On compactifications preserving the dimension of spaces

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Received 6 March 2000; received in revised form 7 March 2001

Abstract

Compactifications preserving the dimension of a normal space are described in the partially ordered set of all compactifications. © 2002 Elsevier Science B.V. All rights reserved.

AMS classification: 55M10; 54F45

Keywords: Normal space; Compactification; Dimension; Topological weight

1. Introduction

Let $X$ be a normal (Hausdorff) space. The dimension of $X$ defined by means of finite open coverings is denoted by the symbol $\dim X$. Throughout this article we assume that $\dim X = n$. The symbol $w(X)$ means the topological weight of $X$, i.e., the minimal power of an open base of $X$.

In this paper we try to get some compactifications of a space $X$ which preserve the dimension of $X$. It is known that the dimension $\dim \beta X$ of the Čech compactification $\beta X$ of a normal space $X$ is equal to the dimension $\dim X$ of the space $X$. But usually the topological weight $w(\beta X)$ of $\beta X$ is bigger than the weight $w(X)$ of $X$. As it is shown in [2] any normal space $X$ has at least one compactification $Y$ for which $w(Y) = w(X)$ and $\dim Y = \dim X$.

In connection with these facts the following question arises naturally: How many compactifications of this kind there are? Our general theorem gives, in particular, an answer to this question.

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**Theorem.** Let $Y$ be a compactification of $X$ and $\kappa = w(Y)$ be the topological weight of the space $Y$. Then there exists a compactification $Z$ of $X$ such that:

(i) $\dim Z = \dim X = n$,
(ii) $w(Z) = w(Y) = \kappa$,
(iii) $Y \leq Z$.

Here the inequality $Y \leq Z$ means that there exists a continuous mapping $Z \to Y$ which is identity on $X \subset Z$ (and $X \subset Y$).

**Remark.** Applying a well-known factorization theorem of Mardešić [1] to the natural mapping of the Čech compactification $\beta X$ one can readily receive a similar result in which $\dim Z \leq n$. Thus the contribution of our theorem consists in strengthening this inequality to the equality $\dim Z = n$.

There exist many examples illustrating this theorem. For instance, let $X$ be the set of all points $(x, y)$ of the plane $\mathbb{R}^2$ with rational coordinates $x$ and $y$ for which $0 \leq x \leq 1$ and $0 \leq y \leq 1$ and let $Y$ be the square $0 \leq x \leq 1, 0 \leq y \leq 1$. The space $Y$ is the compactification of $X$ and $\dim Y = 2$. But the space $X$ is countable and hence $\dim X = 0$. Both spaces $X, Y$ have countable weight. According to the theorem there exists a compactification $Z$ of countable weight for $X$ such that $\dim Z = 0$ and $Y \leq Z$. The space $Y$ is also a compactification of $\tilde{X} = Y \setminus X$, $\dim \tilde{X} = 1$. Concerning the same theorem there exists a compactification $\tilde{Z}$ of $\tilde{X}$ with the countable weight and such that $\dim \tilde{Z} = 1$ and $Y < \tilde{Z}$.

It is known that any completely regular space $X$ has a compactification $Y$ for which $w(Y) = w(X)$ (see [4]). Thus we have the following result.

**Corollary.** For any normal space $X$ there exists a compactification $Z$ such that $\dim Z = \dim X$ and $w(Z) = w(X)$.

Now it would be proper to remember here some definitions and results which will be of use in our work.

2. Preliminaries

For two coverings $\alpha$ and $\beta$ of some space $X$ the symbol $\alpha < \beta$ means that $\beta$ is a refinement of $\alpha$. For a subset $A \subset X$, the star $St_\alpha A$ of $A$ with respect to a covering $\alpha = \{A_\lambda\}$ is defined as follows:

$$St_\alpha A = \left\{ \bigcup_\lambda A_\lambda : A_\lambda \in \alpha \text{ and } A_\lambda \cap A \neq \emptyset \right\}$$

