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Quasideterminants

Israel Gelfand,[a] Sergei Gelfand,[b,*] Vladimir Retakh,[a,1] and Robert Lee Wilson[a]

[a] *Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA*
[b] *American Mathematical Society, 201, Charles Street, Providence, RI 02904-2213, USA*

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Abstract

The determinant is a main organizing tool in commutative linear algebra. In this review we present a theory of the quasideterminants defined for matrices over a division ring.

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0. Introduction

The ubiquitous notion of a determinant has a long history, both visible and invisible. The determinant has been a main organizing tool in commutative linear algebra and we cannot accept the point of view of a modern textbook [FIS] that "determinants … are of much less importance than they once were".

Attempts to define a determinant for matrices with noncommutative entries started more than 150 years ago and include several great names. For many years the

*Corresponding author. Fax: +1-401-331-3842.

E-mail addresses: igelfand@math.rutgers.edu (I. Gelfand), sxg@ams.org (S. Gelfand), vretakh@math.rutgers.edu (V. Retakh), rwilson@math.rutgers.edu (R. Lee Wilson).

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most famous examples of matrices of noncommutative objects were quaternionic matrices and block matrices. It is not surprising that the first noncommutative determinants or similar notions were defined for such structures.

Cayley [C] was the first to define, in 1845, the determinant of a matrix with noncommutative entries. He mentioned that for a quaternionic matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ the expressions $a_{11}a_{22} - a_{12}a_{21}$ and $a_{11}a_{22} - a_{21}a_{12}$ are different and suggested choosing one of them as the determinant of the matrix A. The analog of this construction for 3×3-matrices was also proposed in [C] and later developed in [J]. This "naive" approach is now known to work for quantum determinants and some other cases. Different forms of quaternionic determinants were considered later by Study [St], Moore [Mo] and Dyson [Dy].

There were no direct "determinantal" attacks on block matrices (excluding evident cases) but important insights were given by Frobenius [Fr] and Schur [Schur].

A theory of determinants of matrices with general noncommutative entries was in fact originated by Wedderburn in 1913. In [W] he constructed a theory of noncommutative continued fractions or, in modern terms, "determinants" of noncommutative Jacobi matrices.

In 1926–1928 Heyting [H] and Richardson [Ri,Ri1] suggested analogs of a determinant for matrices over division rings. Heyting is known as a founder of intuitionist logic and Richardson as a creator of the Littlewood–Richardson rule. Heyting tried to construct a noncommutative projective geometry. As a computational tool, he introduced the "designant" of a noncommutative matrix. The designant of a 2×2-matrix $A = (a_{ij})$ is defined as $a_{11} - a_{12}a_{22}^{-1}a_{21}$. The designant of an $n \times n$-matrix is defined then by a complicated inductive procedure. The inductive procedures used by Richardson were even more complicated. It is important to mention that determinants of Heyting and Richardson in general are rational functions (and not polynomials!) in matrix entries.

The idea to have nonpolynomial determinants was strongly criticized by Ore [O]. In [O] he defined a polynomial determinant for matrices over an important class of noncommutative rings (now known as Ore rings).

The most famous and widely used noncommutative determinant is the Dieudonne determinant. It was defined for matrices over a division ring R by Dieudonne in 1943 [D]. His idea was to consider determinants with values in $R^*/[R^*, R^*]$ where R^* is the monoid of invertible elements in R. The properties of Dieudonne determinants are close to those of commutative ones, but, evidently, Dieudonne determinants cannot be used for solving systems of linear equations.

An interesting generalization of commutative determinants belongs to Berezin [B,Le]. He defined determinants for matrices over so-called super-commutative algebras. In particular, Berezin also understood that it is impossible to avoid rational functions in matrix entries in his definition.

Other famous examples of noncommutative determinants developed for different special cases are: quantum determinants [KS,Ma], Capelli determinants [We], determinants introduced by Cartier–Foata [CF,F] and Birman–Williams [BW], etc. As we explain later (using another universal notion, that of quasideterminants) these determinants and the determinants of Dieudonne, Study, Moore, etc., are related to each other much more than one would expect.

The notion of quasideterminants for matrices over a noncommutative division ring was introduced in [GR,GR1,GR2]. Quasideterminants are defined in the "most noncommutative case", namely, for matrices over free division rings. We believe that quasideterminants should be one of main organizing tools in noncommutative algebra giving them the same role determinants play in commutative algebra. The quasideterminant is not an analog of the commutative determinant but rather of a ratio of the determinant of an $n \times n$-matrix to the determinant of an $(n-1) \times (n-1)$-submatrix.

The main property of quasideterminants is a "heredity principle": let A be a square matrix over a division ring and (A_{ij}) a block decomposition of A (into submatrices of A). Consider the A_{ij}'s as elements of a matrix X. Then the quasideterminant of the matrix X will be a matrix B, and (under natural assumptions) the quasideterminant of B is equal to a suitable quasideterminant of A. Since determinants of block matrices are not defined, there is no analog of this principle for ordinary (commutative) determinants.

Quasideterminants have been effective in many areas including noncommutative symmetric functions [GKLLRT], noncommutative integrable systems [EGR,EGR1, RS], quantum algebras and Yangians [GR,GR1,GR2,KL,Mol,Mol1,MolR], and so on [P,RRV,RSh,Sch]. Quasideterminants and related quasi-Plücker coordinates are also important in various approaches to noncommutative algebraic geometry (e.g., [K,KR,SvB]).

Many areas of noncommutative mathematics (Ore rings, rings of differential operators, theory of factors, "quantum mathematics", Clifford algebras, etc.) were developed separately from each other. Our approach shows an advantage of working with totally noncommutative variables (over free rings and division rings). It leads us to a large variety of results, and their specialization to different noncommutative areas implies known theorems with additional information.

The price one pays for this is a huge number of inversions in rational noncommutative expressions. The minimal number of successive inversions required to express an element is called the height of this element. This invariant (inversion height) reflects the "degree of noncommutativity" and it is of a great interest by itself.

Our experience shows that in dealing with noncommutative objects one should not imitate the classical commutative mathematics, but follow "the way it is" starting with basics. In this paper we concentrate on two problems: noncommutative Plücker coordinates (as a background of a noncommutative geometry) and the noncommutative Bezout and Viète theorems (as a background of noncommutative algebra). We apply the obtained results to the theory of noncommutative symmetric functions started in [GKLLRT].

We have already said that the universal notion of a determinant has a long history, both visible and invisible. The visible history of determinants comes from the fact that they are constructed from another class of universal objects: matrices.

The invisible history of determinants is related with the Heredity principle for matrices: matrices can be viewed as matrices with matrix entries (block matrices) and some matrix properties come from the corresponding properties of block matrices.

In some cases, when the matrix entries of the block matrix commute, the determinant of a matrix can be computed in terms of the determinants of its blocks, but in general it is not possible: the determinant of a matrix with matrix entries is not defined because the entries do not commute. In other words, the determinant does not satisfy the Heredity principle.

Quasideterminants are defined for matrices over division rings and satisfy the Heredity Principle. Their definition can be specialized for matrices over a ring (including noncommutative rings) and can be connected with different "famous" determinants. This reflects another general principle: in many cases noncommutative algebra can be made simpler and more natural than commutative algebra.

The survey describes the first 10 years of development of this very active area, and we hope that future work will bring many new interesting results.

The paper is organized as follows. In Section 1 a definition of quasideterminants is given and the main properties of quasideterminants (including the Heredity principle) are described.

In Section 2 we discuss an important example: quasideterminants of quaternionic matrices. These quasideterminants can be written as polynomials with real coefficients in the matrix entries and their quaternionic conjugates.

As we already mentioned, mathematics knows a lot of different versions of noncommutative determinants. In Section 3 we give a general definition of determinants of noncommutative matrices (in general, there are many determinants of a fixed matrix) and show how to obtain some well-known noncommutative determinants as specializations of our definition.

In Section 4 we introduce noncommutative versions of Plücker and flag coordinates for rectangular matrices over division rings.

In Section 5 we discuss two related classical problems for noncommutative polynomials in one variable: how to factorize a polynomial into products of linear polynomials and how to express the coefficients of a polynomial in terms of its roots.

This results obtained in Section 5 lead us to a theory of noncommutative symmetric functions (Section 6) and to universal quadratic algebras associated with so-called pseudo-roots of noncommutative polynomials and noncommutative differential polynomials (Section 7).

In Section 8 we present another approach to the theory of noncommutative determinants, traces, etc., and relate it to the results presented in Sections 3 and 5.

Some applications to noncommutative continued fractions, characteristic functions of graphs, noncommutative orthogonal polynomials and integrable systems are given in Section 9.

1. General theory and main identities

1.1. The division ring of rational functions in free variables

Throughout the paper we will work with rings of fractions of various noncommutative rings. There are several ways to define rings of fractions in the noncommutative case. We will use the approach developed by Amitsur, Bergman and P. M. Cohn (for a detailed exposition see, e.g., [Co]). The advantage of this approach is that it is constructive; its disadvantage is that it does not look very natural.

First, we define the free division ring generated by a finite set. Let $X = \{x_1, \dots, x_m\}$ be a finite set. Define $\mathcal{F}(X)$ as the free algebra generated by $m+2$ elements $0, 1, x_1, \dots, x_m$, unary operations $a \mapsto -a$, $a \mapsto a^{-1}$, and binary operations $+$ and $\times$, so that $\mathcal{F}(X)$ contains such elements as $(x-x)^{-1}$ and even 0^{-1}. No commutativity, associativity, distributivity, or other conditions are imposed, so, that, in particular, elements $(x_1 + x_2) \times x_3$ and $x_1 \times x_3 + x_2 \times x_3$ are distinct, and three elements

$$(-x_1)^{-1}, \quad (0 - x_1)^{-1}, \quad -x_1^{-1} \times 1^{-1},$$

are also distinct. Elements of $\mathcal{F}(X)$ are called *formulas* or rational expressions over X.

Denote by $\mathcal{P}(X)$ the subset of $\mathcal{F}(X)$ consisting of formulas without division, i.e., without operation $(\ )^{-1}$.

Now let R be a $\mathbb{Q}$-algebra. By a partial homomorphism of $\mathcal{F}(X)$ to R we mean a pair (G, β) consisting of a subset $G \subset \mathcal{F}(X)$ and a map $\beta : G \to R$ such that

(i) $0, 1 \in G$ and $\beta(0) = 0$, $\beta(1) = 1$,
(ii) if $a_1 = -b$, $a_2 = b + c$, $a_3 = b \times c$ are elements in G then $b, c \in G$ and $\beta(a_1) = -\beta(b)$, $\beta(a_2) = \beta(b) + \beta(c)$, $\beta(a_3) = \beta(b)\beta(c)$.
(iii) Let $b \in G$ and let $\beta(b)$ be invertible in R. Then $b^{-1} \in G$ and $\beta(b^{-1}) = (\beta(b))^{-1}$.

Let again R be a $\mathbb{Q}$-algebra and $\alpha : X \to R$ an arbitrary map. We say that a partial homomorphism (G, β) of $\mathcal{F}(X)$ to R is an extension of a map α if, in addition to (i)–(iii), the following condition is satisfied.

(iv) For $i = 1, \dots, m$ we have $x_i \in G$ and $\beta(x_i) = \alpha(x_i)$.

Clearly, for an arbitrary α, conditions (i), (ii), (iv) determine a natural extension $(\mathcal{P}(X), \alpha_{\mathcal{P}})$, and for any other extension (G, β) we have $G \supset \mathcal{P}(X)$, $\beta|_{\mathcal{P}(X)} = \alpha_{\mathcal{P}}$.

For two extension (G_1, β_1) and (G_2, β_2) of α define their intersection (G, β) as follows:

$$G \text{ is the set of all } a \in G_1 \cap G_2 \text{ such that } \beta_1(a) = \beta_2(a),$$

$$\beta(a) = \beta_1(a) = \beta_2(a) \quad \text{for } a \in G.$$

Clearly, (G, β) is again an extension of α. Therefore, the intersection of all extensions of α is again an extension. We call it the *canonical extension* of α and denote by $(G_0, \bar{\alpha})$, or simply $\bar{\alpha}$.

Since each $\alpha : X \to R$ admits at least one extension (for example, $(\mathcal{P}(X), \alpha_{\mathcal{P}})$), the definition of the canonical extension makes sense.

If $(G_0, \bar{\alpha})$ is the canonical extension of α and $a \in G_0$ we say that α *can be evaluated at* a.

Let D be a division ring over $\mathbb{Q}$. Denote by $M(X, D)$ the set of all maps $\alpha : X \to D$. Clearly, $M(X, D)$ is isomorphic to D^m, where $m = \operatorname{card} X$.

