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Anisotropic 2D mesh adaptation in hp-adaptive FEM

Arkadiusz Szymczak[a], Anna Paszyńska[b], Maciej Paszyński[a], David Pardo[c]

*[a]Department of Computer Science
AGH University of Science and Technology,
Al. Mickiewicza 30, 30-059 Cracow, Poland
[b]Faculty of Physics, Astronomy and Applied Computer Science
Jagiellonian University, Cracow, Poland
[c]University of the Basque Country (UPV/EHU) and
IKERBASQUE (Basque Foundation for Sciences), Bilbao, Spain*

Abstract

The paper presents a grammar for anisotropic two-dimensional mesh adaptation in hp-adaptive Finite Element Method with rectangular elements. It occurs that a straightforward approach to modeling this process via grammar productions leads to potential deadlock in h-adaptation of the mesh. This fact is shown on a Petri net model of an exemplary adaptation. Therefore auxiliary productions are added to the grammar in order to ensure that any sequence of productions allowed by the grammar does not lead to a deadlock state. The fact that the enhanced grammar is deadlock-free is proven via a corresponding Petri net model. The proof has been performed by means of reachability graph construction and analysis. The paper is enhanced with numerical simulations of magnetolluric measurements where the deadlock problem occured.

Keywords: anisotropic mesh adaptation, hp-adaptive finite element method, grammar, Petri net, deadlock

1. Introduction

In this paper we present a Petri nets based approach for testing and preventing deadlock during mesh transformations for two dimensional *hp*-adaptive Finite Element Method (*hp*-FEM) computations [1, 2]. The *hp*-adaptation generates a sequence of meshes providing exponential convergence of the numerical error with respect to the number of degrees of freedom. It has multiple applications including material science [3, 5, 4], propagation of electromagnetic waves, in particular with oil-industry applications [6, 7, 8, 9] as well as heat transfer problems [5, 4].

The paper presents a grammar for anisotropic two-dimensional mesh adaptation in hp-adaptive Finite Element Method with rectangular elements. h-adaptation of the mesh elements is governed by the following set of rules:

- element interiors and element edges are broken separately;

- an element edge may be broken for the Nth time only when all its adjacent element interiors have been broken for the Nth time;

- an element interior may be broken for the $(N + 1)^{th}$ time only when all its adjacent element edges have been broken for the Nth time;

- an element may be broken either horizontally or vertically or in both directions as a single operation (we consider anisotropic mesh refinements)

The first three rules follows from the fact that we enforce 1-irregularity rule [1] over the mesh, to prevent the presence of double constrained nodes. According to the *1-irregularity rule* a finite element can be broken only once without breaking the adjacent large elements. The rule prevents unbroken element edges from being adjacent to more than two finite elements on one side. When an unbroken edge is adjacent to one large finite element on one side and two smaller finite elements on the other side, the approximation over these two smaller elements is constrained by the approximation over the larger element. To avoid a technical nightmare with constrained approximation over multiple constrained edges, the 1-irregularity rule is commonly used in the *h* adaptive algorithms.

The next section presents a grammar governing the process of h-adaptation such that all the above rules are fulfilled. It occurs that straightforward translation of the above rules to grammar productions can lead to a deadlock in a sequence of mesh element adaptations allowed by this grammar. This fact is shown by means of a Petri net modeling the grammar-driven h-adaptation.

The subsequent section presents an enhancement to the defined grammar. The new productions are added to remove the deadlock by canceling one of the anisotropic refinements and replacing it by the isotropic one.

The analysis of the Petri net modeling the h-adaptation governed by the enhanced grammar and its reachability graph proves that the enhanced grammar is deadlock-free.