For two coverings $\alpha$ and $\beta$ of $X$ the symbol $\alpha \prec^* \beta$ means that for any point $x \in X$ the set $St_\beta x$ is contained in a set $A_\lambda \in \alpha$. Such $\beta$ is called a star-refinement of the covering $\alpha$. It is known that any finite open covering $\alpha$ of a completely regular space $X$ always has some star-refinements.
The intersection $\alpha \wedge \beta$ of two coverings $\alpha$ and $\beta$ of $X$ is defined as follows:

$$\alpha \wedge \beta = \{ U_i \cap V_j : U_i \in \alpha, V_j \in \beta \}$$

The multiplicity of a covering $\alpha = \{ A_i \}$ is the maximal integer $k$ such that one can find $k$ sets $A_{i1}, \ldots, A_{ik}$ in $\alpha$ for which $\bigcap_{i=1}^{k} A_{is} \neq \emptyset$.

**Uniform structures**

A system $\Sigma$ of open coverings of $X$ is called a uniform structure of $X$, if the following conditions are satisfied:

(W1) For any $\alpha \in \Sigma$ there is a covering $\beta \in \Sigma$, for which $\alpha <^* \beta$.

(W2) For any $\alpha, \beta \in \Sigma$ there exists a covering $\gamma \in \Sigma$ for which $\alpha \wedge \beta < \gamma$.

(W3) For any point $x \in X$ and any neighbourhood $Ox$ of $x$ there exists some $\alpha \in \Sigma$ for which $x \in St_\alpha x \subset Ox$.

The symbol $\Sigma_1 \leq \Sigma_2$ means that for any $\alpha \in \Sigma_1$ there exists $\beta \in \Sigma_2$ for which $\alpha < \beta$.

(The structure $\Sigma_1$ is called coarser than $\Sigma_2$, while the structure $\Sigma_2$ is called finer than $\Sigma_1$.)

The structures $\Sigma_1$ and $\Sigma_2$ are equivalent if $\Sigma_1 \leq \Sigma_2$ and at the same time $\Sigma_2 \leq \Sigma_1$.

Some part $\Sigma'$ of a partially ordered set $\Sigma$ (i.e., $\Sigma' \subset \Sigma$) is called a cofinal part of $\Sigma$ if for any $\lambda \in \Sigma$ there exists some $\lambda' \in \Sigma'$ such that $\lambda \leq \lambda'$.

The cofinal part $\Sigma'$ of a uniform structure $\Sigma$ is also a uniform structure (which is, obviously equivalent to $\Sigma$).

It is evident that the system of all open coverings of a compact space $Y$ is a uniform structure. It can be seen as in [3] that all uniform structures of a compact space $Y$ are equivalent. Furthermore, if $Y$ is a compactification of $X$, they define by restrictions to the space $X$ equivalent structures of $X$ (consisting of finite coverings).

It is also known that any uniform structure of $X$ consisting of finite coverings defines a compactification $Y$ of $X$, and equivalent structures define the same compactification $Y$ (see [3]).

In this article we use only uniform structures consisting of finite open coverings.

If $Y$ is the compactification of $X$ corresponding to a structure $\Sigma$, then all coverings $\alpha \in \Sigma$ have extensions on $Y$ [3]. Any extension of $\alpha$ of this kind has the same multiplicity as $\alpha$.

Let $\Sigma_1$, $\Sigma_2$ be two uniform structures of $X$ which are the restrictions of some uniform structures defined on two compactifications $Y_1$, $Y_2$ of the space $X$. It is evident that the relation $Y_1 \leq Y_2$ implies $\Sigma_1 \leq \Sigma_2$. Moreover, if $\Sigma_1$, $\Sigma_2$ are some uniform structures of $X$ and $Y_1$, $Y_2$ are the corresponding compactifications of $X$, then $\Sigma_1 \leq \Sigma_2$ if and only if $Y_1 \leq Y_2$ (see [3]).

**Remark.** Our definition of a uniform structure is equivalent to the usual one although it differs somewhat from the usual one (cf., for example, [3]): any uniform structure in our sense is simply a cofinal part (or a basis) of the usual one. Our terminology appears to be convenient for some applications.
3. Auxiliary statements and proof of the theorem

Let $X$ be a completely regular Hausdorff space and let $\Sigma$ be a uniform structure of finite open coverings of $X$.

