Sampling theorem for bandlimited Hardy space functions generated by Regge problem

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\textbf{A B S T R A C T}

The Whittaker–Shannon–Kotel’nikov (WSK) sampling theorem provides a reconstruction formula for the Paley–Wiener class of bandlimited functions. It is known that the WSK sampling theorem can be obtained from a sampling theorem associated with a linear, self-adjoint, first order boundary-value problem.

In the present paper, a generalization of the WSK sampling theorem is extended to the Hardy space of functions in the upper half-plane, $H^2_+$. A notion of bandlimitedness in $H^2_+$ is defined and a sampling theorem for this class of bandlimited functions is obtained.

Analogous to the Paley–Wiener class of bandlimited functions, which is the image under the Fourier transformation of functions with compact supports, the class of bandlimited functions in the Hardy space is the image of vector-functions with compact supports under an integral transformation, which realizes the incoming spectral representation in scattering theory. The boundary-value problem that generates the sampling points is of the second order and non-self-adjoint (the Regge problem). Its eigenvalues (the sampling points) are known in scattering theory as resonances and eigenfunctions are related to the resonance states. It is shown that the sampling points are complex numbers in the upper half-plane that are uniformly separated and symmetric with respect to the imaginary axis.

One of the novelties of the paper is that it sheds light on a connection between three different subjects that do not seem to have much in common, i.e., sampling theorems, $H^p$ spaces, and the Lax–Phillips scattering theory.

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1. Introduction

The Whittaker–Shannon–Kotel’nikov (WSK) sampling theorem can be stated as follows:

\textbf{Theorem 1 (Whittaker–Shannon–Kotel’nikov).} If a function $f$ is bandlimited to $[-\sigma, \sigma]$ in the sense of Paley–Wiener, i.e., it is representable as

\[ f(t) = \int_{-\sigma}^{\sigma} e^{-ixt} g(x) \, dx, \quad t \in \mathbb{R}, \quad (1.1) \]

for some function $g \in L^2(-\sigma, \sigma)$, then $f$ can be reconstructed from its samples, $f(k\pi/\sigma)$. The construction formula is

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E-mail addresses: azayed@condor.depaul.edu (A.I. Zayed), marianna.shubov@euclid.unh.edu (M.A. Shubov).
\[ f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\sigma}\right) \frac{\sin(\sigma t - k\pi)}{(\sigma t - k\pi)}, \quad t \in \mathbb{R}, \]  

with the series being absolutely and uniformly convergent on compact subsets of \( \mathbb{R} \). See, e.g., [26, p. 16].

The Paley–Wiener space of functions bandlimited to \([-\sigma, \sigma]\) will be denoted by \( PW_\sigma \). The points \( t_k = k\pi/\sigma \) are called the sampling points and the functions

\[ S_k(t) = \frac{\sin\sigma(t-t_k)}{\sigma(t-t_k)} = \text{sinc}(\sigma(t-t_k)/\pi), \quad \text{sinc}(z) = \begin{cases} \sin\pi z/\pi z, & z \neq 0, \\ 1, & z = 0, \end{cases} \]

are called the sampling functions. The series in Eq. (1.2) can be written as a Lagrange interpolation formula

\[ f(t) = \sum_{k=-\infty}^{\infty} f(t_k) \frac{G(t)}{(t-t_k)G'(t_k)}, \]  

where \( t_k = k\pi/\sigma \) and \( G(t) = \sin\sigma t = \sigma t \prod_{k=1}^{\infty} (1 - t^2/t_k^2) \).

The series (1.3) is a special case of the Lagrange interpolation formula

\[ \sum_{k} f(z_k) \frac{G(z)}{G'(z_k)(z-z_k)}, \]  

where \( G(z) \) is an entire function whose zeros are located exactly at the points \( \{z_k\} \). The range of \( k \) is usually either the integers or the non-negative integers. The value \( t_0 \) is often reserved for \( t_0 = 0 \).

There are a number of generalizations of the WSK theorem, some of which are akin to boundary-value problems involving linear differential operators. It is known that the WSK sampling theorem can be obtained from a sampling theorem associated with a linear, self-adjoint, first order boundary-value problem. More generally, any regular, self-adjoint, boundary-value problem with simple eigenvalues associated with \( n \)th order linear differential operator gives rise to a sampling theorem for a class of entire functions of order \( 1/n \); see [2,23].

In the present work, we generalize the WSK theorem in a different direction by extending it to a class of functions that is in some sense more general than the class of entire functions, namely to the Hardy space \( H^2_+ \) of functions analytic in the upper-half plane (lower half-plane).

In order to introduce a notion of bandlimitedness in \( H^2_+ \), we need a specific integral operator that comes from the following boundary-value problem. Consider a one-dimensional wave equation on the semi-axis \((0, \infty)\) with a non-smooth density coefficient. We then rewrite this problem as an evolution equation. In our case, the dynamics generator for the evolution problem, \( \mathcal{L} \), is a non-self-adjoint matrix differential operator in a two-component Hilbert space, \( \mathcal{H} \), of Cauchy data.

The Regge problem consists of investigating the spectrum of \( \mathcal{L} \) and proving expansions with respect to its eigenfunctions. It is an important observation in the Lax–Phillips scattering theory that the physical evolution problem is equivalent to the model operator (the shift operator) restricted to a specific subspace \( \mathcal{K} \) of the Hardy space \( H^2_+ \). \( K = H^2_+ \ominus \mathcal{S}(k)H^2_+ \), with \( \mathcal{S}(k) \) being the scattering matrix. It turns out that the operator \( T \), realizing such a representation of the evolution problem, can be constructed explicitly as an integral operator whose kernel is associated with the Regge problem. The operator \( T \) plays a central role in the sampling theorem because it is used to introduce a space of bandlimited functions in \( H^2_+ \). In addition, the Regge problem delivers two more important results: (a) the set of the resonances, i.e., the set of complex points \( \{k_n\}_{n \in \mathbb{Z}} \) for which the Regge problem has non-trivial solutions, is exactly the set of the sampling points in our sampling theorem, and (b) the resonances are the eigenvalues of the operator \( \mathcal{L} \); the corresponding eigenfunctions are called the resonance states. They form a Riesz basis in a specific subspace \( \mathcal{K} \) of the state space \( \mathcal{H} \). It is the operator \( T \) that connects \( \mathcal{K} \subset \mathcal{H} \) and \( \mathcal{K} \subset H^2_+ \), i.e., \( T(\mathcal{K}) = \mathcal{K} \). This relation is of crucial importance for the sampling theorem.

The sampling points in the WSK theorem are real and uniformly distributed. In a more general setup, the sampling points can be complex but their real parts cannot deviate very far from the integers. Unlike the sampling points for the class \( PW_\sigma \) of bandlimited functions, the sampling points for the bandlimited Hardy space functions are complex numbers in the upper half-plane that are symmetric with respect to the imaginary axis and satisfy the Carleson condition. Furthermore, their imaginary parts tend to a constant \( C > 0 \).

One of the novelties of this paper is that it sheds light on a connection between three subjects that do not seem to have much in common, sampling theory, \( H^p \) spaces, and the Lax–Phillips scattering theory.

The paper is organized as follows. For the reader’s convenience, in Section 2 we survey some facts related to Kramer’s sampling theorem and the boundary-value problems associated with Lagrange-type interpolation. Section 3.1 deals with subsets of the Hardy space \( H^2_+ \) and their interpolation sequences and in Section 3.2 we introduce the initial boundary-value problem for the wave equation with a non-smooth density coefficient. We then rewrite this problem as an evolution equation in the state space and describe its dynamics generator whose eigenfunctions play an important role in the construction of the integral transformation \( T \).
In Section 3.3, we outline the main elements of the Lax-Phillips scattering theory, i.e., we describe explicitly the incoming $D_-$ and outgoing $D_+$ subspaces of the state space $\mathcal{H}$ of Cauchy data. We introduce an important subspace $\mathcal{K} \subset \mathcal{H}$, which is orthogonal to both $D_-$ and $D_+$, i.e., $\mathcal{K} = \mathcal{H} \oplus [D_+ \oplus D_-]$. It is the subspace $\mathcal{K}$ that plays a key role in the definition of the bandlimitness. We introduce the Jost solution, $F$, and construct the scattering matrix $S$. Section 3.4 deals with the functional model induced by the standard shift operator restricted to a specific subspace $\mathcal{K}$ of the Hardy space $H^2_1$, $K = H_1 + S H^2_1$. If one takes $S$ in the representation for $K$ precisely the same as the scattering matrix from Section 3.3, then the two pictures are connected. This connection is established in Section 4.1, where we construct an integral operator $T$ realizing the incoming spectral representation with the following important property: $T(\mathcal{K}) = \mathcal{K}$. Finally in Section 4.2 we conclude the paper by proving the sampling theorem.

To derive the Lagrange-type interpolation formula, we need the Riesz basis property of the resonance states in the energy space of Cauchy data for our initial boundary-value problem. This result, which is important in its own right, follows uniformly on compact subsets of $\pi$. The spectrum of the generator of the above semi-group is completely determined by the properties of $S$. The characteristic function of the generator is precisely the function $S$ that occurs in the formula for the subspace $\mathcal{K}$. We refer to the monograph [17] for a comprehensive survey on the shift operator.