2. Grammar - straightforward approach

Productions of the first version of the grammar are presented in Figure 1. Productions (BI), (BIH), (BIV) denote breaking a mesh element interior in both directions, horizontally and vertically respectively. Production (BE) denotes breaking a mesh element edge. We utilize here simplified graphic representation of element edges and interiors. For full graph grammar description of mesh refinements procedures we refer to [11, 12], while for the graph grammar description of the solver algorithm we refer to [10, 13, 14].
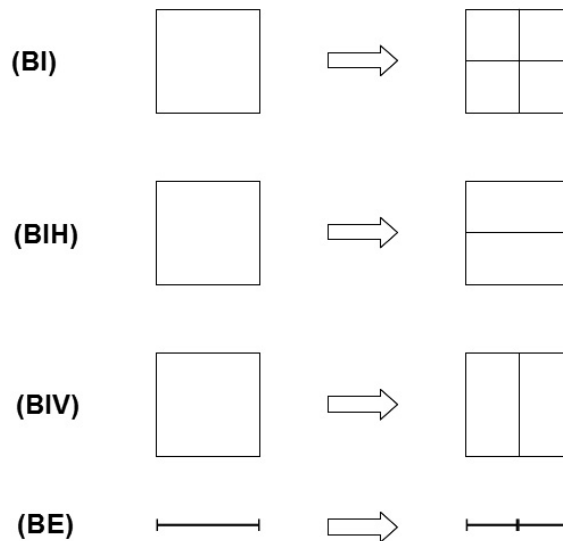

Figure 1: Productions of the first version of the grammar.

Let us consider adaptation of the exemplary mesh shown in Figure 2.
Petri net modeling the h-adaptation process of the exemplary mesh is presented in Figure 3.
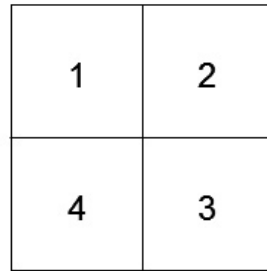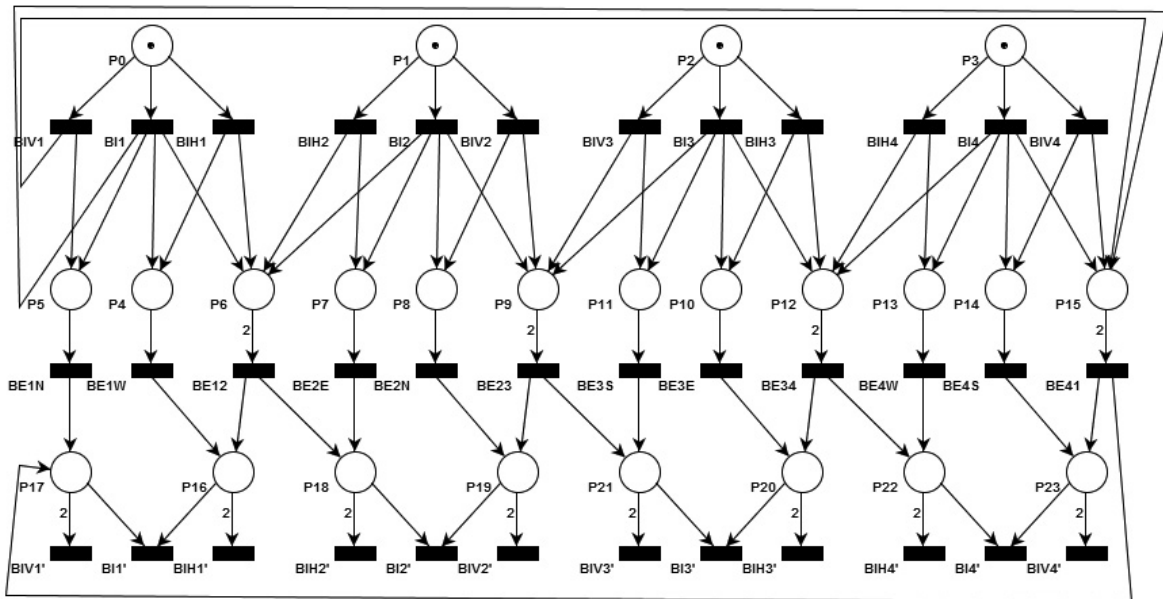
Figure 2: Exemplary mesh.

Figure 3: Petri net modeling h-adaptation driven by the first version of the grammar.

