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On strong solutions for positive definite jump diffusions

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Abstract

We show the existence of unique global strong solutions of a class of stochastic differential equations on the cone of symmetric positive definite matrices. Our result includes affine diffusion processes and therefore extends considerably the known statements concerning Wishart processes, which have recently been extensively employed in financial mathematics.

Moreover, we consider stochastic differential equations where the diffusion coefficient is given by the αth positive semidefinite power of the process itself with $0.5 < \alpha < 1$ and obtain existence conditions for them. In the case of a diffusion coefficient which is linear in the process we likewise get a positive definite analogue of the univariate GARCH diffusions.

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1. Introduction

A result of the general theory for affine Markov processes on the cone S_d^+ of symmetric positive semidefinite matrices developed in [13] is that for a $d \times d$ matrix-valued standard

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Brownian motion B, $d \times d$ matrices Q and β, a symmetric constant drift b, and a positive linear drift $\Gamma : S_d^+ \to S_d^+$, *weak global solutions* exist for the stochastic differential equation (SDE)

$$dX_t = \sqrt{X_t} dB_t Q + Q^\top dB_t^\top \sqrt{X_t} + (X_t \beta + \beta^\top X_t + \Gamma(X_t) + b)dt, \qquad (1.1)$$
$$X_0 = x \in S_d^+,$$

whenever $b - (d-1)Q^\top Q \in S_d^+$. Above, $\sqrt{X}$ denotes the unique positive semidefinite square root of a matrix $X \in S_d^+$. For $\Gamma = 0$, solutions to the SDE (1.1) are called Wishart processes and their existence has been considered in detail in the fundamental paper by Bru [8]. Further probabilistic investigations on properties of Wishart processes have been carried out in [19,20,26], for instance, and references therein.

In the present paper, we focus on the existence of global *strong solutions* of (1.1) and generalisations of it including jumps and more general diffusion coefficients. Because of the non-Lipschitz diffusion at the boundary of the cone, this problem is a quite delicate one – a priori it is only clear that a unique local solution of (1.1) exists until X_t hits the boundary of S_d^+, since the SDE is locally Lipschitz in the interior of S_d^+. Furthermore, known results for pathwise uniqueness, for instance, that of the seminal paper of Yamada and Watanabe [45, Corollary 3], are essentially one dimensional, and therefore do not apply. Hence, the present setting seems to be more complicated than, for instance, the canonical affine one (concerning diffusions on $\mathbb{R}_+^m \times \mathbb{R}^n$, [23, Lemma 8.2]).

Positive semidefinite matrix-valued processes are increasingly used in finance, particularly for stochastic modelling of multivariate stochastic volatility phenomena in equity and fixed income models; see [9,10,14–17,24,25,27,40]. See also [13] and the references therein. Most papers mentioned use Bru's class of Wishart diffusions, as this results in multivariate analogues of the popular Heston stochastic volatility model and its extensions, or Ornstein–Uhlenbeck type processes [40] giving a multivariate generalisation of the popular model of [3] or a combination of these [31]. This motivated the research of [13] on positive semidefinite affine processes including all the aforementioned models and generalising the results of [21], which covered all of these models in the univariate setting. Appropriate multivariate models are especially important for issues like portfolio optimisation, portfolio risk management and the pricing of options depending on several underlyings, which are heavily influenced by the dependence structure.

Clearly S_d^+-valued processes model the covariances, not the correlations, which are, however, preferable when interpreting the dependence structure. The results of the present paper are particularly relevant when one wants to derive correlation dynamics (see, e.g., [9,10]), because one needs to assume boundary non-attainment conditions for a rigorous derivation.

The name "Wishart process" is, unfortunately, not always used in the same way in the literature. We follow the above cited applied papers in finance and call any solution to (1.1) with $\Gamma = 0$ a "Wishart process" whereas in most of the previous probabilistic literature a "Wishart process" also requires $\beta = 0$ and the "Wishart processes with drift" of [20] are not even special cases of our "Wishart processes". For $\Gamma = \beta = 0$ and $b = nQ^T Q$ with $n \in \mathbb{N}$ one may also speak of a "squared Ornstein–Uhlenbeck process". In the univariate case the name "Wishart process" is not used; instead one typically uses the name "Cox–Ingersoll–Ross process" in the financial literature and "squared Bessel process" in the probability literature.

However, in this paper we do not limit ourselves to the analysis of (1.1). Instead we consider, as a special case of a considerably more general result, a similar SDE allowing for a general (not necessarily linear) drift Γ and an additional jump part of finite variation. This implies that many Lévy-driven SDEs on S_d^+ like the positive semidefinite Ornstein–Uhlenbeck (OU) type processes (see [4,39]) or the volatility process of a multivariate COGARCH process (see [43]), where the existence of global strong solutions has previously been shown by pathwise arguments, are special cases of our setting. Thus our results allow us to consider certain "jump diffusions" (in the sense of [12]), namely mixtures of such jump processes and Wishart diffusions, in applications.

It should be noted that [8] also contains results on strong solutions for Wishart processes (see our upcoming Proposition 3.1 and Remark 4.8); however, they are derived under strong parametric restrictions, because her method requires an application of Girsanov's theorem. The latter is based on a martingale criterion, which in the matrix-valued setting seems hard to verify. Also, the general result (with a non-vanishing linear drift) only holds until the first time when two of the eigenvalues of the process collide. Our approach generalises her method of proof for the case $\beta = 0$ (vanishing linear drift) and avoids change of measure techniques.

The most general result of our paper, Theorem 3.4, also opens the way to using positive semidefinite extensions of the univariate GARCH diffusions of [36] or of so-called generalised Cox–Ingersoll–Ross models (cf. e.g. [6,22]), where the square root in the diffusion part of (1.1) is replaced by the αth positive semidefinite power with $\alpha \in [1/2, 1]$ (see Corollary 3.5).

The remainder of the paper is structured as follows. In the next section we summarise some notation and preliminaries. In Section 3 we state our main result, Theorem 3.4, and its corollaries applying to Wishart processes, and matrix-variate generalised Cox–Ingersoll–Ross and GARCH diffusions. Moreover, we compare our results to the work of Bru which is recalled in Proposition 3.1. In the following section we gradually develop the proof of our result. Our method relies on a generalisation of the so-called *McKean argument*, but avoids the use of Girsanov's theorem. In Section 4.1 we thus provide a self-contained proof of a generalisation of *McKean's argument* and then deliver the proof of Theorem 3.4 in Section 4.2. We conclude the paper with some final remarks in Section 5.

