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Higher order bad loci

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Abstract

Zero-schemes on smooth complex projective varieties, forcing all elements of ample and free linear systems to be reducible, are studied. Relationships among the minimal length of such zero-schemes, the positivity of the line bundle associated with the linear system, and the dimension of the variety are established. Bad linear spaces are also investigated.

MSC: 14C20; 14E25

1. Introduction

Given a linear system on a smooth, complex, projective variety X with $\dim X \geq 2$, it is often of importance to find an irreducible element passing through a given set of points. In these circumstances, generality assumptions are not useful. One is naturally brought to consider sets of points that are able to break all elements of the given linear system. The first steps in the treatment of this phenomenon were conducted in [1,2], where the notions of *bad point* and *bad locus* were introduced and studied.

Let L be an ample line bundle on X, spanned by $V \subseteq H^0(X, L)$. A point $x \in X$ is *bad* for the linear system $|V|$ if all elements of $|V|$ containing x are reducible or non-reduced. The existence of a bad point for an ample and free linear system is shown to be exclusively a two-dimensional phenomenon, while bad points do not occur for very ample linear systems.

The notion of bad point on a projective n-fold X can be generalized in different directions. One can view a single bad point as a reduced zero-scheme of length one and therefore generalize the notion to zero-schemes of any length. On the other hand, recalling that bad points occur only on surfaces, one could view a bad point as a linear space of codimension two, i.e. a sub-manifold Λ of X, isomorphic to $\mathbb{P}^{n-2}$, such that $L_{|\Lambda} = \mathcal{O}_{\mathbb{P}^{n-2}}(1)$.

In this work the more general notions of *higher order bad locus*, *bad zero-scheme*, and *bad linear space* are introduced. A zero-scheme ξ is *bad* for the linear system $|V|$ if all elements of $|V|$ containing (scheme theoretically)

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ξ are reducible or non-reduced. The minimum length of a bad zero-scheme for the pair (X, V) is introduced as a numerical character denoted by $b = b(X, V)$, see Section 3 for details. Similarly, b_0 denotes the minimum length of a bad, reduced zero-scheme.

The main goal of the first sections of this work is to investigate relationships among b_0, b, the dimension of X, and the positivity of L.

A crucial point is whether a bad zero-scheme of minimal length imposes independent conditions on $|V|$. To answer this question positively, one would need, for any zero-scheme ξ not imposing independent conditions, to find a subscheme $\eta \subset \xi$, imposing to $|V|$ linearly independent conditions that are equivalent to the ones imposed by ξ. While this happens in several instances, e.g. when ξ is reduced, this fact seems doubtful in general. To overcome this problem, avoiding duplication of statements, a notion of *suitable* pairs (V, ξ) is introduced, see Section 2 for details.

In [1, Theorem 2, (i)] it is shown that if $b(= b_0) = 1$ for an ample and free linear system, then $\dim X = 2$. Theorem 22, under the assumption that there exists a suitable pair (V, ξ), gives the bound $\dim X \le b+1$, generalizing the result above. On the other hand, the corresponding inequality with b_0 instead of b holds with no further assumption, see Remark 23.

Bringing the positivity of L into the picture, one can assume that L is k-very ample, i.e., every zero-scheme of length $k+1$ imposes independent conditions on sections of L. Then, Theorem 30 gives a stronger bound if $k \ge 2$.

In [3, Theorem 1.7.9] a characterization of the case $b = 2$ for L very ample (1-very ample) is given. In particular X must be a surface and bad zero-schemes of length 2 are contained in a line ℓ. Proposition 33 generalizes this results under the assumption that L is k-very ample and b is realized by a reduced zero-scheme of length $k+1$. In this case X must also be a surface and the bad, reduced zero-schemes of length $k+1$ are contained in a rational normal curve of degree k.

A similar characterization for $k = 2$ and $b = 3$, where b may be realized by a non-reduced zero-scheme, is given in Proposition 27.

Sharpness of all bounds is illustrated by a series of examples.

Bad linear spaces are discussed in the final section. It turns out that they must necessarily have codimension two and that they are inherited by hyperplane sections. These two facts are combined to show that bad linear spaces do not occur at all for very ample linear systems.

2. Notation and background

Throughout this article X denotes a smooth, connected, projective variety of dimension n, n-fold for short, defined over the complex field $\mathbb{C}$. Its structure sheaf is denoted by $\mathcal{O}_X$ and the canonical sheaf of holomorphic n-forms on X is denoted by K_X. Cartier divisors, their associated line bundles and the invertible sheaves of their holomorphic sections are used with no distinction. Mostly additive notation is used for their group.

Let $S^t X$ be the t-th symmetric power of X and $X^{[t]}$ be the Hilbert scheme of zero-subschemes of X of length t. Let $X^{[t]}_{(1,\ldots,1)}$ be the stratum of reduced zero-subschemes of length t. We denote by $X^{[t]}_{(1+r,1,1,\ldots,1)}$, for $0 \le r \le \min\{t-1, n\}$, the set of zero-subschemes ξ of length t such that $\mathrm{Supp}(\xi) = \{x_1, x_2, \ldots, x_{t-r}\}$, and $\mathcal{I}_\xi = \mathfrak{a} \cdot \mathfrak{m}_2 \cdot \cdots \cdot \mathfrak{m}_{t-r}$ where $\mathfrak{m}_i$ is the maximal ideal of $\mathcal{O}_{X,x_i}$ and $\mathfrak{a} = (u_i u_j, u_{r+1}, \ldots, u_n \mid 1 \le i \le j \le r)$, $u_1, \ldots u_n$ denoting local coordinates at x_1.

Given a zero-scheme $\xi \in X^{[t]}_{(1,\ldots,1)}$, we sometimes identify ξ and its support $\mathrm{Supp}(\xi)$; for example we write $x \in \xi$ to mean $x \in \mathrm{Supp}(\xi)$. For any coherent sheaf $\mathcal{F}$ on X, $h^i(X, \mathcal{F})$ is the complex dimension of $H^i(X, \mathcal{F})$. When the ambient variety is understood, we often write $H^i(\mathcal{F})$ and $h^i(\mathcal{F})$ respectively for $H^i(X, \mathcal{F})$ and $h^i(X, \mathcal{F})$. Let L be a line bundle on X. If L is ample, the pair (X, L) is called a *polarized variety*. For a subspace $V \subseteq H^0(X, L)$ the following notations are used:

$|V|$, the linear system associated with V;

$|V \otimes \mathcal{I}_Z|$, with a slight abuse of notation, the linear system of divisors in $|V|$ which contain, scheme theoretically, the subscheme Z of X;

$\mathrm{Bs}|V|$, the base locus of the linear system $|V|$;

φ_V, the rational map given by $|V|$.

If $V = H^0(L)$ we write L instead of V in all of the above. Let L be a line bundle generated by its global sections. When the linear span of $\varphi_L(\xi)$ is a $\mathbb{P}^k$ for every zero-scheme $\xi \in X^{[k+1]}$ we say that L is k-very ample. Equivalently:

Definition 1. Let k be a non-negative integer. A line bundle L on X is k-very ample if the restriction map $H^0(X, L) \to H^0(L \otimes \mathcal{O}_\xi)$ is surjective for every zero-scheme $\xi \in X^{[k+1]}$.

The second Bertini theorem, see for example [4], is the main tool to handle linear systems whose elements are all reducible. The following remark on the dimension of the base locus of such linear systems follows easily from that theorem and will be useful to us.

Remark 2. Let L be a line bundle on a smooth variety X, spanned by a subspace $V \subseteq H^0(L)$. Assume Z is a non-trivial subscheme of X such that $|V \otimes \mathcal{I}_Z|$ does not have a fixed component and it is composed with a pencil (i.e. the rational map associated with $|V \otimes \mathcal{I}_Z|$ has one-dimensional image). Two generic fibers of the pencil are divisors on X and their intersection lies in the base locus $\mathrm{Bs}(|V \otimes \mathcal{I}_Z|)$ of $|V \otimes \mathcal{I}_Z|$. This implies that $\dim(\mathrm{Bs}(|V \otimes \mathcal{I}_Z|)) = n - 2$.

For ample linear systems we also have the following observation on the dimension of the base locus.

Remark 3. Let L be an ample line bundle on a smooth variety X, spanned by a subspace $V \subseteq H^0(X, L)$. Assume ξ is a zero-scheme on X which imposes k independent conditions on $|V|$, i.e. $\dim(|V \otimes \mathcal{I}_\xi|) = \dim(|V|) - k$. Consider the cohomology sequence $0 \to H^0(L \otimes \mathcal{I}_\xi) \to H^0(L) \xrightarrow{r_\xi} H^0(L \otimes \mathcal{O}_\xi)$, and let $W = r_\xi(V)$, where $\dim W = k$. Choose a basis for V, taking into account the decomposition $V \simeq \mathrm{Ker}(r_\xi) \oplus W$, to describe the map φ_V. Then one sees that $\varphi_V(\mathrm{Bs}(|V \otimes \mathcal{I}_\xi|)) \subseteq \mathbb{P}(W) = \mathbb{P}^{k-1}$. As L is ample, φ_V does not contract any positive dimensional subvariety and thus $\dim(\mathrm{Bs}(|V \otimes \mathcal{I}_\xi|)) \le k - 1$.