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The novelty of Kramer’s work is not so much in the proof of his theorem as in his observation of how to generate the WSK sampling theorem is a special case of Kramer’s sampling theorem. Indeed, if $I = [-\sigma, \sigma]$, $K(x, t) = e^{\text{i}xt}$, and $t_n = \frac{\pi}{\sqrt{\sigma}}$, then $[e^{\text{i}xt_n}]_{t_n = -\infty}^{\infty}$ is a complete orthogonal set in $L^2(I)$, and $S_n^*(t) = \text{sinc}(\sigma(t - t_n)/\pi)$. Hence Eqs. (2.1) and (2.2) are reduced to (1.1) and (1.2).

The novelty of Kramer’s work is not so much in the proof of his theorem as in his observation of how to generate the kernel function $K(x, t)$ and the sampling points $\{t_n\}_{n \in \mathbb{Z}}$. Kramer noticed that the kernel function and the sampling points could be found from certain boundary-value problems. More precisely, consider the differential operator $L$

$$L = p_0(x) \frac{d^n}{dx^n} + \cdots + p_{n-1}(x) \frac{d}{dx} + p_n(x), \quad x \in I,$$
where \( p_k(x) \) is a complex-valued function with \( n - k \) continuous derivatives, \( k = 0, 1, \ldots, n \), for any \( x \in I = (a, b) \), and \( p_0(x) \neq 0 \) for any \( x \in (a, b) \), with \( -\infty < a < b < \infty \) and let \( U_j(y) \) be linear forms defined by

\[
U_j(y) = \sum_{k=1}^{n} (\alpha_{j,k} y^{(k-1)}(a) + \beta_{j,k} y^{(k-1)}(b)), \quad j = 1, 2, \ldots, n.
\]

Consider the following regular, self-adjoint boundary-value problem:

\[
Ly = -ty, \quad x \in I, \tag{2.4}
\]

\[
U_j(y) = 0, \quad j = 1, \ldots, n. \tag{2.5}
\]

The eigenfunctions of such a problem form a complete orthogonal family in \( L^2(I) \). [15, p. 82], Kramer's observation is that if the problem (2.4)–(2.5) possesses a function \( \phi(x, t) \) that generates the eigenfunctions of the problem \( \{\phi_n(x)\} \) when the eigenvalue parameter \( t \) is replaced by the eigenvalues \( \{\lambda_n\} \), i.e., \( \phi(x, t_n) = \phi_n(x) \), then one can take the sampling points to be \( \{t_n\} \) and the kernel function \( K(x, t) \) to be \( \phi(x, t) \). For example, the WSK theorem arises from a very simple first order boundary-value problem:

\[
iy' = -ty, \quad x \in (-\sigma, \sigma), \quad y(-\sigma) = y(\sigma). \tag{2.6}
\]

It is easily seen that the eigenvalues and the eigenfunctions are \( \{t_n = n\pi / \sigma\} \), \( \{e^{it_n x}\} \), and the kernel function is \( K(x, t) = e^{it_n x} \).

It is customary in the theory of boundary-value problems to denote the eigenvalue parameter by \( \lambda \) and the eigenvalues by \( \lambda_n \), therefore, from now on we shall denote the sampling points by \( \lambda_n \) whenever the sampling expansion is associated with a boundary-value problem. Kramer's observation leads to the following definition.

**Definition 3.** We say that a boundary-value problem has the Kramer property if it possesses a function \( \phi(x, \lambda) \), entire in \( \lambda \), that satisfies \( L\phi(x, \lambda) = -\lambda \phi(x, \lambda) \) and generates the eigenfunctions of the problem \( \{\phi_n(x)\} \) when the parameter \( \lambda \) is replaced by the eigenvalues \( \{\lambda_n\} \), i.e., \( \phi(x, \lambda_n) = \phi_n(x) \).

Evidently, the Kramer property implies that the eigenvalues are simple. Under what conditions the boundary-value problem (2.4)–(2.5) has the Kramer property is still an open question. Some partial but important answers have been obtained in recent years and on which we shall elaborate a little more later.

### 2.2. Boundary-value problems and Lagrange-type interpolation

Although the connection between sampling theorems and boundary-value problems has been the focus of many research papers in the last few years [1–3,7–9,14,16,24,25,28], our focus is more on the connection between boundary-value problems and Lagrange-type interpolation.

The series in (2.2) and (2.3) does not resemble, and in general, is not a Lagrange-type interpolation series since it cannot always be put in the form (1.4). Nevertheless, if the sampling points and functions are obtained from the self-adjoint boundary-value problem (2.4)–(2.5), then (2.2) can be brought closer to the Lagrange-type interpolation series, provided that the problem has the Kramer property. For, if \( \phi(x, \lambda) \) is a function that generates the eigenfunctions of the problem, then \( L\phi(x, \lambda) = -\lambda \phi(x, \lambda) \), \( L\phi_n(x) = -\lambda_n \phi_n(x) \), and hence because the eigenvalues are real

\[
\int_{a}^{b} \left[ \phi_n(x) L\phi(x, \lambda) - \phi(x, \lambda) \overline{L\phi_n(x)} \right] dx = (\lambda_n - \lambda) \int_{a}^{b} \phi(x, \lambda) \overline{\phi_n(x)} \; dx.
\]

But on the other hand, by Lagrange's identity [5, p. 80] for differential operators, we have for any two functions \( u(x) \) and \( v(x) \) in \( C^n[a, b] \),

\[
\overline{v(x)} Lu(x) - u(x) \overline{Lv(x)} = \frac{d}{dx} [u(x), v(x)], \tag{2.6}
\]

where

\[
[u, v](x) = \sum_{m=1}^{n} \sum_{j+k=m-1} (-1)^j u^{(k)}(x) \left( p_{n-m}(x) \overline{v(x)} \right)^{(j)}, \quad j, k \geq 0.
\]

Therefore,

\[
G_{k}(\lambda) = (\lambda_k - \lambda) \int_{a}^{b} \phi(x, \lambda) \overline{\phi_k(x)} \; dx = [\phi, \phi_k](b) - [\phi, \phi_k](a). \tag{2.7}
\]
which implies that $G_k$ is entire because $\phi(x, \lambda)$ is an entire function in $\lambda$. Clearly, $G_k(\lambda_k) = 0$ and moreover, $G_k(\lambda_m) = 0$ if $k \neq m$ by the orthogonality of the eigenfunctions $\{\phi_k(\lambda)\}$. Hence $G_k(\lambda_m) = 0$ for all $m$. Differentiating $G_k(\lambda)$ leads to

$$G'_k(\lambda) = (\lambda_k - \lambda) \int_a^b \frac{\partial \phi(x, \lambda)}{\partial \lambda} \phi_k(x) \, dx - \int_a^b \phi(x, \lambda) \overline{\phi_k(x)} \, dx,$$

and by setting $\lambda = \lambda_k$, we obtain

$$G'_k(\lambda_k) = -\|\phi_k\|^2.$$  \hfill (2.8)

Hence, by combining (2.7), (2.8) and (2.3), we obtain

$$S^*_k(\lambda) = \frac{G_k(\lambda)}{(\lambda - \lambda_k)G'_k(\lambda_k)},$$

which in turn, upon its substitution in (2.2), leads to

$$f(\lambda) = \sum_k f(\lambda_k) \frac{G_k(\lambda)}{(\lambda - \lambda_k)G'_k(\lambda_k)}.$$  \hfill (2.9)

Eq. (2.9) is similar, but not exactly the same as (1.4) since in (1.4) all the functions, $G_k(\lambda)$, are the same and equal to a function $G(\lambda)$, which, without loss of generality, may be taken as the canonical product of its zeros; see [26, p. 177]. Therefore, if

$$G(\lambda) = \begin{cases} \prod_j (1 - \frac{\lambda}{\lambda_j}) & \text{if zero is not an eigenvalue,} \\ \lambda \prod_j (1 - \frac{\lambda}{\lambda_j}) & \text{if zero is an eigenvalue,} \end{cases}$$

then the series (2.9) becomes a Lagrange-type interpolation series

$$f(\lambda) = \sum_k f(\lambda_k) \frac{G(\lambda)}{(\lambda - \lambda_k)G'_k(\lambda_k)}.$$  \hfill (2.10)

For the convergence of the infinite products, see [26, p. 173]. The above discussion motivates the following definition.

**Definition 4.** A boundary-value problem having the Kramer property is said to possess the Lagrange-type interpolation property if its associated sampling series is a Lagrange-type interpolation series as in (2.10). We denote this class of boundary-value problems by $L$.

What kind of boundary-value problem belongs to the class $L$? It was shown by A. Zayed, P. Butzer, and G. Hinsen in [29] (see also [27] for more general results), that the following regular Sturm–Liouville problem:

$$y'' - q(x)y = -\lambda y, \quad x \in [a, b],$$

$$\cos \alpha y(a) + \sin \alpha y'(a) = 0, \quad \cos \beta y(b) + \sin \beta y'(b) = 0,$$

where $\alpha, \beta \in \mathbb{R}$ and $q \in C[a, b]$ belongs to $L$.

It was conjectured in [4] and proved in [23] that regular, self-adjoint, boundary-value problems with simple eigenvalues have the Lagrange-type interpolation property. It should be noted that the class of functions covered by sampling theorems associated with regular, self-adjoint, boundary-value problems with simple eigenvalues consists of entire functions of order $1/n$, in $\lambda$, i.e., in $f = \sqrt[n]{\lambda}$, where $n$ is the order of the differential operator.