2. Notation and the general set-up

We assume given an appropriate filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}_+})$ satisfying the usual hypotheses (complete and right-continuous filtration) and rich enough to support all processes occurring. For short, we sometimes write just Ω when actually referring to this filtered probability space. B is a $d \times d$ standard Brownian motion on Ω and $d \in \mathbb{N}$ always denotes the dimension. Furthermore, we use the following notation, definitions and setting:

- $\mathbb{R}_+ := [0, \infty)$, M_d is the set of real-valued $d \times d$ matrices and I_d is the identity matrix.
- $S_d \subset M_d$ is the space of symmetric matrices, and $S_d^+ \subset S_d$ is the cone of symmetric positive semidefinite matrices in S_d and S_d^{++} its interior, i.e. the positive definite matrices. The partial order on S_d induced by the cone is denoted by $\preceq$, and $x \succ 0$, if and only if $x \in S_d^{++}$. We endow S_d with the scalar product $\langle x, y\rangle := \mathrm{Tr}(xy)$, where $\mathrm{Tr}(A)$ denotes the trace of $A \in M_d$. $\|\cdot\|$ denotes the associated norm, and $d(x, \partial S_d^+) = \inf_{y\in\partial S_d^+} \|x - y\|$ is the distance from $x \in S_d^+$ to the boundary ∂S_d^+.
- The usual tensor (Kronecker) product of two matrices A, B is denoted by $A \otimes B$ and the vectorisation operator mapping M_d to $\mathbb{R}^{d^2}$ by stacking the columns of a matrix A below each other is denoted by $\mathrm{vec}(A)$ (see [29, Chapter 4] for more details).
- A function $f : S_d^{++} \to U$ with U being (a subset of) a normed space is called *locally Lipschitz* if $\|f(x) - f(y)\| \le K(C)\|x - y\| \,\forall x, y \in C$ for all compacts $C \subset U$. f is said to have *linear growth* if $\|f(x)\|^2 \le K(1 + \|x\|^2) \,\forall x \in S_d^{++}$.

- An S_d-valued càdlàg adapted stochastic process X is called S_d^+-increasing if $X_t \succeq X_s$ a.s. for all $t > s \geq 0$. Such a process is necessarily of finite variation on compacts by [4, Lemma 5.21] and hence a semimartingale. We call it of *pure jump type* provided $X_t = X_0 + \sum_{0<s\leq t} \Delta X_s$, where $\Delta X_s = X_s - X_{s-}$.

For the necessary background on stochastic analysis we refer the reader to one of the standard references like [30,41,42]. Moreover, we frequently employ stochastic integrals where the integrands or integrators are matrix or even linear operator valued. Thus, we briefly explain how they have to be understood. Let $(A_t)_{t\in\mathbb{R}^+}$ in M_d, $(B_t)_{t\in\mathbb{R}^+}$ in M_d be càdlàg and adapted processes and $(L_t)_{t\in\mathbb{R}^+}$ in M_d be a semimartingale (i.e. each element is a semimartingale). Then we denote by $\int_0^t A_{s-} dL_s B_{s-}$ the matrix C_t in M_d which has ijth element $C_{ij,t} = \sum_{k=1}^d \sum_{l=1}^d \int_0^t A_{ik,s-} B_{lj,s-} dL_{kl,s}$. Equivalently such an integral can be understood in the sense of [35] by identifying it with the integral $\int_0^t \mathbf{A}_{s-} dL_s$ with $\mathbf{A}_t$ being for each fixed t the linear operator $M_d \to M_d, X \mapsto A_t X B_t$ and L being a semimartingale in the Hilbert space M_d. Stochastic integrals of the form $\int_0^t K(X_{s-}) dJ_s$ with J being a semimartingale in M_d (coordinatewise or equivalently as in [35, Section 10] where the equivalence easily follows from [35, Section 10.9] and by noting that on a finite dimensional Hilbert space all norms are equivalent) and $K(x) : M_d \to M_d$ a linear operator for all x can be understood again as in [35]. Alternatively, one can equivalently identify M_d with $\mathbb{R}^{d^2}$ using the vec-operator and $K(x)$ with a matrix in M_{d^2,d^2} and then define the stochastic integral coordinatewise as above.

3. The statement of the main results

3.1. Wishart diffusions with jumps

In order to illustrate the context of our result and, because it is of most relevance in applications, we discuss first the special case of Wishart diffusions with jumps. For $Q \in M_d$, $\delta > d - 1$, $\beta \in M_d$ and an M_d-valued standard Brownian motion B, a Wishart process is the strong solution of the equation

$$\begin{aligned} dX_t &= \sqrt{X_t} dB_t Q + Q^\top dB_t^\top \sqrt{X_t} + (X_t \beta + \beta^\top X_t + \delta Q^\top Q) dt, \\ X_0 &= x \in S_d^{++}, \end{aligned} \tag{3.1}$$

on the maximal stochastic interval $[0, T_x)$, where T_x is naturally defined as

$$T_x = \inf\{t > 0 : X_t \in \partial S_d^+\}.$$

That such a unique local strong solution, which does not explode before or at time T_x, exists follows from standard SDE theory, since all the coefficients in (3.1) are locally Lipschitz and of linear growth on S_d^{++}. To be more precise, this follows by appropriately localising the usual results as e.g. in [41, Chapter V] or by variations of the proofs in [35, Chapter 3]. A localisation procedure adapted particularly to certain convex sets like S_d^+ is presented in detail in [44, Section 6.7].

The following is a summary of the results [8, Theorem 2, 2' and 2''] – the, to the best of our knowledge, only known results regarding strong existence of Wishart processes:

Proposition 3.1. *Suppose that $\delta \geq d + 1$.*

(i) *If $Q = I_d$ and $\beta = 0$, then $T_x = \infty$.*
Suppose additionally that the d eigenvalues of x are distinct.

