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Limit theorem for the statistical solution of Burgers equation

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Abstract

In this work we study limit theorems for the Hopf–Cole solution of the Burgers equation when the initial value is a functional of some Gaussian processes. We use the Gaussian chaos decomposition, and we get "Gaussian scenario" with new normalization factors. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The one-dimensional Burgers equation without force has the form

$$\partial_t u + u\partial_x u = \frac{v}{2}\partial_{xx}^2 u, \quad x \in \mathbb{R},\ t > 0. \tag{1}$$

Here $v > 0$ is the viscosity and the initial value $u(x,0) = -\mathrm{d}\xi(x)/\mathrm{d}x$ is given, and satisfies $\xi(x) = \mathrm{o}(x^2)$ as $|x| \to +\infty$. If we introduce a potential function ψ defined as $u = -\partial_x\psi$, then the Hopf–Cole substitution $\psi = v\ln\theta$ shows that θ satisfies the heat equation $\partial_t\theta = (v/2)\partial_{xx}^2\theta$. Using this fact one can write down for the solution $u = u(x,t,v)$ the explicit expression

$$u(x,t,v) = \frac{I(x,\ t,v)}{J(x,\ t,v)}, \tag{2}$$

where

$$I(x,\ t,v) = \int_{-\infty}^{\infty} \frac{x-y}{t} g\left(\frac{x-y}{\sqrt{tv}}\right) \exp\left\{\frac{\xi(y)}{v}\right\} \mathrm{d}y, \tag{3}$$

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and

$$J(x,t,v)=\int_{-\infty}^{\infty} g\left(\frac{x-y}{\sqrt{tv}}\right)\exp\left\{\frac{\xi(y)}{v}\right\}\,\mathrm{d}y. \tag{4}$$

The function g is the density of the standard Gaussian random variable $\mathcal{N}(0,1)$.

Many authors have investigated the solution $u(x,t,v)$ on different types of initial conditions which are stationary Gaussian processes. It is well-known (Albeverio et al., 1994; Bulinskii and Molchanov, 1991) that, if $v=1$ and $\xi(x)$ is a stationary zero-mean Gaussian process, with covariance function $B(x)=\mathbb{E}[\xi(x)\xi(0)]$ is assumed to satisfy

$$\int_{\mathbb{R}} |B(x)|\,\mathrm{d}x<\infty, \tag{5}$$

then the solution $u(x,t,1)$ obeys the "Gaussian scenario", that is, as $L\to\infty$,

$$L^{3/2}u(Lx,L^2t,1)\Rightarrow X(x,t), \tag{6}$$

where X is a Gaussian process, and $\Rightarrow$ stands for the weak convergence of the finite-dimensional distributions. In the case where

$$B(x)=1-\frac{\lambda_2}{2}x^2+\frac{\lambda_4}{4!}x^4+o(x^4)\quad \text{as}\quad x\to 0\quad \text{and}\quad B(x)=o(1/\ln x)$$
$$\text{as}\quad x\to\infty, \tag{7}$$

Molchanov et al. (1995) have proved that, as $L\to+\infty$,

$$L\sqrt{2\ln L}\,u(Lx,L^2\sqrt{2\ln L}\,t,1/L^2\sqrt{2\ln L})\ \Rightarrow\ \frac{x-y_{j(x,t)}}{t},$$

where $y_{j(x,t)}$ is defined via a Poisson process $(y_j,u_j)_{j\in\mathbb{Z}}$ on $\mathbb{R}^2$ with the intensity $\mathrm{e}^{-u}\,\mathrm{d}y\,\mathrm{d}u$, by maximizing the difference

$$\max_j\left[u_j-\frac{(x-y_j)^2}{2t}\right]=u_{j(x,t)}-\frac{(x-y_{j(x,t)})^2}{2t}. \tag{8}$$

On the other hand, Surgailis (1996) has proved that if $B(0):=\lambda_0>1$,

$$B(x)=\lambda_0-\frac{\lambda_2}{2}x^2+o(x^2)\ \text{as}\ x\to 0\ \text{and}\ B(x)=o(1/\ln x)\ \text{as}\ x\to\infty, \tag{9}$$

then

$$L\sqrt{2\ln L}\,u(Lx,L^2\sqrt{2\ln L}\,t,1/\sqrt{2\ln L})\Rightarrow v(x,t),$$

where

$$v(x,t)=\sum_{i\in\mathbb{Z}} t^{-1}(x-y_i)\exp\left\{\sqrt{\lambda_0}u_i-\frac{(x-y_i)^2}{2t}\right\}$$
$$\times\left\{\sum_{i\in\mathbb{Z}}\exp\left\{\sqrt{\lambda_0}u_i-\frac{(x-y_i)^2}{2t}\right\}\right\}^{-1}.$$

See also Hu and Woyczynski (1995) for other cases. In our work we unify these results and we broaden this list. This is the aim of Section 2 Theorem 2.1. In Section 3 we study the case when $\xi(x)=\int_{\mathbb{R}} f(x,y)\,\mathrm{d}W(y)-\frac{1}{2}\int_{\mathbb{R}} f^2(x,y)\,\mathrm{d}y$, where W is the Gaussian white noise, and we obtain the "Gaussian scenario".

2. General scaling limit behavior

We set, for $L>0$, for positive function $T(L)$, and for functions $A(L)$, $b(L)$,

$$V(Ly,v)=\exp\{v^{-1}(\xi(Ly)-b(L))\}-A(L).$$

The parameter L may depend on v. We suppose that for all $(x,t)\in\mathbb{R}\times\mathbb{R}_+$, $t\neq 0$, there exists scaling $\beta(x,t,L)>0$, such that for any $\varepsilon\in\{0,1\}$, as $L\to\infty$,

$$\beta(x,t,L)\int_{\mathbb{R}}V(Ly,v)g\left(\frac{L(y-x)}{\sqrt{T(L)tv}}\right)((x-y)/t)^{\varepsilon}\,\mathrm{d}y \;\Rightarrow\; (Z^{x,t},((x-y)/t)^{\varepsilon})$$
$$:=\int_{\mathbb{R}}((x-y)/t)^{\varepsilon}Z^{x,t}(\mathrm{d}y), \tag{10}$$

where $Z^{x,t}$ is a generalized process.

The following theorem gives a convergence of finite-dimensional distributions of the rescaled solutions of (1).