Numbers in the transition names denote the corresponding mesh element to which a given production is applied. For instance, transition *BI1* corresponds to the execution of production **(BI)** on mesh element number 1. Likewise, transition *BE12* corresponds to the execution of productionn **(BE)** on the common edge of elements number 1 and 2. A single letter *X* at the end of *BEX* transition name positions the element edge in relation to the corresponding element interior. *W* stands for West, *N* for North, *E* - East and *S* - South. For instance, transition *BE1N* corresponds to the execution of production *(BE)* on the northern edge of element number 1. The prim sign at the end of a transition name indicates that firing such transition models execution of a corresponding grammar production in the subsequent iteration of mesh adaptation. Any of the next iteration transitions being active means that the first iteration of adaptation has not ended with a deadlock. Such a Petri net precisely models all possible flows of one iteration of h-adaptation of the exemplary mesh. However, it is very complex in terms of the number of states it allows. Therefore a simplified version of the Petri net, as presented in Figure 4, has been developed.

The sets of places and transitions have been limited to only such places and transitions that are relevant to deadlock occurrence. Moreover, one auxiliary place and one auxiliary transition have been added to make the Petri net live if not the inherent deadlock. From the application point of view, however, it would suffice to end the simulation (sequence of fired transitions) once any of transitions *BE12, BE23, BE34, BE41* gets activated (meaning no deadlock has occurred).
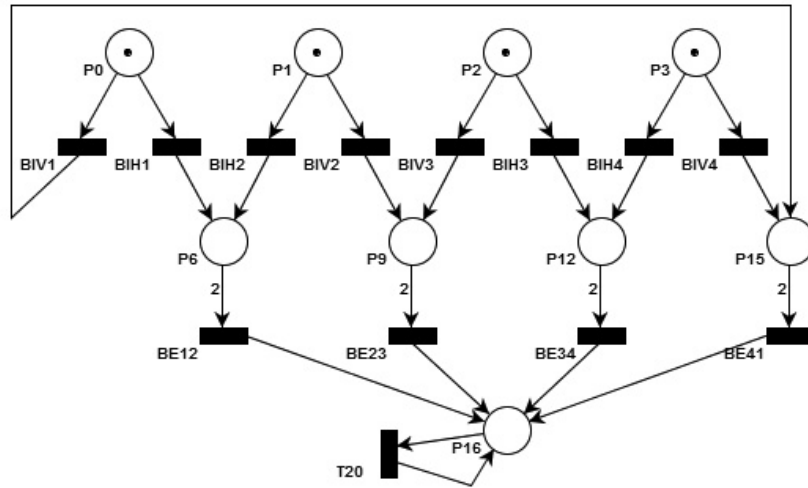
Figure 4: Simplified Petri net for the first version of the grammar.

Remark 1. The grammar is not deadlock-free.

It is clearly visible that the following sequence of fired transitions: *BIH1, BIV2, BIH3, BIV4* leads to a dead state. In this state, all the internal edges are constrained and cannot be further broken without violating the mesh 1-irregularity rule.

3. Enhanced grammar

Figure 5 presents productions that have been added to the previously defined grammar.
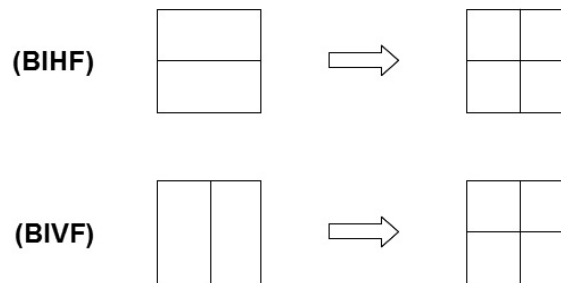

Figure 5: New productions in the enhanced graph grammar.

Figure 6 presents the Petri net modeling h-adaptation process of the exemplary mesh from Figure 2.

Similarly to the first version of the grammar and the corresponding Petri net, also the enhanced version is too complex for deadlock analysis. Therefore also this Petri net has been modified analogously to the previous one. The simplified Petri net is presented in Figure 7.