(ii) *If* $Q \in S_d^{++}$, *and* $-\beta \in S_d^+$ *such that* β *and* Q *commute, then there exists a solution* $(X_t)_{t\in\mathbb{R}_+}$ *of* (3.1) *until the first time* τ_x *when two of the eigenvalues of* X_t *collide.*

(iii) *If* $\beta = \beta_0 I_d$ *and* $Q = \gamma I_d$, *where* $\beta_0, \gamma \in \mathbb{R}$, *then* $T_x = \infty$ *for the solution of* $(X_t)_{t\in\mathbb{R}_+}$ *of* (3.1).

Consequently, for the respective choices of parameters, there exist unique global strong S_d^{++}*-valued solutions of the SDE* (3.1) *on* $[0, \tau_x)$ *or on all of* $[0, \infty)$.

The upcoming general Theorem 3.4 implies the following result for a generalisation of the Wishart SDE allowing for additional jumps and a non-linear drift Γ.

Corollary 3.2. *Suppose that* $b \in S_d$, $Q \in M_d$, $\beta \in M_d$, *and let*

- J *be an* S_d*-valued càdlàg adapted process which is* S_d^+*-increasing and of pure jump type,*
- $\Gamma : S_d^{++} \to S_d^+$ *be a locally Lipschitz function of linear growth and*
- $K : S_d^{++} \to L(S_d^+, S_d^+)$ *(the linear operators on* S_d *mapping* S_d^+ *into* S_d^+*) be a locally Lipschitz function of linear growth.*

If $b \succeq (d+1)Q^\top Q$, *then the SDE*

$$\begin{aligned} dX_t &= \sqrt{X_{t-}}dB_tQ + Q^\top dB_t^\top \sqrt{X_{t-}} \\ &\quad + (X_{t-}\beta + \beta^\top X_{t-} + \Gamma(X_{t-}) + b)dt + K(X_{t-})dJ_t, \\ X_0 &= x \in S_d^{++}, \end{aligned} \tag{3.2}$$

has a unique adapted càdlàg global strong solution $(X_t)_{t\in\mathbb{R}_+}$ *on* S_d^{++}. *In particular we have* $T_x := \inf\{t \ge 0 : X_{t-} \in \partial S_d^+ \text{ or } X_t \notin S_d^{++}\} = \inf\{t \ge 0 : X_{t-} \in \partial S_d^+\} = \infty$ *almost surely.*

Proof. For the term on the right hand side of the upcoming condition (3.3) we obtain

$$\mathrm{Tr}(2\beta) + \mathrm{Tr}(\Gamma(x)x^{-1}) + \mathrm{Tr}((b-(d+1)Q^\top Q)x^{-1}) \ge 2\,\mathrm{Tr}(\beta),$$

noting that x^{-1}, $\Gamma(x)$ and $b-(d+1)Q^\top Q$ are positive semidefinite and that S_d^+ is a self-dual cone, which implies that $\mathrm{Tr}(zy) \ge 0$ for any $z, y \in S_d^+$. Setting $c(t) = 2\,\mathrm{Tr}(\beta)$, an application of Theorem 3.4 concludes. □

By choosing Γ linear and $J = 0$, we obtain a result for (1.1) which considerably generalises Proposition 3.1.

Remark 3.3. (i) In the univariate case the condition $b \succeq (d+1)Q^\top Q$ is known to be also necessary for boundary non-attainment (see [42, Chapter XI]).

(ii) A possible choice for J is a matrix subordinator without drift (see [2]), i.e. an S_d^+-increasing Lévy process. On choosing $\Gamma \neq 0$ in (3.2) appropriately, our results also apply to SDEs involving matrix subordinators with a non-vanishing drift.

(iii) Setting $Q = 0$, $\Gamma = 0$, K to the identity and b equal to the drift of the matrix subordinator, Eq. (3.2) becomes the SDE of a positive definite OU type process [4,39]. Likewise, it is straightforward to see that the SDE of the volatility process Y of the multivariate COGARCH process of [43] is a special case of (3.2).

(iv) An OU type process on the positive semidefinite matrices is necessarily driven by a Lévy process of finite variation having positive semidefinite jumps only (this follows by slightly adapting the arguments in the proof of [39, Theorem 4.9]). This entails that a generalisation of the above result to a more general jump behaviour requires additional technical restrictions.

3.2. The general SDE and existence result

The main result of this paper is the following general theorem concerning non-attainment of the boundary of S_d^+ and the existence of a unique global strong solution for a generalisation of the SDE (1.1). The proof of this result is gradually developed in the next sections.

Theorem 3.4. *Let*

- $F, G : \mathbb{R}_+ \times S_d^{++} \to M_d$ *be functions such that* $G^\top \otimes F$ *given by* $G^\top \otimes F(t,x) = (G(t,x))^\top \otimes F(t,x)$ *is locally Lipschitz and of linear growth,*
- $H : \mathbb{R}_+ \times S_d^{++} \to S_d$ *be locally Lipschitz and of linear growth,*
- J *be an* S_d*-valued càdlàg adapted process which is* S_d^+*-increasing and of pure jump type, and*
- $K : S_d^{++} \to L(S_d^+, S_d^+)$ *(the linear operators on* S_d *mapping* S_d^+ *into* S_d^+*) be a locally Lipschitz function of linear growth.*

Suppose that there exists a function $c : \mathbb{R}_+ \to \mathbb{R}$ *which is locally integrable, i.e.* $\int_0^s |c(t)|dt < \infty$ *for all* $s \in \mathbb{R}^+$*, such that*

$$c(t) \le \mathrm{Tr}(H(t,x)x^{-1}) - \mathrm{Tr}(f(t,x)x^{-1})\mathrm{Tr}(g(t,x)x^{-1}) - \mathrm{Tr}(f(t,x)x^{-1}g(t,x)x^{-1}) \tag{3.3}$$

for all $x \in S_d^{++}$ *and* $t \in \mathbb{R}_+$ *where* $f(t,x) := F(t,x)F(t,x)^\top$, $g(t,x) = G(t,x)^\top G(t,x)$.

Then the SDE

$$\begin{aligned} dX_t &= F(t,X_{t-})dB_t G(t,X_{t-}) + G(t,X_{t-})^\top dB_t^\top F(t,X_{t-})^\top \\ &\quad + H(t,X_{t-})dt + K(X_{t-})dJ_t, \\ X_0 &= x \in S_d^{++}, \end{aligned} \tag{3.4}$$

has a unique adapted càdlàg global strong solution $(X_t)_{t\in\mathbb{R}_+}$ *on* S_d^{++}.