Theorem 2.1. (i) *Suppose that* $v=1$, *and* $T(L)=L^2$. *If the scaling* $\beta(x,t,L):=\beta(L)$ *does not depend on* (x,t), *then* $Z^{x,t}(\mathrm{d}y)=g((y-x)/\sqrt{t})Z(\mathrm{d}y)$, *where* Z *is a generalized process. Moreover, if for all* φ *belonging to the Schwartz space* $\mathcal{S}(\mathbb{R})$,

$$\lim_{L\to\infty}\int_{\mathbb{R}}\varphi(y)\exp\{\xi(Ly)-b(L)\}\,\mathrm{d}y=a\int_{\mathbb{R}}\varphi(y)\,\mathrm{d}y, \tag{11}$$

then, as $L\to\infty$, *the finite-dimensional distributions of the two parameter random fields,*

$$L\beta(L)u(Lx,L^2t,1),\quad x\in\mathbb{R}\text{ and }t>0,$$

converge to the corresponding finite-dimensional distributions of the random field

$$a^{-1}\int_{\mathbb{R}}\frac{x-y}{t}g\left(\frac{x-y}{\sqrt{t}}\right)Z(\mathrm{d}y),\quad x\in\mathbb{R}\text{ and }t>0.$$

(ii) *If* $A(L)=0$, *then the finite-dimensional distributions of the two parameter random field,*

$$T(L)L^{-1}u(Lx,T(L)t,v),\quad x\in\mathbb{R}\text{ and }t>0,$$

converge to the corresponding finite-dimensional distributions of the random field

$$\frac{\int_{\mathbb{R}}(x-y)/(t)Z^{x,t}(\mathrm{d}y)}{\int_{\mathbb{R}}Z^{x,t}(\mathrm{d}y)},\quad x\in\mathbb{R}\text{ and }t>0.$$

Proof. The proof of (i) is similar to the proof of Theorem 2.1 in Funaki et al. (1995), and it is based on the well-known following lemma.

Lemma 2.1. *If* $\{X_t,t>0\}$ *and* $\{Y_t,t>0\}$ *are families of random variables such that* $X_t\to X$ *in law and* $Y_t\to c$ *in probability where* c *is a real constant, then* $X_t+Y_t\to X+c, X_tY_t\to cX, X_t/Y_t\to X/c$ (*if* $c\neq 0$), *in law as* $t\to+\infty$.

Now we come back to the proof of (ii). In this case $V(Ly,v)=\exp\{v^{-1}(\xi(Ly)-b(L))\}$, and from (2), (3) and (4) we have

$$I(Lx,T(L)t,v)=\frac{L^2}{T(L)}\int_{\mathbb{R}}\frac{x-y}{t}V(Ly,v)g\left(\frac{L(y-x)}{\sqrt{T(L)tv}}\right)\,\mathrm{d}y\exp(v^{-1}b(L)),$$

and

$$J(Lx,T(L)t,v)=L\int_{\mathbb{R}}V(Ly,v)g\left(\frac{L(y-x)}{\sqrt{T(L)tv}}\right)\,\mathrm{d}y\exp(v^{-1}b(L)).$$

We have for all $\alpha_i,\gamma_i,x_i\in\mathbb{R}$, $t_i>0$; $1\leqslant i\leqslant n$,

$$\left[\sum_{i=1}^{n}\alpha_i\beta(x_i,t_i,L)T(L)L^{-2}I(Lx_i,T(L)t_i,v)\right.$$
$$\left.+\sum_{i=1}^{n}\gamma_iL^{-1}\beta(x_i,t_i,L)J(Lx_i,T(L)t_i,v)\right]\exp(-v^{-1}b(L))$$
$$=\int_{\mathbb{R}}\left\{\sum_{i=1}^{n}\left(\alpha_i\frac{x_i-y}{t_i}+\gamma_i\right)\beta(x_i,t_i,L)V(Ly,v)g\left(\frac{L(y-x_i)}{\sqrt{T(L)t_iv}}\right)\right\}\,\mathrm{d}y.$$

From (10) the member of the left-hand side converges weakly, as $L\to\infty$, to

$$\int_{\mathbb{R}}\left\{\sum_{i=1}^{n}\left(\alpha_i\frac{x_i-y}{t_i}+\gamma_i\right)Z^{x_i,t_i}(\mathrm{d}y)\right\}$$
$$=\sum_{i=1}^{n}\alpha_i\int_{\mathbb{R}}\frac{x_i-y}{t_i}Z^{x_i,t_i}(\mathrm{d}y)+\sum_{i=1}^{n}\gamma_i\int_{\mathbb{R}}Z^{x_i,t_i}(\,\mathrm{d}y).$$

It follows that

$$T(L)L^{-1}\frac{I(Lx,T(L)t,v)}{J(Lx,T(L)t,v)}=T(L)L^{-1}u(Lx,T(L)t,v)\Rightarrow\frac{\int_{\mathbb{R}}(x-y)/(t)Z^{x,t}(\mathrm{d}y)}{\int_{\mathbb{R}}Z^{x,t}(\mathrm{d}y)},$$

whence the claimed result. □

Remark 2.1. If $A(L)=A$, and $\beta(L)\to\infty$, as $L\to\infty$, then (11) is satisfied, with $a=A$.

Corollary 2.1. *Let $\xi(x)$ be a stationary zero-mean Gaussian process.*

(1) *If $\xi(x)$ satisfies* (5), *then part* (i) *of Theorem* 2.1 *is satisfied with*

$$b(L)=0,\quad A(L)=0,\qquad\beta(L)=\sqrt{L},\quad a=\exp\{\mathbb{E}[\xi(0)^2]/2\},$$

and $Z(\mathrm{d}y)=cW(\mathrm{d}y)$; where W is the Gaussian white noise on $\mathbb{R}$ and $c=a(\int_{\mathbb{R}}(\mathrm{e}^{B(z)}-1)\,\mathrm{d}z)^{1/2}$.