The goals of this paper are:

1. To extend the sampling theorem to the Hardy space $H_+^2$ of functions analytic in the upper-half plane. To achieve this goal, we introduce a subspace, $K$, of $H_+^2$, on which the sampling theorem can be applied. This space $K$, which we call the spaces of bandlimited Hardy functions, plays the role of and shares similar properties to those of the class $PW_\sigma$ of bandlimited functions. Here an important point should be emphasized. Since $f \in H_+^2$ if and only if

$$f(z) = \int_0^\infty \overline{F(t) e^{i\sigma t}} \, dt,$$
for some function $F(t) \in L^2(0, \infty)$, it is tempting to declare $f$ bandlimited in $H^2_+$ if $F$ has support $[0, a]$ for some $0 < a < \infty$, or

$$f(z) = \int_0^a F(t)e^{izt} \, dt,$$

as in (1.1). However, this definition is futile because such $f(z)$ is entire and is not a genuine member of $H^2_+$. Thus, another approach is needed.

2. To describe the boundary-value problem associated with this new sampling theorem. In fact, the boundary-value problem plays a fundamental role in determining the space $K$. In view of our discussion in Section 2.2, this new boundary-value problem cannot be regular, self-adjoint, with simple eigenvalues because such a problem generates a sampling theorem associated with entire functions. In fact, we will show that the boundary-value problem is not self-adjoint and in addition it has the spectral parameter $\lambda$ appearing in the boundary conditions.

3. To shed light on a connection between three fields of mathematical analysis that do not seem to have much in common, sampling theory, $H^p$ spaces, and the Lax–Phillips scattering theory. To the best of our knowledge, this connection has not been observed before.

3. Preliminaries

For the reader’s convenience we recall some facts from the theory of $H^p$ spaces and the Lax–Phillips scattering theory that will be used in the sequel. Most of the material on $H^p$ can be found in [6,10,20].

3.1. Hardy spaces

A function $f(z)$ analytic in the upper-half plane, $\Im z > 0$, belongs to the Hardy space $H^p_+$, $1 \leq p < \infty$ if there is a constant $C$ such that

$$\int_{-\infty}^{\infty} |f(x+iy)|^p \, dx \leq C,$$

for all $y > 0$. The class $H^\infty_+$ consists of all bounded analytic functions in the upper-half plane. The Hardy space $H^p_-$ of functions analytic in the lower half plane can be defined similarly by replacing $y > 0$ by $y < 0$.

Let $f \in H^p_+$ and $\{z_k\}$ be its zeros, where $\Im z_k > 0$. Then [6, p. 191] $f(z) = b(z)g(z)$ where $g$ is a non-vanishing function in $H^p_+$ with $|g(x)| = |f(x)|$ a.e., and $b(z)$ is the Blaschke product

$$b(z) = \left(\frac{z-i}{z+i}\right)^m \prod_k \frac{|z_k^2 + 1|}{z_k^2 + 1} \frac{z - z_k}{z - \overline{z_k}},$$

where $m$ is a non-negative integer, $z_k \neq i$. The Blaschke product, which exists for $f \in H^p_+$, $1 \leq p < \infty$, converges in $\Im z > 0$ if and only if

$$\sum_k \frac{\Im z_k}{1 + |z_k|^2} < \infty.$$

If $f \in H^p_+$, then $f(x) = \lim_{y \to 0^+} f(x + iy)$ exists a.e. If $f \in H^2_+$, and we denote the Fourier transform of its boundary function by $\hat{f}$, then $\hat{f}(t) = 0$ for almost all $t < 0$. A theorem of Paley and Wiener states that $f \in H^2_+$ if and only if

$$f(z) = \int_0^\infty F(t)e^{izt} \, dt,$$

for some function $F \in L^2(0, \infty)$, and $F = \hat{f}$.

Let $A$ be a class of functions analytic in some domain $\mathcal{D}$ and $Z = \{z_n\}$ be a set of points in $\mathcal{D}$. $Z$ is said to be a zero set for $A$ if there is a nonzero function $f \in A$ that vanishes exactly at $\{z_n\}$, i.e., $f(z_n) = 0$ for all $n$ and $f(z') \neq 0$ if $z' \neq z_n$ for all $n$. A sequence $Z$ is said to be a uniqueness set for $A$ if the only function $f \in A$ that vanishes on $Z$ is the zero function. This implies that if $Z$ is a uniqueness set for a linear space $A$, $f \in A$ and $f(z_n) = g(z_n)$ for all $n$, then $f \equiv g$. It also follows from the definitions that a sequence is a set of uniqueness if and only if it is not a zero set.

A sequence $Z = \{z_n\}$ is said to be an interpolation sequence for $A$ if for a given sequence of complex numbers $W = \{w_n\}$ there is a function $f \in A$ such that $f(z_n) = w_n$ for all $n$. For the problem to have a solution, the sequence $W$ must satisfy
conditions consistent with the class $A$. Conditions on $Z$ and $W$ to guarantee that the interpolation problem has a solution in $H^p_+$ can be found in [10, Ch. IX]. For $H^p_+$, $1 \leq p \leq \infty$, every interpolation sequence is a zero set.

A sequence $Z = [\lambda_n]_{n \in \mathbb{Z}}$ in the upper half-plane is said to be uniformly separated if there is a number $\delta > 0$ such that

$$\inf_{k \in \mathbb{Z}} \prod_{j=-\infty, j \neq k}^{\infty} \left| \frac{\lambda_k - \lambda_j}{\lambda_k - \lambda_j} \right| \geq \delta > 0.$$ 

This condition is also called the Carleson condition in reference to L. Carleson who has shown that $Z$ is uniformly separated if and only if it is an interpolation sequence for $H^\infty_+$.

Here we note that the function $\sin \pi z$ shows that the set of integers $\mathbb{Z}$ is a zero set for the class $E_\pi$ of entire functions of exponential type $\pi$, nevertheless, $\mathbb{Z}$ is a uniqueness set for a subclass of $E_\pi$, namely the class of bandlimited functions, $PW_\pi$. Similarly, we shall show that although uniformly separated sequences are zero sets for $H^2_+$, they are uniqueness sets for the subclass, $K$, of $H^2_+$.

### 3.2. Scattering theory

The propagation of acoustic waves in a nonhomogeneous three-dimensional medium is governed by the wave equation; see [12].

$$\rho(x)u_{tt} - \Delta u = 0, \quad (3.1)$$

where $\Delta$ is the Laplacian, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\rho$ is the density of the medium and the modulus of elasticity of the medium is assumed to be constant and equal to 1. Let $a > 0$ and consider Eq. (3.1) in the case where $\rho$ is spherically symmetric: $\rho = \rho(r), \quad r = ||x|| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. We assume that $\rho(r) = 1$ for $r > a$. Furthermore, we assume that $\rho$ is smooth and positive for $0 < r < a$ and has either a finite jump on the sphere $r = a$, but such that $\rho(a - 0) \neq 0$, or it has a singularity corresponding to infinite rarefaction ($\rho(a - 0) = 0$), or infinite condensation ($\rho(a - 0) = \infty$).

After separation of variables in spherical coordinates and setting $u(x, t) = Y_\ell(\phi, \theta)u_\ell(r, t)$, where $Y_\ell(\phi, \theta)$ is the spherical harmonic of degree $\ell$, Eq. (3.1) splits into an infinite sequence of one-dimensional wave equations. In fact, if we look for a solution of (3.1) in the form $u(x, t) = e^{ikt}Y_\ell(\phi, \theta)y(r)/r$, and then replace $u_\ell$ by the function $v_\ell(r, t) = ru_\ell(r, t)$, we obtain

$$\frac{\partial^2 v_\ell}{\partial t^2} + L_\ell v_\ell = 0, \quad \ell = 0, 1, 2, \ldots, \quad (3.2)$$

where $L_\ell$ is the differential operator

$$L_\ell v = -\frac{1}{\rho(r)} \left[ \frac{\partial^2 v}{\partial r^2} - \frac{\ell(\ell + 1)}{r^2} v \right]. \quad (3.3)$$

Each equation in (3.2) is called a partial wave equation with $L_\ell$ being a partial operator. Consider the following boundary-value problem for the differential operator $L_\ell$:

$$L_\ell y(r) = k^2 y(r), \quad r \in [0, \infty), \quad (3.4)$$

$$y(0) = 0, \quad (3.5)$$

$$y(r) = e^{ika}H^2_{\ell+1/2}(kr)\sqrt{kr}, \quad \text{for } a < r, \quad (3.6)$$

where $H^2_{\ell+1/2}(z) = \sqrt{\pi/2} \exp[i(\ell/2 + 1/4 + \pi/4)]\mathcal{H}_{\ell+1/2}(z)$ and $\mathcal{H}_{\ell+1/2}(z)$ is the Hankel function of the second type.

The first condition follows from the fact that $u(x, t)$ must be bounded at $x = 0$, while the second condition is equivalent to the Sommerfeld radiation condition $y(r) = e^{-ik(r-a)}(1 + 0(r^{-1}))$; see [18,19]. If $\rho(r) = 1$ in the region $r > a$, Eq. (3.4) has two linearly independent solutions which are proportional to the Hankel functions $H^1_{\ell+1/2}(kr)\sqrt{kr}, \quad j = 1, 2$, of the first and second types respectively. These two solutions differ in their behavior at infinity. We choose the second type which will give us resonances in the upper half-plane.