In particular, we have $T_x := \inf\{t \ge 0 : X_{t-} \in \partial S_d^+ \text{ or } X_t \notin S_d^{++}\} = \inf\{t \ge 0 : X_{t-} \in \partial S_d^+\} = \infty$ *almost surely.*

3.3. Positive definite extensions of generalised Cox–Ingersoll–Ross processes and GARCH diffusions

In the univariate case, generalised Cox–Ingersoll–Ross (GCIR) processes given by the SDE $dx_t = (b + ax_t)dt + qx_t^\alpha dB_t$ with $b \ge 0, q > 0, a \in \mathbb{R}$ and $\alpha \in [1/2, 1]$ are – as discussed in the introduction – of relevance in financial modelling. $\alpha = 1/2$ corresponds, of course, to the already discussed Bessel case, whereas $\alpha = 1$ gives the so-called GARCH diffusions. Given the popularity of the Wishart based models in modern finance, it seems natural to consider also positive semidefinite extensions of the GCIR processes. An application of our general theorem to the case where $F(X) = X^\alpha$, $G(X) = Q$ with $\alpha \in [1/2, 1]$ yields:

Corollary 3.5. (i) *Suppose that* $\alpha \in [1/2, 1]$, $b \in S_d$, $Q \in M_d$, $\beta \in M_d$, *and let*

- J *be an* S_d*-valued càdlàg adapted process which is* S_d^+*-increasing and of pure jump type,*
- $\Gamma : S_d^{++} \to S_d^+$ *be a locally Lipschitz function of linear growth and*
- $K : S_d^{++} \to L(S_d^+, S_d^+)$ *(the linear operators on* S_d *mapping* S_d^+ *into* S_d^+*) be a locally Lipschitz function of linear growth.*

Suppose that for all $x \in S_d^{++}$

$$\operatorname{Tr}(\Gamma(x)x^{-1} + bx^{-1}) \geq \operatorname{Tr}(x^{2\alpha-1})\operatorname{Tr}(Q^\top Q x^{-1}) + \operatorname{Tr}(x^{2\alpha-2}Q^\top Q). \tag{3.5}$$

Then the SDE

$$\begin{aligned} dX_t &= X_{t-}^\alpha B_t Q + Q^\top dB_t^\top X_{t-}^\alpha + (X_{t-}\beta + \beta^\top X_{t-} + \Gamma(X_{t-}) + b)dt \\ &\quad + K(X_{t-})dJ_t, \\ X_0 &= x \in S_d^{++}, \end{aligned} \tag{3.6}$$

has a unique adapted càdlàg global strong solution $(X_t)_{t\in\mathbb{R}_+}$ *on* S_d^{++}. *In particular we have* $T_x := \inf\{t \geq 0 : X_{t-} \in \partial S_d^+ \text{ or } X_t \notin S_d^{++}\} = \inf\{t \geq 0 : X_{t-} \in \partial S_d^+\} = \infty$ *almost surely.*

(ii) *Any of the following sets of conditions implies* (3.5)*:*

(a) $b + \Gamma(x) \succeq \operatorname{Tr}(x^{2\alpha-1})Q^\top Q + x^{\alpha-1/2}Q^\top Q x^{\alpha-1/2}$ *for all* $x \in S_d^{++}$.
(b) $b + \Gamma(x) \succeq \operatorname{Tr}(x^{2\alpha-1})Q^\top Q + \lambda_{Q^\top Q}x^{2\alpha-1}$ *for all* $x \in S_d^{++}$ *with* $\lambda_{Q^\top Q}$ *denoting the largest eigenvalue of* $Q^\top Q$.
(c) $\alpha = 1$ *and* $b + \Gamma(x) \succeq \operatorname{Tr}(x)Q^\top Q + \lambda_{Q^\top Q}x$ *for all* $x \in S_d^{++}$.
(d) $b \succeq 0$ *and* $\Gamma(x) \succeq 2\operatorname{Tr}(x^{2\alpha-1})Q^\top Q$ *for all* $x \in S_d^{++}$.
(e) $b \succeq 0$ *and* $\Gamma(x) \succeq 2\left(\operatorname{Tr}(x) + d(2\alpha-1)^{2-2\alpha}\right)Q^\top Q$ *for all* $x \in S_d^{++}$ *(and setting* $0^0 := 1$ *for* $\alpha = 1/2$*).*
(f) $b \succeq 0$ *and* $\Gamma(x) \succeq 2(\operatorname{Tr}(x) + d)Q^\top Q$ *for all* $x \in S_d^{++}$.
(g) $\alpha > 1/2$, $d = 1$, $\Gamma(x) \geq 0$ *for all* $x \in \mathbb{R}_+$ *and* $b > 0$.

Proof. One easily calculates the right hand side of (3.3) to be equal to $\operatorname{Tr}(2\beta + \Gamma(x)x^{-1} + bx^{-1}) - \operatorname{Tr}(x^{2\alpha-1})\operatorname{Tr}(Q^\top Q x^{-1}) - \operatorname{Tr}(x^{2\alpha-2}Q^\top Q)$ and hence (i) follows from Theorem 3.4.

Turning to the proof of (ii), using the self-duality of S_d^+ as in the proof of Corollary 3.2 gives (a). Next we observe that $Q^\top Q \preceq \lambda_{Q^\top Q} I_d$ and, hence, $x^{\alpha-1/2}Q^\top Q x^{\alpha-1/2} \preceq \lambda_{Q^\top Q}x^{2\alpha-1}$. This gives (b), and (c) is simply the special case for $\alpha = 1$.

Since for $A, B \in S_d^+$ we have that $\operatorname{Tr}(AB) \leq \operatorname{Tr}(A)\operatorname{Tr}(B)$ due to the Cauchy–Schwarz inequality and the elementary inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for all $a, b \in \mathbb{R}_+$, we have that $\operatorname{Tr}(x^{2\alpha-2}Q^\top Q) \leq \operatorname{Tr}(x^{2\alpha-1})\operatorname{Tr}(Q^\top Q x^{-1})$. Hence, (3.5) is implied by $\operatorname{Tr}(\Gamma(x)x^{-1} + bx^{-1}) \geq 2\operatorname{Tr}(x^{2\alpha-1})\operatorname{Tr}(Q^\top Q x^{-1})$. Using, once again, the self-duality gives (d).