(2) *If $\xi(x)$ satisfies* (9), *then part* (ii) *of Theorem* 2.1. *is satisfied with*

$$v=1/\sqrt{2\ln L},\quad T(L)=L^2\sqrt{2\ln L},$$

$$b(L)=\sqrt{\lambda_0}(\sqrt{2\ln L}+\ln[(\lambda_2/\lambda_0)^{1/2}/2\pi]/\sqrt{2\ln L})$$

and

$$\beta(x,t,L) = (\lambda_2/2\pi\lambda_0)^{1/2} L\sqrt{2\ln L},$$
$$Z^{x,t}(\mathrm{d}y) = g\left(\frac{x-y}{\sqrt{t}}\right)\sum_{i\in\mathbb{Z}} \pi^{-1/2}\exp(\sqrt{\lambda_0}u_i)\delta(y-y_i).$$

Here $\sum_i \delta_{(y_i,u_i)}$ *is a random Poisson measure on* $\mathbb{R}^2$ *with intensity* $\mathrm{e}^{-u}\,\mathrm{d}u\,\mathrm{d}y$.

(3) *If* $\xi(x)$ *satisfies* (7) *then part* (ii) *of Theorem* 2.1 *is satisfied with*

$$v = \frac{1}{L^2\sqrt{2\ln L}}, \quad T(L) = L^2\sqrt{2\ln L}, \quad b(L) = \sqrt{2\ln L} + \ln[(\lambda_2)^{1/2}/2\pi]/\sqrt{2\ln L}$$

and the normalization $\beta(x,t,L)$ *is given by*

$$1/\beta(x,t,L) = \int_{\Delta_{j(x,t)}(L)} \exp(v^{-1}\{\xi(Ly)-b(L)\})g\left(\frac{L(x-y)}{\sqrt{t}}\right)\mathrm{d}y,$$

where $\Delta_{j(x,t)}(L) = \{y\in\mathbb{R} : |y-y_{j(x,t)}| \leqslant \frac{1}{L\sqrt{2\ln L}}\}$, *and* $Z^{x,t}(\mathrm{d}y) = \delta(y-y_{j(x,t)})$. *Here* $y_{j(x,t)}$ *is given by* (8).

The proofs of (1), (2), and (3) are due, respectively, to Albeverio et al. (1994), Surgailis (1996) and Molchanov et al. (1995).

3. The Gaussian scenario

In the sequel $v = 1$, and $T(L) = L^2$. The initial value is a functional of the one-dimensional Gaussian white noise W with intensity the Lebesgue measure. More precisely, we consider the setting where $\xi(x)$ is such that the random variables $\exp\{\xi(x)\}$ belong to $L^2(\mathbb{P})$, the Hilbert space of square integrable functions with respect to $\mathbb{P}$, the measure of the underlying probability space supporting W.

We are interested in the asymptotic behavior of the ratio $u(Lx, L^2t, 1) := u(Lx, L^2t)$ as $L\to+\infty$. In view to obtain the normalization $\beta(L)$, the centering constants $b(L)$, and $A(L)$, we employ the Gaussian chaos decomposition Itô (1951). The nonlinear stochastic functional $\exp\{\xi(x)\}$ has the chaos decomposition

$$\exp\{\xi(x)\} = \sum_{k=0}^{\infty}\frac{1}{k!}I_k(f_k(x)), \tag{12}$$

where for $k\geqslant 1$, $I_k(f_k(x)) = \int\cdots\int_{\mathbb{R}^k} f_k(x,y_1,\ldots,y_k)\,\mathrm{d}W(y_1)\ldots\mathrm{d}W(y_k)$ is the stochastic multiple Wiener integral, and for all $x\in\mathbb{R}$, $f_k(x) : y = (y_1,\ldots,y_k) \to f_k(x,y)$ is a symmetric function belonging to $L^2(\mathbb{R}^k, \mathrm{d}x^{\otimes k})$. The latter space is endowed with the natural scalar product denoted $\langle\cdot,\cdot\rangle$, and with the norm $||\cdot|| = \langle\cdot,\cdot\rangle^{1/2}$. If $\xi(x)$ is a Gaussian process then (12) is reduced to the Hermite expansion. The Gaussian chaos decomposition has a long history of application to the Burgers, and Navier-Stokes turbulence, both in the mathematical, and in the fluid dynamic communities, for references see e.g. Funaki et al. (1995).

We set for $k\geqslant 0$, and φ belonging to the Schwartz space $\mathscr{S}(\mathbb{R})$,

$$v_k(\varphi, L) = \frac{1}{k!}\int_{\mathbb{R}} \varphi(y) I_k(f_k(Ly))\,\mathrm{d}y.$$

From (12) we get

$$v(\varphi,L) := \int_{\mathbb{R}} \varphi(y)\exp\{\xi(Ly)\}\,\mathrm{d}y = \sum_{k=0}^{\infty} v_k(\varphi,L).$$

From the well-known formula Itô (1951),

$$\mathbb{E}\left[I_k(f_k(x_1))I_j(f_j(x_2))\right] = k!\delta_k^j\langle f_k(x_1), f_k(x_2)\rangle,$$

we have

$$\mathbb{E}[v(\varphi,L)] = v_0(\varphi,L), \quad \text{and} \quad \mathrm{Var}(v(\varphi,L)) = \sum_{k=1}^{+\infty}\mathrm{Var}(v_k(\varphi,L)).$$

For $k\geqslant 1$, we set $B_k(x_1,x_2) = \langle f_k(x_1), f_k(x_2)\rangle$, and we get

$$Var(v_k(\varphi,L)) = \frac{1}{k!}\int\int_{\mathbb{R}^2}\varphi(y_1)\varphi(y_2)B_k(Ly_1,Ly_2)\,\mathrm{d}y_1\,\mathrm{d}y_2.$$

Theorem 3.1. *Suppose that* $\mathbb{E}[\exp\{\xi(x)\}] := A$, *a constant which does not depend on* x, *and there exist a positive measurable function* $\beta(L)\to\infty$, *as* $L\to\infty$, *a measure* μ *belonging to the space of tempered distributions* $\mathscr{S}'(\mathbb{R}^2)$ *such that*: *for all* G *belonging to the Schwartz space* $\mathscr{S}(\mathbb{R}^2)$, *as* $L\to\infty$,

$$[\mathrm{H}_1]: \quad \int\int_{\mathbb{R}^2}\beta(L)^2B_1(Lx,Ly)G(x,y)\,\mathrm{d}x\,\mathrm{d}y \to \int\int_{\mathbb{R}^2}G(x,y)\mu(\,\mathrm{d}x,\mathrm{d}y),$$

$$[\mathrm{H}_2]: \quad \beta(L)^2\sup_{k\geqslant 2}\int\int_{\mathbb{R}^2}|B_k\,(Lx,Ly)\,G(x,y)|\,\mathrm{d}x\,\mathrm{d}y \to 0.$$