Definition 1.1.1. (i) The *domain* $\operatorname{dom} a$ of an element $a \in \mathcal{F}(X)$ is subset of $M(X, D)$ consisting of the maps $\alpha : X \to D$ such that α can be evaluated at a.

(ii) An element $a \in \mathcal{F}(X)$ is called *nondegenerate* if $\operatorname{dom} a \neq \emptyset$, and *degenerate* otherwise.

(iii) Two elements $a_1, a_2 \in \mathcal{F}(X)$ are called *equivalent* if they are both nondegenerate and $\bar{\alpha}(a_1) = \bar{\alpha}(a_2)$ for all $\alpha \in \operatorname{dom} a_1 \cap \operatorname{dom} a_2$.

For example, for $x \in X$, the elements $x - x \in \mathcal{F}(X)$ is nondegenerate and equivalent to $0 \in \mathcal{F}(X)$ whereas $(x - x)^{-1}$ is degenerate. Another example: for $x \in X$, the element $a_1 = (1 - x)^{-1} + (1 - x^{-1})^{-1}$ is equivalent to $a_2 = 1$.

Theorem 1.1.2 (Cohn [Co, Section 7.2]). (i) *If* $a_1, a_2 \in \mathcal{F}(X)$ *are both nondegenerate, then* $\operatorname{dom}_1 \cap \operatorname{dom} a_2 \neq \emptyset$.

(ii) *Assume, in addition, that D is a division ring with the center* $\mathbb{Q}$. *Then the equivalence classes of elements in* $\mathcal{F}(X)$ *form a division ring, called* $F_D(X)$.

(iii) *If division rings* D_1 *and* D_2 *with center* $\mathbb{Q}$ *are infinite dimensional over* $\mathbb{Q}$, *then the projections* $\mathcal{F}(X) \to F_{D_1}$ *and* $\mathcal{F}(X) \to F_{D_2}$ *induce an isomorphism* $F_{D_1} \sim F_{D_2}$.

Part (iii) of Theorem 1.1.2 allows us to identify the division rings $F_D(X)$ for all division rings D infinite-dimensional over $\mathbb{Q}$. We denote this division ring by $F(X)$ and called it the *free division ring* generated by X. For example, if X consists of one element x, then $F(X) = \mathbb{Q}(x)$ is the field of rational functions over $\mathbb{Q}$ in one variable.

Elements $f \in F(X)$ are called (noncommutative) *rational functions* in variables $x \in X$, and any element $a \in \mathcal{F}(X)$ in the equivalence class f is called a *rational expression* of the function f.

Remark. Similar results hold if $\mathbb{Q}$ is replaced by an arbitrary field k of characteristic 0 (for example, by $\mathbb{C}$).

The next proposition shows that for an arbitrary $\mathbb{Q}$-algebra R evaluations of a map $\alpha : X \to R$ on two equivalent elements coincide.

Proposition 1.1.3. *Let* R *be a* $\mathbb{Q}$-*algebra*, $\alpha : X \to R$ *a map, and* $(G_0, \bar{\alpha})$ *the canonical extension of* α. *If* $a_1, a_2 \in \mathcal{F}(X)$ *are equivalent and both lie in* $G_0 \subset \mathcal{F}(X)$, *then* $\bar{\alpha}(a_1) = \bar{\alpha}(a_2)$.

Definition 1.1.4. Let $\alpha : X \to R$ and let $(G_0, \bar{\alpha})$ be the canonical extension of α. We say that α can be evaluated at $f \in F(X)$ if there exists $a \in \mathcal{F}(X)$ in the equivalence class of f such that $a \in G_0$, and in this case we define the value of α at f by the formula $\alpha(f) = \bar{\alpha}(a)$.

Proposition 1.1.3 shows that this definition makes sense.

Cohn has shown (see [Co, Section 7.2]) that the division ring $F(X)$ can be characterized by a universality property as follows.

Theorem 1.1.5. *There exists a division ring $F(X)$ over $\mathbb{Q}$ and a monomorphism of algebras $\theta : \mathbb{Q}\langle X \rangle \to F(X)$ with the following property.*

If D is an arbitrary division ring and

$$\varphi : \mathbb{Q}\langle X \rangle \to D,$$

a homomorphism, then there is a unique pair (R, ψ) consisting of a subring $R \subset F(X)$ containing $\theta(\mathbb{Q}\langle X \rangle)$ and a homomorphism

$$\psi : R \to D$$

such that $\varphi = \psi\theta$ and if $a \in R$ and $\psi(a) \neq 0$, then $a^{-1} \in R$.

The pair $(F(X), \theta)$ is determined uniquely up to a unique isomorphism.

To conclude this subsection, we recall the definition of inversion height (see, for example, [Re]).

Definition 1.1.6. (i) The inversion height of a formula $a \in \mathcal{F}(X)$ is the maximal number of nested inversions in a.

(ii) The inversion height of an element $f \in F(X)$ is the smallest inversion height of a formula in the equivalence class f.

Examples. (i) The inversion height of a polynomial in generators $x \in X$ equals zero.

(ii) The inversion height of the ratio of two polynomials PQ^{-1} equals 0 if P is right divisible by Q (i.e., there exists a polynomial R such that $P = RQ$), and 1 otherwise.

In the next two examples, let x, y, z be three different elements of X.

(iii) Consider the elements $a_1, a_2 \in \mathcal{F}(X)$ given by the formulas $a_1 = (1-x)^{-1} + (1-x^{-1})^{-1}$ and $a_2 = x^{-1} + x^{-1}(z^{-1}y^{-1} - x^{-1})^{-1}x^{-1}$. Let f_1 and f_2 be the corresponding elements in $F(X)$. Then the inversion height of a_1 and a_2 equals 2. On the other hand, in $F(X)$ we have $a_1 = 1$ and $a_2 = (x - yz)^{-1}$. Hence, the height of f_1 equals 0 and the height of f_2 equals 1.

(iv) The height of the element $f \in F(X)$ given by the formula $(x - yw^{-1}z)^{-1}$ equals 2.

1.2. Definition of quasideterminants

Let I, J be two finite sets of the same cardinality n and $\mathcal{X}$ be the set of n^2 elements x_{ij}, $1 \leqslant i,j \leqslant n$. Denote by $F(\mathcal{X})$ the free division ring generated by $\mathcal{X}$ (see 1.1). Let X be the $n \times n$-matrix over $F(\mathcal{X})$ with rows indexed by elements of I, columns indexed by elements of J, and the (i,j)th entry equal to $x_{ij} \in F(\mathcal{X})$.

Proposition 1.2.1. *The matrix X is invertible over $F(\mathcal{X})$.*

Proof. The proof is by induction in n. Let us assume, for simplicity, that $I = J = \{1, \dots, n\}$.

For $n = 1$, $X = (x_{11})$ and the inverse matrix $Y = X^{-1}$ is $Y = (y_{11})$, where $y_{11} = (x_{11})^{-1}$.

Let $n \geqslant 2$. Represent $X = (x_{ij})$ as a 2×2 block matrix according to the decompositions $\{1, \dots, n\} = \{1, \dots, n-1\} \cup \{n\}$ of I and J,

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

so that X_{11}, X_{12}, X_{21}, X_{22} are matrices of order $(n-1) \times (n-1)$, $(n-1) \times 1$, $1 \times (n-1)$, and 1×1, respectively. Then one can directly verify that the matrix Y given in the same block decomposition

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$$

by the formulas

$$Y_{11} = (X_{11} - X_{12}X_{22}^{-1}X_{21})^{-1},$$

$$Y_{12} = -X_{11}^{-1}X_{12}(X_{22} - X_{21}X_{11}^{-1}X_{12})^{-1},$$

$$Y_{21} = -X_{22}^{-1}X_{21}(X_{11} - X_{12}X_{22}^{-1}X_{21})^{-1},$$

$$Y_{22} = (X_{22} - X_{21}X_{11}^{-1}X_{12})^{-1},$$

is the inverse to X.

Let I, J be as in 1.2.1 and let Y be the matrix inverse to X, as in Proposition 1.2.1. Notice that each entry y_{ij} of Y is a nonzero element of the division ring $F(\mathcal{X})$. $\square$

Definition 1.2.2 (Quasideterminant of a matrix with formal entries). For $i \in I$, $j \in J$ the (i,j)th quasideterminant $|X|_{ij}$ of X is the element of $F(\mathcal{X})$ defined by the formula

$$|X|_{ij} = (y_{ji})^{-1}$$

where $Y = X^{-1} = (y_{ij})$, see Proposition 1.2.1.

From the proof of Proposition 1.2.1, we obtain the following recurrence relations for $|X|_{ij}$.

First of all, if $n = 1$, so that $I = \{i\}$, $J = \{j\}$, then $|X|_{ij} = x_{ij}$.

Next, let $n \geqslant 2$ and let X^{ij} be the $(n-1) \times (n-1)$-matrix obtained from X by deleting the ith row and the jth column. Then

$$|X|_{ij} = x_{ij} - \sum x_{ii'} (|X^{ij}|_{j'i'})^{-1} x_{j'j}. \tag{1.2.1}$$

Here the sum is taken over $i' \in I \backslash \{i\}$, $j' \in J \backslash \{j\}$.

Remark. In part (ii) of Definition 1.2.1, X^{ij} is the matrix with formal entries $x_{i'j'}$ indexed by elements $i' \in I \backslash \{i\}$, $j' \in J \backslash \{j\}$, and $(|X^{ij}|_{j'i'})^{-1}$ is the inverse of the quasideterminant $|X^{ij}|_{i'j'}$ in the corresponding free division ring $F(\mathcal{X}') \subset F(\mathcal{X})$, where $\mathcal{X}' = \{x_{i'j'},\ i' \in I \backslash \{i\},\ j' \in J \backslash \{j\}\}$.

Examples 1.2.3. (a) For the 2×2-matrix $X = (x_{ij})$, $i,j = 1,2$, there are four quasideterminants:

$$|X|_{11} = x_{11} - x_{12} \cdot x_{22}^{-1} \cdot x_{21}, \quad |X|_{12} = x_{12} - x_{11} \cdot x_{21}^{-1} \cdot x_{22},$$

$$|X|_{21} = x_{21} - x_{22} \cdot x_{12}^{-1} \cdot x_{11}, \quad |X|_{22} = x_{22} - x_{21} \cdot x_{11}^{-1} \cdot x_{12}.$$

(b) For the 3×3-matrix $X = (x_{ij})$, $i,j = 1,2,3$, there are 9 quasideterminants. One of them is

$$\begin{aligned} |X|_{11} = {} & x_{11} - x_{12}(x_{22} - x_{23}x_{33}^{-1}x_{32})^{-1}x_{21} - x_{12}(x_{32} - x_{33} \cdot x_{23}^{-1}x_{22})^{-1}x_{31} \\ & - x_{13}(x_{23} - x_{22}x_{32}^{-1}x_{33})^{-1}x_{21} - x_{13}(x_{33} - x_{32} \cdot x_{22}^{-1}x_{23})^{-1}x_{31}. \end{aligned}$$

The action of the product of symmetric groups $S_n \times S_n$ on $I \times J$, $|I| = |J| = n$, induces the action of $S_n \times S_n$ on the set of variables $\{a_{ij}\}$, $i \in I$, $j \in J$, and the corresponding action on the free division ring $F(\mathcal{X})$. We denote this latter action by $f \mapsto (\sigma, \tau)f$, $\sigma, \tau \in S_n$.

The following proposition shows that the definition of the quasideterminant is compatible with this action.

Proposition 1.2.4. *For $(\sigma, \tau) \in S_n \times S_n$ we have $(\sigma, \tau)(|X|_{ij}) = |X|_{\sigma(i)\tau(j)}$.*

In particular, the stabilizer subgroup of $|X|_{ij}$ under the action of $S_n \times S_n$ is isomorphic to $S_{n-1} \times S_{n-1}$.

Proposition 1.2.4 shows that in the definition of the quasideterminant, we do not need to require I and J to be ordered or a bijective correspondence between I and J to be given.

We go now to the definition of quasideterminants over a ring R with unit. Let $A = (a_{ij})$, $i \in I$, $j \in J$, be a matrix over R. Such a matrix determines the map $\alpha_A : \mathcal{X} \to R$, $\mathcal{X} = \{x_{ij}\}$, given by the formula $\alpha_A(x_{ij}) = a_{ij}$.

Definition 1.2.5 (Quasideterminant of a matrix over a ring). Let $i \in I$, $j \in J$, and the formal quasideterminant $|X|_{ij} \in F(\mathcal{X})$ can be evaluated at α_A in the sense of Definition 1.1.4. Then we say that the (ij)th quasideterminant $|A|_{ij}$ of A exists and is equal to $\alpha_A(|X|_{ij})$. Otherwise, we say that $|A|_{ij}$ does not exist.