The fact that, given a zero-scheme of length b, one can always find a zero-subscheme of any length $a \le b$ is going to be important. For clarity in the exposition we report below a self contained proof.

Lemma 4. *Let K be an algebraically closed field. Let ξ be a zero-scheme over K of length $b > 0$. Let $0 < a < b$, then there is a zero-subscheme $\xi' \subset \xi$ of length a.*

Proof. Let $\xi = \mathrm{Spec}(A)$. As $\dim(A) = 0$, A is an Artinian ring and thus it is the product of local Artinian rings. Let $A = B_1 \times \cdots \times B_k$, where $\mathrm{Supp}(B_i) = x_i$ and thus $\mathrm{Supp}(\xi) = \{x_1, \dots, x_k\}$. Assume $\mathrm{length}(\mathrm{Spec}(B_i)) = b_i$, i.e. $b = b_1 + \cdots + b_k$. If ξ is a reduced subscheme, i.e. $b_1 = \cdots = b_k = 1$, then a subscheme ξ' as desired is given by the ring $B_1 \times \cdots \times B_a$.

Assume that $b_i \ge 2$ for some i. To prove the assertion of the lemma it is enough to show that, for every i, we can find a subscheme $\eta \subset \mathrm{Spec}(B_i)$ with $\mathrm{length}(\eta) = b_i - 1$. Let $\mathfrak{m}_i$ be the maximal ideal of the local ring B_i and let h be the smallest integer so that ${\mathfrak{m}_i}^h = 0$. Notice that $1 < h \le b_i$. Because $\mathrm{length}(\mathrm{Spec}(B_i)) = b_i$, it must be ${\mathfrak{m}_i}^{b_i} = 0$. Otherwise B_i would have a filtration of length $\ge b_i + 1$:

$${\mathfrak{m}_i}^{b_i} \subset {\mathfrak{m}_i}^{b_i - 1} \subset \cdots \subset \mathfrak{m}_i \subset B_i,$$

which would contradict the assumption that $\dim_K(B_i) = b_i$. Let $x \in {\mathfrak{m}_i}^{h-1}$. The ideal (x) of B_i, generated by x, is a one-dimensional K vector space. Consider the surjective map $\phi : B_i \to (x)$ defined by sending 1 to x. Since $\phi(\mathfrak{m}_i) = 0$ the map ϕ factors through $B_i/\mathfrak{m}_i = K$ (because K is algebraically closed) and thus K maps surjectively onto (x).

This implies that the quotient $B_i/(x)$ defines a subscheme

$$\eta = \mathrm{Spec}(B_i/(x)) \subset \mathrm{Spec}(B_i),$$

where

$$\mathrm{length}(\eta) = \dim_K(B_i/(x)) = \dim_K(B_i) - \dim_K(x) = b_i - 1.$$

The following two lemmata deal with changes in positivity of a k-very ample line bundle when blowing up at reduced zero-schemes. A detailed study can be found in [5, 4].

Lemma 5. *Let L be a k-very ample line bundle on a projective n-fold X, with $k \ge 0$. Let $\pi : \tilde{X} \to X$ be the blow-up of X at $\xi \in X^{[t]}_{(1,\dots,1)}$, for any $0 \le t \le k$, with exceptional divisors $E_1, \dots, E_t$. Then the line bundle $\mathcal{L} = \pi^*(L) - E_1 - \cdots - E_t$ is globally generated on $\tilde{X}$.*

Proof. Let $\mathcal{I}_\xi = \mathfrak{m}_{x_1} \cdot \mathfrak{m}_{x_2} \cdot \dots \cdot \mathfrak{m}_{x_t}$ be the ideal defining the reduced zero-scheme ξ. For every point $y \in (\tilde{X} \setminus \cup E_i)$ consider the zero-scheme of length $t+1$ defined by the ideal $\mathcal{I}_{\xi'} = \mathcal{I}_\xi \cdot \mathfrak{m}_{\pi(y)}$. It is $\operatorname{Supp}(\xi') = \operatorname{Supp}(\xi) \cup \pi(y)$. Because L is k-very ample, there is a section of L vanishing at $\operatorname{Supp}(\xi)$ and not vanishing at $\pi(y)$. This implies that there is a section of $\mathcal{L}$ not vanishing at y.

If $y \in E_i$ for some i, it corresponds to a tangent direction τ to X at $\pi(E_i) = x_i$. For simplicity let us fix $i = 1$. Choose local coordinates $\{u_1, \dots, u_n\}$ around x_1 and assume that the tangent direction corresponds to the coordinate u_1. The zero-scheme $\xi' \in X^{[t+1]}_{(2,1,\dots,1)}$ defined by the ideal $\mathcal{I}_{\xi'} = (u_1^2, u_2, \dots, u_n) \cdot \mathfrak{m}_{x_2} \cdot \dots \cdot \mathfrak{m}_{x_t}$ has length $t+1 \le k+1$, and $\operatorname{Supp}(\xi') = \operatorname{Supp}(\xi)$. Because L is k-very ample, the map

$$H^0(X, L) \to H^0(L \otimes \mathcal{O}_{\xi'})$$

is onto. Therefore there is a section $s \in H^0(X, L)$ which vanishes at $\operatorname{Supp}(\xi)$ and such that $ds(\tau) \neq 0$. Let $D = (s)_0$ and let $s' \in H^0(\mathcal{L})$ be the section corresponding to $\pi^*(D) - E_1 - \dots - E_k$. Thus $s'(y) \neq 0$.

Lemma 6 *([5, 4.1]). Let L be a k-very ample line bundle on a projective manifold X, with $k \ge 1$. Let $\pi : \tilde{X} \to X$ be the blow-up of X at $\xi \in X^{[t]}_{(1,\dots,1)}$, for any $t \le k-1$, with exceptional divisors $E_1, \dots, E_t$. Then the line bundle $\mathcal{L} = \pi^*(L) - E_1 - \dots - E_t$ is very ample.*

Remark 7. In the same context as Lemma 6, $\tilde{X}$ can be naturally identified with a closed subscheme of $X^{[t+1]}$ and, under this identification, $\mathcal{L}$ corresponds to a very ample line bundle of the subscheme. Moreover, if $\dim X = 2$, then such line bundle extends to a very ample line bundle on the whole $X^{[t+1]}$. Indeed, let $\xi \in X^{[t]}_{(1,\dots,1)}$ and for all $l \ge t$ let

$$X^{[l]}_\xi = \{\eta \in X^{[l]} \text{ such that } \operatorname{Supp}(\xi) \subseteq \operatorname{Supp}(\eta)\}.$$

It is $X^{[t]}_\xi = \{\xi\}$ and $X^{[t+1]}_\xi \cong \tilde{X}$. Let $s_\xi : X \to S^{t+1}X$ be the map defined by $s_\xi(x) = (x, \operatorname{Supp}(\xi))$, and $\rho : X^{[t+1]} \to S^{t+1}X$ be the Hilbert–Chow morphism. It is:

$$\rho^{-1}(s_\xi(X)) \cong X^{[t+1]}_\xi.$$

Thus the following diagram is commutative:

$$\begin{array}{ccccc}
\tilde{X} & = & X^{[t+1]}_\xi & \longrightarrow & X^{[t+1]} \\
{\scriptstyle \pi}\downarrow & & \downarrow & & \downarrow \\
X & = & X & \longrightarrow & S^{t+1}X.
\end{array}$$

By Lemma 6 the line bundle $\mathcal{L} = \pi^*(L) - E_1 - \dots - E_t$ defines an embedding of $X^{[t+1]}_\xi$.

In the case $\dim(X) = 2$ the line bundle $\mathcal{L}$ can be equivalently described as follows. Let $\zeta_{t+1} \subset X^{[t+1]} \times X$ be the universal family with projection maps $p_1 : \zeta_{t+1} \to X^{[t+1]}$ and $p_2 : \zeta_{t+1} \to X$. Consider the rank $(t+1)$ vector bundle $L^{[t+1]} = p_{1*}(p_2^* L)$. Then

$$\mathcal{L} = \det(L^{[t+1]})|_{X^{[t+1]}_\xi}.$$

In other words $\det(L^{[t+1]})$ extends $\mathcal{L}$. Note that because $\dim(X) = 2$ and L is $(t+1)$-very ample then the line bundle $\det(L^{[t+1]})$ is very ample, [6,7].

Let L be a line bundle on a smooth complex projective variety X of dimension $n \ge 2$ and let ξ be a zero-dimensional subscheme of X. Let $\mathcal{I}_\xi$ be the ideal sheaf of ξ and consider the exact sequence

$$0 \to L \otimes \mathcal{I}_\xi \to L \to L \otimes \mathcal{O}_\xi \to 0.$$

Consider the induced homomorphism $H^0(X, L) \to H^0(L \otimes \mathcal{O}_\xi)$. We denote by r_ξ its restriction to V.

Definition 8. Let X, L, V, ξ, and r_ξ be as above. The pair (V, ξ) is *suitable* if, whenever r_ξ is not surjective, there exists a subscheme $\eta \subset \xi$, such that r_η is surjective and $r_\eta(V) = \rho(r_\xi(V))$, where $\rho : H^0(L \otimes \mathcal{O}_\xi) \to H^0(L \otimes \mathcal{O}_\eta)$ is the obvious restriction homomorphism. This is equivalent to requiring that η imposes linearly independent conditions on V and $|V \otimes \mathcal{I}_\eta| = |V \otimes \mathcal{I}_\xi|$.