**Definition 5.** We say that a complex number $\tilde{k}$ is a resonance if there exists a value $l = 0, 1, 2, \ldots$, such that the corresponding problem (3.4)-(3.6) with $k = \tilde{k}$ has a non-trivial solution. The corresponding solution is called a quasi-eigenfunction.

Throughout the rest of this article we will focus on the case $\ell = 0$ and relegate the other cases to later work. The boundary-value problem (3.4)-(3.6) now takes the following form of the Sturm–Liouville problem:

$$-\frac{1}{\rho(r)} y''(r) = k^2 y(r), \quad r \in [0, \infty), \quad (3.7)$$

$$y(0) = 0, \quad (3.8)$$

$$y(r) = e^{ika(r)}, \quad a \leq r. \quad (3.9)$$
Recall that (3.7)–(3.9) is not a standard Sturm–Liouville problem since the spectral parameter, $k$, enters the second boundary condition. The last condition may be replaced with the well-known impedance condition: $(y' + iky)(a) = 0$.

In our work the density $\rho$ might be singular either at the end-points of the interval $[0,a]$ or at some interior points. However, it is a well-established fact that the asymptotical distribution of the resonances strongly depends on the density behavior in vicinities of the end-points; see [18,19,21]. This is exactly the reason for the classification below. The weight function $\rho$ is assumed to possess the following properties:

1. $\rho \in C^2(0, a)$ and $\rho$ may have certain singularities at $r = 0$ and $r = a$.
2. One of the following three conditions is satisfied in a vicinity of $r = 0$:
   - (a) $\rho(0) > 0$, $\rho \in C^2[0, a)$;
   - (b) $\rho(r) = r^{2\theta - 2}\rho_1(r)$, where $1 < 2q < 2$, $\rho_1 \in C^2[0, a)$, and $\rho_1(r) > 0$ for $r \in (0, a]$;
   - (c) the same conditions as (b), but $1 < q < \infty$.
3. One of the following three conditions is satisfied in a vicinity of $r = a$:
   - (a) $\rho(a) > 0$, $\rho \in C^2[0, a)$, $\rho(a - 0) \neq 1$;
   - (b) $\rho(r) = (a - r)^{2\theta - 2}\rho_1(r)$, where $1 < 2q < 2$, $\rho_1 \in C^2[0, a)$, and $\rho_1(r) > 0$ for $r \in (0, a]$;
   - (c) the same conditions as (b), but $1 < q < \infty$.

In cases 2(b) and 3(b), $\rho$ has a singularity at $r = 0$ and $r = a$, which physically means there is an infinite condensation of matter; in the cases 2(c) and 3(c), $\rho$ has a zero at $r = 0$ and $r = a$, which physically means an infinite rarefaction of matter.

**Definition 6.** We say that $\rho$ is a density function of type I if it satisfies condition (1), one of the conditions 2(a), 2(b) or 2(c), and condition 3(a). Types II and III are defined by the same conditions except that 3(a) is replaced by 2(b) or 3(c) respectively.

The above definition reflects the fact that the leading term in the asymptotic representation of the resonances does not depend on the behavior of the density at $r = 0$ and is uniquely determined by the behavior of $\rho$ in a vicinity of $r = a$.

Since the resonances play an important role in the sampling theorem, we will recall some of their properties. The problem (3.7)–(3.9) defines a countable set of resonances $\{\lambda_n\}_{n=1}^\infty$, such that $\Im \lambda_n > 0$, with $\Im \lambda_n \to C$ and $\{|n|\Im \lambda_n| \to \infty\}$ as $|n| \to \infty$. It turns out that if $0 < \rho(a - 0) \neq 1$, then $C > 0$, but if $\rho(a - 0) = 0$ or $\rho(a - 0) = \infty$, then $C = 0$; see [21,22].

The resonances are symmetric with respect to the imaginary axis, lie in a strip parallel to the real axis, and are isolated with $\inf_{n>0} |k_n - k_0| > 0$. They are simple, except possibly for a finite number of them. They are uniformly separated, i.e., they satisfy the Carleson condition

$$\inf_i \prod_{j=\infty, j \neq i}^{\infty} \frac{|k_j - k_i|}{|k_j - k_i|} = \delta > 0. \quad (3.10)$$

Equally important to the resonances are the resonance states, which also play a fundamental role in the derivation of the sampling theorem. The resonance states are two-component functions which belong to the energy space and can be constructed from the quasi-eigenfunctions. They are generated from the vibrating string equation

$$u_{tt}(t, r) - (\rho(r))^{-1}u_{rr}(t, r) = 0, \quad 0 \leq r \leq a < \infty, \quad (3.11)$$

and form a Riesz basis in the corresponding Hilbert space.

Let us consider the set of smooth two-component functions $V = (v_0, v_1)$ (data) on the interval $[0, a]$ such that the component $v_0$ vanishes in a neighborhood of the point 0. The closure of this set with respect to the energy metric

$$\|V\|^2_{\mathcal{H}_a} = \frac{1}{2} \int_0^a \left[ |v_0'(r)|^2 + (\rho(r)|v_1(r)|^2 \right] dr \quad (3.12)$$

is a Hilbert space with an inner product

$$\langle V, U \rangle_{\mathcal{H}_a} = \frac{1}{2} \int_0^a \left[ v_0'(r)\bar{u}_0'(r) + \rho(r)v_1(r)\bar{u}_1(r) \right] dr. \quad (3.13)$$

which we denote by $\mathcal{H}_a$. In $\mathcal{H}_a$ we consider an operator $\mathcal{L}$ defined by the differential expression

$$\mathcal{L} = -\frac{1}{\rho} \frac{d^2}{dr^2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad (3.14)$$

on the domain.
where $H^i(0,a)$, $i = 1,2$, being the standard Sobolev spaces.

We note that $\mathcal{L}$ is a non-self-adjoint operator on $\mathcal{H}_0$. It can be verified directly that its adjoint is defined by the same differential expression (3.14) on the domain

$$\mathcal{D}(\mathcal{L}^*) = \{ V \in \mathcal{H}_0: v_0 \in H^2(0,a), v_1 \in H^1(0,a), v_1(0) = 0, v_0'(a) + v_1(a) = 0 \}.$$  

(3.15)

3.3. Lax–Phillips scheme

The operator $\mathcal{L}$ naturally arises in the Lax–Phillips development of scattering theory as an infinitesimal generator of a contraction semigroup associated with the scattering problem for a wave equation. Let us briefly outline this scattering problem. Let $\rho$ be an extension of the density function $\tilde{\rho}$ to the positive semi-axis, i.e., $\tilde{\rho}(r) = \rho(r)$ for $r \in [0,a]$ and $\tilde{\rho}(r) = 1$ for $r \in (a,\infty)$; see (3.11). Let us consider the following wave equation on the semi-axis, $r \in [0,\infty)$:

$$u_t(t,r) - (\tilde{\rho}(r))^{-1} u_{rr}(t,r) = 0.$$  

(3.17)

Eq. (3.17) generates a group $U(t)$ of transformations of the Cauchy data

$$U_t: (u(0,r), u_t(0,r)) \rightarrow (u(t,r), u_t(t,r)), \quad -\infty < t < \infty.$$  

(3.18)

The group $U(t)$ is unitary in the space $\mathcal{H}$ of data obtained by completion with respect to the energy metric of the smooth data that vanish near $r = 0$ and at infinity

$$\|\mathcal{U}\|^2_{\mathcal{H}} = \frac{1}{2} \int_0^\infty \left[ |u_0'(r)|^2 + \tilde{\rho}(r)|u_1(r)|^2 \right] dr.$$  

(3.19)

To simplify the notation, in what follows we will use $\rho$ to denote the density and keep in mind that $\rho = 1$ for $r > a$.

Let us equip (3.17) with the following boundary and initial conditions:

$$u(t,0) = 0, \quad u(0,r) = u_0(r), \quad u_t(0,r) = u_1(r).$$  

(3.20)

Following [12], we introduce in $\mathcal{H}$ the outgoing, $\mathcal{D}_+$, and incoming, $\mathcal{D}_-$, subspaces. $\mathcal{D}_+$ is a subspace of all initial data such that the corresponding solution of Eq. (3.17) with (3.20) vanishes for $r - t \leq a$; $\mathcal{D}_-$ is a subspace of all initial data $(u_0, u_1)$ such that the corresponding solution of (3.17) and (3.20) vanishes for $r + t \leq a$. The subspaces $\mathcal{D}_{\pm}$ have the following properties:

(i) $U_t \mathcal{D}_+ \subset \mathcal{D}_+$ for $t \geq 0$;  
(ii) $\bigcap_{t \geq 0} U_t \mathcal{D}_+ = \bigcap_{t \leq 0} U_t \mathcal{D}_- = \emptyset$;  
(iii) $\bigcup_{t \in \mathbb{R}} U_t \mathcal{D}_\pm = \mathcal{H}$;  
(iv) $\mathcal{D}_+ \perp \mathcal{D}_-.$  

(3.21)

The proof of these properties can be found in [18,21]. It turns out that the subspace $\mathcal{K}$ defined by

$$\mathcal{K} = \mathcal{H} \ominus (\mathcal{D}_+ \oplus \mathcal{D}_-)$$  

(3.22)

is of a special importance to us. An explicit description of the subspaces $\mathcal{D}_-$ and $\mathcal{D}_+$ can be given as follows:

$$\mathcal{D}_- = \{ V: \text{supp} \ V \subseteq [a,\infty), v_0 \in H^1(0,\infty), v_1 = v_0' \},$$

$$\mathcal{D}_+ = \{ V: \text{supp} \ V \subseteq [a,\infty), v_0 \in H^1(0,\infty), v_1 = -v_0' \}.$$  

(3.23)

Let $\mathcal{P}$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}$. The operators

$$\mathcal{Z}_t = \mathcal{P} U_t \mathcal{K}$$  

(3.24)

form a strongly continuous semigroup of contractions on $\mathcal{K}$. Let $(i\mathcal{B})$ be an infinitesimal generator of $\mathcal{Z}_t$, i.e., $\mathcal{Z}_t = e^{i\mathcal{B}t}$. It is shown in [21] that $\mathcal{B}$ is an unbounded non-self-adjoint operator with a purely discrete spectrum. The eigenvalues of $\mathcal{B}$ are called resonances. They coincide with the zeroes of the scattering matrix in the Lax–Phillips scattering theory; see [18,19,21,22]. It is also shown in [21] that the resonances from Definition 5 coincide with the eigenvalues of $\mathcal{B}$.