According to this definition, the quasideterminant $|A|_{ij}$ of a matrix A over R is an element of R.

According to Definition 1.2.2 and Proposition 1.2.4, the quasideterminant $|A|_{ij}$ can be computed as follows. Denote by r_i^j the row submatrix of length $n-1$ obtained from ith row of A by deleting the element a_{ij}, and by c_j^i the column submatrix of height $n-1$ obtained from jth column of A by deleting the element a_{ij}.

Proposition 1.2.6. *Let $|I|, |J| > 1$ and assume that the $(n-1) \times (n-1)$-matrix A^{ij} is invertible over R. Then*

$$|A|_{ij} = a_{ij} - r_i^j (A^{ij})^{-1} c_j^i. \tag{1.2.2}$$

Remark. For a generic matrix A, to find the quasideterminant $|A|_{ij}$, one should take the formula to $|X|_{ij}$, substitute $x_{ij} \mapsto a_{ij}$, and verify that all inversions exist in R. However, in special cases (for example, when some of the entries of A equal zero), one might need to replace the formula for the quasideterminant by an equivalent formula and only then to substitute $x_{ij} \mapsto a_{ij}$. Here is an example.

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix},$$

where a_{21} and a_{32} are invertible in R. The quasideterminant $|A|_{13}$ cannot be defined using formula (1.2.1) since the rational expression $a_{12}(a_{22} - a_{21}a_{31}^{-1}a_{32})^{-1}a_{23}$ is not defined. However, if we replace this expression in formula (1.2.1) by the equivalent expression $a_{12}a_{32}^{-1}a_{31}(a_{22}a_{32}^{-1}a_{31} - a_{21})^{-1}a_{23}$, the new formula is defined for the matrix A and the corresponding rational function given the quasideterminant $|A|_{13}$.

Let us note also that the since the submatrix

$$A^{13} = \begin{pmatrix} a_{21} & a_{22} \\ 0 & a_{32} \end{pmatrix}$$

is invertible, the quasideterminant $|A|_{13}$ can be defined using formula (1.2.2).

Sometimes it is convenient to adopt a more graphic notation for the quasideterminant by boxing the element a_{ij}. For $A = (a_{ij})$, $i,j = 1, \dots, n$, we write

$$|A|_{pq} = \begin{vmatrix} a_{11} & \dots & a_{1q} & \dots & a_{1n} \\ & \dots & & \dots & \\ a_{p1} & \dots & \boxed{a_{pq}} & \dots & a_{pn} \\ & \dots & & \dots & \\ a_{n1} & \dots & a_{nq} & \dots & a_{nn} \end{vmatrix}.$$

If A is a generic $n \times n$-matrix (in the sense that all square submatrices of A are invertible), then there exist n^2 quasideterminants of A. However, a nongeneric matrix may have k quasideterminants, where $0 \leqslant k \leqslant n^2$. Example 1.2.3(a) shows that each of the quasideterminants $|A|_{11}$, $|A|_{12}$, $|A|_{21}$, $|A|_{22}$ of a 2×2-matrix A is defined whenever the corresponding element a_{22}, a_{21}, a_{12}, a_{11} is invertible.

Remark. The definition of the quasideterminant can be generalized to define $|A|_{ij}$ for a matrix $A = (a_{ij})$ in which each a_{ij} is an invertible morphism $V_j \to V_i$ in an additive category C and the matrix A^{pq} of morphisms is invertible. In this case the quasideterminant $|A|_{pq}$ is a morphism from the object V_q to the object V_p.

The next example shows that the notion of a quasideterminant is not a generalization of a determinant over a commutative ring, but rather a generalization of a ratio of two determinants.

Example. If the elements a_{ij} of the matrix A commute, then

$$|A|_{pq} = (-1)^{p+q} \frac{\det A}{\det A^{pq}}.$$

We will show in Section 3 that similar expressions for quasideterminants can be given for quantum matrices, quaternionic matrices, Capelli matrices and other cases listed in the Introduction.

In general quasideterminants are not polynomials in their entries, but (noncommutative) rational functions. The following theorem was conjectured by Gelfand and Retakh, and proved by Reutenauer [Re] in a slightly different form.

Theorem 1.2.7. *Quasideterminants of the $n \times n$-matrix $X = (x_{ij})$ with formal entries have the inversion height $n - 1$ over the free division ring $F(\mathcal{X})$, $\mathcal{X} = \{x_{ij}\}$.*

In the commutative case determinants are finite sums of monomials with appropriate coefficients. As is shown in [GR1,GR2], in the noncommutative case quasideterminants of a matrix $X = (x_{ij})$ with formal entries x_{ij} can be identified with formal power series in the matrix entries or their inverse. A simple example of this type is described below.

Let $X = (x_{ij})$, $i, j = 1, \ldots, n$, be a matrix with formal entries. Denote by E_n the identity matrix of order n and by Γ_n the complete oriented graph with vertices $\{1, 2, \ldots, n\}$, with the arrow from i to j labeled by x_{ij}. A path p : $i \to k_1 \to k_2 \to \cdots \to k_t \to j$ is labeled by the word $w = x_{ik_1} x_{k_1 k_2} x_{k_2 k_3} \ldots x_{k_t j}$.

Denote by P_{ij} the set of words labelling paths going from i to j, i.e. the set of words of the form $w = x_{ik_1} x_{k_1 k_2} x_{k_2 k_3} \ldots x_{k_t j}$. A simple path is a path p such that $k_s \neq i, j$ for every s. Denote by $P_{ij}{}'$ the set of words labelling simple paths from i to j.

Let R be the ring of formal power series in x_{ij} over a field. From [Co, Section 4], it follows that there is a canonical embedding of R in a division ring D such that the image of R generates D. We identify R with its image in D.

Proposition 1.2.8. *Let i, j be two distinct integers between* 1 *and n. The rational functions $|I_n - X|_{ii}$, $|I_n - X|_{ij}^{-1}$ are defined in D and*

$$|I_n - X|_{ii} = 1 - \sum_{w \in P'_{ii}} w, \quad |I_n - X|_{ij}^{-1} = \sum_{w \in P_{ij}} w.$$

Example. For $n = 2$,

$$|I_2 - X|_{11} = 1 - x_{11} - \sum_{p \geqslant 0} x_{12} x_{22}^p x_{21}.$$

For some matrices of special form over a ring, quasideterminants can be expressed as polynomials in the entries of the matrix. The next proposition shows that this holds, in particular, for the so-called almost triangular matrices. Such matrices play an important role in many papers, including [DS,Ko,Gi].

Proposition 1.2.9. *The following quasideterminant is a polynomial in its entries*:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \ldots & \boxed{a_{1n}} \\ -1 & a_{22} & a_{23} & \ldots & a_{2n} \\ 0 & -1 & a_{33} & \ldots & a_{3n} \\ & & \ldots & & \\ 0 & & \ldots & -1 & a_{nn} \end{vmatrix}$$
$$= a_{1n} + \sum_{1 \leqslant j_1 < j_2 < \cdots < j_k < n} a_{1j_1} a_{j_1+1, j_2} a_{j_2+1, j_3} \ldots a_{j_k+1, n}.$$

Remark. Denote the expression on right-hand side by $P(A)$. Note that $(-1)^{n-1} P(A)$ equals to the determinant of the almost upper-triangular matrix over a commutative ring. For noncommutative almost upper triangular matrices, Givental [Gi] (and others) defined the determinant as $(-1)^{n-1} P(A)$.

Example. For $n = 3$ we have

$$P(A) = a_{13} + a_{11}a_{23} + a_{12}a_{33} + a_{11}a_{22}a_{33}.$$

1.3. Transformation properties of quasideterminants

Let $A = (a_{ij})$ be a square matrix of order n over a ring R.

(i) The quasideterminant $|A|_{pq}$ does not depend on permutations of rows and columns in the matrix A that do not involve the pth row and the qth column. This follows from Proposition 1.2.3.

(ii) *The multiplication of rows and columns.* Let the matrix $B = (b_{ij})$ be obtained from the matrix A by multiplying the ith row by $\lambda \in R$ from the left, i.e., $b_{ij} = \lambda a_{ij}$ and $b_{kj} = a_{kj}$ for $k \neq i$. Then

$$|B|_{kj} = \begin{cases} \lambda |A|_{ij} & \text{if } k = i, \\ |A|_{kj} & \text{if } k \neq i \text{ and } \lambda \text{ is invertible.} \end{cases}$$

Let the matrix $C = (c_{ij})$ be obtained from the matrix A by multiplying the jth column by $\mu \in R$ from the right, i.e. $c_{ij} = a_{ij}\mu$ and $c_{il} = a_{il}$ for all i and $l \neq j$. Then

$$|C|_{i\ell} = \begin{cases} |A|_{ij}\mu & \text{if } l = j, \\ |A|_{i\ell} & \text{if } l \neq j \text{ and } \mu \text{ is invertible.} \end{cases}$$

(iii) *The addition of rows and columns.* Let the matrix B be obtained from A by replacing the kth row of A with the sum of the kth and lth rows, i.e., $b_{kj} = a_{kj} + a_{lj}$, $b_{ij} = a_{ij}$ for $i \neq k$. Then

$$|A|_{ij} = |B|_{ij}, \quad i = 1, \dots k-1, k+1, \dots n, \ j = 1, \dots, n.$$

Let the matrix C be obtained from A by replacing the kth column of A with the sum of the kth and lth columns, i.e., $c_{ik} = a_{ik} + a_{il}$, $c_{ij} = a_{ij}$ for $j \neq k$. Then

$$|A|_{ij} = |C|_{ij}, \quad i = 1, \dots, n, \quad , \dots, \ell - 1, \ell + 1, \dots n.$$

1.4. General properties of quasideterminants

1.4.1. Two involutions (see [GR4])

For a square matrix $A = (a_{ij})$ over a ring R, denote by $IA = A^{-1}$ the inverse matrix (if it exists), and by $HA = (a_{ji}^{-1})$ the Hadamard inverse matrix (also if it exists). It is evident that if IA exists, then $I^2A = A$, and if HA exists, then $H^2A = A$.

Let $A^{-1} = (b_{ij})$. According to Theorem 1.2.1, $b_{ij} = |A|_{ji}^{-1}$. This formula can be rewritten in the following form.

Theorem 1.4.1. *For a square matrix A over a ring R,*

$$HI(A) = (|A|_{ij}) \tag{1.4.1}$$

provided that all quasideterminants $|A|_{ij}$ exist.

1.4.2. Homological relations (see [GR])

Let $X = (x_{ij})$ be a square matrix of order n with formal entries. For $1 \leqslant k, l \leqslant n$ let X^{kl} be the submatrix of order $n-1$ of the matrix X obtained by deleting the kth row and the lth column. Quasideterminants of the matrix X and the submatrices are connected by the following *homological relations.*

Theorem 1.4.2. (i) *Row homological relations*:

$$-|A|_{ij} \cdot |A^{i\ell}|_{sj}^{-1} = |A|_{i\ell} \cdot |A^{ij}|_{s\ell}^{-1}, \quad s \neq i.$$

(ii) *Column homological relations*:

$$-|A^{kj}|_{it}^{-1} \cdot |A|_{ij} = |A^{ij}|_{kt}^{-1} \cdot |A|_{kj}, \quad t \neq j.$$

The same relations hold for matrices over a ring R provided the corresponding quasideterminants exist and are invertible.

A consequence of homological relations is that the ratio of two quasideterminants of an $n \times n$ matrix (each being a rational function of inversion height $n-1$) actually equals a ration of two rational functions each having inversion height $<n-1$.

1.4.3. Heredity

Let $A = (a_{ij})$ be an $n \times n$ matrix over a ring R, and let

$$A = \begin{pmatrix} A_{11} & \dots & A_{1s} \\ & & \\ A_{s1} & \dots & A_{ss} \end{pmatrix} \tag{1.4.2}$$

be a block decomposition of A, where each A_{pq} is a $k_p \times l_q$ matrix, $k_1 + \dots + k_s = l_1 + \dots + l_s = n$. Let us choose p' and q' such that $k_{p'} = l_{q'}$, so that $A_{p'q'}$ is a square matrix.