Lemma 9. *Let X, L, V, be as in the definition above. Let $\xi \in X^{[t]}$. Then (V, ξ) is suitable in each of the following cases:*

(1) *ξ reduced;*
(2) *$\xi \in X^{[t]}_{(1+r,1,1,\ldots,1)}$;*
(3) *$V = H^0(X, L)$, L is $(k-2)$-very ample and $t = \text{length}(\xi) \le k$.*

Proof. Assume that r_ξ is not surjective, i.e. $\xi \in X^{[t]}$ does not impose t independent conditions. To prove (1), let $\text{Supp}(\xi) = \{x_1, \ldots, x_t\}$ and let $\mathcal{I}_\xi = \mathfrak{m}_1 \cdot \cdots \cdot \mathfrak{m}_t$ where $\mathfrak{m}_i$ is the maximal ideal of $\mathcal{O}_{X,x_i}$. Consider the vector subspace

$$\text{Im}(r_\xi) \subset H^0(L \otimes \mathcal{O}_\xi) \cong \oplus_1^t \mathcal{O}_X/\mathfrak{m}_i \cong \mathbb{C}^t.$$

After extending a basis of $\text{Im}(r_\xi)$ to $H^0(L \otimes \mathcal{O}_\xi)$ one can assume that $\text{Im}(r_\xi) \cong \oplus_1^s \mathcal{O}_X/\mathfrak{m}_i \cong \mathbb{C}^s$, where $s < t$. Then the reduced zero-subscheme $\eta = \{x_1, \ldots, x_s\}$ imposes independent conditions on V and it is what we need.

To prove (2) let $\text{Supp}(\xi) = \{x_1, \ldots, x_{t-r}\}$ and let $u_1, \ldots, u_n$ denote local coordinates at x_1. Recall that $\mathcal{I}_\xi = \mathfrak{a} \cdot \mathfrak{m}_2 \cdot \cdots \cdot \mathfrak{m}_{t-r}$ where $\mathfrak{m}_i$ is the maximal ideal of $\mathcal{O}_{X,x_i}$ and $\mathfrak{a} = (u_i u_j, u_{r+1}, \ldots, u_n \mid 1 \le i \le j \le r)$. It is

$$H^0(L \otimes \mathcal{O}_\xi) \cong \mathbb{C}^t,$$

where the first summand on the right hand side, which is isomorphic to $\mathcal{O}_X/\mathfrak{m}_1$, is contained in $\text{Im}(r_\xi)$ as V spans L. Notice that the restriction map $\rho : H^0(X, L) \to H^0(L \otimes \mathcal{O}_\xi)$ acts on global sections $s \in H^0(X, L)$ as follows: $\rho(s) = (s(x_1), \frac{\partial s}{\partial u_1}(x_1), \ldots, \frac{\partial s}{\partial u_r}(x_1), s(x_2), \ldots, s(x_{t-r}))$. Because only first derivatives at x_1 appear, one can proceed as in the proof of (1) by completing the generator of the first summand above to a basis of $\text{Im}(r_\xi)$.

To prove (3), note that as L is $(k-2)$ very ample, then necessarily $t = k$. By Lemma 4 there exists a subscheme $\eta \subset \xi$ of length $k-1$. Consider the following commutative diagram:

$$\begin{array}{ccc} H^0(X, L) & \xrightarrow{r_\xi} & H^0(X, L \otimes \mathcal{O}_\xi) \cong \mathbb{C}^k \\ \| & & \downarrow \rho \\ H^0(X, L) & \xrightarrow{r_\eta} & H^0(X, L \otimes \mathcal{O}_\eta) \cong \mathbb{C}^{k-1}. \end{array}$$

As L is $(k-2)$-very ample, r_η is surjective. Therefore $H^0(X, L \otimes \mathcal{O}_\eta) \cong \text{Im}(r_\xi)$, and η is the required subscheme.

The above lemma is mostly intended as a useful tool to enhance readability of a number of results expressed in terms of suitable pairs contained in Sections 3 and 4.

3. Higher order bad loci

The definition of bad locus, introduced in [1] and further studied in [2], can be fairly naturally generalized to subsets of $X^{[t]}$ as follows.

Definition 10. Let X be a complex, non-singular, projective variety. Let L be a line bundle on X spanned by a subspace $V \subseteq H^0(X, L)$.

(1) The t-th bad locus of (X, V), for $t \ge 1$ is:

$$\mathcal{B}_t(X, V) = \{\xi \in X^{[t]} \mid |V \otimes \mathcal{I}_\xi| \ne \emptyset \text{ and } \forall D \in |V \otimes \mathcal{I}_\xi|,\ D \text{ is reducible or non-reduced}\}.$$

(2) The reduced t-th bad locus of (X, V), for $t \ge 1$ is:

$$\mathcal{B}^0_t(X, V) = \mathcal{B}_t(X, V) \cap X^{[t]}_{(1,\ldots,1)}.$$

We write $\mathcal{B}_t(X, L)$ and $\mathcal{B}^0_t(X, L)$ if $V = H^0(X, L)$.

An element $\xi \in \mathcal{B}_t(X, V)$ is called a *bad* zero-scheme for the linear system $|V|$.

There is a clear relationship among the $\mathcal{B}_t$'s:

Lemma 11. *If $\mathcal{B}_t(X, V) \neq \emptyset$ then $\mathcal{B}_k(X, V) \neq \emptyset$ for every $k \geq t$.*

Proof. Let $\xi \in \mathcal{B}_t(X, V)$. If $\dim|V \otimes \mathcal{I}_\xi| \geq 1$ let x be any point in $X \setminus \mathrm{Supp}(\xi)$. Otherwise let D be the unique element in $|V \otimes \mathcal{I}_\xi|$ and let x be any point in $D \setminus \mathrm{Supp}(\xi)$. Consider the zero-scheme ξ', of length $t+1$, obtained by adding the reduced point x to ξ. It is $\mathrm{Supp}(\xi') = \mathrm{Supp}(\xi) \cup \{x\}$, $\mathcal{O}_{\xi',y} = \mathcal{O}_{\xi,y}$ for every $y \in \mathrm{Supp}(\xi)$ and $\mathcal{O}_{\xi',x} = \frac{\mathcal{O}_X}{\mathfrak{m}_x}$. Because $|V \otimes \mathcal{I}_{\xi'}| \subseteq |V \otimes \mathcal{I}_\xi|$ all the divisors in $|V \otimes \mathcal{I}_{\xi'}|$ are reducible and $\xi' \in \mathcal{B}_{t+1}(X, V)$.

The above lemma suggests the following definitions.

Definition 12. Let (X, V) be as above. The b-index of the pair (X, V) is:

$$b(X, V) = \begin{cases} \infty & \text{if } \mathcal{B}_t(X, V) = \emptyset \text{ for every } t \geq 1 \\ \min\{t \mid \mathcal{B}_t(X, V) \neq \emptyset\} & \text{otherwise.} \end{cases}$$

The reduced b-index of the pair (X, V) is:

$$b_0(X, V) = \begin{cases} \infty & \text{if } \mathcal{B}^0_t(X, V) = \emptyset \text{ for every } t \geq 1 \\ \min\{t \mid \mathcal{B}^0_t(X, V) \neq \emptyset\} & \text{otherwise.} \end{cases}$$

We write b and b_0, respectively, for $b(X, V)$ and $b_0(X, V)$, when the pair (X, V) is clear from the context.

Remark 13. It follows immediately from the above definition that $b(X, L) \leq b_0(X, L)$. Moreover, $\mathcal{B}_b(X, V) \cap X^{[b]}_{(1,\ldots,1)} \neq \emptyset$ if and only if $b(X, V) = b_0(X, V)$. Notice also that if there are no suitable pairs (V, ξ), then $b < b_0$.

Remark 14. If $|V|$ contains a reducible element D, then $b(X, V) < \infty$. Indeed, let A be an irreducible component of D. A zero-scheme $\xi \subset A$ can be constructed with $r = \mathrm{length}(\xi) > \dim|V|_{|A}$ and sufficiently general to have $|V \otimes \mathcal{I}_\xi| = A + |V - A|$. Then $\xi \in \mathcal{B}_r(X, V)$, hence $b(X, V) \leq r$.

Suppose $\mathrm{Pic}(X) = \mathbb{Z}[L]$ where L is ample and spanned by V. Then $b(X, V) = \infty$. Indeed $|V|$ cannot contain any reducible element. If $A + B$ were such an element it would be $A = aL$ and $B = bL$ for $a, b \geq 1$. This would give $L = (a + b)L$ which is a contradiction. Recall that Barth–Larsen's Theorem can provide plenty of such examples.

Remark 15. Let $W \subseteq V \subseteq H^0(X, L)$ be two subspaces which generate L. Then clearly $\mathcal{B}_t(X, V) \subseteq \mathcal{B}_t(X, W)$ and thus $b(X, W) \leq b(X, V)$ and $b_0(X, W) \leq b_0(X, V)$. This is illustrated in the following example.