Since the trace is the sum of the eigenvalues, $\lambda \geq \lambda^{2\alpha-1}$ for all $\lambda \geq 1$ and $\alpha \in [1/2, 1]$ and $\lambda^{2\alpha-1} \leq \lambda + \max_{\lambda\in[0,1]}\left\{\lambda^{2\alpha-1} - \lambda\right\}$ for all $\lambda \in [0, 1)$ and $\alpha \in [1/2, 1]$, we immediately obtain (e) from (d), because $\max_{\lambda\in[0,1]}\left\{\lambda^{2\alpha-1} - \lambda\right\} = (2\alpha-1)^{2-2\alpha}$. In turn (f) follows from (e) noting that $\max_{\lambda\in[0,1]}\left\{\lambda^{2\alpha-1} - \lambda\right\} \in [0, 1]$.

Turning to (g) we have for the right hand side of (3.3) in the univariate case

$$\ell(x) = 2\beta + \Gamma(x)/x + b/x - 2Q^2/x^{2-2\alpha}.$$

Now one notes that the second term is non-negative and that for $b > 0$ the term $b/x - 2Q^2/x^{2-2\alpha}$ is bounded from below on $\mathbb{R}^+$, because $\lim_{x\to 0,\, x>0} x^{-1}/x^{2\alpha-2} = \infty$. Hence, Corollary 3.2concludes. □

In the different cases of (ii) a valid choice of b and Γ is always obtained by taking them equal to the right hand side of the inequalities. It should be noted that (c) shows that in the positive semidefinite GARCH diffusion generalisation, one can always take a linear drift. Likewise, (e) and (f) show that a linear drift is possible for the generalised CIR. For $\alpha = 1/2$ the case (d) is again sharp in the univariate setting, but for general dimensions it is a stronger condition than the one given in Corollary 3.2.

The last case (g) in particular recovers the well-known univariate result for $dx_t = (b + ax_t)dt + qx_t^\alpha dB_t$ with $b \geq 0, q > 0, a \in \mathbb{R}$ and $\alpha \in [1/2, 1]$. In our matrix-variate case for $\alpha > 1/2$ a result similar to the univariate one, namely that a strictly positive constant drift is all that is needed to ensure boundary non-attainment, seems to be out of reach. If one tried to use arguments similar to (e) in general, one would need something like $\mathrm{Tr}(bx^{-1}) \geq k\mathrm{Tr}(x^{2\alpha-1})\mathrm{Tr}(Q^\top Qx^{-1}) + K$ with some constants $k > 0$ and K to ensure (3.5). However, when the process comes close to the boundary of the cone, this only means that at least one eigenvalue gets close to zero. Hence, $\mathrm{Tr}(bx^{-1})$ and $\mathrm{Tr}(Q^\top Qx^{-1})$ should then go to infinity at a comparable rate. However, all the other eigenvalues of x may still be arbitrarily large and so there is no appropriate upper bound on the term $\mathrm{Tr}(x^{2\alpha-1})$.

4. Proofs

In this section we gradually prove our main result. As a priori all processes involved are only defined up to a stopping time, we collect first some basic definitions regarding stochastic processes defined on stochastic intervals following mainly [33].

Definition 4.1. Suppose that $A \in \mathcal{F}$ and let T be a stopping time.

- A random variable X on A is a mapping $A \to \mathbb{R}$ which is measurable with respect to the σ-algebra $A \cap \mathcal{F}$.
- A family $(X_t)_{t\in\mathbb{R}_+}$ of random variables on $\{t < T\}$ is called a stochastic process on $[0, T)$. If X_t is $\{t < T\} \cap \mathcal{F}_t$-measurable for all $t \in \mathbb{R}_+$, then X is said to be adapted.
- An adapted process M on $[0, T)$ is called a continuous local martingale on the interval $[0, T)$ if there exists an increasing sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ and a sequence of continuous martingales $(M^{(n)})_{n\in\mathbb{N}}$ (in the usual sense on $[0, \infty)$) such that $\lim_{n\to\infty} T_n = T$ a.s. and $M_t = M_t^{(n)}$ on $\{t < T_n\}$. Other local properties for adapted processes on $[0, T)$ are defined likewise.
- A semimartingale on $[0, T)$ is the sum of a càdlàg local martingale on $[0, T)$ and an adapted càdlàg process of locally finite variation on $[0, T)$.
- For a continuous local martingale on $[0, T)$, the quadratic variation is the $\mathbb{R} \cup \{\infty\}$-valued stochastic process $[M, M]$ defined by

$$[M, M]_t = \sup_{n\in\mathbb{N}} [M^{(n)}, M^{(n)}]_{t\wedge T_n} \quad \text{for all } t \in \mathbb{R}_+.$$

4.1. McKean's argument

In this section we finally establish Proposition 4.3 which generalises an argument of [34, p. 47, Problem 7] concerning continuous local martingales on stochastic intervals used, for instance, in [7,8,37]. We keep the tradition of referring to it as *McKean's argument*. Since it may also be helpful in other situations, we state our result and its proof in detail.

Lemma 4.2. *Let M be a continuous local martingale on a stochastic interval $[0, T)$. Then on $\{T > 0\}$ it holds almost surely that either $\lim_{t\uparrow T} M_t$ exists in $\mathbb{R}$ or that $\limsup_{t\uparrow T} M_t = -\liminf_{t\uparrow T} M_t = \infty$.*

Proof. Combine [33, Theorem 3.5] with analogous arguments of the proof of [42, Chapter V, Proposition 1.8]. □

Proposition 4.3 (*McKean's Argument*). *Let $Z = (Z_s)_{s\in\mathbb{R}_+}$ be an adapted càdlàg $\mathbb{R}^+\setminus\{0\}$-valued stochastic process on a stochastic interval $[0, \tau_0)$ such that $Z_0 > 0$ a.s. and $\tau_0 = \inf\{0 < s \le \tau_0 : Z_{s-} = 0\}$. Suppose that $h : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}$ is continuous and satisfies the following:*

(i) *For all $t \in [0, \tau_0)$, we have $h(Z_t) = h(Z_0) + M_t + P_t$, where*
  (a) *P is an adapted càdlàg process on $[0, \tau_0)$ such that $\inf_{t\in[0,\tau_0\wedge T)} P_t > -\infty$ a.s. for each $T \in \mathbb{R}^+ \setminus \{0\}$,*
  (b) *M is a continuous local martingale on $[0, \tau_0)$ with $M_0 = 0$,*

(ii) *and $\lim_{z\downarrow 0} h(z) = -\infty$.*

Then $\tau_0 = \infty$ a.s.