Let also $X = (x_{pq})$ be a matrix with formal variables and $|X|_{p'q'}$ be the $p'q'$-quasideterminant of X. In the formula for $|X|_{p'q'}$ as a rational function in variable x_{pq} we can substitute each variable x_{pq} with the corresponding matrix A_{pq}, obtaining a rational expression $F(A_{pq})$. Let us note that all matrix operations in this rational expression formally make sense, i.e., in each addition, the orders of summands coincide, in each multiplication, the number of columns of the first multiplier equals the number of rows of the second multiplier, and each matrix that has to be inverted is a square matrix. Let us assume that all matrices in this rational expression for that need to be inverted, are indeed invertible over R. Computing $F(A_{pq})$, we obtain an $k_{p'} \times l_{q'}$ matrix over R, whose rows are naturally numbered by indices

$$i = k_1 + \cdots + k_{p'-1} + 1, \ldots, k_1 + \cdots + k_{p'} \tag{1.4.3}$$

and columns are numbered by indices

$$j = l_1 + \cdots + l_{q'-1} + 1, \ldots, l_1 + \cdots + l_{q'}. \tag{1.4.4}$$

We denote this matrix by $|X|_{p'q'}(A)$.

Let us note that under our assumptions, $k_{p'} = l_{q'}$, so that $|X|_{p'q'}(A)$ is a square matrix over R.

Theorem 1.4.3. *Let the index i lies in the range* (1.4.3) *and the index j lies in the range* (1.4.4). *Let us assume that the matrix* $|X|_{p'q'}(A)$ *is defined. Then each of the quasideterminants* $|A|_{ij}$ *and* $||X|_{p'q'}(A)|_{ij}$ *exist if and only if the other exists, and in this case*

$$|A|_{ij} = ||X|_{p'q'}(A)|_{ij}. \tag{1.4.5}$$

Example 1. Let in (1.4.2) $s = 2, p' = q' = 1$ and $k_1 = l + 1 = 1$. Then formula (1.4.5) becomes the inductive definition of the quasideterminant $|A|_{ij}$ (see Definition 1.2.5).

Example 2. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

Take the decomposition $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ of A into four 2×2 matrices, so that $A_{11} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $A_{12} = \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix}$, $A_{21} = \begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}$, $A_{22} = \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix}$. Let us use formula (1.4.5) to find the quasideterminant $|A|_{13}$. We have

$$\begin{aligned} |X|_{12}(A) &= A_{12} - A_{11} A_{21}^{-1} A_{22} \\ &= \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}^{-1} \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} \\ &= \begin{pmatrix} a_{13} - \ldots & a_{14} - \ldots \\ a_{23} - \ldots & a_{24} - \ldots \end{pmatrix}. \end{aligned}$$

Denote the matrix in the right-hand side of this formula by $\begin{pmatrix} c_{13} & c_{14} \\ c_{23} & c_{24} \end{pmatrix}$. Then

$$|A|_{13} = \begin{vmatrix} c_{13} & c_{14} \\ c_{23} & c_{24} \end{vmatrix}_{13},$$

or, in other notation,

$$|A|_{13} = \begin{vmatrix} \boxed{c_{13}} & c_{14} \\ c_{23} & c_{24} \end{vmatrix}.$$

1.4.4. A generalization of the homological relations

Homological relations admit the following generalization. For a matrix $A = (a_{ij})$, $i \in I, j \in J$, and two subsets $L \subset I$, $M \subset J$ denote by $A^{L,M}$ the submatrix of the matrix A obtained by deleting the rows with the indexes $\ell \in L$ and the columns with the indexes $m \in M$. Let A be a square matrix, $L = (\ell_1, \ldots, \ell_k), M = (m_0, \ldots, m_k)$. Set $M_i = M \backslash \{m_i\}$, $i = 0, \ldots, k$.

Theorem 1.4.4 (Gelfand et al. [GR1,GR2]). *For $p \notin L$ we have*

$$\sum_{i=0}^{k} |A^{L,M_i}|_{pm_i} \cdot |A|^{-1}_{\ell m_i} = \delta_{p\ell},$$

$$\sum_{i=0}^{k} |A|^{-1}_{m_i \ell} \cdot |A^{M_i,L}|_{m_i p} = \delta_{\ell p},$$

provided the corresponding quasideterminants are defined and the matrices $|A|^{-1}_{m_i\ell}$, $|A|^{-1}_{\ell m_i}$ are invertible over R.

1.4.5. Quasideterminants and Kronecker tensor products

Let $A = (a_{ij})$, $B = (b_{\alpha\beta})$ be matrices over a ring R. Denote by $C = A \otimes B$ the Kronecker tensor product, i.e., the matrix with entries numbered by indices $(i\alpha, j\beta)$, and with the $(i\alpha, j\beta)$th entry equal to $c_{i\alpha, j\beta} = a_{ij} b_{\alpha\beta}$.

Proposition 1.4.5. *If quasideterminants $|A|_{ij}$ and $|B|_{\alpha\beta}$ are defined, then the quasideterminant $|A \otimes B|_{i\alpha, j\beta}$ is defined and*

$$|A \otimes B|_{i\alpha, j\beta} = |A|_{ij} |B|_{\alpha\beta}.$$

Note that in the commutative case the corresponding identity determinants is different. Namely, if A is a $m \times m$-matrix and B is a $n \times n$-matrix over a commutative ring, then $\det(A \otimes B) = (\det A)^n (\det B)^m$.

1.4.6. Quasideterminants and matrix rank

Let $A = (a_{ij})$ be a matrix over a division ring.

Proposition 1.4.6. *If the quasideterminant $|A|_{ij}$ is defined, then the following statements are equivalent.*

(i) $|A|_{ij} = 0$,
(ii) *the ith row of the matrix A is a left linear combination of the other rows of A;*
(iii) *the jth column of the matrix A is a right linear combination of the other columns of A.*

Example. Let $i, j = 1, 2$ and $|A|_{11} = 0$, i.e., $a_{11} - a_{12}a_{22}^{-1}a_{21} = 0$. Therefore, $a_{11} = \lambda a_{21}$, where $\lambda = a_{12}a_{22}^{-1}$. Since $a_{12} = (a_{12}a_{22}^{-1})a_{22}$, the first row of A is proportional to the second row.

There exists the notion of linear dependence for elements of a (right or left) vector space over a division ring. So there exists the notion of the row rank (the dimension of the left vector space spanned by the rows) and the notion of the column rank (the dimension of the right vector space spanned by the columns) and these ranks are equal [Ja,Co]. This also follows from Proposition 1.4.6.

By definition, an r-quasiminor of a square matrix A is a quasideterminant of an $r \times r$-submatrix of A.

Proposition 1.4.7. *The rank of the matrix A over a division algebra is $\geqslant r$ if and only if at least one r-quasiminor of the matrix A is defined and is not equal to zero.*

1.5. Basic identities

1.5.1. Row and column decomposition

The following result is an analogue of the classical expansion of a determinant by a row or a column.

Proposition 1.5.1. *Let A be a matrix over a ring R. For each $k \neq p$ and each $\ell \neq q$ we have*

$$|A|_{pq} = a_{pq} - \sum_{j \neq q} a_{pj}(|A^{pq}|_{kj})^{-1}|A^{pj}|_{kq},$$

$$|A|_{pq} = a_{pq} - \sum_{i \neq p} |A^{iq}|_{pi}(|A^{pq}|_{i\ell})^{-1}a_{iq},$$

provided all terms in right-hand sides of these expressions are defined.

As it was pointed out in [KL], Proposition 1.5.1 immediately follows from the homological relations (Theorem 1.4.2).

1.5.2. Sylvester's identity

Let $A = (A_{ij}),\ i,j = 1,\dots,n,$ be a matrix over a ring R and $A_0 = (a_{ij}),\ i,j = 1,\dots,k$, a submatrix of A that is invertible over R. For $p,q = k+1,\dots,n$ set

$$c_{pq} = \begin{vmatrix} & A_0 & & a_{1q} \\ & & & \vdots \\ & & & a_{kq} \\ a_{p1} & \dots & a_{pk} & a_{pq} \end{vmatrix}_{pq}.$$

These quasideterminants are defined because matrix A_0 is invertible.

Consider the $(n-k)\times(n-k)$ matrix

$$C = (c_{pq}), \quad p,q = k+1,\dots,n.$$

The submatrix A_0 is called the *pivot* for the matrix C.

Theorem 1.5.2 (see Gelfand and Retakh [GR]). *For $i,j = k+1,\dots,n$,*

$$|A|_{ij} = |C|_{ij}$$

The commutative version of Theorem 1.5.2 is the following Sylvester's theorem.

Theorem 1.5.3. *Let $A = (a_{ij}),\ i,j = 1,\dots,n,$ be a matrix over a commutative ring. Suppose that the submatrix $A_0 = (a_{ij}),\ i,j = 1,\dots,k,$ of A is invertible. For $p,q = k+1,\dots,n$ set*

$$\tilde{b}_{pq} = \det\begin{pmatrix} A_0 & & & a_{1q} \\ & & & \vdots \\ & & & a_{kq} \\ a_{p1} & \dots & a_{pk} & a_{pq} \end{pmatrix},$$

$$\tilde{B} = (\tilde{b}_{pq}), \quad p,q = k+1,\dots,n.$$

Then

$$\det A = \frac{\det \tilde{B}}{(\det A_0)^{n-k-1}}.$$

Remark 1. A quasideterminant of an $n\times n$-matrix A is equal to the corresponding quasideterminant of a 2×2-matrix consisting of $(n-1)\times(n-1)$-quasiminors of the matrix A, or to the quasideterminant of an $(n-1)\times(n-1)$-matrix consisting of 2×2-quasiminors of the matrix A. One can use any of these procedures for an inductive definition of quasideterminants. In fact, Heyting [H] essentially defined the quasideterminants $|A|_{nn}$ for matrices $A = (a_{ij})$, $i,j = 1, \ldots, n$, in this way.

Remark 2. Theorem 1.5.2 can be generalized to the case where A_0 is a square submatrix of A formed by some (not necessarily consecutive and not necessarily the same) rows and columns of A. In particular, in the case where $A_0 = (a_{ij}), i,j = 2, \ldots, n-1$, Theorem 1.5.2 is an analogue of a well-known commutative identity which is called the ''Lewis Carroll identity'' (see, for example, [Ho]).

1.5.3. Inversion for quasiminors

The following theorem was formulated in [GR]. For a matrix $A = (a_{ij})$, $i \in I$, $j \in J$, over a ring A and subsets $P \subset I$, $Q \subset J$ denote by A_{PQ} the submatrix

$$A_{PQ} = (a_{\alpha\beta}), \quad \alpha \in P, \ \beta \in Q.$$

Let $|I| = |J|$ and $B = A^{-1} = (b_{rs})$. Suppose that $|P| = |Q|$.

Theorem 1.5.4. *Let $k \notin P, \ell \notin Q$. Then*

$$|A_{P\cup\{k\},Q\cup\{\ell\}}|_{k\ell} \cdot |B_{I\backslash P,J\backslash Q}|_{\ell k} = 1.$$

Set $P = I\backslash\{k\}, Q = J\backslash\{\ell\}$. Then this theorem leads to the already mentioned identity

$$|A|_{k\ell} \cdot b_{\ell k} = 1.$$

Example. Theorem 1.5.4 implies the following identity for principal quasiminors. Let $A = (a_{ij})$, $i,j = 1, \ldots, n$ be an invertible matrix over R and $B = (b_{ij}) = A^{-1}$. For a fixed k, $1 \leqslant k \leqslant n$, set $A_{(k)} = (a_{ij})$, $i,j = 1, \ldots, k$ and $B^{(k)} = (b_{ij})$, $i,j = k, \ldots, n$. Then

$$|A_{(k)}|_{kk} \cdot |B^{(k)}|_{kk} = 1.$$

1.5.4. Multiplicative properties of quasideterminants

Let $X = (x_{pq}), Y = (y_{rs})$ be $n \times n$-matrices. The following statement follows directly from Definition 1.2.2.

Theorem 1.5.5. *We have*

$$|XY|_{ij}^{-1} = \sum_{p=1}^{n} |Y|_{pj}^{-1} |X|_{ip}^{-1}.$$

1.6. Noncommutative linear algebra

In this section we use quasideterminants to noncommutative generalizations of basic theorems about systems of linear equations (see [GR,GR1,GR2]).

1.6.1. Solutions of systems of linear equations

Theorem 1.6.1. *Let $A = (a_{ij})$ be an $n \times n$ matrix over a ring R. Assume that all the quasideterminants $|A|_{ij}$ are defined and invertible. Then*

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = \xi_1 \\ \dots \\ a_{n1}x_1 + \dots + a_{nn}x_n = \xi_n \end{cases}$$

for some $x_i \in R$ if and only if

$$x_i = \sum_{j=1}^{n} |A|_{ji}^{-1} \xi_j. \quad i = 1, \dots, n.$$

1.6.2. Cramer's rule

Let $A_\ell(\xi)$ be the $n \times n$-matrix obtained by replacing the ℓth column of the matrix A with the column $(\xi_1, \dots, \xi_n)$.