Remark 2. The enhanced grammar is deadlock-free.

Reachability graph has been generated for given Petri net and given initial marking (shown in the figure). The initial marking reflects the intention of braking each mesh element once. Subsequently it has been analyzed if there exists any path in the reachability graph between the initial state and a dead state. There is no such path in the reachability graph (the Petri net is live) what implies that the grammar modeled by the Petri net is deadlock-free.
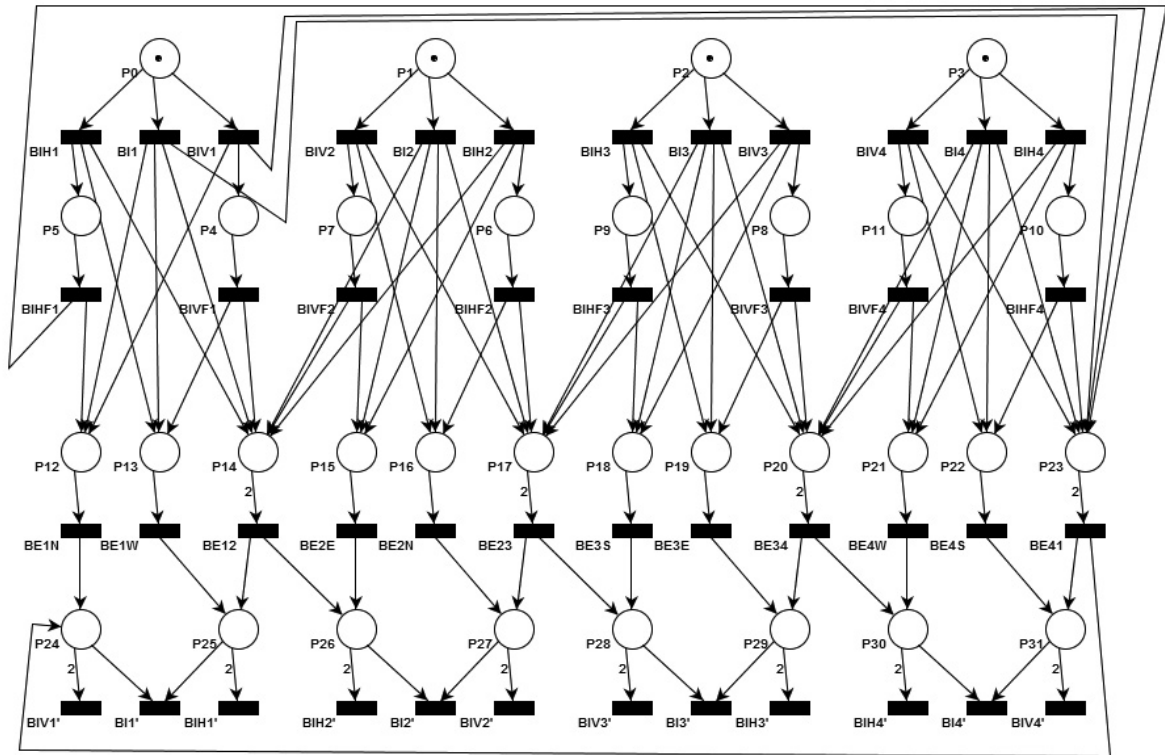
Figure 6: Petri net modeling h-adaptation driven by the enhanced grammar.

4. Numerical example

We illustrate the deadlock problem by simulating a two-dimensional (2D) magnetotelluric (MT) problem. This technique is used to determine a resistivity map of the Earth's subsurface by performing electromagnetic (EM) measurements. The main difference of MT with respect to usual measurement acquisition scenarios is that MT uses natural sources generated within the ionesphere, and it does not require artificial sources. Thus, acquisition of MT measurements is rather inexpensive, and it is possible to cover large areas. Applications of MT measurements include hydrocarbon (oil and gas) exploration and finding suitable regions for storage of $CO2$.

