. This method gave good results in different tests.

5. Example

As the sample illustrating the stated method of identification we consider the system

$$\dot{x}_1(t) = a_{1,1}(t)x_1(t) + a_{1,2}(t)x_1(t)x_2(t) + y_1(t),$$

$$\dot{x}_2(t) = a_{2,1}(t)x_1(t) + a_{2,2}(t)x_2^2(t) + a_{2,3}(t)x_1^2(t)x_2(t) + y_2(t),$$

in which it is required to determine approximately five unknown functions. In the tests we assumed

$$a_{1,1}(t) = t - 1, \quad a_{1,2}(t) = 1, \quad a_{2,1}(t) = -2t, \quad a_{2,2}(t) = t - 1, \quad a_{2,3}(t) = 1,$$

and selected pairs of input and output vectors as the "experiment results" (for example, $x_1(t) = t$, $x_2(t) = \exp(-t)$, $x_1(0) = 0$, $x_2(0) = 1$, $y_1(t) = 1 + t - t\exp(-t) - t^3$, $y_2(t) = (1 - t)\exp(-2t) + 2t^2 - (t^2 + 1)\exp(-t)$, etc.). Then we modelled "errors of measurement device", adding random values from $[-\epsilon, \epsilon]$ to the components of output vectors.

We took [0, 2] as the considered segment. Using ten pairs of input and output vectors, we received the approximate solutions with different m, ϵ and the accuracy of Fourier–Chebyshev coefficient calculation. Further, these solutions were compared to the exact functions in forty points of [0, 2]. Let us denote by Δ the maximal error between the obtained solution and the exact function in these points. For example, we received the next results.

With $m = 6$, $\epsilon = 0.01$, calculating Fourier–Chebyshev coefficients with the help of FFT-algorithm using $L = 128$ values of expanding functions, we received $\Delta = 0.058$ with $\alpha \approx 0.0008$.

With $m = 8$, $\epsilon = 0.001$, $L = 256$, we received $\Delta = 0.017$ with $\alpha \approx 0.00012$.

6. Conclusion

The described method allows us in a number of cases to solve a complicated problem of identification of continuous systems with nonlinear elements of sufficiently common structure. For adaptive systems working in real time, this method cannot be applied, because of low speed of the program work. However, it may find effective application in laboratory investigations of the wide class of nonlinear systems.

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