Theorem 1.6.2. *In notation of Theorem* 1.6.1, *if the quasideterminants $|A|_{ij}$ and $|A_j(\xi)|_{ij}$ are defined, then*

$$|A|_{ij} x_j = |A_j(\xi)|_{ij}.$$

1.6.3. Cayley–Hamilton theorem

Let $A = (a_{ij})$, $i, j = 1, \dots, n$, be a matrix over a ring R. Denote by E_n the identity matrix of order n.

Let t be a formal variable. Set $f_{ij} = |tE_n - A|_{ij}$ for $1 \leqslant i, j \leqslant n$. Then $f_{ij}(t)$ is a rational function in t. Define the matrix function $\tilde{f}_{ij}(t)$ by replacing in $f_{ij}(t)$ each element a_{ij} with the matrix $\tilde{a}_{ij} = a_{ij}E_n$ of order n and the variable t by the matrix A. The functions $f_{ij}(t)$ are called the characteristic functions of the matrix A.

The following theorem was stated in [GR1,GR2].

Theorem 1.6.3. $\tilde{f}_{ij}(A) = 0$ *for all* $i, j = 1, \dots, n$.

2. Important example: quaternionic quasideterminants

As an example, we compute here quasideterminants of quaternionic matrices.

2.1. Norms of quaternionic matrices

Let $\mathbb{H}$ be the algebra of quaternions. Algebra $\mathbb{H}$ is an algebra over the field of real numbers $\mathbb{R}$ with generators $\mathbf{i}, \mathbf{j}, \mathbf{k}$ such that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = \mathbf{k}, \mathbf{jk} = \mathbf{i}, \mathbf{ki} = \mathbf{j}$. It follows from the definition that $\mathbf{ij} + \mathbf{ji} = 0$, $\mathbf{ik} + \mathbf{ki} = 0$, $\mathbf{jk} + \mathbf{kj} = 0$.

Algebra $\mathbb{H}$ possesses a standard anti-involution $a \mapsto \bar{a}$: if $a = x + y\mathbf{i} + z\mathbf{j} + t\mathbf{k}$, $x, y, z, t \in \mathbb{R}$, then $\bar{a} = x - y\mathbf{i} - z\mathbf{j} - t\mathbf{k}$. It follows that $a\bar{a} = x^2 + y^2 + z^2 + t^2$. The multiplicative functional $v : \mathbb{H} \to \mathbb{R}_{\geqslant 0}$ where $v(a) = a\bar{a}$ is called the norm of a. One can see that $a^{-1} = \frac{\bar{a}}{v(a)}$ for $a \neq 0$.

We will need the following generalization of the norm v to quaternionic matrices. Let $M(n, \mathbb{H})$ be the $\mathbb{R}$-algebra of quaternionic matrices of order n. There exists a unique multiplicative functional $v : M(n, \mathbb{H}) \to \mathbb{R}_{\geqslant 0}$ such that

(i) $v(A) = 0$ if and only if the matrix A is noninvertible,

(ii) If A' is obtained from A by adding a left-multiple of a row to another row or a right-multiple of a column to another column, then $v(A') = v(A)$.

(iii) $v(E_n) = 1$ where E_n is the identity matrix of order n.

The number $v(A)$ is called the *norm* of the quaternionic matrix A.

For a quaternionic matrix $A = (a_{ij})$, $i, j = 1, \dots, n$, denote by $A^* = (\bar{a}_{ji})$ the conjugate matrix. It is known that $v(A)$ coincides with the Dieudonne determinant of A and with the Moore determinant of AA^* (see [As] and Sections 3.2–3.4 below). The norm $v(A)$ is a real number and it is equal to an alternating sum of monomials of order $2n$ in the a_{ij} and $\bar{a}_{ij}$. An expression for $v(A)$ is given by Theorem 2.1.2 below.

Let $A = (a_{ij})$, $i, j = 1, \dots, n$, be a quaternionic matrix. Let $I = \{i_1, \dots, i_k\}$, $J = \{j_1, \dots, j_k\}$ be two ordered sets of natural numbers such that all i_p and all j_p are distinct. Set

$$z_{I,J} = a_{i_1 j_1} \bar{a}_{i_2 j_1} a_{i_2 j_2} \dots a_{i_k j_k} \bar{a}_{i_1 j_k}.$$

Denote by $\mu_i(A)$ the sum of all $z_{I,J}(A)$ such that $i_1 = i$. One can easily see that $\mu_i(A)$ is a real number since with each monomial $z_{I,J}$ it contains the conjugate monomial $\overline{z_{I,J}} = z_{I',J'}$, where $I' = \{i_1, i_k, i_{k-1} \dots, i_2\}$, $J = \{j_k, j_{k-1}, \dots, j_1\}$.

Proposition 2.1.1. *The sum $\mu_i(A)$ does not depend on i.*

Example. For $n = 1$ the statement is obvious. For $n = 2$ we have

$$\mu_1(A) = a_{11}\bar{a}_{21}a_{22}\bar{a}_{12} + a_{12}\bar{a}_{22}a_{21}\bar{a}_{11},$$

$$\mu_2(A) = a_{22}\bar{a}_{12}a_{11}\bar{a}_{21} + a_{21}\bar{a}_{11}a_{12}\bar{a}_{22}.$$

Note that for two arbitrary quaternions x, y we have $xy + \bar{y}\bar{x} = 2\Re(xy) = 2\Re(yx) = yx + \bar{x}\bar{y}$, where $\Re(a)$ is the real part of the quaternion a. By setting $x = a_{11}\bar{a}_{21}$, $y = a_{22}\bar{a}_{12}$ one see that $\mu_1(A) = \mu_2(A)$.

Proposition 2.1.1 shows that we may omit the index i in $\mu_i(A)$ and denote it by $\mu(A)$.

Let $A = (a_{ij}),\ i,j = 1, \dots, n$ be a matrix. We call an (unordered) set of square submatrices $\{A_1, \dots, A_s\}$ where $A_p = (a_{ij}),\ i \in I_p,\ j \in J_p$ a *complete set* if $I_p \cap I_q = J_p \cap J_q = \emptyset$ for all $p \neq q$ and $\bigcup_p I_p = \bigcup_p J_p = \{1, \dots, n\}$.

Theorem 2.1.2. *Let $A = (a_{ij}),\ i,j = 1, \dots, n$ be a quaternionic matrix. Then*

$$v(A) = \sum (-1)^{k_1 + \cdots + k_p - p} \mu(A_1) \dots \mu(A_p),$$

where the sum is taken over all complete sets $(A_1, \dots, A_p)$ of submatrices of A, k_i is the order of the matrix A_i.

Example. For $n = 2$ we have

$$v(A) = v(a_{11})v(a_{22}) + v(a_{12})v(a_{21}) - (a_{11}\bar{a}_{21}a_{22}\bar{a}_{12} + a_{12}\bar{a}_{22}a_{21}\bar{a}_{11}).$$

Corollary 2.1.3. *Let A be a square quaternionic matrix. Fix an arbitrary $i \in \{1, \dots, n\}$. Then*

$$v(A) = \sum (-1)^{k(B_1)-1} v(B_1)\mu(B_2),$$

where the sum is taken over all complete sets of submatrices (B_1, B_2) such that B_2 contains an element from the ith row, $k(B_1)$ the order of B_1, and $v(B_1) = 1$ if $B_2 = A$.

2.2. *Quasideterminants of quaternionic matrices*

This section contains results from [GRW1].

Let $A = (a_{ij}),\ i,j = 1, \dots, n$, be a quaternionic matrix. Let $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_k\}$ be two ordered sets of natural numbers $1 \leqslant i_1, i_2, \dots, i_k \leqslant n$ and $1 \leqslant j_1, j_2, \dots, j_k \leqslant n$ such that all i_p are distinct and all j_p are distinct. For $k = 1$ set $m_{I,J}(A) = a_{i_1 j_1}$. For $k \geqslant 2$ set

$$m_{I,J}(A) = a_{i_1 j_2}\bar{a}_{i_2 j_2}a_{i_2 j_3}\bar{a}_{i_3 j_3}a_{i_3 j_4} \dots \bar{a}_{i_k j_k}a_{i_k j_1}.$$

If the matrix A is Hermitian, i.e., $a_{ji} = \bar{a}_{ij}$ for all i,j, then

$$m_{I,J}(A) = a_{i_1 j_2}a_{j_2 i_2}a_{i_2 j_3}a_{j_3 i_3}a_{i_3 j_4} \dots a_{j_k i_k}a_{i_k j_1}.$$

To a quaternionic matrix $A = (a_{pq}),\ p, q = 1, \dots, n$, and to a fixed row index i and a column index j we associate a polynomial in $a_{pq}, \bar{a}_{pq}$, which we call the (i,j)th double permanent of A.

Definition 2.2.1. The (i,j)th double permanent of A is the sum

$$\pi_{ij}(A) = \sum m_{I,J}(A),$$

taken over all orderings $I = \{i_1, \ldots, i_n\}, J = \{j_1, \ldots, j_n\}$ of $\{1, \ldots, n\}$ such that $i_1 = i$ and $j_1 = j$.

Example. For $n = 2$

$$\pi_{11}(A) = a_{12}\bar{a}_{22}a_{21}.$$

For $n = 3$

$$\pi_{11}(A) = a_{12}\bar{a}_{32}a_{33}\bar{a}_{23}a_{21} + a_{12}\bar{a}_{22}a_{23}\bar{a}_{33}a_{31} + a_{13}\bar{a}_{33}a_{32}\bar{a}_{22}a_{21} + a_{13}\bar{a}_{23}a_{22}\bar{a}_{32}a_{31}.$$

For a submatrix B of A denote by B^c the matrix obtained from A by deleting all rows and columns containing elements from B. If B is a $k \times k$-matrix, then B^c is a $(n-k) \times (n-k)$-matrix. B^c is called the complementary submatrix of B.

Quasideterminants of a matrix $A = (a_{ij})$ are rational functions of elements a_{ij}. Therefore, for a quaternionic matrix A, its quasideterminants are polynomials in a_{ij} and their conjugates, with coefficients that are rational functions of a_{ij} always taking rational values. The following theorem gives expressions for these polynomials.

Theorem 2.2.2. *If the quasideterminant $|A|_{ij}$ of a quaternionic matrix is defined, then*

$$v(A^{ij})|A|_{ij} = \sum (-1)^{k(B)-1} v(B^c)\pi_{ij}(B),$$

where the sum is taken over all square submatrices B of A containing $a_{ij}, k(B)$ is the order of B, and we set $v(B^c) = 1$ for $B = A$.

Recall that according to Proposition 1.2.6 the quasideterminant $|A|_{ij}$ is defined if the matrix A^{ij} is invertible. In this case $v(A^{ij})$ is invertible, so that formula (2.2.1) indeed gives an expression for $|A|_{ij}$.

The right-hand side in (2.2.1) is a linear combination with real coefficients of monomials of lengths $1, 3, \ldots, 2n-1$ in a_{ij} and $\bar{a}_{ij}$. The number $\mu(n)$ of such monomials for a matrix of order n is $\mu(n) = 1 + (n-1)^2\mu(n-1)$.

Example. For $n = 2$

$$v(a_{22})|A|_{11} = v(a_{22})a_{11} - a_{12}\bar{a}_{22}a_{21}.$$

For $n = 3$

$$\begin{aligned} v(A^{11})|A|_{11} = & v(A^{11})a_{11} - v(a_{33})a_{12}\bar{a}_{22}a_{21} - v(a_{23})a_{12}\bar{a}_{32}a_{31} \\ & - v(a_{32})a_{13}\bar{a}_{23}a_{21} - v(a_{22})a_{13}\bar{a}_{33}a_{31} + a_{12}\bar{a}_{32}a_{33}\bar{a}_{23}a_{21} \\ & + a_{12}\bar{a}_{22}a_{23}\bar{a}_{33}a_{31} + a_{13}\bar{a}_{33}a_{32}\bar{a}_{22}a_{21} + a_{13}\bar{a}_{23}a_{22}\bar{a}_{32}a_{31}. \end{aligned}$$

The example shows how to simplify the general formula for quasideterminants of matrix of order 3 (see Section 1.2) for quaternionic matrices.

The following theorem, which is similar to Corollary 2.1.3, shows that the coefficients in formula (2.2.1) are uniquely defined.

Theorem 2.2.3. *Let quasideterminants $|A|_{ij}$ of quaternionic matrices are given by the formula*

$$\xi(A^{ij})|A|_{ij} = \sum (-1)^{k(B)-1} \xi(B^c) \pi_{ij}(B)$$

and all coefficients $\xi(C)$ depend of submatrix C only, then $\xi(C) = v(C)$ for all square matrix C.

Example. For $n = 2$ set $a_{11} = 0$. Then $\xi(a_{22})a_{12}a_{22}^{-1}a_{21} = a_{12}\bar{a}_{22}a_{21}$. This implies that $\xi(a_{22}) = \bar{a}_{22}a_{22} = v(a_{22})$.