We employ a *hp*-adaptive finite element method for simulations of MT measurements [15]. We consider the measurement acquisition scenario described in Figure 8, composed from a source modelled as a plane wave operating at 100 Hz coming from the top part of the domain, a layer of air, a background earth material with resistivity equal to $200\Omega m$, and three target zones with resistivities equal to $1000\Omega m$, $2000\Omega m$, and $50\Omega m$, respectively. This 2D model problem is governed by Maxwell's equations, and we incorporate a Perfectly Matched Layer (PML) in order to truncate the computational domain. Receivers are located along the air-earth interface. For simplicity, in this paper we only consider one receiver position on top of the $2000\Omega m$ resistivity layer. For that position, we employ the so-called *hp* goal-oriented adaptive strategy [16], [17].

The version without fixing the deadlock problem breaks down with just 6,245 unknowns, delivering an error of 0.03%, while for reliable analysis of MT results 0.001% accuracy is needed. The final mesh delivering 0.001% accuracy is presented in Figure 10. The corresponding final solution is presented in Figure 11. The solution obtained on the mesh with accuracy 0.03% where the deadlock happened is presented in Figure 12. It is clear that the accuracy is too low to estimate the properties of formation layers, since the solution over the part of domain representing the air is dominating. Finally, Figures 13 and 14 present the mesh with the deadlock problem.
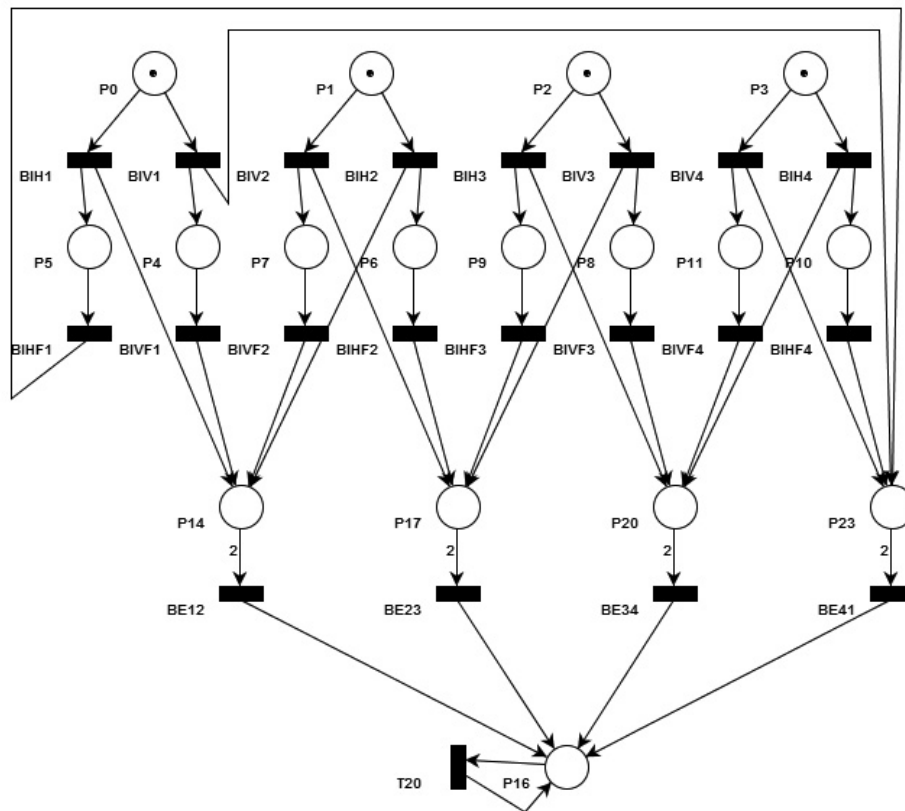
Figure 7: Simplified Petri net for the enhanced grammar.

5. Conclusions

In this paper a grammar for anisotropic two-dimensional mesh adaptation in hp-adaptive Finite Element Method with rectangular elements has been presented.

In the first straightforward approach the h-adaptation rules are exactly translated to grammar productions. The corresponding Petri net model shows that h-adaptation governed by such a grammar can enter a deadlock state.