Example 16. Let $(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ and $\xi \in X^{[2]}$. As $|L \otimes \mathcal{I}_\xi|$ contains always irreducible conics, it is $b(X, L) \geq 3$. Let now η consist of three distinct points on a line. As all elements of $|L \otimes \mathcal{I}_\eta|$ are reducible, it is $b(X, L) = b_0(X, L) = 3$. Now let x_0, x_1, x_2 be homogeneous coordinates on $\mathbb{P}^2$ and consider the following vector subspace of $H^0(X, L)$: $U := \langle x_0^2 + x_1^2 + x_2^2, x_0x_1, x_0x_2, x_1x_2\rangle$. Note that U spans L and $\varphi_U : \mathbb{P}^2 \to \mathbb{P}^3$ is a birational morphism whose image, Σ, is Steiner's Roman surface. Theorem 1.1 in [2], since $\varphi_U(X)$ is neither $\mathbb{P}^2$ nor a cone, implies $\mathcal{B}_1(X, U) = \emptyset$. Notice that φ_U maps all three points $e_0 = (1 : 0 : 0)$, $e_1 = (0 : 1 : 0)$, $e_2 = (0 : 0 : 1)$ to $(1 : 0 : 0 : 0)$, the triple point of Σ. In other words,

$$|U - e_0| = |U - e_1| = |U - e_2|.$$

Now, let ξ be the zero-scheme consisting of e_i and another point p, possibly infinitely near, lying on the line $\langle e_i, e_j\rangle$. Then every conic in $|U \otimes \mathcal{I}_\xi|$ is reducible, containing e_i, e_j, p, hence the line $\langle e_i, e_j\rangle$. This shows that $\xi \in \mathcal{B}_2(X, U)$ and therefore $b(X, U) = b_0(X, U) = 2 < b(X, L)$. Finally, let W be either the vector subspace $\langle x_0^2, x_1^2, f(x_0, x_1, x_2)\rangle$, where f is a general form of degree 2, or $\langle x_0^2, x_1^2, x_2^2\rangle$. Then the pairs (X, W) correspond to Example 7.2 (i) and Example 7.3 (jjj) in [2], respectively. Recalling that $\mathcal{B}(X, W) = \{e_2\}$ or $\{e_0, e_1, e_2\}$ respectively, in both cases we have $b(X, W) = b_0(X, W) = 1 < b(X, U) < b(X, L)$.

The following proposition generalizes some points of [1, Theorem 2].

Proposition 17. *Let X be a smooth n-dimensional variety. Let L be an ample line bundle on X, spanned by a subspace $V \subseteq H^0(X, L)$. Let $\xi \in \mathcal{B}_b(X, V)$, and suppose $|V \otimes \mathcal{I}_\xi|$ has finite base locus. Then*

(i) $n = 2$;

(ii) *there is an ample line bundle A on X with $h^0(A) \geq 2$ such that every $D \in |V \otimes \mathcal{I}_\xi|$ is of the form $D = A_{b_1} + \cdots + A_{b_r}$, for some $r \geq 2$, with A_{b_i} varying in a rational pencil $B \subseteq |A|$;*
(iii) *with r as in* (ii)*, for all $x \in \mathrm{Bs}|V \otimes \mathcal{I}_\xi|$, x is a point of multiplicity $r \geq 2$ for all $D \in |V \otimes \mathcal{I}_\xi|$. In particular this is true for all $x \in \mathrm{Supp}(\xi)$, i.e. $\mathcal{I}_{\xi,x} \subseteq \mathfrak{m}_x^r$, where $r \geq 2$;*
(iv) $\mathrm{Supp}(\xi) \subseteq A_{b_i} \cap A_{b_j} = \mathrm{Bs}|V \otimes \mathcal{I}_\xi|$*, for all distinct A_{b_i} and A_{b_j} appearing in the expression of a general $D \in |V \otimes \mathcal{I}_\xi|$.*

Proof. As $|V \otimes \mathcal{I}_\xi|$ has finite base locus, Bertini's second theorem and Remark 2 give (i) and that the image C of the rational map $\varphi_{V \otimes \mathcal{I}_\xi}$ is one-dimensional. After resolving its indeterminacies, taking Stein's factorization we get the following diagram:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\varphi}} & C \subset \mathbb{P} \\ {\scriptstyle\alpha}\downarrow & \nearrow{\scriptstyle\beta} & \\ B & & \end{array}$$

where $\tilde{X}$ is a suitable blow-up of X, B is a smooth curve, α has connected fibres and β is a finite morphism. Note that $B \simeq \mathbb{P}^1$ because there is at least one exceptional divisor in $\tilde{X}$, mapping surjectively to B via $\tilde{\varphi}$. Let $r = \deg \beta \deg C$. Then every $D \in |V \otimes \mathcal{I}_\xi|$ can be written as $D = A_{b_1} + \cdots + A_{b_r}$, where each A_{b_i} is the image on X of a fibre of α. Thus, $r \geq 2$. Notice that all A_{b_i}'s are linearly equivalent as they vary in the rational pencil B. It follows that $h^0(A) \geq 2$, $L \sim rA$ (linearly equivalent) and A is ample. This proves (ii). Let $x \in \mathrm{Bs}|V \otimes \mathcal{I}_\xi|$. As $x \in D$ for all $D \in |V \otimes \mathcal{I}_\xi|$, and there are no fixed components, x must belong to infinitely many elements A_b, $b \in B$, hence to all of them. Thus x is a point of multiplicity greater or equal to r for all D. In particular, if $x \in \mathrm{Supp}(\xi)$, $\mathcal{I}_{\xi,x} \subseteq \mathfrak{m}_x^r$. This proves (iii). Moreover $\mathrm{Supp}(\xi) \subseteq \mathrm{Bs}|V \otimes \mathcal{I}_\xi| \subseteq A_{b_i} \cap A_{b_j}$ for any $i, j = 1, \ldots, r$. To prove (iv), it is then enough to show that for a general $D \in |V \otimes \mathcal{I}_\xi|$ it is $\mathrm{Bs}|V \otimes \mathcal{I}_\xi| \supseteq A_{b_i} \cap A_{b_j}$ for any $i, j = 1, \ldots, r$. This follows from Bertini's first theorem because every point $y \in A_{b_i} \cap A_{b_j}$ is a singular point for D, which is generally chosen.

The following lemma shows that, in the case of suitable pairs, the linear span of a bad zero-scheme of minimal length is always of maximal dimension.

Lemma 18. *Let X be a smooth n-dimensional variety. Let L be an ample line bundle on X, spanned by a subspace $V \subseteq H^0(X, L)$. Assume $b_0(X, V) < \infty$.*
(i) *Let $\xi \in \mathcal{B}_b(X, V)$. If (V, ξ) is suitable, then ξ imposes exactly b independent conditions on $|V|$;*
(ii) $b(X, V) \leq b_0(X, V) \leq \dim |V|$.

Proof. To prove (i), assume that $\dim(|V \otimes \mathcal{I}_\xi|) = \dim(|V|) - m > \dim(|V|) - b$. As (V, ξ) is suitable, there exists a zero-scheme $\eta \subset \xi$ with $\mathrm{length}(\eta) < b$, such that $|V \otimes \mathcal{I}_\eta| = |V \otimes \mathcal{I}_\xi|$. This is impossible because it would imply $\eta \in \mathcal{B}_t(X, V)$ with $t < b$. If ξ in the argument above is reduced then $b = b_0$, as noted in Remark 13. Moreover in this case η is reduced and the right side of inequality (ii) follows immediately from (i). Remark 13 completes the proof.

Remark 19. Notice that replacing $\mathcal{B}_b(X, V)$ with $\mathcal{B}^0_{b_0}(X, V)$ and b with b_0, the same argument as in the proof of Lemma 18 shows that (i) holds for every $\xi \in \mathcal{B}^0_{b_0}(X, V)$.

The upper bound in Lemma 18 (ii) above can be strict, see Example 16 above with $V = H^0(X, L)$ and Example 20 below. Nonetheless it can be attained in some cases, showing that it is optimal. This can be seen in Example 21.

Example 20. Let $X = \mathbb{P}^3$ with homogeneous coordinates x_0, x_1, x_2, x_3, and consider the vector subspace V of $H^0(\mathcal{O}_{\mathbb{P}^3}(2))$ generated by the monomials $x_0^2, x_1^2, x_2^2, x_3^2$. Note that V spans $L = \mathcal{O}_{\mathbb{P}^3}(2)$ and defines a morphism $\mathbb{P}^3 \to \mathbb{P}^3$ of degree 8. Let $\{e_0, e_1, e_2, e_3\}$ be the standard basis for $\mathbb{C}^4$. Now, let $\xi = \{e_0, e_1\}$. Then $|V \otimes \mathcal{I}_\xi|$ is the pencil of quadrics generated by x_2^2 and x_3^2. Every such quadric is reducible, hence $\xi \in \mathcal{B}^0_2(X, V)$. Note that $|V \otimes \mathcal{I}_\xi|$ has no fixed component and is composed with the pencil $|\mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathcal{I}_\ell|$, ℓ being the linear span of ξ. In conclusion we have

$$b(X, V) \leq b_0(X, V) = 2 = \dim |V| - 1,$$

and thus $b(X, V) = 2$, as $\mathcal{B}(X, V) = \emptyset$, see [1, Theorem 2].