In what follows, we need special solutions of Eq. (3.7).
(a) A solution of Eq. (3.7) satisfying the boundary condition (3.9) will be called the *Jost solution* and will be denoted by $F(k, r)$. The function

$$M(k) = \lim_{r \to 0} F(k, r)$$  \hspace{1cm} (3.25)

is traditionally called the *Jost function*.

(b) The set of functions \{F(k, r), F(-k, r)\} forms a fundamental set of solutions for Eq. (3.7).

(c) It is convenient to introduce another solution of the same Eq. (3.7) that satisfies the condition

$$\lim_{r \to 0} r^{-1} \Phi(k, r) = -1.$$  \hspace{1cm} (3.26)

The solution $\Phi$ satisfies the equation and the zero boundary condition at $r = 0$; condition (3.26) is just a normalization condition. Obviously, $\Phi$ is an eigenfunction of the continuous spectrum of the operator $L = (\rho(r))^{-1} \frac{d^2}{dr^2}$ considered on the semi-axis, $r \in [0, \infty)$, with the zero boundary condition at $r = 0$.

It can be readily seen that the solution $\Phi$ may be represented in the form

$$\Phi(k, r) = \frac{1}{2ik} [M(-k)F(k, r) - M(k)F(-k, r)], \quad k \in \mathbb{R}. \quad (3.27)$$

If we introduce the reflection coefficient

$$S(k) = \frac{M(k)}{M(-k)}, \quad k \in \mathbb{R}, \quad (3.28)$$

then

$$\Phi(k, r) = \frac{1}{2ik} M(-k) \left[ F(k, r) - S(k)F(-k, r) \right], \quad k \in \mathbb{R}. \quad (3.29)$$

The reflection coefficient is unitary on the real axis and meromorphic on the complex plane. Under our assumptions on the weight $\rho$, the operator $L$ has no discrete spectrum. This fact is equivalent to the Jost function $M(k)$ having no zeros for $\Im(k) \leq 0$. Using (3.28) we obtain that $S(k) = S(-k)$ that yields the symmetry of the roots of $S$ with respect to the imaginary axis.

Using classical methods of spectral analysis for differential operators [15], one can establish a bijective correspondence between the space of square-integrable functions on the interval $0 \leq k < \infty$ with the weight function $\sigma(k) = 2k^2\pi^{-1}|M(-k)|^{-2}$ and the space $L^2(0, \infty), \{g \in L^2(0, \infty) \text{ if } g\sqrt{\sigma} \in L^2(0, \infty)\}$. This correspondence is given by the formulas

$$\tilde{f}(k) = \int_0^\infty \Phi(k, x) \rho(x) f(x) \, dx, \quad f(x) = \int_0^\infty \Phi(k, x) \sigma(k) \tilde{f}(k) \, dk. \quad (3.30)$$

Let us introduce the following functions:

$$\Psi_-(k, x) = 2ik \frac{\Phi(k, x)}{M(-k)}, \quad \Psi_+(k, x) = -2ik \frac{\Phi(k, x)}{M(k)}. \quad (3.31)$$

Our goal is to rewrite (3.30) in terms of $\Psi_\pm$. To this end, let us multiply the first equation of (3.30) by the factor $(\sqrt{2\pi})^{-1}2ik(M(-k))^{-1}$ and have

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \frac{2ik}{M(-k)} \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \Psi_-(k, x) \rho(x) \tilde{f}(x) \, dx. \quad \tilde{f}(k) \quad (3.30)$$

Now let us rewrite the second equation in (3.30) in terms of $\tilde{f}$ and have

$$f(x) = \int_0^\infty \Phi(k, x) \frac{2k^2}{\pi M(-k)M(k)} \tilde{f}(k) \, dk \quad (3.30)$$

$$= \int_0^\infty \Phi(k, x) \frac{2k^2}{\pi M(-k)M(k)} \frac{\sqrt{2\pi} M(-k)}{2ik} \tilde{f}(k) \, dk \quad (3.30)$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty -\frac{2ik}{M(k)} \Phi(k, x) \tilde{f}(k) \, dk = \frac{1}{\sqrt{2\pi}} \int_0^\infty \Psi_-(k, x) \tilde{f}(k) \, dk. \quad (3.30)$$
So, the inversion formulas take the form

\[
\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \Psi_{+}(k, x) \rho(x) f(x) \, dx,
\]

\[
f(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \overline{\Psi_{+}(k, x)} \tilde{f}(k) \, dk.
\]

(3.32)

Representations (3.32) yield

\[
\frac{1}{2\pi} \rho(x) \int_{0}^{\infty} \Psi_{+}(k, x) \Psi_{-}(k, \tilde{x}) \, dk = \delta(x - \tilde{x}),
\]

\[
\frac{1}{2\pi} \int_{0}^{\infty} \overline{\Psi_{+}(k, x)} \Psi_{-}(k, x) \rho(x) \, dx = \delta(k - \tilde{k}).
\]

(3.33)

Now let us return to the semigroup \( Z_t \) defined in (3.24). In the Lax–Phillips scattering theory, the scattering matrix appears as the characteristic function of the contraction semigroup \( Z_t \). It has an analytic continuation from the upper half plane with respect to \( k \) into the entire complex plane with poles located precisely at the points \( \{ \tilde{k}_n \} \), where \( \{ \tilde{k}_n \} \) is the set of the resonances, \( \Im \tilde{k}_n > 0 \). We now discuss the so-called functional model which provides an equivalent realization of the above picture. The unitary equivalence is established by the so-called spectral representation \( T \) that maps \( \mathcal{H} \) into \( L^2(\mathbb{R}) \).

### 3.4. Functional model

Let \( U_0^t \) be defined on \( L^2(\mathbb{R}) \) by:

\[
(U_0^t f)(k) = e^{ikt} f(k).
\]

(3.34)

Let \( S \) be a function holomorphic in the upper half-plane, that admits a representation as a Blaschke product [13,21] (i.e. \( S \) is a pure inner function with no singular factor)

\[
S(k) = \prod_{n=-\infty}^{\infty} \frac{\tilde{k}_n}{\tilde{k}_n - k}.
\]

(3.35)

In the subspace \( K = H^2_+ \oplus S(k)H^2_- \) (which is trivial only if \( S \) is a unitary constant) we consider a strongly continuous semigroup of contractions

\[
Z_t = P_K U_0^t |_K, \quad t \geq 0.
\]

(3.36)

If \( B \) is the generator of \( Z_t \), then the adjoint semigroup is given by the formula

\[
Z_t^* = e^{-iB^*t}, \quad t \geq 0.
\]

(3.37)

It can be seen that properties (3.21), which are satisfied by the subspaces \( D_+ \) and \( D_- \) of \( \mathcal{H} \), are also satisfied by the two subspaces, \( D_+ = S(k)H^2_+ \) and \( D_- = H^2_- \), which are the outgoing and incoming subspaces for the group \( U_0^t \). These two subspaces of \( L^2(\mathbb{R}) \) are orthogonal and generate \( L^2(\mathbb{R}) \):

\[
\bigcup_{t<0} e^{ikt}D_+ = \bigcup_{t>0} e^{ikt}D_- = L^2(\mathbb{R}).
\]

The orthogonal complement \( K = L^2(\mathbb{R}) \oplus [D_+ \oplus D_-] \) is called the spectral invariant subspace of the problem. The semigroup

\[
Z_t = P_K e^{ikt} |_K
\]

(3.38)

is the model semigroup with the characteristic function \( S \); see [18,21].

Let \( \tilde{k}_n \) be a simple root of \( S \), then the normalized eigenfunctions of the operators \( B \) and \( (-B^*) \) corresponding to the eigenvalues \( k_n \) and \( (-\tilde{k}_n) \) are:

\[
\Psi_{k_n}(k) = \sqrt{2\Im k_n} \frac{S(k)}{k - k_n}, \quad \Phi_{-\tilde{k}_n}(k) = \sqrt{2\Im \tilde{k}_n} \frac{1}{k - \tilde{k}_n}.
\]

(3.39)
4. Bandlimitedness and sampling theorem

In this section we present our main results. We recall that in the previous section we introduced the boundary-value problem that generates a set of the sampling points for the sampling theorem in the Hardy space $H^2_+$. The eigenfunctions do not form an orthogonal basis, but they form a Riesz basis for a subspace $K$ of the energy space $\mathcal{H}$. The eigenfunctions of the adjoint problem form a biorthogonal Riesz basis for $\mathcal{K}$.