Subsequently an enhanced grammar has been developed by adding two auxiliary productions to the original grammar. The analysis based on the generated reachability graph of the corresponding Petri net model has proven that the enhanced grammar is deadlock-free.

Acknowledgments

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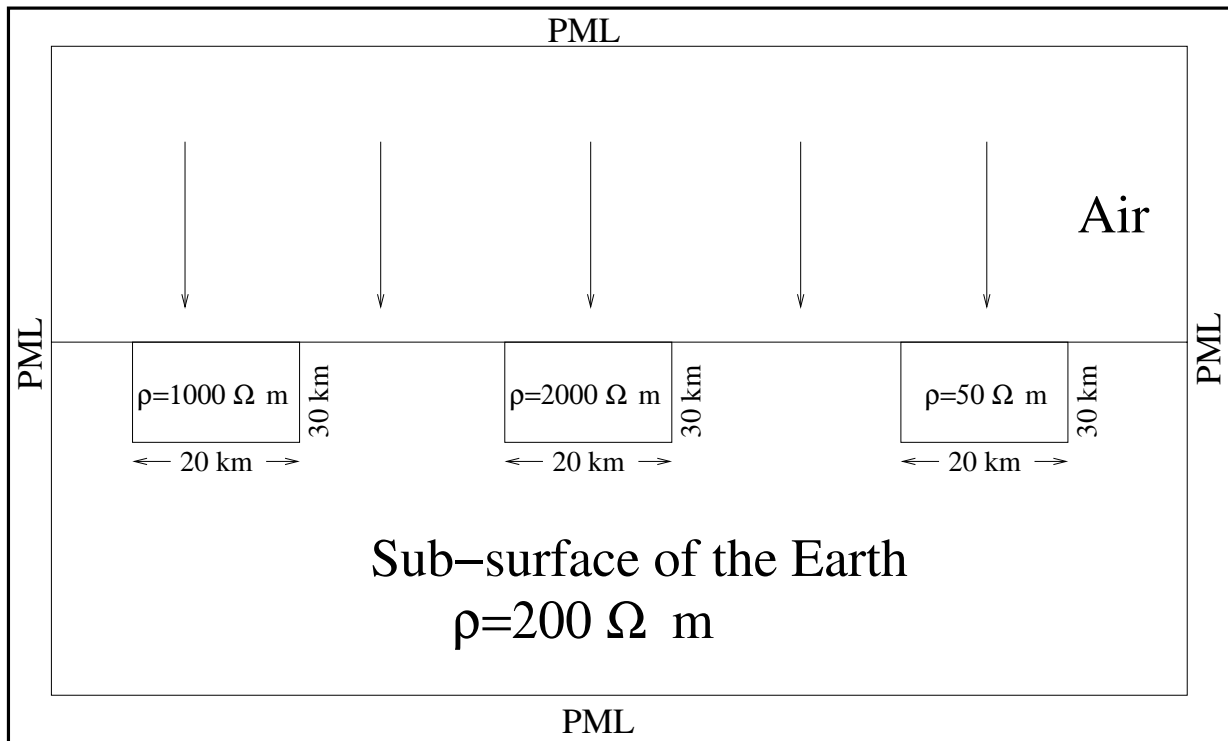
Figure 8: Geometry for the magnetolluric problem being solved.