Example 21. Let X be a Del Pezzo surface with $K_X^2 = 2$ and let $L = -K_X$. It is well known that L is ample and spanned and $\varphi_L : X \to \mathbb{P}^2$ is a double cover, branched along a smooth plane quartic curve. Let $\xi = \{x_1, x_2\}$ be a zero-scheme consisting of two distinct points of the ramification divisor such that $< \varphi_L(x_1), \varphi_L(x_2) >$ is a bitangent line to the branch quartic curve. Then $|L \otimes \mathcal{I}_\xi|$ consists of a single element G having double points at x_1 and x_2. In fact $G = \Gamma_1 + \Gamma_2$, where Γ_1, Γ_2 are two (-1)-curves meeting exactly at x_1, x_2. Thus $b_0(X, L) \le 2$ and it follows from Corollary 1.3 in [2] that $b_0(X, L) = 2 = \dim |L|$.

One of the major results of [1] is the fact that the existence of a bad point forces the variety to be a surface. The following theorem generalizes this result to include higher order bad loci

Theorem 22. *Let X be a smooth n-dimensional variety. Let L be an ample line bundle on X, spanned by a subspace $V \subseteq H^0(X, L)$. Assume there exists $\xi \in \mathcal{B}_b(X, V)$, such that (V, ξ) is suitable. Then:*

(1) $n \le b + 1$;
(2) *If $n = b + 1$, for every $\xi \in \mathcal{B}_b(X, V)$ such that (V, ξ) is suitable, the linear system $|V \otimes \mathcal{I}_\xi|$ has no fixed component and it is composed with a pencil.*

Proof. Let $\xi \in \mathcal{B}_b(X, V)$. If $b = 1$, (1) and (2) follow respectively from [1, Theorem 2, (i) and (ii)]. So let $b \ge 2$. By Lemma 18, ξ imposes exactly b conditions on $|V|$. According to Bertini's second theorem, the linear system $|V \otimes \mathcal{I}_\xi|$ either has a fixed component Σ or it is composed with a pencil. In the first case, Remark 3 gives $\dim(\Sigma) = n - 1 \le b - 1$, i.e. (1) holds as a strict inequality and thus (2) is proven. If $|V \otimes \mathcal{I}_\xi|$ is composed with a pencil, Remarks 2 and 3 give

$$n - 2 = \dim(\mathrm{Bs}(|V \otimes \mathcal{I}_\xi|)) \le b - 1,$$

which completes the proof of (1).

Remark 23. If there are no suitable pairs (V, ξ), which by Remark 13 implies $b < b_0$, the same argument used in the proof of Theorem 22, replacing $\mathcal{B}_b(X, V)$ with $\mathcal{B}^0_{b_0}(X, V)$ and Lemma 18 with Remark 19, gives the same statement as in Theorem 22 for b_0 and for all $\xi \in \mathcal{B}^0_{b_0}(X, V)$.

Example 20 shows that case (2) of Theorem 22 is effective. On the other hand, the following example illustrates the fact that if $n < b + 1$ then the linear system $|V \otimes \mathcal{I}_\xi|$ for all $\xi \in \mathcal{B}_b(X, V)$ may have a fixed component.

Example 24. Let X be an n-fold and let L be an ample line bundle on X, spanned by a subspace $V \subset H^0(X, L)$.

Assume that X contains an L-hyperplane, i.e. a divisor $F \simeq \mathbb{P}^{n-1}$ such that $L^{n-1} \cdot F = 1$. Let $\xi = \{x_1, x_2, \ldots, x_n\}$ be a reduced zero-scheme on F such that the linear span of $\varphi_V(\xi)$ has dimension $n - 1$, i.e. $\varphi_V(\xi)$ spans the entire $\varphi_V(F)$. Then $\xi \in \mathcal{B}^0_n$ with $|V \otimes \mathcal{I}_\xi|$ having F as fixed component. It follows that $b(X, V) \le b_0(X, V) \le n$. On the other hand Theorem 22 gives $b(X, V) \ge n$, hence $b(X, V) = n$.

In particular, if $(X, L) = (\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ is the n-dimensional scroll of a very ample vector bundle E over a smooth curve C then $b(X, L) = b_0(X, L) = n$. Notice that as (X, L) is a scroll, $D \in |L|$ is reducible if and only if D contains a fibre $F = \mathbb{P}^{n-1}$ of the scroll.

4. Bad loci and higher order embeddings

The study of $\mathcal{B}_t(X, L)$, so tightly connected with linear systems containing zero-schemes of any length, is very naturally conducted in the context of higher order embeddings. The focus of this section is on the properties of bad loci of complete linear systems associated with k-very ample line bundles.

Let X be a smooth complex variety and let L be a k-very ample line bundle on X. Assume $\mathcal{B}_t(X, L)$ is not empty for some t. Recall that, by Lemma 9, for all $\xi \in X^{[t]}$, $t \le k + 2$, the pair $(H^0(X, L), \xi)$ is suitable.

In [1,2], the case of a spanned (0-very ample) line bundle with non-empty $\mathcal{B}_1(X, L)$ was treated.

When L is very ample (1-very ample) it is $b_0 \ge b \ge 2$, see [1, Corollary 2], and a complete characterization of the case $b = 2$ is given in [3, Theorem 1.7.9]. In this case $n = 2$ and, for all $D \in |L|$ containing a bad zero-scheme of length 2, $D = \ell + R$, where ℓ is a fixed line containing the bad zero-scheme, $\mathcal{O}_X(R)$ is spanned, and R and ℓ intersect transversally. Notice that the same characterization holds for (X, V), $V \subset H^0(X, L)$, where $|V|$ is a very ample linear system. Notice also that this shows that the natural phenomenon described in Example 24 is the only possibility when L is very ample and $n = 2$. The following proposition generalizes the lower bound on b in terms of k-very ampleness.

Proposition 25. *Let L be a k-very ample line bundle on a projective n-fold X, $n \geq 2$, with $k \geq 1$. Then $b(X, L) \geq k + 1$ if either $n \geq 3$ or $n = 2$ and there exists $\eta \in \mathcal{B}_b(X, L)$ with $x \in \mathrm{Supp}(\eta)$ such that η is reduced at x, i.e. $h^0(\mathcal{O}_{\eta,x}) = 1$.*

Proof. Assume by contradiction that $b \leq k$ and let $\xi \in \mathcal{B}_b(X, L)$. Notice that the base scheme of $|L \otimes \mathcal{I}_\xi|$ is ξ. Indeed, if such a scheme, Z, strictly contained ξ as a scheme, then there would be a zero-scheme ξ' of X, containing ξ, of length $b+1$, such that $|L \otimes \mathcal{I}_\xi| = |L \otimes \mathcal{I}_{\xi'}|$. But this contradicts the k-very ampleness of L, since $b+1 \leq k+1$. Then the assertion follows from Proposition 17, (i) and (iii).

Remark 26. Equality in the statement of Proposition 25 does indeed happen. Let $(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k))$. If $k = 1$ then $b(X, L) = \infty$ by Remark 14. Assume $k \geq 2$. L is k-very ample. Let $\xi = \{x_1, \ldots, x_{k+1}\}$ be a reduced zero-scheme contained in a line ℓ. Then $\xi \in \mathcal{B}^0_{k+1}(X, L)$ and thus Proposition 25 gives $b_0(X, L) = b(X, L) = k + 1$.

In line with [3, 1.7.9] where the case $b = 2$ is fully described, the following proposition gives a complete characterization of the case $b = 3$. The role of the line ℓ for $b = 2$ is here played by a smooth conic.

Proposition 27. *Let L be a k-very ample line bundle on a projective n-fold X, with $n \geq 2$ and $k \geq 2$. Assume $b(X, L) = 3$. Then $n = k = 2$ and either*

(a) *$(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ or*
(b) *for all $\xi \in \mathcal{B}_3(X, L)$, $|L \otimes \mathcal{I}_\xi|$ has a fixed component Γ which is a smooth conic containing ξ.*

Proof. We first show that $k = 2$. Let $\xi \in \mathcal{B}_3(X, L)$. Because $b(X, L) = 3$, Proposition 25 gives $k = 2$ except possibly when $n = 2$ and ξ is supported on a single point. By contradiction, assume $n = 2$, $k \geq 3$, and let $\mathrm{Supp}(\xi) = \{y\}$. The same argument as in the proof of Proposition 25 shows that ξ is the base scheme of $|L \otimes \mathcal{I}_\xi|$. Then, by Proposition 17, all $D \in |L \otimes \mathcal{I}_\xi|$ are of the form $D = A_{b_1} + \cdots + A_{b_r}$, where A_{b_i} varies in a rational pencil $B \subseteq |A|$, and $r = 2$ as $b(X, L) = 3$. Notice that, for a general D, A_{b_i} is smooth at y. Otherwise y would be a point of multiplicity ≥ 4 for all D, contradicting $b(X, L) = 3$. Because $h^0(\mathcal{O}_\xi) = h^0(\mathcal{O}_{\xi,y}) = 3$, it is $A^2 \leq 1$ and hence $A^2 = 1$, A being ample. Indeed if $A^2 \geq 2$ then Proposition 17, (iv), noting that $\mathrm{Supp}(\xi) = \mathrm{Bs}|L \otimes \mathcal{I}_\xi|$, implies that all A_{b_i} have an assigned tangent at y. Then, locally at y, each A_{b_i} has an equation of the form $z_2 + F_i(z_1, z_2) = 0$ where F_i has no terms of degree less than 2. Therefore $H^0(\mathcal{O}_{\xi,y}) \supseteq \langle 1, z_1, z_2, z_1^2, z_1 z_2 \rangle$, which is a contradiction. It follows that L is a 3-very ample line bundle, with $L^2 = (2A)^2 = 4$, which is impossible. To see this, blow up X at two general points $x_1, x_2 \in X$, and, on the new surface $\tilde{X}$, consider the very ample line bundle $\mathcal{L} = \pi^*(L) - E_1 - E_2$ as in Lemma 6. Note that $\mathcal{L}^2 = 2$, while the Picard number of $\tilde{X}$ is at least 3, which is clearly impossible. Thus $k = 2$. The proof now splits into 2 cases, according to the cardinality of $\mathrm{Supp}(\xi)$.