The subspace $K$ is isomorphic to the subspace $\mathcal{K}$ of the Hardy space $H^2_+$. The space $K$ is our space of bandlimited functions in $H^2_+$. Analogous to the space $PW_\sigma$ of bandlimited functions, the space $K$ can also be defined in terms of an integral transform of functions supported on a finite interval as in (1.1).

We derive the sampling theorem and show that the sampling formula is a Lagrange-type interpolation formula in which the sampling points are the resonances introduced in the previous section.

4.1. Realization of incoming spectral representation

The goal of this subsection is to give an explicit description of:

(a) the subspace $\mathcal{K} = \mathcal{H} \ominus (D_+ \ominus D_-)$; see Eq. (3.22);
(b) the incoming spectral representation and the operator $T$ realizing it;
(c) the infinitesimal generator of the contraction semigroup $Z_t$.

First we address goal (a). We will show that $\mathcal{K}$ consists of data $V \in \mathcal{H}$ such that $v_1(x) = 0$ for $x > a$ and $v_0(x) = v_0(a)$ for $x \geq a$. Indeed, from the definition of the subspace $\mathcal{K}$ it follows that if $V \in \mathcal{K}$, then $V \perp D_-$, i.e., for any $W \in D_-$, we have since supp $W \subset [a, \infty)$,

$$\int_{a}^{\infty} (v_0'(x) w_0(x) + v_1(x) \overline{w_1(x)}) \rho(x) \, dx = 0. \tag{4.1}$$

Using (3.23), we reduce the above equation to $\int_{a}^{\infty} (v_0' + v_1) \overline{w_1} \, dx = 0$. Since the set of compactly supported functions is dense in $L^2(a, \infty)$, we get

$$v_0'(x) + v_1(x) = 0, \quad x \geq a. \tag{4.2}$$

Similarly, if we use the orthogonality relation that $V \perp D_+$, we get

$$v_0'(x) - v_1(x) = 0, \quad x \geq a. \tag{4.3}$$

Combining the last two equations, we obtain $v_1 = 0$ for $x \geq a$ and $v_0 = \text{constant}$ for $x \geq a$. Since $v_0$ is a continuous function, we get $v_0(x) = v_0(a), x \geq a$.

(b) For the incoming spectral representation, we recall that in the subspace $\mathcal{K}$ there exists a semigroup of contractions $Z_t, t \geq 0$, generated by the wave equation. In this representation, $\mathcal{K}$ goes over to the subspace $K = H^2_+ \ominus S(k)H^2_+$ with $S$ being introduced in (3.35). The semigroup $Z_t$ in this representation coincides with the "model" semigroup $P_k e^{it\mathcal{H}}_{|K}, t \geq 0$, with $P_k$ being an orthogonal projection in $H^2_+$ onto $K$; see Eq. (3.36).

Now we are in a position to prove an important result on the operator $T$ realizing the incoming spectral representation.

**Theorem 7.** Let $\mathcal{H}^0$ be a dense subset of the energy space $\mathcal{H}$ consisting of compactly supported functions, i.e., $V = (v_0(x), v_1(x))^T \in \mathcal{H}^0$ with supp $V \subset (0, \infty)$. Let $T$ be an integral operator defined by the formula

$$T(V)(k) \equiv \bar{V}(k) = \frac{1}{2\sqrt{2\pi}} \int_{0}^{\infty} \Psi_-(k, x) \rho(x) \left[ ikv_0(x) + v_1(x) \right] \, dx$$

$$= \frac{1}{2\sqrt{2\pi}} \left( \mathcal{F}(k, \cdot), \bar{V}(\cdot) \right)_{\mathcal{H}}, \tag{4.4}$$

with $\bar{V} = (-\overline{v_0}, \overline{v_1}) \in \mathcal{H}$, $\Psi_-(k, \cdot)$ being given by (3.31) and

$$\mathcal{F}(k, x) = \begin{pmatrix} \frac{1}{ik} \Psi_-(k, x) \\ -\Psi_-(k, x) \end{pmatrix}. \tag{4.5}$$

Then $T$ is a mapping from $\mathcal{H}^0$ into $L^2(\mathbb{R}), T : \mathcal{H}^0 \to L^2(\mathbb{R})$, whose closure is an isometric mapping of $\mathcal{H}$ onto $L^2(\mathbb{R})$ with the following properties:
\[ T(U,V)(k) = e^{ikt} \tilde{W}(k), \quad (4.6) \]
\[ T(D_-) = H^2_-, \quad T(D_+) = S(k)H^2_+, \quad K = T(K) = H^2_+ \otimes S(k)H^2_+. \quad (4.7) \]

**Proof.** First we prove that \( T \) is an isometric mapping of \( \mathcal{H} \) into \( L^2(\mathbb{R}) \). On smooth compactly supported data, we have

\[
\|\tilde{W}\|_{L^2(\mathbb{R})}^2 = \frac{1}{4} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} \psi_-(k,x) \rho(x) [ikv_0(x) + v_1(x)] dx \\
\times \int_{-\infty}^{\infty} \psi_-(k,x') \rho(x') [ikv_0(x') + v_1(x')] dx' \equiv \sum_{j=1}^{4} J_j. \tag{4.8}
\]

Let us consider each of the four integrals in (4.8). Changing the order of integration, we have for \( J_1 \)

\[
J_1 = \frac{1}{4} \int_{-\infty}^{\infty} dx dx' \int_{-\infty}^{\infty} \psi_-(k,x) \psi_-(k,x') v_0(x) v_0(x') \rho(x) \rho(x') dk.
\tag{4.9}
\]

Since \( \psi''_-(k,x) = -k^2 \rho(x) \psi_-(k,x) \), then (4.9) can be transformed to

\[
J_1 = -\frac{1}{4} \int_{-\infty}^{\infty} dx dx' \int_{-\infty}^{\infty} \psi''_-(k,x) \psi_-(k,x') v_0(x) v_0(x') \rho(x) \rho(x') dk.
\tag{4.10}
\]

Using the first orthogonality relation of (3.33) and noting that the integral there over \( k \) is from 0 to \( \infty \), we simplify (4.10) to have

\[
J_1 = -\frac{1}{2} \int_{0}^{\infty} dx dx' \int_{-\infty}^{\infty} v_0''(x) v_0(x') \delta(x-x') \rho(x) \rho(x') dx dx' = \frac{1}{2} \|v_0''\|_{L^2(0,\infty)}^2.
\tag{4.11}
\]

Consider the next integral \( J_2 \)

\[
J_2 = \frac{1}{4} \int_{-\infty}^{\infty} dx dx' \int_{-\infty}^{\infty} ik \psi_-(k,x) \psi_-(k,x') \rho(x) v_0(x) v_1(x') \rho(x') dx dx'.
\tag{4.12}
\]

Taking into account that \( \psi_-(k,x) \psi_-(k,x') \) is an even function of \( k \) for each \( x \) and \( x' \), we obtain that \( J_2 = 0 \). Since \( J_3 = J_2 \), we get \( J_3 = 0 \). Finally we evaluate \( J_4 \) and have

\[
J_4 = \frac{1}{4} \int_{-\infty}^{\infty} dx dx' \int_{-\infty}^{\infty} v_1(x) \psi_-(k,x') \psi_-(k,x) \rho(x) \rho(x') dx dx' = \frac{1}{2} \|v_1''\|_{L^2(0,\infty)}^2.
\tag{4.13}
\]

Collecting together (4.10)-(4.13) we obtain from (4.8) that \( \|\tilde{W}\|_{L^2(\mathbb{R})} = \|V\|_{\mathcal{H}} \). The fact that \( T \) maps \( \mathcal{H} \) onto \( L^2(-\infty, \infty) \) is a consequence of the completeness of the system \( \{\psi_-(k,x)\} \).

Now we prove the first two relations of (4.7). Namely, let \( V \in D_- \), i.e., \( V = (v_0, v'_0) \) and \( \text{supp} \ v_0 \subset (a, \infty) \). For such data we have

\[
T(V)(k) = \frac{1}{2\sqrt{2\pi}} \int_{a}^{\infty} \psi_-(k,x) [ikv_0(x) + v'_0(x)] \rho(x) dx.
\tag{4.14}
\]
We notice that for \( x > a \) the following representation holds; see Eqs. (3.29) and (3.31):

\[
\Psi_-(k, x) = e^{-ik(x-a)} - S(k)e^{ik(x-a)}.
\]  

(4.15)

With (4.15) formula (4.14) can be transformed to

\[
T(V)(k) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty \left[ e^{-ik(x-a)} - S(k)e^{ik(x-a)} \right] \left[ ik\psi_0(x) + \psi_0'(x) \right] \rho(x) \, dx
\]

\[
= -\frac{1}{2\sqrt{2\pi}} \int_0^\infty (d/dx) \left[ e^{-ik(x-a)} + S(k)e^{ik(x-a)} \right] \psi_0(x) \, dx
\]

\[
+ \frac{1}{2\sqrt{2\pi}} \int_0^\infty \left[ e^{-ik(x-a)} - S(k)e^{ik(x-a)} \right] \psi_0'(x) \, dx
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ik(x-a)} \psi_0'(x) \, dx.
\]  

(4.16)

This result means that when \( \psi'_0 \) runs over \( L^2(a, \infty) \), then \( T(V) \) runs over \( H^2_+ \). The first relation of (4.7) is shown.