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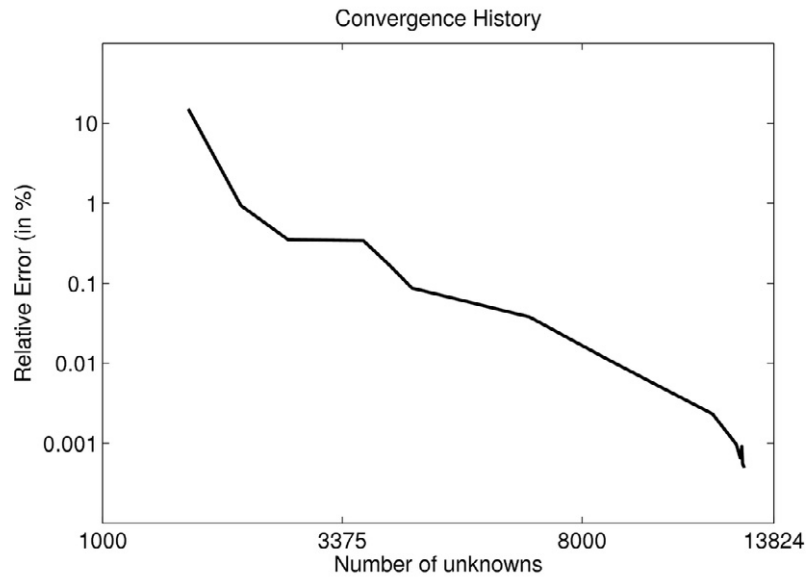
Figure 9: Convergence history for *hp*-adaptive finite element method simulations after removing the deadlock problem.
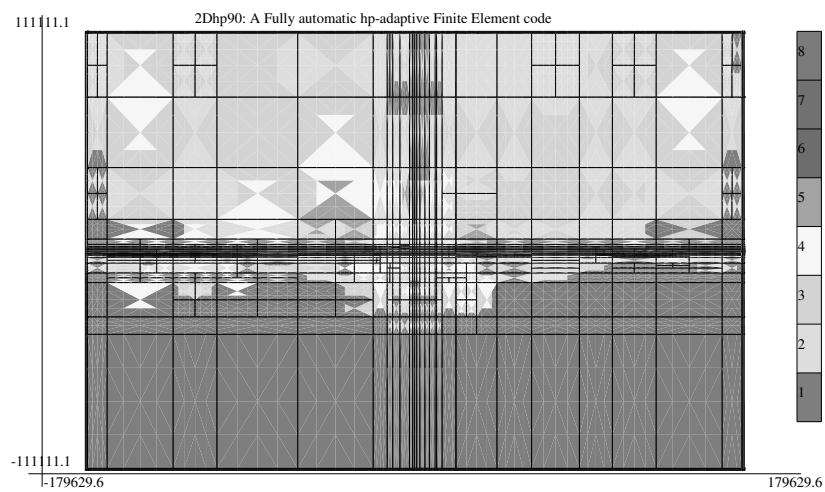

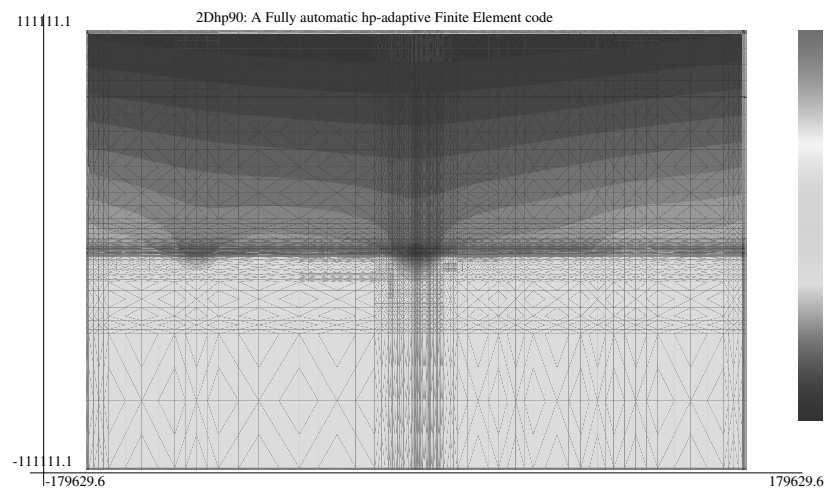Figure 10: Global view of the final *hp* refined mesh without removing the deadlock