Above, $\tau_0 = \inf\{0 < s \le \tau_0 : Z_{s-} = 0\}$ is not to be understood as the definition of τ_0, but it means that the already defined stopping time τ_0 is also the first hitting time of Z_{s-} at zero. Since Z is only defined up to time τ_0, one cannot take the infimum over $\mathbb{R}^+$.

Proof. Since $h(Z_t)_- = h(Z_{t-}) = h(Z_0) + P_{t-} + M_{t-}$ and P_{t-} is a.s. bounded from below on compacts, we have $\tau_0 = \inf\{s > 0 : M_{s-} = -\infty\}$ and further $\tau_0 > 0$ due to the right continuity of Z. Assume, by contradiction, that $\tau_0 < \infty$ on a set $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$. Hence, $\lim_{t\nearrow\tau_0} M_t = -\infty$ on A and this contradicts Lemma 4.2. □

4.2. Proof of Theorem 3.4

Before we provide a proof of Theorem 3.4, we recall some elementary identities from matrix calculus and provide some further technical lemmata. For a differentiable function $f : M_d \to \mathbb{R}$, we denote by ∇f the usual gradient written in coordinates as $(\frac{\partial f}{\partial x_{ij}})_{ij}$.

Lemma 4.4. *On S_d^{++}, we have*

(i) $\nabla \det(x) = \det(x)(x^{-1})^\top = \det(x)x^{-1}$,
(ii) $\frac{\partial^2}{\partial x_{ij}\partial x_{kl}} \det(x) = \det(x)[(x^{-1})_{kl}(x^{-1})_{ij} - (x^{-1})_{il}(x^{-1})_{jk}]$.

Proof. The first identity in (i) can be found in [32, Section 9.10] and the second is an immediate consequence of restricting to symmetric matrices. Now (ii) follows using $\frac{\partial}{\partial x_{kl}} x^{-1} = -x^{-1}\left(\frac{\partial}{\partial x_{kl}} x\right)x^{-1}$ and finally the symmetry

$$\frac{\partial}{\partial x_{kl}x_{ij}} \det(x) = \frac{\partial}{\partial x_{kl}}\left(\det(x)(x^{-1})_{ji}\right) = \det(x)\left((x^{-1})_{lk}(x^{-1})_{ji} + \frac{\partial}{\partial x_{kl}}(x^{-1})_{ji}\right)$$
$$= \det(x)\left((x^{-1})_{lk}(x^{-1})_{ji} - (x^{-1})_{jk}(x^{-1})_{li}\right). \quad \square$$

For a semimartingale X we denote by X^c as usual its continuous part. All semimartingales in the following will have a discontinuous part of finite variation, i.e. $\sum_{0<s\le t} \|\Delta X_s\|$ is finite for all $t \in \mathbb{R}^+$. Thus we define $X_t^c = X_t - \sum_{0<s\le t} \Delta X_s$ and note that the quadratic variation of a semimartingale is that of its local continuous martingale part plus the sum of its squared jumps.

The continuous quadratic variation of X solving (3.4) is only influenced by the Brownian terms and, hence, we have a general version of [8, Equation (2.4)] which is proved just as [1, Lemma 2]:

Lemma 4.5. *Consider the solution X_t of (3.4) on $[0, T_x)$. Then*

$$\frac{d[X_{ij}, X_{kl}]^c_t}{dt} = (FF^\top(t, X_{t-}))_{ik}(G^\top G(t, X_{t-}))_{jl} + (FF^\top(t, X_{t-}))_{il}(G^\top G(t, X_{t-}))_{jk} + (FF^\top(t, X_{t-}))_{jk}(G^\top G(t, X_{t-}))_{il} + (FF^\top(t, X_{t-}))_{jl}(G^\top G(t, X_{t-}))_{ik}.$$

Here $G^\top G(t, x) := G(t, x)^\top G(t, x)$ and $FF^\top(t, x) := F(t, x)F(t, x)^\top$ to ease notation.

Moreover, we shall need the following result where a Brownian motion on a stochastic interval $[0, T)$ is defined as a continuous local martingale on $[0, T)$ with $[\beta, \beta]_t = t$.

Lemma 4.6. *Let X_t be a continuous S_d^+-valued adapted càdlàg stochastic process on a stochastic interval $[0, T)$ with T being a predictable stopping time and suppose that $h : M_d \to M_d$. Then there exists a one-dimensional Brownian motion β^h on $[0, T)$ such that*

$$\mathrm{Tr}\left(\int_0^t h(X_{u-})dB_u\right) = \int_0^t \sqrt{\mathrm{Tr}(h(X_{u-})^\top h(X_{u-}))}d\beta^h_u \tag{4.1}$$

holds on $[0, T)$.

Proof. We define for $t \in [0, T)$,

$$\beta^h_t := \sum_{i,j=1}^d \int_0^t \frac{h(X_{u-})_{ij}}{\sqrt{\mathrm{Tr}(h(X_{u-})^\top h(X_{u-}))}} dB_{u,ji},$$

and since the numerator equals zero, whenever the denominator vanishes, we use the convention that $\frac{0}{0} = 1$. Clearly for each i, j and for all $u \in [0, T)$ we have

$$\left|\frac{h(X_{u-})_{,ij}}{\sqrt{\mathrm{Tr}(h(X_{u-})^\top h(X_{u-}))}}\right| \leq 1$$

which ensures that β^h is well-defined, square integrable and a continuous local martingale on $[0, T)$ by stopping at a sequence of stopping times announcing T. Furthermore, by construction

$$[\beta^h, \beta^h]_t = \sum_{i,j=1}^d \int_0^t \frac{h(X_{u-})^2_{ij}}{\mathrm{Tr}(h(X_{u-})^\top h(X_{u-}))} du = t$$

and therefore β^h is a Brownian motion on $[0, T)$.