Now let \( V \in \mathcal{D}_+ \), i.e., \( V = (\psi_0, -\psi'_0) \). For such \( V \) calculations similar to (4.16) yield

\[
T(V)(k) = \frac{S(k)}{\sqrt{2\pi}} \int_0^\infty e^{ik(x-a)} \psi_0'(x) \, dx.
\]  

(4.17)

This result means that when \( \psi'_0 \) runs over \( L^2(a, \infty) \), then \( TV \) runs over \( S(k)H^2_+ \). The second relation of (4.7) is also shown.

Finally to show that (4.6) holds, it suffices to establish that

\[
T(LV)(k) = kT(V)(k) = k\psi(k), \quad V \in \mathcal{D}(\mathcal{L}).
\]  

(4.18)

Here \( L \) is the generator for the group \( U_t \). It is given by formula (3.14) with the operator \( L = (\rho(r))^{-1} \frac{d^2}{dr^2} \) considered on the semi-axis, \( r \in [0, \infty) \), with the zero boundary condition at \( r = 0 \). Thus, we have from (3.14)

\[
T(LV)(k) = T(V_1)(k), \quad \text{where } V_1(x) = (-iv_1(x), -i[\rho(x)]^{-1}v'_0(x)).
\]  

(4.19)

Evaluating \( TV_1 \), we have

\[
T(V_1)(k) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty \Psi_-(k, x)[ik(-iv_1(x)) - i[\rho(x)]^{-1}v'_0(x)] \rho(x) \, dx
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_0^\infty \Psi_-(k, x)v_1(x) \rho(x) \, dx - i \int_0^\infty \Psi_-(k, x)v'_0(x) \rho(x) \, dx.
\]  

(4.20)

Using \( \Psi''(k, x) = -k^2 \rho(x)\Psi_-(k, x) \), we integrate the second integral of (4.20) by parts twice to obtain for \( \psi_0 \) (supp \( \psi_0 \subset (a, \infty) \))

\[
\int_0^\infty \Psi_-(k, x)v'_0(x) \, dx = \int_0^\infty \Psi''_0v_0(x) \, dx = -\int_0^\infty k^2 \Psi_-(k, x)v_0(x) \rho(x) \, dx.
\]  

(4.21)

Substituting (4.21) into (4.20), we arrive at the desired result

\[
T(V_1)(k) = \frac{1}{2\sqrt{2\pi}} \left\{ k \int_0^\infty \Psi_-(k, x)v_1(x) \rho(x) \, dx + ik^2 \int_0^\infty \Psi_-(k, x)v_0(x) \rho(x) \, dx \right\}
\]

\[
= kT(V)(k).
\]  

(4.22)

The proof is complete. \( \Box \)

Finally, we address goal (c) in the following proposition, whose proof can be found in [21].
Proposition 8. The generator $iB$ of the semigroup $Z_t$ from (3.24) coincides with the non-self-adjoint operator $L$ described by (3.14) and (3.15).

4.2. Sampling theorem

First, we emphasize the similarity between the two notions of bandlimitedness. Indeed, the space $PW_0$ is the image under the Fourier transformation of a subset of $L^2(\mathbb{R})$ consisting of functions with supports in $(-\sigma, \sigma)$, i.e., $\mathcal{F}[L^2(-\sigma, \sigma)] = PW_0$. Likewise, $K$ is the image under the transformation $\mathcal{T}$ of functions that are essentially supported on a finite interval, i.e.,

$$\mathcal{T}(K) = K, \quad K = H_+^2 \otimes S(k)H_+^2.$$ 

We notice here that Lagrange’s identity (2.6) holds for the operator $L$ defined in (3.14) and (3.15) and its adjoint $L^*$, i.e., for $U \in \mathcal{D}(L)$ and $V \in \mathcal{D}(L^*)$ we have

$$\langle \mathcal{L}U, V \rangle_K = \langle U, L^*V \rangle_K.$$  \hspace{1cm} (4.23)

To proceed to the sampling theorem, we need the following result on the spectral distribution of the eigenvalues of the operator $L$, which will be the sampling points; see [21].

Theorem 9. For all three types of the density $\rho$, (see Definition 6) the Jost function $M(\lambda)$ is an entire function of exponential type $\int_0^\infty \sqrt{\rho(t)} \, dt$. This function has a countable set of zeroes $\{\lambda_n\}_{n \in \mathbb{Z}}$ lying in a strip parallel to the real axis. All the zeroes are isolated, i.e., $\text{inf}_n \lambda_n - \lambda_m > 0$, and simple, except for at most a finite number of them. Moreover $M(\lambda)$ is a sine-type function, i.e., for each line parallel to the real axis and lying outside the strip, containing the eigenvalues, there exist two constant $C_2 > C_1 > 0$ such that on this line $C_1 \leq |M(\lambda)| \leq C_2$.

The asymptotic representations of the resonances for all three types of the density imply that the resonances form a set of complex numbers that has two features: (1) the set of resonances is uniformly separated; (2) this set is located in a strip of the complex plane that is parallel to the real axis.

It is also known that the model semigroup $Z_t = P e^{it\mathcal{A}} |_{\mathcal{K}}$ considered in the subspace $K$ of the Hardy space has a set of root vectors that is a Riesz basis in $K$. Due to the unitary equivalence described in Theorem 7 we immediately obtain that the set of the root vectors of the operator $L$ forms a Riesz basis in the subspace $\mathcal{K}$ of the energy space $\mathcal{H}$.

Now we describe the sets of eigenfunctions of the operator $L$ and its adjoint ($L^*$). Let $F_n$ be a two component eigenfunction of $L$ and $F_n^*$ be an eigenfunction of $L^*$. The following formulas are valid for the root sets:

$$\{F_n(x)\}_{n \in \mathbb{Z}} = \left\{ \begin{pmatrix} \frac{1}{i\pi} F(\lambda, x) \\ F(\lambda, x) \end{pmatrix}_{\lambda = \lambda_n} \right\}_{n \in \mathbb{Z}} = \left\{ \begin{pmatrix} \frac{1}{i\pi} F_n(x) \\ F_n(x) \end{pmatrix} \right\}_{n \in \mathbb{Z}},$$  \hspace{1cm} (4.24)

$$\{F_n^*(x)\}_{n \in \mathbb{Z}} = \left\{ \begin{pmatrix} \frac{1}{-i\pi} F(\mu_n, x) \\ F(\mu_n, x) \end{pmatrix}_{\mu_n = -\lambda_n} \right\}_{n \in \mathbb{Z}} = \left\{ \begin{pmatrix} \frac{1}{-i\pi} F_n^*(x) \\ F_n^*(x) \end{pmatrix} \right\}_{n \in \mathbb{Z}}.$$  \hspace{1cm} (4.25)

where $F(\lambda, x)$ is the Jost solution given by (3.25).

The two sets $\{F_n\}_{n \in \mathbb{Z}}$ and $\{F_n^*\}_{n \in \mathbb{Z}}$ being the eigenfunctions of adjoint operators, are mutually biorthogonal, i.e.,

$$\langle F_m, F_n^* \rangle_{\mathcal{H}} = \delta_{n,m} C_m,$$  \hspace{1cm} (4.26)

where $\delta_{n,m}$ is the Kroneker delta and $\{C_m\}_{m \in \mathbb{Z}}$ is a sequence of positive constants satisfying:

$$0 < \inf_{m \in \mathbb{Z}} C_m \leq \sup_{m \in \mathbb{Z}} C_m < \infty.$$ 

Moreover, since $\{F_n\}_{n \in \mathbb{Z}}$ is a Riesz basis in $\mathcal{H}$, the same fact is valid for $\{F_n^*\}_{n \in \mathbb{Z}}$.

Definition 10. A function $G \in \mathcal{H}^2_+$ is said to be bandlimited if it belongs to the range of the operator $\mathcal{T}$ restricted to the subspace $\mathcal{K}$.

Now we can prove our main result.