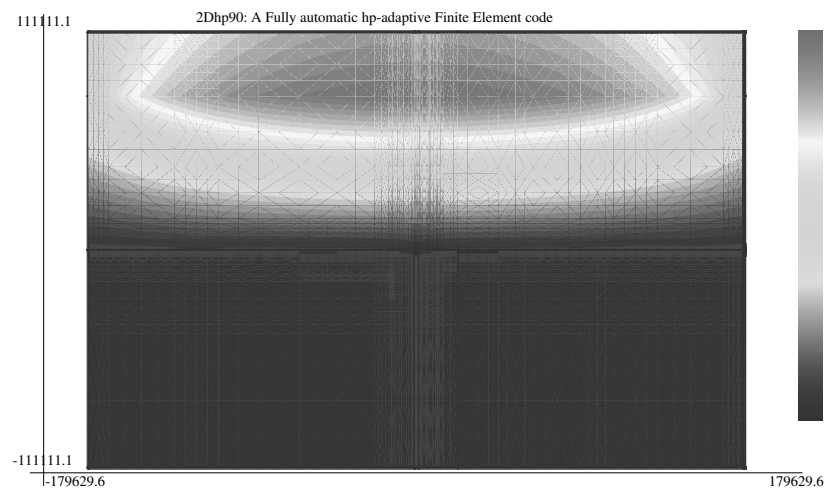Figure 11: Solution over the final mesh with 0.001% accuracy

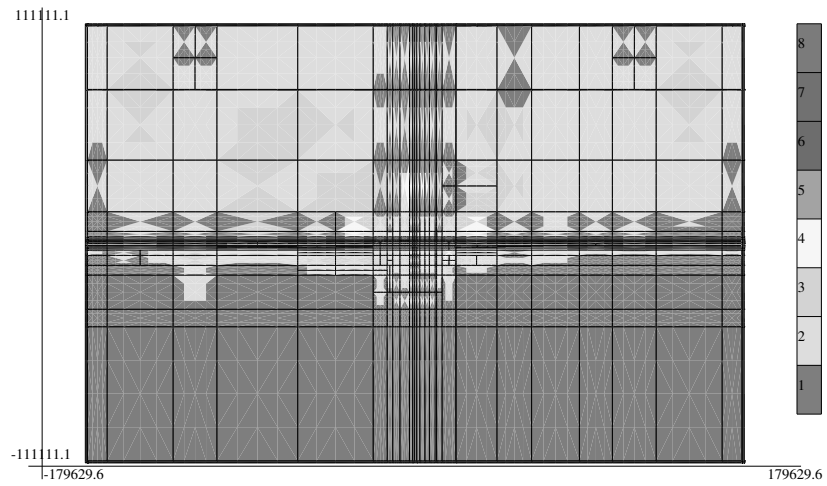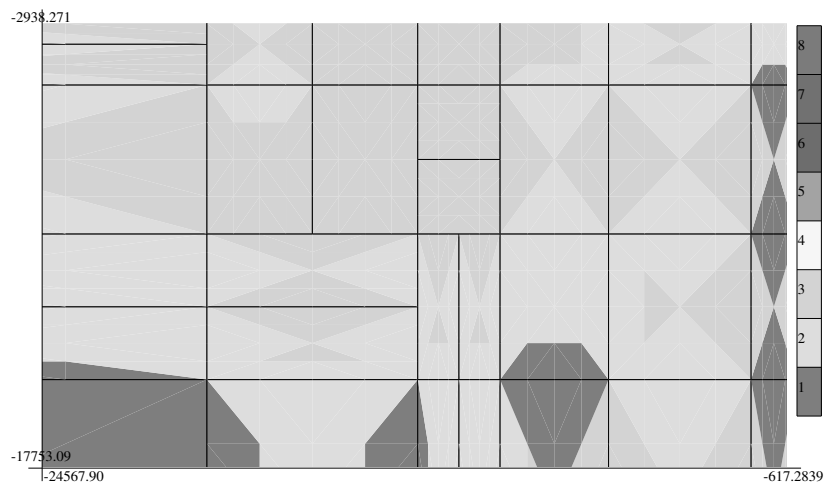Figure 12: Solution over the mesh with 0.03% accuracy where the deadlock problem occured

Figure 13: Global view on the *hp* refined mesh with deadlock

Figure 14: Amplification with the factor 100 towards the deadlock area