3. Noncommutative determinants

Noncommutative determinants were defined in different and, sometimes, not related situations. In this section we present some results from [GR,GR1, GR2,GRW1] describing a universal approach to noncommutative determinants and norms of noncommutative matrices based on the notion of quasideterminants.

3.1. Noncommutative determinants as products of quasiminors

Let $A = (a_{ij}),\ i,j = 1, \dots, n$, be a matrix over a division ring R such that all square submatrices of A are invertible. For $\{i_1, \dots, i_k\}, \{j_1, \dots, j_k\} \subset \{1, \dots, n\}$ define $A^{i_1 \dots i_k, j_1 \dots j_k}$ to be the submatrix of A obtained by deleting rows with indices $i_1, \dots, i_k$ and columns with indices $j_1, \dots, j_k$. Next, for any orderings $I = (i_1, \dots, i_n), J = (j_1, \dots, j_n)$ of $\{1, \dots, n\}$ set

$$D_{I,J}(A) = |A|_{i_1 j_1} |A^{i_1 j_1}|_{i_2 j_2} |A^{i_1 i_2, j_1 j_2}|_{i_3 j_3} \dots a_{i_n j_n}.$$

In the commutative case $D_{I,J}(A)$ is, up the sign, the determinant of A. When A is a quantum matrix $D_{I,J}(A)$ differs from the quantum determinant of A by a factor depending on q [GR,GR1,KL]. The same is true for some other noncommutative algebras. This suggests to call $D_{I,J}(A)$ the (I,J)-predeterminants of A. From the "categorical point of view" the expressions $D_{I,\tilde{I}}(A)$ where $I=(i_1,i_2,\dots,i_n),\tilde{I}=(i_2,i_3,\dots,i_n,i_1)$ are particularly important. We denote $D_I(A)=D_{I,\tilde{I}}(A)$. It is also convenient to have the basic predeterminant

$$\Delta(A)=D_{\{12\dots n\},\{23\dots n1\}}. \tag{3.1.1}$$

We use the homological relations for quasideterminants to compare different $D_{I,J}$. Here we restrict ourselves to elementary transformations of I and J.

Let $I=(i_1,\dots,i_p,i_{p+1},\dots,i_n)$ and $J=(j_1,\dots,j_p,j_{p+1},\dots,j_n)$. Set $I'=(i_1,\dots,i_{p+1},i_p,\dots,i_n),J'=(j_1,\dots,j_{p+1},j_p,\dots,j_n)$. Set also

$$X=|A|_{i_1,j_1}|A^{i_1,j_1}|_{i_2,j_2}\dots|A^{i_1\dots i_{p-2},j_1,\dots,j_{p-2}}|_{i_{p-1},j_{p-1}},$$

$$Y=|A^{i_1\dots i_{p+1},j_1,\dots,j_{p+1}}|_{i_{p+2},j_{p+2}}\dots a_{i_n,j_n},$$

$$u=|A^{i_1\dots i_p,j_1,\dots,j_p}|_{i_{p+1},j_{p+1}},$$

$$w_1=|A^{i_1\dots i_{p-1}i_{p+1},j_1,\dots,j_p}|_{i_p,j_{p+1}},$$

$$w_2=|A^{i_1,\dots i_p,j_1,\dots,j_{p-1}}|_{i_{p+1},j_{p+1}}.$$

Proposition 3.1.1. *We have*

$$D_{I,J'}=-D_{I,J}Y^{-1}u^{-1}w_2^{-1}uw_2Y,$$

$$D_{I',J}=-Xuw_1^{-1}X^{-1}D_{I,J}Y^{-1}u^{-1}w_1Y.$$

Let C be a commutative ring with a unit and $f:\ R\to C$ be a multiplicative map, i.e. $f(ab)=f(a)f(b)$ for all $a,b\in R$.

Let $I=(i_1,\dots,i_n),J=(j_1,\dots,j_n)$ be any orderings of $(1,\dots,n)$. For an element σ from the symmetric group of nth order set $\sigma(I)=(\sigma(i_1),\dots,\sigma(i_n))$. Let $p(\sigma)$ be the parity of σ.

Proposition 3.1.1 immediately implies the following theorem.

Theorem 3.1.2. *In notations of Section* 3.1 *we have*

$$f(D_{I,J}(A))=f(-1)^{p(\sigma_1)+p(\sigma_2)}f(D_{\sigma(I),\sigma(J)}(A)).$$

It follows that $f(D_{I,J}(A)$ is uniquely defined up to a power of $f(-1)$. We call $f(D_{1\dots n,1\dots n})(A))$ the f-determinant A and denote it by $fD(A)$. Note that if f is a homomorphism then f-determinant $fD(A)$ equals to the usual determinant of the commutative matrix $f(A)$.

Corollary 3.1.3. *We have*

$$fD(AB) = fD(A) \cdot fD(B).$$

When R is the algebra of quaternions and $f(a) = v(a) = a\bar{a}$, or, in other words, f is the quaternionic norm, then one can see that $fD(a)$ is the matrix quaternionic norm $v(A)$ (see Section 2.1).

In Theorems 3.1.4–3.1.6 we present formulas for determinants of triangular and almost triangular matrices. A matrix $A = (a_{ij}), i,j = 1, \ldots, n,$ is called an upper-triangular matrix if $a_{ij} = 0$ for $i > j$. An upper-triangular matrix A is called a generic upper-triangular matrix if every square submatrix A consisting of the rows $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k$ and the columns $j_1 \leqslant j_2 \leqslant \cdots \leqslant j_k$ such that $i_1 \leqslant j_1, i_2 \leqslant j_2, \ldots, i_k \leqslant j_k$, is invertible.

Theorem 3.1.4. *Let $A = (a_{ij}),\ i,j = 1, \ldots, n$, be a generic upper-triangular matrix. The determinants $D_{i_1 i_2 \ldots i_n}(A)$ are defined if and only if $i_1 = n$. In this case*

$$D_{ni_2\ldots i_{n-1}}(A) = a_{nn} \cdot |A^{n,i_2}|_{i_2 n}^{-1} \cdot a_{i_2 i_2} \cdot |A^{n,i_2}|_{i_2 n} \cdot |A^{ni_2,i_2 i_3}|_{i_3 n}^{-1} \cdot a_{i_3 i_3} |A^{ni_2,i_2 i_3}|_{i_3 n} \cdot \ldots$$
$$\cdot |A^{ni_2 i_3 \ldots i_{n-1}, i_2 i_3 \ldots i_n}|_{i_n n}^{-1} \cdot a_{i_n i_n} \cdot |A^{ni_2 i_3 \ldots i_{n-1}, i_2 i_3 \ldots i_n}|_{i_n n}.$$

In particular,

$$D_{n,n-1\ldots 2,1}(A) = a_{nn} a_{n-1,n}^{-1} a_{n-1,n-1} a_{n-1,n} \ldots a_{1n}^{-1} a_{11} a_{1n}.$$

A matrix $A = (a_{ij}), i,j = 1, \ldots, n,$ is called an almost upper-triangular matrix if $a_{ij} = 0$ for $i > j+1$. An almost upper-triangular matrix A is called a Frobenius matrix if $a_{ij} = 0$ for all $j \neq n$ and $i \neq j+1$, and $a_{j+1j} = 1$ for $j = 1, \ldots, n-1$.

Theorem 3.1.5. *If A is invertible upper-triangular matrix, then*

$$D_{1,n,n-1\ldots 2}(A) = |A|_{1n} a_{n,n-1} a_{n-1,n-2} \ldots a_{21}.$$

By Proposition 1.2.7, the determinant $D_{1,n,n-1\ldots 2}(A)$ of an upper-triangular matrix A is polynomial in a_{ij}.

Let $p(I)$ be the parity of the ordering $I = (i_1, \ldots, i_n)$.

Theorem 3.1.6. *If A is a Frobenius matrix and the determinant $D_I(A)$ is defined, then $D_I(A) = (-1)^{p(I)+1} a_{1n}$.*

Now let R be a division ring, $R^* = R \backslash \{0\}$ the monoid of invertible elements in R and $\pi : R^* \to R^*/[R^*, R^*]$ the canonical homomorphism. To the abelian group $R^*/[R^*, R^*]$ we adjoin the zero element 0 with obvious multiplication, and denote the obtained semi-group by $\tilde{R}$. Extend π to a map $R \to \tilde{R}$ by setting $\pi(0) = 0$.

We recall here the classical notion of the Dieudonne determinant (see [D,A]). There exists a unique homomorphism

$$\det : M_n(R) \to \tilde{R}$$

such that

(i) $\det A' = \tilde{\mu} \det A$ for any matrix A' obtained from $A \in M_n(R)$ by multiplying one row of A from the left by μ;

(ii) $\det A'' = \det A$ for any matrix A'' obtained from A by adding one row to another;

(iii) $\det(E_n) = 1$ for the identity matrix E_n.

The homomorphism det is called the Dieudonne determinant.

It is known that $\det A = 0$ if $\operatorname{rank}(A) < n$ (see [A, Chapter 4]). The next proposition gives a construction of the Dieudonne determinant in the case where $\operatorname{rank}(A) = n$.

Proposition 3.1.7. *Let A be an $n \times n$-matrix over a division ring R. If $\operatorname{rank}(A) = n$, then*

(i) *There exist orderings I and J of $\{1, \dots, n\}$ such that $D_{I,J}(A)$ is defined.*
(ii) *If $D_{I,J}(A)$ is defined, then the Dieudonne determinant is given by the formula $\det A = p(I)p(J)\pi(D_{I,J}(A))$, where $p(I)$ is the parity of the ordering I.*

Note that Draxl [Dr] introduced the Dieudonne predeterminant, denoted $\delta\varepsilon\tau$. For a generic matrix A over a division ring there exists the Gauss decomposition $A = UDL$ where U, D, L are upper-unipotent, diagonal, and lower-unipotent matrices. Then Draxl $\delta\varepsilon\tau(A)$ is defined as the product of diagonal elements in D from top to the bottom. For nongeneric matrices Draxl used the Bruhat decomposition instead of the Gauss decomposition.

Proposition 3.1.8. *$\delta\varepsilon\tau(A) = \Delta(A)$, where $\Delta(A)$ is given by* (3.1.1).

Proof (For a generic A). Let $y_1, \dots, y_n$ be the diagonal elements in D from top to the bottom. As shown in [GR1,GR2] (see also 4.9), $y_k = |A^{12\dots k-1,12\dots k-1}|_{kk}$. Then $\delta\varepsilon\tau(A) = y_1 y_2 \dots y_n = \Delta(A)$. □

Below we consider below special examples of noncommutative determinants.

3.2. Dieudonne determinant for quaternions

Let $A = (a_{ij}),\ i,j = 1, \dots, n$, be a quaternionic matrix. If A is not invertible, then the Dieudonne determinant of A equals zero. By Proposition 3.1.7, if A is invertible, there exist orderings $I = (i_1, \dots, i_n), J = (j_1, \dots, j_n)$ of $\{1, \dots, n\}$ such that the following expressions are defined:

$$D_{I,J}(A) = |A|_{i_1j_1}|A^{i_1j_1}|_{i_2,j_2}|A^{i_1i_2,j_1j_2}|_{i_3j_3}\dots a_{i_n j_n}.$$

By Theorem 2.2.2, $D_{I,J}(A)$ can be expressed as a polynomial in a_{ij} and $\overline{a_{ij}}$ with real coefficients.

In the quaternionic case the Dieudonne determinant D coincides with the map

$$\det : M_n(\mathbb{H}) \to \mathbb{R}_{\geqslant 0}$$

(see [As]).

The following proposition generalizes a result in [VP].

Proposition 3.2.1. *In the quaternionic case for each* I, J *we have*

$$\det A = v(D_{I,J}(A))^{1/2}$$

(*the positive square root*).

The proof of Proposition 3.2.1 follows from the homological relations for quasideterminants.

3.3. Moore determinants of Herimitian quaternionic matrices

A quaternionic matrix $A = (a_{ij}),\ i,j = 1, \dots, n$, is called Hermitian if $a_{ji} = \bar{a}_{ji}$ for all i,j. It follows that all diagonal elements of A are real numbers and that the submatrices A^{11}, $A^{12,12}$, ... are Hermitian.

The notion of determinant for Hermitian quaternionic matrices was introduced by Moore in 1922 [M,MB]. Here is the original definition.