(1) *Then we have for all* $\varphi\in\mathscr{S}(\mathbb{R})$, *as* $L\to\infty$,

$$\beta(L)\int_{\mathbb{R}}[\exp\{\xi(Ly)\}-A]\varphi(y)\,\mathrm{d}y = \beta(L)(v(\varphi,L)-v_0(\varphi,L)) \to \int_{\mathbb{R}}\varphi(y)Z(\mathrm{d}y),$$

where Z *is a Gaussian field whose covariance function is*

$$\mathrm{cov}((Z,\varphi_1),(Z,\varphi_2)) = \int\int_{\mathbb{R}^2}\varphi_1(y_1)\varphi_2(y_2)\mu(\mathrm{d}y_1,\mathrm{d}y_2).$$

(2) *It follows that*

$$L\beta(L)u(Lx,L^2t,1) \Rightarrow A^{-1}\int_{\mathbb{R}}\frac{x-y}{t}g\left(\frac{x-y}{\sqrt{t}}\right)Z(\mathrm{d}y).$$

Proof. The proof of part (1) is a consequence of Lemma 2.1. Indeed, thanks to the hypotheses $[\mathrm{H}_1]$ and $[\mathrm{H}_2]$, we have, as $L\to\infty$, the convergence in law of $\beta(L)v_1(\varphi,L)$ to the Gaussian process $\int_{\mathbb{R}}\varphi(y)Z(\mathrm{d}y)$, and the convergence of $\beta(L)(v(\varphi,L)-v_0(\varphi,L)-v_1(\varphi,L))$ to 0 in $L^2(\mathbb{P})$. Hence according to Lemma 2.1 $\beta(L)(v(\varphi,L)-v_0(\varphi,L))$ converges in law to $\int_{\mathbb{R}}\varphi(y)Z(\mathrm{d}y)$.

The proof of part (2) is a consequence of part (i) of Theorem 2.1, and from Remark 2.1.

Corollary 3.1. *Suppose that*

$$\xi(x) = \int_{\mathbb{R}} f(x,y)\,\mathrm{d}W(y) - \frac{||f(x,\cdot)||^2}{2},$$

where f is such that $B_1(x,x) = ||f(x,\cdot)||^2 \leqslant 1$ for all x. If the hypotheses [C_1] *and* [C_2] *below are satisfied*:

[C_1] *there exist a positive measurable function $\beta(L) \to \infty$, as $L \to \infty$, and two positive measures $\mu_1, \mu_2 \in \mathscr{S}'(\mathbb{R}^2)$ such that for any $G \in \mathscr{S}(\mathbb{R}^2)$,*

$$\int_{\mathbb{R}^2} \beta(L)^2 B_1^+(Lx, Ly) G(x,y)\, dx\, dy \to \int_{\mathbb{R}^2} G(x,y)\, d\mu_1(x,y)$$

and

$$\int_{\mathbb{R}^2} \beta(L)^2 B_1^-(Lx, Ly) G(x,y)\, dx\, dy \to \int_{\mathbb{R}^2} G(x,y)\, d\mu_2(x,y),$$

where B_1^+, B_1^- are, respectively, the positive and negative parts of B_1.

[C_2] *For all $\varepsilon > 0$, and for all compact set K, subset of $\mathbb{R}^2$, as $L \to \infty$,*

$$\beta(L)^2 \lambda(\{(x,y) \in K, |B_1(Lx, Ly)| \geqslant \varepsilon\}) \to 0,$$

where λ is the Lebesgue measure.

Then, for all $\varphi \in \mathscr{S}(\mathbb{R})$,

$$\beta(L) \int_{\mathbb{R}} [\exp(\xi(Ly)) - 1] \varphi(y)\, dy$$

converges in distribution to the random variable

$$\int_{\mathbb{R}} \varphi(y) Z(dy) := (Z, \varphi),$$

where $Z(dy)$ is a Gaussian field whose covariance function is

$$\mathrm{cov}((Z,\varphi_1),(Z,\varphi_2)) = \int\int_{\mathbb{R}^2} \varphi_1(y_1)\varphi_2(y_2)\, d\mu(y_1,y_2),$$

where $\mu = \mu_1 - \mu_2$.

Proof. For any $x \in \mathbb{R}$, $\exp(\xi(x)) = \sum_{k=0}^{\infty} (1/k!) I_k(f(x,\cdot)^{\otimes k})$, hence the constant A of Theorem 3.1 is equal to 1. The condition [C_1] implies the condition [H_1] of Theorem 3.1. For $k \geqslant 2$, $|B_k(x,y)| = |B_1(x,y)|^k \leqslant B_2(x,y)\, dx\, dy - a.s.$ So, to get the condition [H_2] of Theorem 3.1 we have to prove that for all $G \in \mathscr{S}(\mathbb{R}^2)$, as $L \to \infty$,

$$\int\int_{\mathbb{R}^2} \beta(L)^2 (B_1(Lx,Ly))^2 G(x,y)\, dx\, dy \to 0.$$

We have for $\varepsilon > 0$, $\int\int_{\mathbb{R}^2} \beta(L)^2 (B_1(Lx,Ly))^2 G(x,y)\, dx\, dy = I + J$, where

$$I = \int\int_{\{|B_1(Lx,Ly)| < \varepsilon\}} \beta(L)^2 (B_1(Lx,Ly))^2 G(x,y)\, dx\, dy,$$

and

$$J = \int\int_{\{|B_1(Lx,Ly)| \geqslant \varepsilon\}} \beta(L)^2 (B_1(Lx,Ly))^2 G(x,y)\, dx\, dy.$$

It follows that there exist two constants c_1 and c_2 such that $|I| \leqslant \varepsilon c_1$ (for L large enough), and

$$|J| \leqslant c_2 \int\int_{\{|B_1(Lx,Ly)| \geqslant \varepsilon\}} \beta(L)^2 |B_1(Lx,Ly) G(x,y)|\, dx\, dy.$$