Case 1. Assume first $\xi = \xi' \cup \{x\}$ where $\mathrm{length}(\xi') = 2$ and $x \notin \mathrm{Supp}(\xi')$. Let $\pi : \tilde{X} \to X$ be the blow-up of X at x, with exceptional divisor E. Then the line bundle $\mathcal{L} = \pi^*(L) - E$ is very ample by Lemma 6. Let $\tilde{\xi}'$ be ξ' pulled back on $\tilde{X}$. Note that there is a bijection between $|\mathcal{L} \otimes \mathcal{I}_{\tilde{\xi}'}|$ and $|L \otimes \mathcal{I}_\xi.|$. Hence $\tilde{\xi}' \in \mathcal{B}_2(\tilde{X}, \mathcal{L})$. From [3, 1.7.9] it follows that $n = 2$ and $|\mathcal{L} \otimes \tilde{\xi}'| = \ell + |R|$, where ℓ is an $\mathcal{L}$-line containing $\tilde{\xi}'$ scheme theoretically. Let $\Gamma = \pi(\ell)$. Because L is 2-very ample it must be $L \cdot \Gamma \geq 2$, with equality holding only if $\Gamma \simeq \mathbb{P}^1$. Because ℓ and E are $\mathcal{L}$-lines, it must be $\nu := \ell \cdot E \in \{0, 1\}$. Therefore it is $1 = \mathcal{L} \cdot \ell = (\pi^*(\Gamma) - \nu E) \cdot (\pi^*(L) - E) = \Gamma \cdot L - \nu \geq 2 - 1 = 1$ which implies $\Gamma \cdot L = 2$, i.e. Γ is a smooth conic. Moreover $\ell \cdot E = 1$, hence x belongs to Γ. Thus we are in case (b).

Case 2. Assume now that $\mathrm{Supp}(\xi) = \{x\}$. Let $\pi : \tilde{X} \to X$ be the blow-up of X at x, with exceptional divisor E. As above the line bundle $\mathcal{L} = \pi^*(L) - E$ is very ample. Note that ξ defines a length 2 zero-scheme η of $\tilde{X}$ supported on E. Again there is a bijection between $|\mathcal{L} \otimes \mathcal{I}_\eta|$ and $|L \otimes \mathcal{I}_\xi|$, and thus $\eta \in \mathcal{B}_2(\tilde{X}, \mathcal{L})$. As before we conclude that $n = 2$ and $|\mathcal{L} \otimes \mathcal{I}_\eta| = \ell + |R|$, where ℓ is an $\mathcal{L}$-line containing η scheme theoretically. If $\ell \neq E$ then as above $\Gamma = \pi(\ell)$ is a smooth conic containing x (hence ξ) and it is a fixed component for $|L \otimes \mathcal{I}_\xi|$. We are again in case (b). Let now $\ell = E$. In view of the quoted bijection $|L \otimes \mathcal{I}_\xi|$ corresponds to $|R|$ and thus all elements of $|R|$ are reducible. According to [3, 1.7.9], $|R|$ is base-point free, hence its generic element is smooth. By Bertini's second theorem $|R|$ is composed with a pencil and thus $R^2 = 0$. The pencil B is rational because E is transverse to all fibers.

From $1 = \mathcal{L} \cdot \ell = \ell^2 + \ell \cdot R = -1 + \ell \cdot R$ we get that $\ell \cdot R = 2$, hence $R = A_b + A_{b'}$, $b, b' \in B$, $\mathcal{L} \cdot A_b = \mathcal{L} \cdot A_{b'} = 1$. This shows that $\tilde{X}$ is fibred over $\mathbb{P}^1$ by $\mathcal{L}$-lines. Moreover $\mathcal{L}^2 = \mathcal{L} \cdot (\ell + R) = 3$, and then it follows that $(\tilde{X}, \mathcal{L})$ is a rational cubic scroll. Thus $(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(a))$, for some $a \geq 2$. Let f be a fiber of the cubic scroll. Then $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1)) = E + f$. On the other hand $R = \mathcal{L} - E$, which implies that $2E + af = \pi^* L = a(E + f)$ and thus $a = 2$. This gives case (a).

The following example illustrates case (b) of Proposition 27.

Example 28. Let X be a non-minimal Del Pezzo surface with $K_X^2 \geq 2$ and let $L = -2K_X$. Then L is 2-very ample, see [8]. Let $E \subset X$ be a (-1)-curve, and let $\xi \in X^{[3]}$, supported on E. Note that $|L \otimes \mathcal{I}_\xi|$ is non-empty, since $H^0(X, L) = 1 + 3K_X^2 \geq 7$. Moreover, it is clear that E is a fixed component of $|L \otimes \mathcal{I}_\xi|$, since $LE = 2$. This shows that $\xi \in \mathcal{B}_3(X, L)$. It thus follows that $b(X, L) \leq 3$. Note however that it cannot be $b(X, L) = 2$, since (X, L) does not contain lines. Therefore $b(X, L) = 3$. Note also that the fixed component E of $|L \otimes \mathcal{I}_\xi|$ is a rational normal curve of degree 2 in the embedding given by L.

The following example shows that when L is very ample, but not 2-very ample, the fixed component of $|L \otimes \mathcal{I}_\xi|$ can be singular.

Example 29. Let X be a Del Pezzo surface with $K_X^2 = 1$ and let $L = -3K_X$. Then L is very ample. However L is not 2-very ample, see [8]. Recall that if X is general in moduli then $|-K_X|$ is a pencil containing 12 irreducible elements having a double point. Let Γ be such an element and let x_1 be its singular point. Let x_2, x_3 be two other distinct points of Γ and consider the reduced zero-scheme ξ of length 3 consisting of x_1, x_2, x_3. Note that $|L \otimes \mathcal{I}_\xi|$ is non-empty, since $H^0(X, L) = 7$. Moreover, for any $D \in |L \otimes \mathcal{I}_\xi|$ we have

$$(D \cdot \Gamma)_{x_1} \geq 2, \quad \text{and} \quad D \cap \Gamma \supset \{x_2, x_3\}.$$

Thus

$$4 \leq D \cdot \Gamma = (-3K_X) \cdot (-K_X) = 3.$$

It thus follows that D is reducible and contains Γ. Therefore $\xi \in \mathcal{B}_3^0(X, L)$ and $b_0(X, L) = 3$. Note however that it cannot be $b(X, L) = 2$, since (X, L) does not contain lines. Therefore $b(X, L) = 3$. Note also that the fixed component of $|L \otimes \mathcal{I}_\xi|$ is a singular plane cubic in the embedding given by L.

As mentioned in the introduction, the main goal of this work is to shed light on the global relationship between n, k and b in the case of a k-very ample line bundle. Theorem 22 and Proposition 25 give lower bounds for b respectively in terms of n and k. The fact that equality in Proposition 25 occurs for $n = 2$, see Remark 26, suggests $b \geq n + k - 1$ as a reasonable bound to expect. The following theorem proves the suggested bound for $n \geq 4$. Unfortunately, the information given by the same argument for $n = 3$ is unsatisfactory.

Theorem 30. *Let L be a k-very ample line bundle on a projective n-fold X, with $n \geq 3$ and $k \geq 2$. Assume there exists $\xi \in \mathcal{B}_b(X, L)$ such that $(H^0(L), \xi)$ is suitable.*

(a) *If $n \geq 4$ then $b(X, L) \geq n + k - 1$. Moreover, if equality holds, then $|L \otimes \mathcal{I}_\xi|$ has no fixed component for all $\xi \in \mathcal{B}_b(X, L)$ such that $(H^0(L), \xi)$ is suitable.*
(b) *If $n = 3$ and $b = k + 1$ then $k \geq 3$. Moreover, for all $\xi \in \mathcal{B}_b(X, L)$ such that $(H^0(L), \xi)$ is suitable, $|L \otimes \mathcal{I}_\xi|$ has no fixed component and every irreducible component of* $\mathrm{Bs}|L \otimes \mathcal{I}_\xi|$ *is a rational normal curve of degree k.*

Proof. As L is very ample, we identify any 0-scheme ξ with its image in the embedding φ_L and by the linear span of ξ we mean that of $\varphi_L(\xi)$.