Let $A = (a_{ij}),\ i,j = 1, \dots, n$, be a matrix over a ring. Let σ be a permutation of $\{1, \dots, n\}$. Write σ as a product of disjoint cycles. Since disjoint cycles commute, we may write

$$\sigma = (k_{11}\dots k_{1j_1})(k_{21}\dots k_{2j_2})\dots(k_{m1}\dots k_{mj_m}),$$

where for each i, we have $k_{i1} < k_{ij}$ for all $j > 1$, and $k_{11} > k_{21} > \dots > k_{m1}$. This expression is unique. Let $p(\sigma)$ be the parity of σ. The Moore determinant $M(A)$ is defined as follows:

$$M(A) = \sum_{\sigma \in S_n} p(\sigma) a_{k_{11},k_{12}} \dots a_{k_{1j_1},k_{11}} a_{k_{21},k_{22}} \dots a_{k_{mj_m},k_{m1}}. \tag{3.3.1}$$

(There are equivalent formulations of this definition; e.g., one can require $k_{i1} > k_{ij}$ for all $j > 1$.) If A is Hermitian quaternionic matrix then $M(A)$ is a real number. Moore determinants have nice features and are widely used (see, for example, [Al,As,Dy1]).

We will show (Theorem 3.3.2) that determinants of Hermitian quaternionic matrices can be obtained using our general approach. First we prove that for a quaternionic Hermitian matrix A, the determinants $D_{I,I'}(A)$ coincide up to a sign.

Recall that $\Delta(A) = D_{I,I'}(A)$ for $I = \{1, \dots, n\}$ and that $\Delta(A)$ is a pre-Dieudonne determinant in the sense of [Dr]. If A is Hermitian, then $\Delta(A)$ is a product of real numbers and, therefore, $\Delta(A)$ is real.

Proposition 3.3.1. *Let $p(I)$ be the parity of the ordering I. Then $\Delta(A) = p(I)p(J)D_{I,J}(A)$.*

The proof follows from homological relations for quasideterminants.

Theorem 3.3.2. *Let A be a Hermitian quaternionic matrix. Then $\Delta(A) = M(A)$ (see (3.3.1)).*

Proof. We use the noncommutative Sylvester formula for quasideterminants (Theorem 1.5.2).

For $i, j = 2, \dots, n$ define a Hermitian matrix B_{ij} by the formula

$$B_{ij} = \begin{pmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{pmatrix}.$$

Let $b_{ij} = M(B_{ij})$ and $c_{ij} = |B_{ij}|_{11}$.

Note that $B = (b_{ij})$ and $C = (c_{ij})$ also are Hermitian matrices. It follows from (3.3.1) that $M(A) = a_{nn}^{2-n}M(B)$. Note, that $M(B) = a_{nn}^{n-1}M(C)$, therefore, $M(A) = a_{nn}M(C)$.

By Theorem 1.5.2, $|A|_{11} = |C|_{11}, |A^{11}|_{22} = |C^{11}|_{22}, \dots$. So,

$$\begin{aligned} &|A^{11}|_{22}|A^{11}|_{22} \dots |A^{12\dots n-1,12\dots n-1}|_{n-1,n-1} \\ &\quad = |C^{11}|_{22}|C^{11}|_{22} \dots |C^{12\dots n-1,12\dots n-1}|_{n-1,n-1}. \end{aligned}$$

The product on the left-hand side equals $\Delta(A)a_{nn}^{-1}$ and the product on right-hand side equals $\Delta(C)$, so $\Delta(A) = \Delta(C)a_{nn} = M(A)$. $\square$

3.4. Moore determinants and norms of quaternionic matrices

Proposition 3.4.1. *For generic matrices A, B we have*

$$\nu(A) = \Delta(A)\Delta(A^*) = \Delta(AA^*).$$

Since AA^* is a Hermitian matrix, one has the following

Corollary 3.4.2. $\nu(A) = M(AA^*)$.

3.5. Study determinants

An embedding of the field of complex numbers $\mathbb{C}$ into $\mathbb{H}$ is defined by an image of $\mathbf{i} \in \mathbb{C}$. Chose the embedding given by $x + y\mathbf{i} \mapsto x + y\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$, where $x, y \in \mathbb{R}$ and identify $\mathbb{C}$ with its image in $\mathbb{H}$. Then any quaternion a can be uniquely written as $a = \alpha + \mathbf{j}\beta$ where $\alpha, \beta \in \mathbb{C}$.

Let $M(n, F)$ be the algebra of matrices of order n over a field F. Define a homomorphism $\theta : \mathbb{H} \to M(2, \mathbb{C})$ by setting

$$\theta(a) = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}.$$

For $A = (a_{ij}) \in M(n, \mathbb{H})$, set $\theta_n(A) = (\theta(a_{ij}))$. This extends θ to homomorphism of matrix algebras

$$\theta_n : M(n, \mathbb{H}) \to M(2n, \mathbb{C}).$$

In 1920, Study [St] defined a determinant $S(A)$ of a quaternionic matrix A of order n by setting $S(A) = \det \theta_n(A)$. Here det is the standard determinant of a complex matrix. The following proposition is well known (see [As]).

Proposition 3.5.1. *For any quaternionic matrix A*

$$S(A) = M(AA^*).$$

The proof in [As] was based on properties of eigenvalues of quaternionic matrices. Our proof based on Sylvester's identity and homological relations actually shows that $S(A) = v(A)$ for a generic matrix A.

3.6. Quantum determinants

Note, first of all, that quantum determinants and the Capelli determinants (to be discussed in Section 3.7) are not defined for all matrices over the corresponding algebras. For this reason, they are not actual determinants, but, rather, "determinant-like" expressions. However, using the traditional terminology, we will talk about quantum and Capelli determinants.

We say that $A = (a_{ij})$, $i, j = 1, \ldots, n$, is a *quantum matrix* if, for some central invertible element $q \in F$, the elements a_{ij} satisfy the following commutation relations:

$$a_{ik}a_{il} = q^{-1}a_{il}a_{ik} \quad \text{for } k < l,$$

$$a_{ik}a_{jk} = q^{-1}a_{jk}a_{ik} \quad \text{for } i < j,$$

$$a_{il}a_{jk} = a_{jk}a_{il} \quad \text{for } i<j, k<l,$$

$$a_{ik}a_{jl} - a_{jl}a_{ik} = (q^{-1} - q)a_{il}a_{jk} \quad \text{for } i<j, k<l. \tag{3.6.1}$$

Denote by $\mathcal{A}(n,q)$ the algebra with generators (a_{ij}), $i,j = 1, \dots, n$, satisfying relations (3.6.1). The center of this algebra is the one-dimensional subspace generated by the so called *quantum determinant* of A.

The quantum determinant $\det_q A$ is defined as follows:

$$\det_q A = \sum_{\sigma \in S_n} (-q)^{-l(\sigma)} a_{1\sigma(1)}a_{2\sigma(2)} \dots a_{n\sigma(n)},$$

where $l(\sigma)$ is the number of inversions in σ.

If A is a quantum matrix, then any square submatrix of A also is a quantum matrix with the same q.

Note that the algebra $\mathcal{A}(n,q)$ admits the ring of fractions.

Theorem 3.6.1 (Galland and Retakh [GR], Krob and Leclerc [KL]). *In the ring of fractions of the algebra $\mathcal{A}(n,q)$ we have*

$$\det_q A = (-q)^{i-j}|A|_{ij} \cdot \det_q A^{ij} = (-q)^{i-j} \det_q A^{ij} \cdot |A|_{ij}.$$

Corollary 3.6.2 (Galland and Retakh [GR], Krob and Leclerc [KL]). *In the ring of fractions of the algebra $\mathcal{A}(n,q)$ we have*

$$\det_q A = |A|_{11}|A^{11}|_{22} \dots a_{nn}$$

and all factors on the right-hand side commute.

An important generalization of this result for matrices satisfying Faddeev–Reshetikhin–Takhtadjan relations is given in [ER].

3.7. Capelli determinants

Let $X = (x_{ij})$, $i,j = 1, \dots, n$ be a matrix of formal commuting variables and X^T the transposed matrix. Let $D = (\partial_{ij}), \partial_{ij} = \partial/\partial x_{ij}$, be the matrix of the corresponding differential operators. Since each of the matrices X, D consists of commuting entries, $\det X$ and $\det D$ make sense. Let us set $X^T D = (f_{ij})$, so that $f_{ij} = \sum_k x_{ki}\partial/\partial x_{kj}$.

Let W be a diagonal matrix, $W = \operatorname{diag}(0, 1, 2, \dots, n)$.

By definition, the Capelli determinant $\det_{\text{Cap}}$ of $X^T D - W$ equals to the sum

$$\sum_{\sigma \in S_n} (-1)^{l(\sigma)} f_{\sigma(1)1}(f_{\sigma(2)2} - \delta_{\sigma(2)2}) \dots (f_{\sigma(n)n} - (n-1)\delta_{\sigma(n)n}).$$

The classical Capelli identity says that the sum is equal to $\det X \det D$.

Set $Z = X^T D - I_n$. It was shown in [GR1,GR2] that the Capelli determinant can be expressed as a product of quasideterminants. More precisely, let $\mathcal{D}$ be the algebra of polynomial differential operators with variables x_{ij}.

Theorem 3.7.1. *In the ring of fractions of the algebra $\mathcal{D}$ we have*

$$|Z|_{11}|Z^{11}|_{22}\dots z_{nn} = \det X \det D$$

and all factors on the left-hand side commute.

It is known [We] that the right-hand side in the theorem is equal to the Capelli determinant.

This theorem can also be interpreted in a different way.

Let $A = (e_{ij}),\ i,j = 1,\dots n$ be the matrix of the standard generators of the universal enveloping algebra $U(gl_n)$. Recall that these generators satisfy the relations

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}.$$

Let E_n be the identity matrix of order n. It is well known (see, for example, [Ho]) that coefficients of the polynomial in a central variable t

$$\det(I_n + tA) := \sum_{\sigma\in S_n}(-1)^{l(\sigma)}(\delta_{\sigma(1)1} + te_{\sigma(1)1})\dots(\delta_{\sigma(n)n} + t(e_{\sigma(n)n} - (n-1)\delta_{\sigma(n)n}))$$

generate the center of $U(gl_n)$.

Theorem 3.7.1 can be reformulated in the following way [GKLLRT].

Theorem 3.7.2. $\det(I_n + tA)$ *can be factored in the algebra of formal power series in t with coefficients in $U(gl_n)$:*

$$\det(I_n + tA) = (1 + te_{11})\begin{vmatrix} 1 + t(e_{11} - 1) & te_{12} \\ te_{21} & \boxed{1 + t(e_{22} - 1)} \end{vmatrix} \cdot \dots$$

$$\cdot \begin{vmatrix} 1 + t(e_{11} - n + 1) & \dots & te_{1n} \\ \dots & \dots & \dots \\ te_{n1} & \dots & \boxed{1 + t(e_{nn} - n + 1)} \end{vmatrix}$$

and the factors on the right-hand side commute with each other.

The above version is obtained by using the classical embedding of $U(gl_n)$ into the Weyl algebra generated by $(x_{ij}, \partial/\partial x_{ij}),\ i,j = 1,\dots,n$, where e_{ij} corresponds to

$$f_{ij} = \sum_{k=1}^{n} x_{ki}\partial/\partial x_{kj}.$$

3.8. Berezinians

Let $p(k)$ be the parity of an integer k, i.e. $p(k) = 0$ if k is even and $p(k) = 1$ if k is odd. A (commutative) super-ring over R^0 is a ring $R = R^0 \oplus R^1$ such that

(i) $a_i a_j \in R^{p(i+j)}$ for any $a_m \in R^m$, $m = 0, 1$,
(ii) $ab = ba$ for any $a \in R^0, b \in R$, and $cd = -dc$ for any $c, d \in R^1$.

Let $A = \left(\begin{smallmatrix} X & Y \\ Z & T \end{smallmatrix}\right)$ be an $(m+n)\times(m+n)$-block-matrix over a super-ring $R = R^0 \oplus R^1$, where X is an $m \times m$-matrix over R^0, T is an $n \times n$-matrix over R^0, and Y, Z are matrices over R^1. If T is an invertible matrix, then $X - YT^{-1}Z$ is an invertible matrix over commutative ring R^0. Super-determinant, or Berezinian, of A is defined by the following formula:

$$\mathrm{Ber}\, A = \det(X - YT^{-1}Z)\det T^{-1}.$$

Note that $\mathrm{Ber}\, A \in R^0$.