For L sufficiently large there exist $M > 0$ such that

$$\int\int_{\{x^2+y^2\geqslant M^2\}} \beta(L)^2|B_1(Lx,Ly)G(x,y)|\,\mathrm{d}x\,\mathrm{d}y\leqslant\varepsilon$$

and for $K=\{(x,y)\in\mathbb{R}^2, x^2+y^2\leqslant M\}$,

$$\int\int_{\{|B_1(Lx,Ly)|\geqslant\varepsilon\}} \beta(L)^2|B_1(Lx,Ly)G(x,y)|\,\mathrm{d}x\,\mathrm{d}y$$

$$\leqslant\varepsilon+\int\int_{K\cap[|B(Lx,Ly)|\geqslant\varepsilon]} \beta(L)^2|B_1(Lx,Ly)G(x,y)|\,\mathrm{d}x\,\mathrm{d}y$$

$$\leqslant\varepsilon+c\beta(L)^2\lambda(\{(x,y)\in K, |B_1(Lx,Ly)|\geqslant\varepsilon\}),$$

where $c>0$ is some constant. From that we have for L large enough,

$$I+J\leqslant\varepsilon c_1+c_2[\varepsilon+c\beta(L)^2\lambda(\{(x,y)\in K,|B_1(Lx,Ly)|\geqslant\varepsilon\})].$$

Now taking account of $[C_2]$ we get the desired result.

Remark 3.1. The case where the initial value $u(x,0)$, is the Brownian motion, is out of the scope of Theorem 3.1, and Corollary 3.1.

In this part we study some situations when Corollary 3.1 is satisfied. A measurable function $L:(0,+\infty)\to(0,+\infty)$ is slowly varying at $+\infty$ if, for each $t>0$, as $x\to+\infty$, $L(tx)/L(x)=1$. It is well known (see for example Appendix III in Galambos, 1978; ch. 0 in Bertoin, 1996) that L has the form

$$L(x)=\exp\left\{u(x)+\int_1^x\frac{e(z)}{z}\mathrm{d}z\right\}, \tag{13}$$

where $u,e:(0,+\infty)\to\mathbb{R}$ are bounded measurable functions with $u(x)\to u^*\in\mathbb{R}$, and $e(x)\to 0$, as $x\to+\infty$.

Corollary 3.2. *Let $f\in L^2(\mathbb{R},\mathrm{d}x)$, and $\xi(x)=\int_{\mathbb{R}}f(x-y)\,\mathrm{d}W(y)-||f||^2/2$. The process $\xi(x)$ is a stationary Gaussian process with covariance function $B_1(x,y)=\int_{\mathbb{R}}f(x-z)f(y-z)\,\mathrm{d}z:=B(x-y)$. Suppose that $B(x-y)=(\Phi(|x-y|)/|x-y|^\alpha)$, where $0<\alpha<1$, and Φ is a slowly varying function such that $0<\Phi(|x|)\leqslant|x|^\alpha$ for all x. Then, for all $\varphi\in\mathscr{S}(\mathbb{R})$,*

$$\frac{L^{\alpha/2}}{\sqrt{\Phi(L)}}\int_{\mathbb{R}}[\exp(\xi(Ly))-1]\varphi(y)\,\mathrm{d}y$$

converges in distribution to $\int_{\mathbb{R}}\varphi(y)Z(\mathrm{d}y):=(Z,\varphi)$, where Z is a Gaussian field whose covariance function is

$$\mathrm{cov}((Z,\varphi_1),(Z,\varphi_2))=\int\int_{\mathbb{R}^2}\varphi_1(y_1)\varphi_2(y_2)\frac{\mathrm{d}y_1\,\mathrm{d}y_2}{|y_1-y_2|^\alpha}.$$

Proof. This result is well known, it has been considered by Albeverio et al. (1994), and Leonenko and Orsingher (1995). Corollary 3.1 gives a simplified proof. Indeed, in this case $B_k(x,y)=((\Phi(|x-y|)/|x-y|^\alpha))^k$, and it is easy to see that $|B_k(x,y)|\leqslant|B_2(x,y)|$ for all $k\geqslant 2$. Let us prove the condition $[C_1]$ with $\beta(L)=\frac{L^{\alpha/2}}{\sqrt{\Phi(L)}}$ and $\mu=(\mathrm{d}x\,\mathrm{d}y/|x-y|^\alpha)$.

We have for all $A>0$, $\int\int_{\mathbb{R}^2} \beta(L)^2 B(L|x-y|)G(x,y)\,\mathrm{d}x\,\mathrm{d}y = I+J$, where

$$I=\int\int_{[L|x-y|>A]} \beta^2(L)B(L|x-y|)G(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

and

$$J=\int\int_{[L|x-y|<A]} \beta^2(L)B(L|x-y|)G(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Let us prove that $I\to\int\int_{\mathbb{R}^2}|x-y|^{-\alpha}G(x,y)\,\mathrm{d}x\,\mathrm{d}y$ as $L\to\infty$. First we have

$$\beta^2(L)B(L|x-y|)=\Phi(L|x-y|)\Phi(L)^{-1}.$$

Using (13), we have for all $\varepsilon>0$, and for L large enough,

$$\begin{aligned}|\Phi(L|x-y|)\Phi(L)^{-1}| &= \exp\left\{u(L|x-y|)-u(L)+\int_L^{L|x-y|}\frac{\mathrm{e}(z)}{z}\,\mathrm{d}z\right\}\\ &\leqslant c\{|x-y|^{\varepsilon}+|x-y|^{-\varepsilon}\},\end{aligned}$$

where c is some constant which does not depend on L. From the dominated convergence theorem we get

$$I\to\int\int_{\mathbb{R}^2}|x-y|^{-\alpha}G(x,y)\,\mathrm{d}x\,\mathrm{d}y \text{ as } L\to\infty.$$

Now let us prove that $J\to 0$, as $L\to\infty$. Since for any $x\in\mathbb{R}$, $|B(x)|\leqslant 1$, and

$$J=\frac{L^{\alpha}}{\Phi(L)}\int\int_{[L|x-y|<A]} B(L|x-y|)G(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

we have the estimate

$$\begin{aligned}|J| &\leqslant \frac{L^{\alpha}}{\Phi(L)}\int\int_{[L|x-y|<A]}|G(x,y)|\,\mathrm{d}x\,\mathrm{d}y\\ &\leqslant \Phi^{-1}(L)\int\int_{[L|x-y|<A]}A^{\alpha}|x-y|^{-\alpha}|G(x,y)|\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

We have for $\varepsilon>0$,

$$\begin{aligned}&\Phi^{-1}(L)\int\int_{[L|x-y|<A]}A^{\alpha}|x-y|^{-\alpha}|G(x,y)|\,\mathrm{d}x\,\mathrm{d}y\\ &\quad\leqslant L^{-\varepsilon}\Phi^{-1}(L)\int\int_{[L|x-y|<A]}A^{\alpha+\varepsilon}|x-y|^{-(\alpha+\varepsilon)}|G(x,y)|\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

But $L^{-\varepsilon}\Phi^{-1}(L)\to 0$, as $L\to\infty$, and for $\alpha+\varepsilon<1$, the integral $\int\int_{\mathbb{R}^2}A^{\alpha+\varepsilon}|x-y|^{-\alpha-\varepsilon}|G(x,y)|\,\mathrm{d}x\,\mathrm{d}y$ is bounded. It follows that $J\to 0$, as $L\to\infty$, whence the condition $[\mathrm{C}_1]$ is satisfied.