**Lemma 1.** In the class of all structures equivalent to $\Sigma$ there exists a uniform structure $\Sigma_1$ of $X$ such that all coverings $\alpha \in \Sigma_1$ can be partially ordered by some order relation $<$, satisfying the following conditions:

(i) for any $\alpha, \beta \in \Sigma_1$ the relation $\alpha < \beta$ implies $\alpha < \beta$ (that is, $\beta$ is a refinement of $\alpha$),

(ii) for any $\alpha, \beta \in \Sigma_1$ there exists a covering $\gamma \in \Sigma_1$ for which $\alpha <' \gamma$ and $\beta <' \gamma$,

(iii) for any $\alpha \in \Sigma_1$ the set of all $\beta \in \Sigma_1$, for which $\beta <' \alpha$, is finite (i.e., any $\alpha \in \Sigma_1$ has only a finite number of coverings $\beta \in \Sigma_1$ preceding $\alpha$).

**Proof.** Let us first consider the particular case in which $X$ is a direct product $I^\iota = \prod_\lambda I_\lambda$ of segments $I = I_\lambda = [0, 1]$. Let $\Sigma_\lambda$ be the uniform structure of the segment $I_\lambda = I$ which consists of a sequence of finite open coverings of $I = I_\lambda$ where each covering is a refinement of the preceding one. It is obvious that $\Sigma_\lambda$ satisfies condition (iii) of the lemma. For a finite set of coverings $\alpha_1, \alpha_2, \ldots, \alpha_s$ in which each $\alpha_i$ belongs to $\Sigma_\lambda$, let $\prod_{i=1}^s \alpha_i$ mean the covering of $I^\iota$ defined in the following way. The sets of the covering $\prod_{i=1}^s \alpha_i$ have the form $\prod_\lambda U_\lambda$, where $U_\lambda \in \alpha_i$ for $\lambda = \lambda_i$, $i = 1, 2, \ldots, s$, and $U_\lambda = I_\lambda$ for all other $\lambda$. (In this construction $\lambda_i \neq \lambda_j$ for $i \neq j$ ($i, j \in \{1, \ldots, s\}$.)

Let $\overline{\Sigma_1}$ be the system of all coverings of $I^\iota$ of the form $\prod_{i=1}^s \alpha_i$ (for any finite sets of indices $\lambda_i$ and integer $s$). The system $\overline{\Sigma_1}$ is a uniform structure of $I^\iota$. To see this it is sufficient to show that $\overline{\Sigma_1}$ is a cofinal part of the structure of all finite open coverings of $I^\iota$, i.e., that any open covering $\alpha$ of $I^\iota$ has a refinement $\overline{\alpha} \in \overline{\Sigma_1}$. Since the space $I^\iota$ is compact there exists a finite open refinement $\beta$ of $\alpha$, consisting of basic open sets $W_1, W_2, \ldots, W_m$ of the space $I^\iota$. This means that each $W_i$ has the form $W_i = \prod_\lambda U_i\lambda$, $i = 1, 2, \ldots, m$, $U_i\lambda \subset I_\lambda$ and $U_i\lambda \neq I_\lambda$ only for some finite set of indices $A_0$. (Since the set of indices $i$ is finite, $i = 1, 2, \ldots, m$, we can assume that $A_0$ is the same for all $i$.) Thus $I^\iota = (\prod_{\lambda \in A_0} I_\lambda) \times (\prod_{\lambda \notin A_0} I_\lambda)$ and $\beta$ has the form $\beta' \times \prod_{\lambda \notin A_0} I_\lambda$, where $\beta'$ is a finite covering of the cube $\prod_{\lambda \in A_0} I_\lambda$, which is the usual cube of a Euclidean space.

Let $\varepsilon$ be the Lebesgue number of the covering $\beta'$ and let $\alpha_\iota \in \Sigma_1$ be a covering of $I_\lambda$ for $\lambda \in A_0$ consisting of sets with diameters less than $\varepsilon/\sqrt{N}$, where $N$ is the (finite) cardinal of $A_0$. Then the covering $\prod_{\lambda \in A_0} \alpha_\iota$ is a refinement of $\beta'$ (because the diameter of each set of the covering $\prod_{\lambda \in A_0} \alpha_\iota$ is less than $\varepsilon$), and the covering $\overline{\alpha} = (\prod_{\lambda \in A_0} \alpha_\iota) \times (\prod_{\lambda \notin A_0} I_\lambda)$ is a refinement of $\beta$. Consequently, $\overline{\alpha}$ is also a refinement of $\alpha$, and at the same time $\overline{\alpha} \in \overline{\Sigma_1}$. Thus, $\overline{\Sigma_1}$ is a uniform structure of $I^\iota$.