Finally by the very definition of β^h, we have

$$\mathrm{Tr}(h(X_{t-})dB_t) = \sum_{i,j=1}^d h(X_{t-})_{ij} dB_{t,ji} = \sqrt{\mathrm{Tr}(h(X_{t-})^\top h(X_{t-}))}d\beta^h_t,$$

which proves identity (4.1). □

Finally, we state a variant of Itô's formula which we later employ. It follows easily from the usual versions like [5, Theorem 3.9.1] by arguments similar to those of [33, Theorem 5.4] and [4, Proposition 3.4].

Lemma 4.7. *Let X be an S_d^{++}-valued semimartingale on a stochastic interval $[0, T)$ and $f : S_d^{++} \to \mathbb{R}$ a function that is twice continuously differentiable. If $X_{t-} \in S_d^{++}$ for all*

$t \in [0, T)$ and $\sum_{0<s\le t} \|\Delta X_s\| < \infty$ for $t \in [0, T)$, then $f(X)$ is a semimartingale on $[0, T)$ and

$$f(X_t) = f(X_0) + \mathrm{Tr}\left(\int_0^t \nabla f(X_{s-})^\top dX_s^c\right) + \frac{1}{2}\sum_{i,j,k,l=1}^d \int_0^t \frac{\partial^2}{\partial x_{ij}\partial x_{kl}} f(X_{s-}) d[X_{ij}, X_{kl}]_s^c + \sum_{0<s\le t} (f(X_s) - f(X_{s-})).$$

We are now prepared to provide a proof of Theorem 3.4. Note that to shorten our formulae we use in the following differential notation and not integral notation as above.

Proof of Theorem 3.4. Since

$$\mathrm{vec}\,(F(t, X_{t-})dB_t G(t, X_{t-})) = \left(G(t, X_{t-})^\top \otimes F(t, X_{t-})\right) \mathrm{vec}(dB_t),$$

it is easy to see that all coefficients of (3.4) are locally Lipschitz and of linear growth. Hence, standard SDE theory implies again the existence of a unique càdlàg adapted non-explosive local strong solution until the first time $T_x = \inf\{t \ge 0 : X_{t-} \in \partial S_d^+ \text{ or } X_t \notin S_d^{++}\}$ that X hits the boundary or jumps out of S_d^{++}. Hence, we have to show that $T_x = \infty$.

By the choice of K and J, all jumps have to be positive semidefinite and hence the solution X cannot jump out of S_d^{++}. This implies that $T_x = \inf\{t \ge 0 : X_{t-} \in \partial S_d^+\}$.

In the following, all statements are meant to hold on the stochastic interval $[0, T_x)$. Note that by the right continuity of X_t, a.s. $T_x > 0$. Moreover, we set $T_n = \inf\{t \in \mathbb{R}_+ : d(X_t, \partial S_d^+) \le 1/n \text{ or } \|X_t\| \ge n\}$. Then $(T_n)_{n\in\mathbb{N}}$ is an increasing sequence of stopping times with $\lim_{n\to\infty} T_n = T_x$, and hence T_x is predictable.

We define the following processes and functions according to the notation of Proposition 4.3:

$$Z_t := \det(X_t), \qquad h(z) := \ln(z), \qquad r_t := h(Z_t). \tag{4.2}$$

Then $T_x = \inf\{t > 0 : r_{t-} = -\infty\}$.

By Lemma 4.4(i) and using the abbreviation $f = FF^\top$, $g = G^\top G$, we obtain

$$\mathrm{Tr}(\nabla(\det(X_{t-}))dX_t^c) = \det(X_{t-})\left[2\sqrt{\mathrm{Tr}\left(f(t, X_{t-})X_{t-}^{-1} g(t, X_{t-})X_{t-}^{-1}\right)}dW_t + \mathrm{Tr}\left(H(t, X_{t-})X_{t-}^{-1}\right)dt\right],$$

with some one-dimensional Brownian motion W on $[0, T_x)$, whose existence is guaranteed by Lemma 4.6. Furthermore, by Lemmas 4.4(ii), 4.5 and elementary calculations we have that

$$\begin{aligned}
&\frac{1}{2}\sum_{i,j,k,l} \frac{\partial^2}{\partial x_{ij}\partial x_{kl}} \det(X_{t-}) d[X_{ij}, X_{kl}]_t^c \\
&= \frac{\det(X_{t-})}{2}\sum_{i,j,k,l}\Big[\big((X_{t-}^{-1})_{kl}(X_{t-}^{-1})_{ij} - (X_{t-}^{-1})_{il}(X_{t-}^{-1})_{jk}\big)\big(f(t, X_{t-})_{ik} g(t, X_{t-})_{jl} \\
&\quad + f(t, X_{t-})_{il} g(t, X_{t-})_{jk} + f(t, X_{t-})_{jk} g(t, X_{t-})_{il} + f(t, X_{t-})_{jl} g(t, X_{t-})_{ik}\big)\Big] \\
&= \det(X_{t-})\Big(\mathrm{Tr}(f(t, X_{t-})X_{t-}^{-1} g(t, X_{t-})X_{t-}^{-1}) \\
&\quad - \mathrm{Tr}(f(t, X_{t-})X_{t-}^{-1})\mathrm{Tr}(g(t, X_{t-})X_{t-}^{-1})\Big)\, dt.
\end{aligned}$$

According to Itô's formula, Lemma 4.7, we therefore obtain by summing the two equations

$$\begin{aligned} dZ_t &= 2\det(X_{t-})\sqrt{\mathrm{Tr}(f(t,X_{t-})X_{t-}^{-1}g(t,X_{t-})X_{t-}^{-1})}dW_t + \det(X_t) - \det(X_{t-}) \\ &\quad + \det(X_{t-})\Big[\mathrm{Tr}(H(t,X_{t-})X_{t-}^{-1}) + \mathrm{Tr}(f(t,X_{t-})X_{t-}^{-1}g(t,X_{t-})X_{t-}^{-1}) \\ &\quad - \mathrm{Tr}(f(t,X_{t-})X_{t-}^{-1})\mathrm{Tr}(g(t,X_{t-})X_{t-}^{-1})\Big]dt. \end{aligned}$$