Now we prove the condition $[\mathrm{C}_2]$. Since $(\Phi(|L|)/L^{\alpha})\to 0$, as $L\to\infty$, then for L large enough, and for some constant A, which does not depend on L, we have

$$[B(L|x-y|)\geqslant\varepsilon]\subset[|x-y|\leqslant AL^{-1}].$$

It follows that for any compact set K and for some constant c,

$$\beta(L)^2\lambda(\{(x,y)\in K, B_1(Lx,Ly)\geqslant\varepsilon\}) \leqslant \beta(L)^2\lambda(\{(x,y)\in K, |x-y|\leqslant AL^{-1}\}) \leqslant c\beta(L)^2L^{-1}\to 0$$

as $L\to\infty$, whence the desired result.

Remarks.

(1) Theorem 3.1 can be rephrased as a statement that the stochastic process $\{\exp(\xi(y)), y\in\mathbb{R}\}$ has a large-scale Gaussian limit in the sense of Dobrushin (1980, p. 169) with normalization $A,\beta(L)$. It is well known (Dobrushin, 1979; Dobrushin and Major, 1979) that necessarily, $\beta(L)=L^{\kappa}/\sqrt{\Phi(L)}$, for some constant $\kappa\in\mathbb{R}$. The function $\Phi(L)$ is slowly varying, as $L\to\infty$, and locally bounded. From that we conclude that if $B(x,y)\to 0$, as x,y go to infinity, and satisfies condition [C_1] of Corollary 3.1 with the normalization $\beta(L)=L^{\kappa}/\sqrt{\Phi(L)}$ with $0<\kappa<\frac{1}{2}$, then condition [C_2] holds.

(2) If instead of $B(x)=(\Phi(|x|)/|x|^{\alpha})$, $\alpha\in\,]0,1[$, we suppose only that $B(x)\sim(\Phi(|x|)/|x|^{\alpha})$, as $x\to\infty$, and $B(x)\leqslant 1$ for any $x\in\mathbb{R}$, we can show, essentially in the same way, that the conditions [C_1] and [C_2] still hold. Such a situation happens if the function f which defines the covariance function B satisfies the following conditions:

(A1) $\int_{\mathbb{R}} f^2(x)\,\mathrm{d}x<\infty$.

(A2) $|f(x)|\leqslant Cx^{H_0-3/2}\Phi(x)$ for almost all $x>0$, and $1-1/2m<H_0<1$ for some integer $m\geqslant 1$. Here Φ is a slowly varying function at $+\infty$.

(A3) $f(x)\sim x^{H_0-3/2}\Phi(x)$ as $x\to\infty$.

(A4) There exists a constant γ satisfying $0<\gamma<\min\{H_0-(1-1/2m),1-H_0\}$, such that

$$\int_{-\infty}^{0}|f(u)f(xy+u)|\,\mathrm{d}u=\mathrm{o}(x^{2H_0-2}\Phi(x))y^{2H_0-2-2\gamma},$$

as $x\to\infty$, uniformly in $y\in(0,t]$, for a given $t>0$. See Taqqu (1979, p. 57, Section 2) for more details.

Corollary 3.3. *Let* $\xi(x)=\int_{\mathbb{R}} f(x,y)\,\mathrm{d}W(y)-||f(x,\cdot)||^2/2$.

(A) *If* $f(x,y)=\Phi(|x|)|x|^{-\alpha}h(y)$, Φ *is slowly varying at* ∞; $0<\alpha<\frac{1}{2}$, *and* $||h||=1$, *then, for all* $\varphi\in\mathscr{S}(\mathbb{R})$, *as* $L\to\infty$,

$$\frac{L^{\alpha}}{\Phi(L)}\int_{\mathbb{R}}[\exp(\xi(Ly))-1]\varphi(y)\,\mathrm{d}y$$

converges in distribution to $\int_{\mathbb{R}}\varphi(y)Z(\mathrm{d}y):=(Z,\varphi)$, *where* Z *is a Gaussian field whose covariance function is*

$$\operatorname{cov}((Z,\varphi_1),(Z,\varphi_2))=\int\int_{\mathbb{R}^2}\varphi_1(y_1)\varphi_2(y_2)\frac{\mathrm{d}y_1\,\mathrm{d}y_2}{|y_1|^{\alpha}|y_2|^{\alpha}}.$$

(B) *Suppose that* $f(x,\cdot)=f(|x|,\cdot)\geqslant 0$, $\forall x\in\mathbb{R}$, *non-increasing with respect to* x, *and for all* $x\neq 0$

$$\left(\int_{\mathbb{R}} f^2(L,z)\,\mathrm{d}z\right)^{-1/2} f(Lx,\cdot)$$

converges in $L^2(\mathbb{R})$ *to some function* $g(x,\cdot)$. *We suppose also that the covariance function* $B_1(x,y) := \langle f(|x|,\cdot), f(|y|,\cdot)\rangle$ *is slowly varying at infinity, namely, for all* $x \neq 0, y \neq 0$,

$$\lim_{L\to\infty} \frac{B_1(Lx,Ly)}{B_1(L,L)} = \int_{\mathbb{R}} g(x,z)g(y,z)\,\mathrm{d}z := m(x,y)$$

and that $m(Lx,Ly) = L^{-\alpha}m(x,y)$, *for some* $0<\alpha<1$.