From Proposition 25 it is $b = k + m$ with $m \geq 1$. Let $\xi \in \mathcal{B}_b(X, L)$ be such that $(H^0(L), \xi)$ is suitable and let Γ be a component of maximal dimension of $\mathrm{Bs}|L \otimes \mathcal{I}_\xi|$. Set $t = \dim \Gamma$. Remark 2 gives $t = n - 1$ or $n - 2$. It is also $\Gamma \subseteq \langle \xi \rangle = \mathbb{P}^{k+m-1}$ because of Lemma 18. Let $\langle \Gamma \rangle = \mathbb{P}^r$ so that $t \leq r$ and

$$r \leq k + m - 1. \tag{1}$$

Assume that

$$n \geq m + 2, \tag{2}$$

or

$$n = m + 1 \text{ and } t = n - 1. \quad (3)$$

In both cases it is:

$$t \geq m. \quad (4)$$

As $k \geq 2$, Γ cannot contain lines. Thus if Γ is a variety of minimal degree, i.e. $\deg \Gamma = \operatorname{codim}_{\langle \Gamma \rangle} \Gamma + 1 = r - t + 1$, then one of the following cases must occur [9, Theorem 1]:

(1) Γ is the Veronese surface in $\mathbb{P}^5$;
(2) Γ is a rational normal curve of degree r.

In the former case (1) implies $m \geq 4$ which contradicts (4). In the latter case it is $r \geq k$ as L is k-very ample. This fact combined with (1) and (4) implies that $r = k$, and $m = 1$, as $t = 1$. The assumption $n \geq 3$ then implies $n = 3$ as in part (b) of the statement.

In view of the above argument we can assume that $\deg \Gamma > r - t + 1$. Let λ be a zero-subscheme of Γ consisting of $r - t + 1$ linearly independent points. Then its linear span $\Lambda = \langle \lambda \rangle$ is a $\mathbb{P}^{r-t}$. As $\operatorname{length}(\Lambda \cap \Gamma) = \deg \Gamma > \operatorname{length}(\lambda)$, there exists a zero-scheme λ' of $\Gamma \cap \Lambda$ with $\operatorname{length}(\lambda') = \operatorname{length}(\lambda) + 1$ and in particular

$$|L \otimes \mathcal{I}_\lambda| = |L \otimes \mathcal{I}_{\lambda'}|. \quad (5)$$

Notice that (1) and (4) imply $r - t + 1 \leq k$, hence $\operatorname{length}(\lambda') \leq k + 1$, contradicting the k-very ampleness of L in view of (5).

Consequently neither of the assumptions (3) and (2) can hold unless the setting is as in part (b) of the statement, in which case Proposition 27 implies $k \geq 3$. Therefore if $n \geq 4$ and $n = m + 1 = b - k + 1$ then $t = n - 2$, i.e. $|L \otimes \mathcal{I}_\xi|$ has no fixed component for all $\xi \in \mathcal{B}_b(X, L)$, such that $(H^0(L), \xi)$ is suitable. Moreover, if $n \geq 4$ it must be $b - k = m > n - 2$ i.e. $b \geq n + k - 1$. This completes the proof of (a).

Remark 31. If $n = 2$, and $b = k + 1$, the k-very ampleness of L forces a potential fixed component Γ of $|L \otimes \mathcal{I}_\xi|$, for all $\xi \in \mathcal{B}_{k+1}(X, L)$, to be a rational normal curve of degree k. This can be seen by using a simple adaptation of the main argument of the proof of Theorem 30. There are no examples known to us of threefolds (X, L) with $b = k + 1$ and $k \geq 3$.

5. Reduced bad zero-schemes

Let L be k-very ample, $k \geq 1$, and assume that $\mathcal{B}_t^0(X, L)$ is not empty for some t. On the basis of [1,2], the naive approach would be to consider the blow-up $\pi : \tilde{X} \to X$ of X at one point in the t-th bad locus, hoping to obtain, inductively, a new pair, $(\tilde{X}, \pi^*(L) - E)$, polarized with a line bundle which is still very positive, and has a non-empty $\mathcal{B}_{t-1}^0$. This is, unfortunately, not a good strategy. The new polarized pair contains a linear $\mathbb{P}^{n-1}$, which is impossible for a k-very ample line bundle if $k \geq 2$. Nonetheless, proceeding with a little care in the same context, it is possible to obtain a new pair, polarized with a very ample line bundle admitting a non-empty $\mathcal{B}_2^0$.

The following example will shed more light on the situation.

Example 32. Let $(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$. Notice that L is 2-very ample. Let ℓ be any line in $\mathbb{P}^2$ and let x_1, x_2, x_3 be any three collinear points on ℓ. A conic through the x_is must contain ℓ and therefore be reducible. Thus the reduced zero-scheme $\xi = \{x_1, x_2, x_3\}$ is contained in $\mathcal{B}_3^0(X, L)$. This is an example of a k-very ample line bundle with non-empty $\mathcal{B}_{k+1}^0$. Notice that $|L \otimes \mathcal{I}_\xi|$ has ℓ as fixed component.

Let now $\pi_1 : \tilde{X}_1 \to X$ be the blow-up of X at x_1 with exceptional divisor E_1. It is $\mathcal{L}_1 = \pi_1^*(L) - E_1 = E_1 + 2f$ where f is the proper transform of a line in X through x_1. Thus $\operatorname{Pic}(\tilde{X}_1) = \mathbb{Z}[E_1] \oplus \mathbb{Z}[f]$. Notice that $\mathcal{L}_1$ is very ample and embeds $\tilde{X}_1$ in $\mathbb{P}^4$ as a rational normal scroll of degree 3. Let y_2 and y_3 on $\tilde{X}_1$ be such that $\pi_1(y_i) = x_i$. Then $\{y_2, y_3\} \in \mathcal{B}_2^0(\tilde{X}_1, \mathcal{L}_1)$. The situation here is exactly as described in [3, Theorem 1.7.9] with $\ell = f$ and $R = E_1 + f$.

Let now $\pi_2 : \tilde{X}_2 \to X$ be the blow-up of X at x_1 and x_2 with exceptional divisors E_1 and E_2. Consider the line bundle $\mathcal{L}_2 = \pi_2^*(L) - E_1 - E_2$. This time $\mathcal{L}_2$ is merely spanned and not ample. Let y_3 on $\tilde{X}_2$ be such that $\pi_2(y_3) = x_3$. Then $y_3 \in \mathcal{B}_1(\tilde{X}_2, \mathcal{L}_2)$. The divisor $\tilde{\ell} = \pi_2^*(\ell) - E_1 - E_2$ is a (-1)-curve through y_3, contracted by

$\varphi_{\mathcal{L}_2}$ to a smooth point on a smooth quadric surface. The situation here is exactly as described in [2, Theorem 1.1 case (d)]. Here $\mathcal{B}_1(\tilde{X}_2, \mathcal{L}_2) = \tilde{\ell}$.

Pairs (X, L) with L being k-very ample, whose b_0 index achieves the lower bound given in Proposition 25, turn out to be only 2-dimensional. As [3, Theorem 1.7.9] and Proposition 27 suggest, one would expect that L k-very ample and $b = k + 1$ should imply, for all $\xi \in \mathcal{B}_b(X, L)$, the existence of a fixed component for $|L \otimes \mathcal{I}_\xi|$ which is a rational normal curve of degree k. The following proposition gives the desired characterization assuming $b_0(X, L) = k + 1$, and $k \geq 2$.

Proposition 33. *Let L be a k-very ample line bundle on an n-dimensional manifold X with $k \geq 2$. Assume $b_0(X, L) = k + 1$. Then* $\dim X = 2$ *and for all $\xi \in \mathcal{B}^0_{k+1}$ the linear system $|L \otimes \mathcal{I}_\xi|$ has a smooth fixed component Γ, embedded by $|L|$ as a rational normal curve of degree k, such that $\xi \subset \Gamma$.*

Proof. Let $\xi = (x_1, \dots, x_{k+1}) \in \mathcal{B}^0_{k+1}(X, L)$. Let $\pi : \tilde{X} \to X$ be the blow-up of X at $x_1, \dots, x_{k-1}$ and $y_i = \pi^{-1}(x_{k+1-i})$ for $i = 1, 2$. Let $\eta = \{y_1, y_2\}$. Lemma 6 implies that $\mathcal{L} = \pi^*(L) - E_1 - \cdots - E_{k-1}$ is very ample. Because $|\mathcal{L} \otimes \mathcal{I}_\eta| = |L \otimes \mathcal{I}_\xi|$, it is $\eta \in \mathcal{B}^0_2(\tilde{X}, \mathcal{L})$. Therefore $(\tilde{X}, \mathcal{L})$ must be as in [3, Theorem 1.7.9], i.e. $\dim X = 2$ and there exists a line ℓ through η with $|\mathcal{L} \otimes \mathcal{I}_\eta| = \ell + |R|$ where $\mathcal{O}_X(R)$ is spanned. Let $\Gamma = \pi(\ell)$. Because L is k-very ample it must be $L \cdot \Gamma \geq k$, with equality holding only if $\Gamma \simeq \mathbb{P}^1$. Because ℓ and E_i are $\mathcal{L}$-lines, it must be $\nu_i := \ell \cdot E_i \in \{0, 1\}$. Therefore it is $1 = \mathcal{L} \cdot \ell = (\pi^*(\Gamma) - \sum_i \nu_i E_i) \cdot (\pi^*(L) - \sum_i E_i) = \Gamma \cdot L - \sum_i \nu_i \geq k - (k-1) = 1$ which implies $\Gamma \cdot L = k$, i.e. Γ is a smooth $\mathbb{P}^1$, embedded by $|L|$ as a rational normal curve of degree k. Moreover $\ell \cdot E_i = 1$ for all i, i.e. all x_i's belong to Γ.

Pairs (X, L) where L is k-very ample, for which $b_0 = k + 2$, can be further described under the assumption that $\dim X = 2$. The case $k = 1$ is analyzed first. The general case is then obtained from it.