Using again Itô's formula, we have

$$\begin{aligned} dr_t &= 2\sqrt{\mathrm{Tr}(f(t,X_{t-})X_{t-}^{-1}g(t,X_{t-})X_{t-}^{-1})}dW_t + \ln(\det(X_t)) - \ln(\det(X_{t-})) \\ &\quad + \Big[\mathrm{Tr}(H(t,X_{t-})X_{t-}^{-1}) - \mathrm{Tr}(f(t,X_{t-})X_{t-}^{-1}g(t,X_{t-})X_{t-}^{-1}) \\ &\quad - \mathrm{Tr}(f(t,X_{t-})X_{t-}^{-1})\mathrm{Tr}(g(t,X_{t-})X_{t-}^{-1})\Big]dt. \end{aligned}$$

Hence, we have $r_t = r_0 + M_t + P_t$, where

$$\begin{aligned} M_t &= 2\int_0^t \sqrt{\mathrm{Tr}(f(s,X_{s-})X_{s-}^{-1}g(s,X_{s-})X_{s-}^{-1})}dW_s, \\ P_t &= \int_0^t \Big[\mathrm{Tr}(H(s,X_{s-})X_{s-}^{-1}) - \mathrm{Tr}(f(s,X_{s-})X_{s-}^{-1}g(s,X_{s-})X_{s-}^{-1}) \\ &\quad - \mathrm{Tr}(f(s,X_{s-})X_{s-}^{-1})\mathrm{Tr}(g(s,X_{s-})X_{s-}^{-1})\Big]ds + \sum_{0<s\le t}(\ln(\det(X_s)) - \ln(\det(X_{s-}))). \end{aligned}$$

We infer that $(M_t^{(n)})_{t\ge 0}$ defined by

$$M_t^{(n)} := 2\int_0^t \sqrt{\mathrm{Tr}(f(s,X_{s-}^{T_n})(X_{s-}^{T_n})^{-1}g(s,X_{s-}^{T_n})(X_{s-}^{T_n})^{-1})}dW_s$$

is a continuous martingale. Obviously, $M_t = M_t^{(n)}$ on $\{t < T_n\}$ and thus M is a continuous local martingale on $[0, T_x)$. Furthermore, $X_s - X_{s-} \succeq 0$ for all $s \in [0, T)$ and hence $\det(X_s) \ge \det(X_{s-})$ using [28, Corollary 4.3.3]. Therefore, we have that $P_t \ge \int_0^t c(s)ds$ on $[0, T_x)$.

Finally, by Proposition 4.3 we have that $T_x = \infty$ a.s., noting that c is assumed to be locally integrable. □

Remark 4.8. Bru's method for proving her Proposition 3.1 for Wishart diffusions consists of the following two steps:

(i) First assume that $\beta = 0$. By applying the original McKean argument twice, one derives that $h(\det(X))$ is a local martingale. This is proved separately for $\delta = d + 1$ and $\delta > d + 1$ by choosing $h(z) = \ln(z)$ in the first case and $h(z) = z^{d+1-\delta}$ in the second one. Therefore, the existence of a unique global strong solution on S_d^{++} is settled.

(ii) One may therefore suppose that X_t is an S_d^{++}-valued solution on $[0,\infty)$ of

$$dX_t = \sqrt{X_t}dB_tQ + Q^\top dB_t^\top\sqrt{X_t} + \delta Q^\top Q dt, \qquad X_0 = x \in S_d^{++}.$$

where $Q \in GL(d)$ and $\delta \ge d + 1$. Now, Girsanov's theorem is applied, which allows us to introduce a drift by changing to an equivalent probability measure. This step generalises a one-dimensional method of Pitman and Yor; see [8, p. 748]. The arguments and calculations involved, which are not presented in detail in [8], appear rather complicated and work seemingly only in the special case given in Proposition 3.1(ii), (iii).

The technical details of [8] concerning strong solutions are explained in more detail in [38].

Our proof above circumvented the problems associated with the use of Girsanov's theorem by extending the approach outlined in (i).

5. Conclusion

In this paper we have extended the previously known sufficient boundary non-attainment conditions for certain Wishart processes to more general SDEs on S_d^{++}, which include affine diffusions with state-independent jumps of finite variation. This allowed us to infer the existence of strong solutions of a large class of affine matrix-valued processes. Moreover, we have thus obtained strong existence results for SDEs which can be considered as positive semidefinite extensions of GARCH diffusions and generalised Cox–Ingersoll–Ross processes.

However, this results in several open questions related to our SDE (1.1) which will we hope be addressed in future work. The following questions are beyond the scope of the present paper, since they are obviously rather non-trivial and apparently need techniques very different to the ones employed here. For $d > 1$ and the Wishart diffusions it is not clear whether the condition $b \succeq (d+1)Q^\top Q$ for the drift is a necessary non-attainability condition or not. Only in the case $\beta = 0$, $\Gamma = 0$, $Q = I_d$ and $b = \delta I_d$ with $\delta \in (d-1, d+1)$ is it known from [20, Theorem 1.4] that the boundary is hit. On the other hand, one knows that in the case $d = 1$, pathwise uniqueness holds, and hence there exists a strong solution for all $b \succeq 0$ (even in the general setting of CBI processes; see [18, Theorem 5.1]). For $d \geq 2$, the situation seems in general to be rather complicated and therefore existence of global strong solutions remains an open problem when $b \not\succeq (d+1)Q^\top Q$ (and the conditions for the existence of weak solutions of [13] are satisfied). Likewise, it is a very interesting, in the case of the GCIR processes with $\alpha > 1/2$, to consider whether a state-dependent drift away from the boundary is really necessary and what happens if one has only a constant drift towards the interior of S_d^+.

Finally, we remark that our method of proof could be generalised to state spaces D other than S_d^+, as long as the existence of an appropriate function $g : D \to \mathbb{R}_+$ is guaranteed, such that $g^{-1}(0) = \partial D$. For instance, arguments similar to (but simpler than) those of the proof of Theorem 3.4 yield a rigorous proof of the non-attainment condition formulated in [11, Section 6] for affine jump diffusions on the canonical state space $\mathbb{R}_+^m \times \mathbb{R}^n$. Here one takes $g(x_1, x_2, \ldots, x_m) = x_1 \cdot x_2 \cdot \cdots \cdot x_m$.

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