Then $B_1(L,L)^{-1/2}\int_{\mathbb{R}}\varphi(y)[\exp(\xi(Ly))-1]\,\mathrm{d}y$ *converges, as* $L\to\infty$, *to* (Z,φ), *where* Z *is a Gaussian field with covariance function* $cov((Z,\varphi_1),(Z,\varphi_2)) = \int\int_{\mathbb{R}^2}\varphi_1(x)\varphi_2(y)m(x,y)\,\mathrm{d}x\,\mathrm{d}y$.

Proof. We use Corollary 3.1 for the proof. In the assertion (A), we have $B_k(x,y) = (\Phi(x)^k\Phi^k(y))/(|x|^{k\alpha}|y|^{k\alpha})$, and it is easy to see, for all $k\geqslant 2$, that $|B_k(x,y)|\leqslant|B_2(x,y)|$.

Let us prove the condition $[C_1]$ of Corollary 3.1, with $\beta(L) = (L^\alpha/\Phi(L))$, and $\mu = (\mathrm{d}x\,\mathrm{d}y/|x|^\alpha|y|^\alpha)$. We split $\int\int_{\mathbb{R}^2}\beta(L)^2B_1(Lx,Ly)G(x,y)\,\mathrm{d}x\,\mathrm{d}y = I+J$, where

$$I = \int\int_{[L(|x|+|y|)>A]} \beta^2(L)B_1(Lx,Ly)G(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

and $J = \int\int_{[L(|x|+|y|)\leqslant A]}\beta^2(L)B_1(Lx,Ly)G(x,y)\,\mathrm{d}x\,\mathrm{d}y$. Let us prove that $I\to\int\int_{\mathbb{R}^2}|x|^{-\alpha}|y|^{-\alpha}G(x,y)\,\mathrm{d}x\,\mathrm{d}y$, as $L\to\infty$. From (13), we have for all $\varepsilon>0$, and for L large enough,

$$|\Phi(L|x|)\Phi(L)^{-1}| = \exp\left\{u(L|x|)-u(L)+\int_L^{L|x|}\frac{\mathrm{e}(z)}{z}\mathrm{d}z\right\}\leqslant c\{|x|^\varepsilon+|x|^{-\varepsilon}\},$$

where c is constant which does not depend on L. From the dominated convergence theorem we get

$$I\to\int\int_{\mathbb{R}^2}|x|^{-\alpha}|y|^{-\alpha}G(x,y)\,\mathrm{d}x\,\mathrm{d}y \quad \text{as } L\to\infty.$$

Now let us prove that $J\to 0$ as $L\to\infty$. Since for any $(x,y)\in\mathbb{R}^2$, $|B_1(x,y)|\leqslant 1$, and

$$J = \frac{L^{2\alpha}}{\Phi^2(L)}\int\int_{[L(|x|+|y|)<A]} B_1(Lx,Ly)G(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

we have the estimate

$$\begin{aligned}|J| &\leqslant \frac{L^{2\alpha}}{\Phi^2(L)}\int\int_{[L(|x|+|y|)<A]}|G(x,y)|\,\mathrm{d}x\,\mathrm{d}y\\ &\leqslant \Phi^{-2}(L)\int\int_{[L(|x|+|y|)<A]}A^{2\alpha}(|x|+|y|)^{-2\alpha}|G(x,y)|\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

We have for $\varepsilon>0$,

$$\begin{aligned}&\Phi^{-2}(L)\int\int_{[L(|x|+|y|)<A]}A^{2\alpha}(|x|+|y|)^{-2\alpha}|G(x,y)|\,\mathrm{d}x\,\mathrm{d}y\\ &\quad\leqslant L^{-\varepsilon}\Phi^{-2}(L)\int\int_{[L(|x|+|y|)<A]}A^{2\alpha+\varepsilon}(|x|+|y|)^{-(2\alpha+\varepsilon)}|G(x,y)|\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

If $2\alpha+\varepsilon<1$, then the integral $\int\int_{\mathbb{R}^2}A^{\alpha+\varepsilon}(|x|+|y|)^{-(2\alpha+\varepsilon)}|G(x,y)|\,\mathrm{d}x\,\mathrm{d}y$ is bounded. From that, and from the fact that $L^{-\varepsilon}\Phi^{-2}(L)\to 0$, as $L\to\infty$, we have $J\to 0$ as $L\to\infty$, whence the condition $[C_1]$ is satisfied.

Now, we prove the condition $[C_2]$. Since $B_1(x,y)\to 0$, as $|x|+|y|\to\infty$, then for L large enough, and for some constant A, which does not depend on L, we have

$$[B_1(Lx,Ly)\geqslant\varepsilon]\subset[|x|+|y|\leqslant AL^{-1}].$$

It follows that for any compact set K, and for some constant c,

$$\begin{aligned}\beta(L)^2\lambda(\{(x,y)\in K, B_1(Lx,Ly)\geqslant\varepsilon\}) &\leqslant \beta(L)^2\lambda(\{(x,y)\in K, |x|+|y|\leqslant AL^{-1}\})\\ &\leqslant c\beta(L)^2L^{-2}\to 0,\end{aligned}$$

as $L\to\infty$, which yields the desired result. □

Now we prove the assertion (B). To prove $[C_1]$ it is sufficient to take $G>0$. Let $a>0, b>0$. We split $B_1(L,L)^{-1}\int\int_{\mathbb{R}^2}B_1(Lx,Ly)G(x,y)\,\mathrm{d}x\,\mathrm{d}y = I+J$, where

$$I=\int\int_{[|x|>a,|y|>b]}B_1(L,L)^{-1}B_1(Lx,Ly)G(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

and

$$J=\int\int_{[|x|\leqslant a]\cup[|y|\leqslant b]}B_1(L,L)^{-1}B_1(Lx,Ly)G(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

From the monotonicity of B, and the dominated convergence theorem we have

$$I\to\int\int_{[|x|>a,|y|>b]}G(x,y)m(x,y)\,\mathrm{d}x\,\mathrm{d}y\quad\text{as } L\to\infty.$$

Now,

$$G(x,y)B_1(L,L)^{-1}B_1(Lx,Ly)1_{[0<|x|<|y|]}\leqslant G(x,y)B_1(L,L)^{-1}B_1(Lx,Lx)1_{[0<|x|<|y|]}$$

and each of these functions converges as $L\to\infty$. Using the same proof as in Corollary 3.2 we show that the integral of the latter function converges. Combining the fact that $\int\int_{[0,1]^2}m(x,y)\,\mathrm{d}x\,\mathrm{d}y<\infty$, see Haan and Resnick (1979), with the following variant of Fatou's lemma: if $0\leqslant h_n\leqslant g_n$ are real-valued functions on some measure space, and $h_n\to h$, $g_n\to g$, then $\int h_n\to\int h$ provided $\int g<\infty$, we can show that

$$J\to\int\int_{[|x|\leqslant a]\cup[|y|\leqslant b]}G(x,y)m(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

which yields $[C_1]$. $[C_2]$ follows from the fact that $B_1(x,y)\to 0$ as $x,y\to\infty$ and from $B_1(L,L)\sim L^{-\alpha}$, as $L\to\infty$.