Since $\overline{\Sigma_1}$ is a uniform structure it satisfies conditions (i) and (ii) of the lemma with respect to the usual order $\leq$. Since $\overline{\Sigma_1}$ satisfies condition (iii) of the lemma for any $\lambda$ (on segment spaces $I_\lambda$), then $\overline{\Sigma_1}$ satisfies condition (iii), too.

Now we consider the general case. The structure $\Sigma$ of the completely regular space $Y$ of $X$ such that $\Sigma$ coincides with the restriction to the space $X$ of some uniform structure $\Sigma'$ of the compact space $Y$, which is isomorphic to the
structure \( \Sigma \) (see Section 2). Hence without any restriction of generality we can suppose that the space \( X \) is compact.

Let us remember that all uniform structures of a compact space \( X \) are equivalent. It is known (see [4]) that there exists an embedding of \( X \) in \( I^\kappa = \prod_\lambda I_\lambda \), where \( \kappa \) is the topological weight \( w(X) \) of the space \( X \). Now we define the structure \( \bar{\Sigma}_1 \) as consisting of all coverings \( \bar{\alpha} \) of the space \( X \) having the form \( \bar{\alpha} = \bar{\alpha} \cap X \), where \( \bar{\alpha} \in \bar{\Sigma}_1 \). The order \( \prec \) is defined as \( \bar{\alpha}_1 \prec \bar{\alpha}_2 \) if and only if \( \bar{\alpha}_1 \prec \bar{\alpha}_2 \) for \( \bar{\alpha}_1, \bar{\alpha}_2 \in \bar{\Sigma}_1 \). It is obvious that \( \Sigma_1 \) satisfies conditions (i), (ii) and it is also evident that condition (iii) of the lemma is valid too. The lemma is proved. \( \square \)

**Remark.** It may happen that \( \alpha_1 \prec \alpha_2 \) (i.e., \( \alpha_2 \) is a refinement of \( \alpha_1 \)) although none of the relations \( \bar{\alpha}_1 \prec \bar{\alpha}_2 \), \( \bar{\alpha}_2 \prec \bar{\alpha}_1 \) are true. This means that the usual relation \( \prec \) is finer than the relation \( \prec^\prime \).

**Lemma 2.** Let \( Y \) be the compactification of \( X \) corresponding to a uniform structure \( \Sigma \) of the space \( X \). Then the structure \( \bar{\Sigma}_1 \) of Lemma 1 can be chosen so that the power \( |\bar{\Sigma}_1| \) of \( \Sigma_1 \) will be equal to \( \kappa = w(Y) \).

**Proof.** According to the proof of Lemma 1 the structure \( \Sigma_1 \) of \( X \) (as well as the corresponding structure of \( Y \)) is the restriction to \( X \) (equivalently to \( Y \)) of the uniform structure \( \bar{\Sigma}_1 \) of the compact space \( I^\kappa \) having the topological weight \( \kappa = w(Y) \), hence \( |\bar{\Sigma}_1| \leq |\Sigma_1| \). Since for any index \( \lambda \) the structure \( \Sigma_\lambda \) of the segment \( I_\lambda \) in the product \( I^\kappa = \prod_\lambda I_\lambda \) is countable then \( |\bar{\Sigma}_1| = \kappa \), hence \( |\Sigma_1| \leq \kappa \). But the collection of open sets belonging to all coverings of the structure of \( Y \) which isomorphic to \( \Sigma_1 \) forms a base of open sets of \( Y \) hence \( |\Sigma_1| \) cannot be less than \( \kappa \). The lemma is proved. \( \square \)

**Lemma 3.** If \( X \) is a normal space with dim \( X = n \) and \( \Sigma_1 \) is a uniform structure of \( X \) as in Lemma 1, then there exist a cofinal part \( \Sigma_0 \) of the structure \( \Sigma_1 \) (with respect to order \( \prec^\prime \)) and a system \( \Sigma \) of open finite coverings of \( X \) having multiplicity \( \leq n + 1 \), such that there is a one-to-one correspondence between \( \Sigma_0 \) and \( \Sigma_1 \) satisfying the following conditions:

(i) If two coverings \( \alpha' \in \Sigma_0 \) and \( \alpha \in \Sigma \) correspond to each other then \( \alpha' \prec \alpha \) (i.e., \( \alpha \) is a refinement of \( \alpha' \)).