Theorem 11. Let $g = (g_0, g_1) \in \mathcal{K}$ and $\tilde{g} = (-\tilde{g}_0, \tilde{g}_1) \in \mathcal{K}$. Denote

$$G(\lambda) = \frac{1}{\sqrt{2\pi}} \langle F(\lambda, \cdot), g^* \rangle_{\mathcal{H}} = (\mathcal{T} \tilde{g})(\lambda).$$  \hspace{1cm} (4.27)
Then $G \in K = H^2_+ \odot S(\lambda)H^2_+$ and it can be reconstructed from its values at the points $\{\lambda_n\}_{n \in \mathbb{Z}}$ using the following Lagrange interpolation-type formula:

$$G(\lambda) = \sum_{n \in \mathbb{Z}} G(\lambda_n) \frac{P(\lambda)}{(\lambda - \lambda_n)P'(\lambda_n)},$$

(4.28)

with $P(\lambda)$ given by

$$P(\lambda) = \frac{1}{2} \left\{ \frac{1}{i \lambda} \left( \frac{d}{dx} \Psi_-(\lambda, x) \right)_{x=a} + \Psi_-(\lambda, a) \right\}.$$

(4.29)

**Proof.** Using the Riesz basis property of the sets $\{\mathcal{F}_n\}_{n \in \mathbb{Z}}$ and $\{\mathcal{F}_n^*\}_{n \in \mathbb{Z}}$ in $K$ and their biorthogonality, for any $g \in K$, we have the following expansion:

$$g(x) = \sum_{n \in \mathbb{Z}} \frac{\langle g, \mathcal{F}_n \rangle_H}{\langle \mathcal{F}_n, \mathcal{F}_n \rangle_H} \mathcal{F}_n^*(x).$$

(4.30)

Since $\tilde{g} \in K$, then by Theorem 7, $G(\lambda) = (T \tilde{g})(\lambda) \in K$. Substituting (4.30) in (4.27) and using the continuity of the inner product, we obtain that

$$G(\lambda) = \frac{1}{\sqrt{2\pi}} \frac{\langle \mathcal{F}(\lambda, \cdot), g \rangle_H}{\langle \mathcal{F}, \mathcal{F} \rangle_H} = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \frac{\langle g, \mathcal{F}_n \rangle_H}{\langle \mathcal{F}_n, \mathcal{F}_n \rangle_H} \langle \mathcal{F}(\lambda, \cdot), \mathcal{F}_n^* \rangle_H. $$

(4.31)

However, directly from (4.27) it can be seen that

$$G(\lambda_n) = \frac{1}{\sqrt{2\pi}} \langle \mathcal{F}(\lambda_n, \cdot), g \rangle_H = \frac{1}{\sqrt{2\pi}} \langle \mathcal{F}_n, g \rangle_H,$$

(4.32)

which yields

$$G(\lambda) = \sum_{n \in \mathbb{Z}} G(\lambda_n) \frac{\langle \mathcal{F}(\lambda, \cdot), \mathcal{F}_n^* \rangle_H}{\langle \mathcal{F}_n, \mathcal{F}_n \rangle_H}. $$

(4.33)

Now we modify the quotient from (4.33). On the one hand, since $\mathcal{F}(\lambda, \cdot) \in \mathcal{D}(L)$, we have

$$\langle (L \mathcal{F})(\lambda, \cdot), \mathcal{F}_n^* \rangle_H = \lambda \langle \mathcal{F}(\lambda, \cdot), \mathcal{F}_n^* \rangle_H.$$ 

(4.34)

On the other hand, for the same inner product, the following expression holds:

$$I = \langle (L \mathcal{F})(\lambda, \cdot), \mathcal{F}_n^* \rangle_H = -\frac{i}{2} \left\{ \int_0^\infty \left( \Psi_-(\lambda, x) \frac{1}{-i \mu_n} (F(\mu_n, x))_x \right) dx + \int_0^\infty \left( \frac{1}{i \lambda} (\Psi_-(\lambda, x))_{xx} F(\mu_n, x) \right) dx \right\}. $$

(4.35)

In (4.35), we have used representation (4.25) for $\mathcal{F}_n^*$. Integrating by parts in (4.35) we obtain:

$$2iI = \left[ \Psi_-(\lambda, x) \frac{1}{-i \mu_n} (F(\mu_n, x))_x \right]_0^a + \left[ \frac{1}{i \lambda} (\Psi_-(\lambda, x))_{xx} F(\mu_n, x) \right]_0^a $$

$$- \int_0^\infty \Psi_-(\lambda, x) \frac{1}{i \mu_n} (F(\mu_n, x))_x dx - \int_0^\infty \left( \frac{1}{i \lambda} (\Psi_-(\lambda, x))_{xx} F(\mu_n, x) \right) dx. $$

(4.36)

To simplify (4.36), we recall that $\Psi_-(\lambda, 0) = F(\mu_n, 0) = 0$ and

$$F''(\mu_n, x) = -\mu_n^2 \rho(x) F(\mu_n, x), \quad F'(\mu_n, a) = -i \mu_n F(\mu_n, a).$$

Thus, using Eq. (4.5) for $\mathcal{F}$ and (4.25) for $\mathcal{F}_n^*$, we obtain

$$I = -\frac{i}{2} \left\{ \Psi_-(\lambda, a) + \frac{1}{i \lambda} \left( \frac{d}{dx} \Psi_-(\lambda, x) \right)_{x=a} \right\} F(\mu_n, a) - \mu_n \langle \mathcal{F}(\lambda, \cdot), \mathcal{F}_n^* \rangle_H.$$ 

(4.37)

We notice here that according to our definition $F(\mu_n, a) \neq 0$. Indeed, assuming that $F(\mu_n, a) = 0$, we would have had a problem on a finite interval with trivial Cauchy data at $x = a$. The solution of such a problem is evidently trivial.
Thus, for the same inner product $\langle \mathcal{L}_{\mathcal{F}, \mathcal{F}^{\pm}_{n}} \rangle_{\mathcal{H}}$ we have two representations given by (4.34) and (4.37). By equating these two representations, we obtain

$$P(\lambda) \equiv \frac{1}{2} \left\{ \frac{1}{i\lambda} \left( \langle \Psi_{-}(\lambda, x) \rangle_{x=a} + \Psi_{-}(\lambda, a) \right) \right\}$$

$$= \frac{i(\lambda + \mu_{n})(\mathcal{F}(\lambda, \cdot), \mathcal{F}^{\pm}_{n})_{\mathcal{H}}}{F(\mu_{n}, a)} - \frac{i(\lambda - \mu_{n})(\mathcal{F}(\lambda, \cdot), \mathcal{F}^{\pm}_{n})_{\mathcal{H}}}{F(\mu_{n}, a)}.$$  \hspace{1cm} (4.38)

From (4.38), it immediately follows that

$$\left( \frac{d}{d\lambda} P(\lambda) \right)_{\lambda = \lambda_{n}} = \frac{i\langle \mathcal{F}(\lambda, \cdot), \mathcal{F}^{\pm}_{n} \rangle_{\mathcal{H}}}{F(\mu_{n}, a)}.$$  \hspace{1cm} (4.39)

Combining (4.38) with (4.39), we obtain that

$$\langle \mathcal{F}(\lambda, \cdot), \mathcal{F}^{\pm}_{n} \rangle_{\mathcal{H}} = \frac{P(\lambda)}{(\lambda - \lambda_{n})P'(\lambda_{n})}.$$  \hspace{1cm} (4.40)

Substituting (4.40) into (4.33) yields (4.28). To complete the proof, we have to show that the series converges uniformly when $\lambda$ runs over any compact subset of the upper half plane.

Let $\mathcal{S}_{N}$ be the following partial sum

$$\mathcal{S}_{N}(\lambda) = \sum_{|n| \leq N} G(\lambda_{n}) \frac{P(\lambda)}{(\lambda - \lambda_{n})P'(\lambda_{n})},$$

then by formula (4.31) we have

$$\left| G(\lambda) - \mathcal{S}_{N}(\lambda) \right|^{2} = \frac{1}{2\pi} \left| \langle \mathcal{F}(\lambda, \cdot), g \rangle_{\mathcal{H}} - \sum_{|n| \leq N} \langle \mathcal{F}_{n}, g \rangle_{\mathcal{H}} \frac{\langle \mathcal{F}(\lambda, \cdot), \mathcal{F}^{\pm}_{n} \rangle_{\mathcal{H}}}{\langle \mathcal{F}_{n}, \mathcal{F}^{\pm}_{n} \rangle_{\mathcal{H}}} \right|^{2}$$

$$= (2\pi)^{-1} \left| \langle \mathcal{F}(\lambda, \cdot), g - \mathcal{S}_{N} \rangle_{\mathcal{H}} \right|^{2}$$

$$\leq (2\pi)^{-1} \left\| \mathcal{F}(\lambda, \cdot) \right\|_{\mathcal{H}}^{2} \left\| g - \mathcal{S}_{N} \right\|_{\mathcal{H}}^{2},$$  \hspace{1cm} (4.41)

where

$$\mathcal{S}_{N}(x) = \sum_{|n| \leq N} \frac{\langle g, \mathcal{F}_{n} \rangle_{\mathcal{H}}}{\langle \mathcal{F}_{n}, \mathcal{F}^{\pm}_{n} \rangle_{\mathcal{H}}} \mathcal{F}^{\pm}_{n}(x).$$  \hspace{1cm} (4.42)

Since

$$\left\| \mathcal{F}(\lambda, \cdot) \right\|_{\mathcal{H}}^{2} = \frac{1}{2} \int_{0}^{a} \left( \frac{1}{|\lambda|^{2}} \left| \Psi_{-}(\lambda, x) \right|^{2} + \rho(x) \left| \Psi_{-}(\lambda, x) \right|^{2} \right) dx$$  \hspace{1cm} (4.43)

is bounded on any compact subset of the upper $\lambda$-half-plane and $\mathcal{S}_{N}$ is the nth partial sum of a series converging to $g$ in $\mathcal{H}$, we have $\|g - \mathcal{S}_{N}\|_{\mathcal{H}} \rightarrow 0$ as $N \rightarrow \infty$ and this completes the proof.  \hspace{0.5cm} $\Box$

**Corollary 12.** The set $\{\lambda_{n}\}$ is a uniqueness set for the class $K$.

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**References**