We finish this work by considering the case where $\langle f(y_1,\cdot),f(y_2,\cdot)\rangle=0$ for $|y_1|+|y_2|\geqslant 1$. This example is out of the scope of Theorem 3.1. The covariance function B_1 is positive and vanishing outside the ball $|x|+|y|\leqslant 1$, then the condition $[C_1]$ holds with $\beta(L)=L$ but $[C_2]$ does not hold. We want to prove the following theorem.

Theorem 3.2. *Under the latter hypothesis we have for all* $\varphi\in\mathscr{S}(\mathbb{R})$

$$L\int_{\mathbb{R}}\varphi(y)\left[\exp\left\{\int f(Ly,u)\,\mathrm{d}W(u)-\frac{B_1(Ly,Ly)}{2}\right\}-1\right]\mathrm{d}y\to\mathcal{N}(0,\varphi^2(0)c),$$

where $c=\int\int_{\mathbb{R}^2}(\exp\{B_1(x,y)\}-1)\,\mathrm{d}x\,\mathrm{d}y$, *and*

$$L^2u(Lx,L^2t)\to c\frac{xZ}{t},$$

where Z *is a normal random variable.*

Proof. We will use the technique of Theorem 1.1 in Hu and Woyczynski (1995). Let us prove that

$$L\int_{-\infty}^{\infty}\varphi(y)\left[\exp\left\{\int f(Ly,u)\,\mathrm{d}W(u)-\frac{B_1(Ly,Ly)}{2}\right\}-1\right]\mathrm{d}y$$

converges in distribution to $\mathcal{N}(0,\varphi(0)^2c)$. Let

$$H(L)=L\int_{-\infty}^{-L}\varphi(y)\left[\exp\left\{\int f(Ly,u)\,\mathrm{d}W(u)-\frac{B_1(Ly,Ly)}{2}\right\}-1\right]\mathrm{d}y.$$

Then

$$\mathbb{E}[|H(L)|^2]\leqslant kL^2\left(\int_{-\infty}^{-L}\varphi(y)\,\mathrm{d}y\right)^2.$$

Using Chebyshev's inequality and since $\varphi\in\mathcal{S}(\mathbb{R})$ we get $H(L)\to 0$ in probability as $L\to\infty$. Similarly, we can get as $L\to\infty$ the convergence in probability to 0 of

$$L\int_{L}^{\infty}\varphi(y)\left[\exp\left\{\int f(Ly,u)\mathrm{d}W(u)-\frac{B_1(Ly,Ly)}{2}\right\}-1\right]\mathrm{d}y.$$

The remainder term is

$$L\int_{-L}^{L}\varphi(y)\left[\exp\left\{\int f(Ly,u)\,\mathrm{d}W(u)-\frac{B_1(Ly,Ly)}{2}\right\}-1\right]\mathrm{d}y$$
$$=I+\sum_{-L^2\leqslant k\leqslant L^2}\eta_k(L),$$

where

$$\eta_k(L)=L\int_{k/L}^{(k+1)/L}\varphi(y)\left[\exp\left\{\int f(Ly,u)\,\mathrm{d}W(u)-\frac{B_1(Ly,Ly)}{2}\right\}-1\right]\mathrm{d}y$$

and I converges in probability to 0. Since $B(Ly_1,Ly_2)=\int_{\mathbb{R}}f(Ly_1,u)f(Ly_2,u)\,\mathrm{d}u=0$ for $y_1\in[k/L,(k+1)/L]$, $y_2\in[j/L,(j+1)/L]$ with $|k-j|>2$, then the sequence $\eta_k(L),-L^2\leqslant k\leqslant L^2$ is 2-dependent sequence. It is useful to recall the following Bulinskii (1987) result: let $X_j(t), j\in U(t)$ be an $m(t)$-dependent field on a finite set $U(t)\subset\mathbb{Z}$, and let, for some $s\in(2,3]$ and all $t>0$,

$$\sup_{j\in U(t)}(\mathbb{E}[|X_j(t)|^s])^{1/s}=\mathrm{C}_s(t)<\infty.$$

Then

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(\delta^{-1}(t)\sum_{j\in U(t)}(X_j(t)-\mathbb{E}[X_j(t)])\leqslant x\right)-F(x)\right|$$
$$\leqslant k_0|U(t)|M_s^s(t)m^{s-1}(t)+M_s(t)m(t)+|U(t)|^{1/2}M_s^2(t)m^{2/s}(t),$$

where $\delta^2(t)=\mathrm{Var}\sum_{j\in U(t)}X_j(t)$, k_0 is some constant, $|U(t)|$ is a number of points in $U(t)$, $M_s(t)=\delta^{-1}(t)\mathrm{C}_s(t)$, and $F(x)$ is the distribution function of $\mathrm{N}(0,1)$.

Now we return to our proof. For $(\eta_k(L),-L^2\leqslant k\leqslant L^2)$ we get $|U(L)|=2L^2, \mathrm{C}_3(L)\leqslant k_1L^{-1}, 0<k_2\leqslant\delta(L)\leqslant k_3,\quad M_3(L)\leqslant k_4L^{-1}$, and then we have

$$k_0|U(L)|M_3^3(L)2^2+M_3(L)2+|U(L)|^{1/2}M_3^2(L)2^{2/3}\leqslant kL^{-1},$$

so that

$$L\int_{-\infty}^{\infty}\varphi(y)\left[\exp\left\{\int f(Ly,u)\mathrm{d}W(u)-\frac{||f(Ly)||^2}{2}\right\}-1\right]\mathrm{d}y\to\mathcal{N}(0,\varphi(0)^2c).$$

Now part (i) of Theorem 2.1 combined with Remark 2.1 achieve the proof.

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