(ii) If \( \alpha', \beta' \in \Sigma_0 \) and \( \alpha' \prec \beta' \), then for the corresponding coverings \( \alpha, \beta \in \Sigma \) it is true that \( \alpha \prec \beta \) (i.e., \( \beta \) is a star-refinement of \( \alpha \)).

**Proof.** Let us consider pairs \( (\alpha', \alpha) \) of coverings in which \( \alpha' \in \Sigma_1 \), \( \alpha' \prec \alpha \), and \( \alpha \) is a finite open covering with multiplicity \( \leq n + 1 \). The set of such pairs is not empty (by the definition of dimension of the space \( X \)). We call a system \( \pi \) of such pairs \( (\alpha', \alpha) \) a marked system, if for \( (\alpha', \alpha), (\beta', \beta) \in \pi \) and \( \alpha' \prec \beta' \) it follows that \( \alpha \prec \beta \). Let \( S \) be the set of all marked systems \( \pi = \{(\alpha', \alpha)\} \). The set \( S \) is not empty: there are systems \( \pi \) consisting of one pair \( (\alpha', \alpha) \) (and such systems are, obviously, marked).

The set \( S \) of marked systems \( \pi \) is partially ordered by the inclusion: \( \pi_1 \prec \pi_2 \) if and only if \( \pi_1 \subseteq \pi_2 \). Let \( S' \) be an arbitrary part of \( S \) which is ordered: for \( \pi_1, \pi_2 \in S' \) there is
π₁ < π₂ or π₂ < π₁. Then obviously π₀ = \bigcup_{π \in S'} π is a marked system, π₀ \in S and π < π₀ for any π \in S'. In accordance with Zorn’s Lemma (see, for example, in [5, § 69]) for the partially ordered sets S having such property the following proposition is valid: for any \( π₁ \in S \) there exists a system π in the set S such that π₁ < π and π is a maximal marked system, i.e., for any π' \in S the condition π < π' implies π' = π.

Thus any marked system π₁ is contained in a maximal marked system π. Let π be a maximal marked system, let \( \Sigma₀ \) be the set of all \( \alpha' \) such that some pair (\( \alpha', \alpha \)) belongs to π, and let \( \Sigma \) be the set of all the corresponding coverings \( \alpha \). Since the system of coverings π is a marked system, \( \Sigma₀ \) and \( \Sigma \) satisfy all the conditions of the lemma. But we have to prove that the system \( \Sigma₀ \) is a cofinal part of \( \Sigma₁ \).

Suppose that this statement is wrong. Then in the structure \( \Sigma₁ \) there exists some covering \( \alpha'₀ \) for which there is no covering \( \alpha' \) in \( \Sigma₀ \) such that \( \alpha'₀ <^* \alpha' \). In accordance with condition (iii) of Lemma 1 \( \Sigma₁ \) contains only a finite number of coverings \( \alpha' \) for which \( \alpha' <^* \alpha'₀ \). Since \( \Sigma₀ \subseteq \Sigma₁ \), this is also valid for \( \Sigma₀ \). Let \( \alpha'_1, \alpha'_2, \ldots, \alpha'_k \) be all the coverings in \( \Sigma₀ \) for which \( \alpha'_i <^* \alpha'_0 \), \( i = 1, \ldots, k \), and let \( \alpha_1, \alpha_2, \ldots, \alpha_k \) be the corresponding coverings of the system \( \Sigma \). Since \( \dim X = n \) there exists a covering \( \alpha₀ \) of the space \( X \) with multiplicity \( \leq n + 1 \) such that \( \alpha'_0 \land \alpha_1 \land \cdots \land \alpha_k <^* \alpha₀ \). Consequently for \( \alpha'_i <^* \alpha'_0 \) in \( \Sigma₀ \) we have \( \alpha_i <^* \alpha₀ \), \( \alpha_i = 1, 2, \ldots, k \). Thus, the system \( \pi' = \pi \cup (\alpha'_0, \alpha₀) \) of pairs \( (\alpha', \alpha) \) contains π as a proper subset and π < π'. However this contradicts the fact that the system π is maximal. The lemma is proved. □