Proposition 34. *Let L be a very ample line bundle on a surface X. Assume $b_0(X, L) = 3$. For all $\xi \in \mathcal{B}^0_3$, if $|L \otimes \mathcal{I}_\xi|$ does not have a fixed component then φ_L embeds X in $\mathbb{P}^N$, in such a way that there exists a linear $\mathbb{P}^2 \subset \mathbb{P}^N$, tangent to $\varphi_L(X)$ at the 3 points $\varphi_L(\mathrm{Supp}(\xi))$.*

Proof. Due to the assumption on the absence of a fixed component, $|L \otimes \mathcal{I}_\xi|$ has finite base locus. Hence, by Proposition 17, every $D \in |L \otimes \mathcal{I}_\xi|$ is of the form $D = A_{b_1} + \cdots + A_{b_r}$, $r \geq 2$, where all A_{b_j}'s belong to a rational pencil B. Moreover, if $\mathrm{Supp}(\xi) = \{x_1, x_2, x_3\}$, Proposition 17, (iii), implies that $r \geq 2$ and that every $D \in |L \otimes \mathcal{I}_\xi|$ has a point of multiplicity ≥ 2 at x_i for $i = 1, 2, 3$. This gives the following chain of equalities:

$$\bigcap_{i=1}^{3} |L \otimes \mathfrak{m}^2_{x_i}| = |L \otimes \mathcal{I}^2_\xi| = |L \otimes \mathcal{I}_\xi|.$$

By Lemma 18 the term on the right is a linear subspace of codimension 3 in $|L|$. On the other hand, each of the linear spaces $|L \otimes \mathfrak{m}^2_{x_i}|$ appearing on the left has codimension 3 in $|L|$, since L is very ample, see for example [10, Proposition 1.3 and Remark 2.3.3]. This means that the three linear subspaces $|L \otimes \mathfrak{m}^2_{x_i}|$ coincide, i.e.,

$$|L \otimes \mathfrak{m}^2_{x_1}| = |L \otimes \mathfrak{m}^2_{x_2}| = |L \otimes \mathfrak{m}^2_{x_3}|.$$

In other words, looking at X embedded by $|L|$, every hyperplane tangent to X at x_1 is tangent also at x_2 and x_3. Equivalently, X embedded by $|L|$ has the same embedded tangent plane at x_1, x_2, x_3.

Proposition 35. *Let $k \geq 1$ and L be a k-very ample line bundle on a surface X. Assume $b_0(X, L) = k + 2$. For all $\xi \in \mathcal{B}^0_{k+2}$, if $|L \otimes \mathcal{I}_\xi|$ does not have a fixed component then there exists a linear $\mathbb{P}^{k+1}$, tangent to $\varphi_L(X)$ at the $k + 2$ points $\varphi_L(\mathrm{Supp}(\xi))$.*

Proof. Let $\xi = \{x_1, \dots, x_{k+2}\} \in \mathcal{B}^0_{k+2}(X, L)$. Let $\eta \in X^{[k-1]}_{(1,\dots,1)}$ be any reduced zero-scheme obtained by choosing $k - 1$ of the $k + 2$ points of ξ. Let $\tau = \xi \setminus \eta = \{x, y, z\}$. Let $\pi : \tilde{X} \to X$ be the blow-up of X at the $k - 1$ points of η with exceptional divisors E_i, let $\mathcal{L} = \pi^*(L) - \sum_{i=1}^{k-1} E_i$, and let $\tilde{\tau} = \{\tilde{x}, \tilde{y}, \tilde{z}\} = \pi^{-1}(\tau)$. According to Lemma 6, $\mathcal{L}$ is very ample on $\tilde{X}$ and the proof of Proposition 34 gives $|\mathcal{L} \otimes \mathcal{I}_{\tilde{\tau}}| = |\mathcal{L} - 2\tilde{x} - 2\tilde{y} - 2\tilde{z}|$. As $|L \otimes \mathcal{I}_\eta| = |\mathcal{L}|$, we have $|L \otimes \mathcal{I}_\xi| = |L \otimes \mathcal{I}_\eta \otimes \mathcal{I}^2_\tau|$. Because the last equality is true no matter how η was chosen, it follows that $|L \otimes \mathcal{I}_\eta| = |L \otimes \mathcal{I}^2_\eta|$. This means that the $\mathbb{P}^{k+1}$ spanned by $\varphi_L(\xi)$ is tangent to $\varphi_L(X)$ at the $k+2$ points $\varphi_L(\mathrm{Supp}(\xi))$.

6. Bad linear spaces

As mentioned in the introduction, one may view a bad point as a *bad linear space* of codimension two, as Andrew Sommese suggested to the first author. In this section we adopt this point of view. After introducing a natural definition of bad linear spaces we show that they must necessarily have codimension two and that they are inherited by hyperplane sections. These two facts are combined to show that bad linear spaces of very ample linear systems do not occur at all.

Definition 36. Let X be a smooth projective n-fold, $n \geq 2$, and let L be an ample line bundle on X spanned by $V \subseteq H^0(X, L)$. Let $\Lambda \subset X$ be an L-linear subspace of codimension ≥ 2, i.e. $(\Lambda, L_{|\Lambda}) = (\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$, for some $r \leq n-2$. Let $\mathcal{I}_\Lambda$ be the ideal sheaf of Λ. We say that Λ is a *bad linear space* for (X, V) if for all $D \in |V \otimes \mathcal{I}_\Lambda|$, D is reducible or non-reduced.

Lemma 37. *Let X be a smooth projective n-fold, $n \geq 2$, and let L be an ample line bundle on X, spanned by a subspace $V \subseteq H^0(X, L)$. If Λ is a bad linear space of (X, V), then* $\mathrm{codim}_X(\Lambda) = 2$.

Proof. Let ξ be a zero-scheme on X consisting of $r+1$ distinct points on Λ, not lying on an L-hyperplane of Λ, so that $|V \otimes \mathcal{I}_\Lambda| = |V \otimes \mathcal{I}_\xi|$ with ξ imposing $r+1$ independent conditions on V. Remark 3 gives $\dim(\mathrm{Bs}|V \otimes \mathcal{I}_\xi|) \leq r$. On the other hand $\Lambda \subset \mathrm{Bs}|V \otimes \mathcal{I}_\xi|$, hence $\dim(\mathrm{Bs}|V \otimes \mathcal{I}_\Lambda|) = r$. In particular, as $r \leq n-2$, $|V \otimes \mathcal{I}_\Lambda|$ has no fixed component. Thus it follows from Remark 2 that $r = \dim \mathrm{Bs}|V \otimes \mathcal{I}_\Lambda| = n-2$.

The following proposition shows that bad linear spaces are inherited by hyperplane sections. To see this, let (X, L) and Λ be as above. Let $x \in \Lambda$. As V spans L, there exists a smooth $Y \in |V|$ not passing through x and in particular $\Lambda \not\subset Y$. Then $\lambda := \Lambda \cap Y$ is an $L_{|Y}$-hyperplane of Y. Let W be the image of V under the restriction homomorphism $H^0(X, L) \to H^0(Y, L_{|Y})$.

Proposition 38. *Let notation be as above. If Λ is a bad linear space for (X, V), then λ is a bad linear space for (Y, W).*

Proof. Let $\rho : V \to W$ be the homomorphism induced by the restriction $H^0(X, L) \to H^0(Y, L_{|Y})$. Clearly ρ is a surjection and its kernel is $\mathbb{C}\langle s_0 \rangle$, where $s_0 \in V$ is a non-trivial section vanishing on Y. So, $\dim(W) = \dim(V) - 1$. Note that $\lambda = \mathbb{P}^{n-3}$, by Lemma 37 and $\rho(V \otimes \mathcal{I}_\Lambda) \subseteq W \otimes \mathcal{I}_\lambda$. Moreover, $\mathrm{Ker}(\rho) \cap (V \otimes \mathcal{I}_\Lambda) = \{0\}$, because any non-trivial element of $\mathrm{Ker}(\rho)$ vanishes exactly on Y, hence it cannot vanish on Λ. Therefore the homomorphism

$$\rho_{|V \otimes \mathcal{I}_\Lambda} : V \otimes \mathcal{I}_\Lambda \to W \otimes \mathcal{I}_\lambda \tag{6}$$

is an injection. On the other hand

$$\begin{aligned} \dim(V \otimes \mathcal{I}_\Lambda) &= \dim(V) - (\dim(\Lambda) + 1) \\ &= \dim(W) - (\dim(\lambda) + 1) \\ &= \dim(W \otimes \mathcal{I}_\lambda). \end{aligned}$$

Hence (6) is an isomorphism, which gives the assertion.

Theorem 39. *Let X be a smooth projective variety of dimension $n \geq 2$ and let $|V|$ be a very ample linear system on X. Then (X, V) cannot contain bad linear spaces.*

Proof. Let S be the smooth surface cut out by $(n-2)$ general elements of $|V|$ and let $|U|$ be the trace of $|V|$ on S. Note that the corresponding linear system $|U|$ is very ample on S. Now, by contradiction let Λ be a bad linear space for (X, V). By an inductive application of Proposition 38 and Lemma 37 we conclude that $p := \Lambda \cap S$ is a bad point of (S, U). This contradicts [1, Corollary 2, (ii)].

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