**Lemma 4.** The system of coverings \( \Sigma \) in Lemma 3 is a uniform structure of the space \( X \).

**Proof.** Let \( \alpha \) be any covering in \( \Sigma \) and let \( \alpha' \) be the corresponding covering in \( \Sigma₀ \). Since \( \Sigma₀ \) is cofinal in \( \Sigma₁ \) (Lemma 3) then according to condition (ii) of Lemma 1 there exists some \( \beta' \in \Sigma₀ \) for which \( \alpha' < \beta' \). Hence in accordance with condition (ii) of Lemma 3 it is valid that \( \alpha <^* \beta \) for the covering \( \beta \) in \( \Sigma \) corresponding to the \( \beta' \in \Sigma₀ \). This gives us that condition (W1) is satisfied.

Let \( \alpha, \beta \) be two coverings in \( \Sigma \) and let \( \alpha', \beta' \) be the corresponding coverings in \( \Sigma₀ \). Since \( \Sigma₀ \) is a cofinal part of \( \Sigma₁ \) (Lemma 3), then according to condition (ii) of Lemma 1, there exists a covering \( \gamma' \in \Sigma₀ \) such that \( \alpha' < \gamma' \) and \( \beta' < \gamma' \). In accordance with condition (ii) of Lemma 3 for the covering \( \gamma \in \Sigma \) which corresponds to \( \gamma' \) the relations \( \alpha <^* \gamma \) and \( \beta <^* \gamma \) are true. So \( \alpha \land \beta < \gamma \). Hence (W2) is satisfied.

For any point \( x \in X \) and any neighbourhood \( U \) of \( x \) in accordance with the condition (W3) of the uniform structure definition there exists a covering \( \alpha' \) in the uniform structure \( \Sigma₀ \) for which \( x \in S_{\alpha'} \subseteq U \). Let the covering \( \alpha \) in \( \Sigma \) correspond to \( \alpha' \). Since \( \alpha' < \alpha \) (see condition (i) of Lemma 3), then \( x \in S_{\alpha} \subseteq S_{\alpha'} \subseteq U \). This means that condition (W3) is valid for the system \( \Sigma \). The lemma is proved. □

Let \( \Sigma₁ \) be a uniform structure of finite open coverings of a space \( X \) which corresponds to some compactification \( Y \) of \( X \).
Proposition. The normal space $X$ has a compactification $Z$ as in the theorem if and only if $X$ has a uniform structure $\Sigma$ of finite open coverings such that:

(i) The power of $\Sigma$ is equal to $\kappa = w(Y)$.

(ii) The multiplicities of the coverings $\alpha \in \Sigma$ are $\leq n + 1$ (and cannot be less for any refinements of some coverings $\alpha \in \Sigma$).

(iii) $\Sigma_1 \subseteq \Sigma$.

Proof. Let $Z$ be a compactification of $X$ as in the theorem. Let $\beta$ be a base of open sets of $Z$ such that the power of $\beta$ is equal to $\kappa$. Let $\Sigma'$ be the system of all finite coverings $\alpha = \{U_i\}$ of $Z$ such that $U_i \in \beta$. It is clear that the power of $\Sigma'$ is equal to $\kappa$. Since $Z$ is a compact space, then for any open covering $\beta$ of $Z$ there exists a covering $\alpha \in \Sigma'$ such that $\beta < \alpha$. Therefore $\Sigma'$ is a cofinal part in the system of all open coverings of $Z$ and hence it is a uniform structure of $Z$.

For any covering $\alpha \in \Sigma'$ there exists a finite open covering $\beta$ of multiplicity $\leq n + 1$ such that $\alpha < \beta$ ($\beta$ is a refinement of $\alpha$) and for each $\alpha \in \Sigma'$ we choose exactly one such $\beta$ (including one of them having no refinements multiplicity less than $n + 1$). We should note that generally the sets in $\beta$ are not in $\beta$. Let $\Sigma''$ be the system of all such coverings $\beta$. It is clear that the power of $\Sigma''$ is equal to $\kappa$. The system $\Sigma''$ is a cofinal part of the set of all open coverings of $Z$. For any covering $\gamma$ of $Z$ there exists a refinement $\alpha$ of $\gamma$, $\alpha \in \Sigma'$, and a pair $\alpha < \beta$, $\beta \in \Sigma''$, hence $\beta$ is a refinement of $\gamma$. As in the case of $\Sigma'$ the system $\Sigma''$ is a uniform structure of $Z$.

Let $\Sigma$ be a uniform structure of $X$ consisting of all coverings $\beta \cap X$, $\beta \in \Sigma''$. Since $\Sigma$ is the uniform structure corresponding to the compactification $Z$ and $Y \subseteq Z$, then $\Sigma_1 \subseteq \Sigma$. The structure $\Sigma$ satisfies the conditions of the proposition.

Now let $\Sigma$ be a uniform structure of $X$ as in the proposition. Let $Z$ be the compactification of $X$ corresponding to this structure. For any open set $U \subseteq X$ let $O(U) = Z \setminus \overline{X \setminus U}$ (the closure operation is in $Z$). For an open covering $\beta = \{U_i\}$ let $O(\beta) = \{O(U_i)\}$. For any $\beta \in \Sigma$ the system $O(\beta)$ of the open sets $O(U_i)$ is a covering of the compactification $Z$ (see [3]). Moreover, the system $O(\Sigma)$ of all coverings $O(\beta)$, $\beta \in \Sigma$, is a uniform structure of the compactification $Z$ (see [3]), hence it is cofinal in the set of all open coverings of $Z$. Since the space $X$ is dense in $Z$, the multiplicity of each covering $O(\beta)$ is equal to the multiplicity of $\beta$, hence $\dim Z \leq n$. But the strict inequality $\dim Z < n$ is impossible otherwise for the coverings of $Z$ in $O(\Sigma)$, as well as for the coverings in $\Sigma$, refinements of multiplicity $< n + 1$ could be found. Thus $\dim Z = \dim X$.

Since $\Sigma_1 \subseteq \Sigma$, then $Y \subseteq Z$. The power of $O(\Sigma)$ is equal to the power of $\Sigma$, hence $w(Z) \leq \kappa$. But the inequality $w(Z) < w(Y)$ is impossible because there is a continuous surjection of $Z$ on $Y$ but the topological weight cannot be increased by any continuous surjections of compact spaces. So the proposition is proved. □

Proof of the theorem. Let $\Sigma'$ be a uniform structure of the space $X$ corresponding to the compactification $Y$ and let $\Sigma_1$ be a uniform structure satisfying the conditions of Lemma 1 and equivalent to $\Sigma'$. In accordance with the previous proposition to prove the theorem it is sufficient to show that the uniform structure $\Sigma$ of the space $X$ inherited for $\Sigma_1$ in
Lemmas 3 and 4 satisfies all the assertions of the proposition. The power of Σ is equal to the power of Σ₀, but the power of Σ₀ is κ = w(Y). Indeed the structure Σ₀ is a part of Σ₁ but the power of Σ₁ is equal to κ by Lemma 2, hence |Σ₀| ≤ κ. Furthermore, the collection of open sets belonging to all coverings of Σ₀ is in one-to-one correspondence with the analogous collection from O(Σ₀) which composes an open base of the compact space Y, hence the strict inequality |Σ₀| < κ = w(Y) is impossible.

According to Lemma 3 the multiplicities of all coverings α ∈ Σ are less or equal n + 1 and some covering α ∈ Σ has no refinements with less than n + 1 multiplicities. Thus, the structure Σ satisfies condition (ii) of the proposition. Condition (iii) of the proposition is a consequence of condition (i) of Lemma 3 and of the equivalent of Σ₀ and Σ₁. The theorem is proved. □

Acknowledgements

The authors thank the referee for the useful remarks and comments. This work was supported by TUBITAK.

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