Theorem 3.8.1. *Let R^0 be a field. Set $J_k = \{1, 2, \ldots, k\}$, $k \leqslant m+n$ and $A^{(k)} = A^{J_k, J_k}$. Then* Ber A *is a product of elements of R^0*:

$$\mathrm{Ber}\, A = |A|_{11} |A^{(1)}|_{22} \ldots |A^{(m-1)}|_{mm} |A^{(m)}|^{-1}_{m+1,m+1} \ldots |A^{(m+n-1)}|^{-1}_{m+n,m+n}.$$

3.9. Cartier–Foata determinants

Let $A = (a_{ij})$, $i, j = 1, \ldots, n$ be a matrix such that the entries a_{ij} and a_{kl} commute when $i \neq k$. In this case Cartier and Foata [CF,F] defined a determinant of A as

$$\det{}_{\mathrm{CF}} A = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)}.$$

The order of factors in monomials $a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)}$ is insignificant.

Let C_n be the algebra over a field F generated by (a_{ij}), $i, j = 1, \ldots, n$, with relations $a_{ij} a_{kl} = a_{kl} a_{ij}$ if $i \neq k$. Algebra C_n admits the ring of fractions.

Theorem 3.9.1. *In the ring of fractions of algebra C_n, let $A = (a_{ij})$, $i, j = 1, \ldots, n$ be a matrix such that the entries a_{ij} and a_{kl} commute when $i \neq k$.*

$$|A|_{pq} = (-1)^{p+q} \det{}_{\mathrm{CF}}(A^{pq})^{-1} \det{}_{CF} A$$

and all factors in (3.9.1) *commute.*

Corollary 3.9.2. *In the ring of fractions of algebra C_n we have*

$$\det_{CF} = |A|_{11}|A^{11}|_{22} \dots a_{nn}$$

and all factors commute.

4. Noncommutative Plücker and Flag coordinates

Most of the results described in this section were obtained in [GR4].

4.1. Commutative Plücker coordinates

Let $k \leqslant n$ and A be a $k \times n$-matrix over a commutative ring R. Denote by $A(i_1, \dots, i_k)$ the $k \times k$-submatrix of A consisting of columns labeled by the indices $i_1, \dots, i_k$. Define $p_{i_1 \dots i_k}(A) := \det A(i_1, \dots, i_k)$. The elements $p_{i_1 \dots i_k}(A) \in R$ are called Plücker coordinates of the matrix A. The Plücker coordinates $p_{i_1 \dots i_k}(A)$ satisfy the following properties:

(i) (invariance) $p_{i_1 \dots i_k}(XA) = \det X \cdot p_{i_1 \dots i_k}(A)$ for any $k \times k$-matrix X over R;

(ii) (skew-symmetry) $p_{i_1 \dots i_k}(A)$ are skew-symmetric in indices $i_1, \dots, i_k$; in particular, $p_{i_1 \dots i_k}(A) = 0$ if a pair of indices coincides;

(iii) (Plücker relations) Let $i_1, \dots, i_{k-1}$ be $k-1$ distinct numbers which are chosen from the set $1, \dots, n$, and $j_1, \dots, j_{k+1}$ be $k+1$ distinct numbers chosen from the same set. Then

$$\sum_{t=1}^{k} (-1)^t p_{i_1 \dots i_{k-1} j_t}(A) p_{j_1 \dots j_{t-1} j_{t+1} \dots j_{k+1}}(A) = 0.$$

Example. For $k = 2$ and $n = 4$ the Plücker relations in (iii) imply the famous identity

$$p_{12}(A)p_{34}(A) - p_{13}(A)p_{24}(A) + p_{23}(A)p_{14}(A) = 0. \tag{4.1.1}$$

Historically, Plücker coordinates were introduced as coordinates on Grassmann manifolds. Namely, let $R = F$ be a field and $G_{k,n}$ the Grassmannian of k-dimensional subspaces in the n-dimensional vector space F^n. To each $k \times n$-matrix A of rank k we associate the subspace of F^n generated by the rows of A. By the invariance property (i), we can view each Plücker coordinate $p_{i_1 \dots i_k}$ as a section of a certain ample line bundle on $G_{k,n}$, and all these sections together define an embedding of $G_{k,n}$ into the projective space $\mathbb{P}^N$ of dimension $N = \binom{k}{n} - 1$. In this sense, Plücker coordinates are projective coordinates on $G_{k,n}$.

4.2. Quasi-Plücker coordinates for $n \times (n+1)$- and $(n+1) \times n$-matrices

Let $A = (a_{ij}),\ i = 1, \dots, n,\ j = 0, 1, \dots, n,$ be a matrix over a division ring R. Denote by $A^{(k)}$ the $n \times n$-submatrix of A obtained from A by removing the kth column and suppose that all $A^{(k)}$ are invertible. Choose an arbitrary $s \in \{1, \dots, n\}$, and denote

$$q_{ij}^{(s)}(A) = |A^{(j)}|_{si}^{-1} |A^{(i)}|_{sj}.$$

Proposition 4.2.1. *The element $q_{ij}^{(s)}(A) \in R$ does not depend on s.*

We denote the common value of $q_{ij}^{(s)}(A)$ by $q_{ij}(A)$ and call $q_{ij}(A)$ the *left quasi-Plücker coordinates* of the matrix A.

Proof of Proposition 4.2.1. Considering the columns of the matrix A as $n+1$ vectors in the right n-dimensional space R^n over R, we see that there exists a nonzero $(n+1)$-vector $(x_1, \dots, x_{n+1}) \in R^{n+1}$ such that

$$A \begin{pmatrix} x_0 \\ \dots \\ x_n \end{pmatrix} = 0.$$

This means that

$$A^{(j)} \begin{pmatrix} x_0 \\ \dots \\ \widehat{x}_j \\ \dots \\ x_n \end{pmatrix} = -\begin{pmatrix} a_{1j} \\ \dots \\ a_{nj} \end{pmatrix} x_j.$$

Since all submatrices $A^{(k)}$ are invertible, each x_I is a nonzero element of R. Cramer's rule and transformations properties for quasideterminants imply that $|A^{(j)}|_{si} x_i = -|A^{(i)}|_{sj} x_j$. Therefore,

$$q_{ij}^{(s)}(A) = |A^{(j)}|_{si}^{-1} |A^{(i)}|_{sj} = -x_i x_j^{-1} \tag{4.2.1}$$

does not depend on s. □

Proposition 4.2.2. *If g is an invertible $n \times n$-matrix over R, then $q_{ij}(gA) = q_{ij}(A)$.*

Proof. We have $gA \begin{pmatrix} x_0 \\ \dots \\ x_n \end{pmatrix} = 0$. Therefore, $q_{ij}(gA) = -x_i x_j^{-1} = q_{ij}(A)$. □

In the commutative case, $q_{ij}(A)$ is a ratio of two Plücker coordinates: $q_{ij}(A) = p_{1,\dots,\widehat{j},\dots,n}/p_{1,\dots,\widehat{i},\dots,n} = \det A^{(j)}/\det A^{(i)}$.

Similarly, we define the right quasi-Plücker coordinates $r_{ij}(B)$ for $(n+1)\times n$-matrix $B = (b_{ji})$. Denote by $B^{(k)}$ the submatrix of B obtained from B by removing the kth row. Suppose that all $B^{(k)}$ are invertible, choose $s \in \{1,\dots,n\}$, and set $r_{ij}^{(s)}(B) = |B^{(j)}|_{is}|B^{(i)}|_{js}^{-1}$.

Proposition 4.2.3. (i) *The element $r_{ij}^{(s)}(B)$ does not depend of s.*

Denote the common value of elements $r_{ij}^{(s)}(B)$ by $r_{ij}(B)$.

(ii) *If g is an invertible $n \times n$-matrix over R, then $r_{ij}(Bg) = r_{ij}(B)$.*

In the commutative case, $r_{ij}(A) = \det B^{(j)}/\det B^{(i)}$.

4.3. *Definition of left quasi-Plücker coordinates. General case*

Let $A = (a_{pq}),\ p = 1,\dots,k,\ q = 1,\dots,n,\ k<n$, be a matrix over a division ring R. Choose $1 \leqslant i,j,i_1,\dots,i_{k-1} \leqslant n$ such that $i \notin I = \{i_1,\dots,i_{k-1}\}$. Let $A(i,j,i_1,\dots,i_{k-1})$ be the $k \times (k+1)$-submatrix of A with columns labeled by $i,j,i_1,\dots,i_{k-1}$.

Definition 4.3.1. Define left quasi-Plücker coordinates $q_{ij}^I(A)$ of the matrix A by the formula

$$q_{ij}^I(A) = q_{ij}(A(i,j,i_1,\dots,i_{k-1})).$$

By Proposition 4.2.1, left quasi-Plücker coordinates are given by the formula

$$q_{ij}^I(A) = \begin{vmatrix} a_{1i}a_{1i_1} & \dots & a_{1,i_{k-1}} \\ & \dots & \\ a_{ki}a_{ki_1} & \dots & a_{ki_{k-1}} \end{vmatrix}_{si}^{-1} \cdot \begin{vmatrix} a_{1j}a_{1,i_1} & \dots & a_{1,i_{k-1}} \\ & \dots & \\ a_{kj}a_{ki_1} & \dots & a_{ki_{k-1}} \end{vmatrix}_{sj}$$

for an arbitrary $s, 1 \leqslant s \leqslant k$.

Proposition 4.3.2. *If g is an invertible $k \times k$-matrix over R, then $q_{ij}^I(gA) = q_{ij}^I(A)$.*

Proof. Use Proposition 4.2.2. □

In the commutative case $q_{ij}^I = p_{jI}/p_{iI}$, where $p_{\alpha_1\dots\alpha_k}$ are the standard Plücker coordinates.

4.4. Identities for the left quasi-Plücker coordinates

The following properties of q_{ij}^I immediately follow from the definition.

(i) q_{ij}^I does not depend on the ordering on elements in I;
(ii) $q_{ij}^I = 0$ for $j \in I$;
(iii) $q_{ii}^I = 1$ and $q_{ij}^I \cdot q_{jk}^I = q_{ik}^I$.

Theorem 4.4.1 (Skew-symmetry). *Let* $N, |N| = k+1$, *be a set of indices*, $i, j, m \in N$. *Then*

$$q_{ij}^{N\setminus\{i,j\}} \cdot q_{jm}^{N\setminus\{j,m\}} \cdot q_{mi}^{N\setminus\{m,i\}} = -1.$$

Theorem 4.4.2 (Plücker relations). *Fix* $M = (m_1, \dots, m_{k-1}), L = (\ell_1, \dots, \ell_k)$. *Let* $i \notin M$. *Then*

$$\sum_{j \in L} q_{ij}^M \cdot q_{ji}^{L\setminus\{j\}} = 1.$$

Examples. Suppose that $k = 2$.
(1) From Theorem 4.4.1 it follows that

$$q_{ij}^{\{\ell\}} \cdot q_{j\ell}^{\{i\}} \cdot q_{\ell i}^{\{j\}} = -1.$$

In the commutative case, $q_{ij}^{\{\ell\}} = \frac{p_{j\ell}}{p_{i\ell}}$ so this identity follows from the skew-symmetry $p_{ij} = -p_{ji}$.
(2) From Theorem 4.4.2 it follows that for any i, j, ℓ, m

$$q_{ij}^{\{\ell\}} \cdot q_{ji}^{\{m\}} + q_{im}^{\{\ell\}} \cdot q_{mi}^{\{j\}} = 1.$$

In the commutative case this identity implies the standard identity (cf. (4.1.1))

$$p_{ij} \cdot p_{\ell m} - p_{i\ell} \cdot p_{jm} + p_{im} \cdot p_{\ell j} = 0.$$

Remark. The products $p_{ij}^{\{\ell\}} p_{ji}^{\{m\}}$ (which in the commutative case are equal to $\frac{p_{j\ell}}{p_{i\ell}} \cdot \frac{p_{im}}{p_{jm}}$) can be viewed as noncommutative cross-ratios.

To prove Theorems 4.4.1 and 4.4.2 we need the following lemma. Let $A = (a_{ij}),\ i = 1, \dots, k,\ j = 1, \dots, n, k < n$, be a matrix over a division ring. Denote by $A_{j_1,\dots,j_\ell}, \ell \leqslant n$, the $k \times \ell$-submatrix $(a_{ij}),\ i = 1, \dots, k,\ j = j_1, \dots, j_\ell$. Consider the $n \times n$-matrix

$$X = \begin{pmatrix} A_{1\dots k} & A_{k+1\dots n} \\ 0 & E_{n-k} \end{pmatrix},$$

where E_m is the identity matrix of order m.

Lemma 4.4.3. *Let $j < k < i$. If $q_{ij}^{1\dots\hat{j}\dots k}(A)$ is defined, then $|X|_{ij}$ is defined and*

$$|X|_{ij} = -q_{ij}^{1\dots\hat{j}\dots k}(A). \tag{4.4.1}$$

Proof. We must prove that

$$|X|_{ij} = -|A_{1\dots\hat{j}\dots ki}|_{si}^{-1} \cdot |A_{1\dots k}|_{sj} \tag{4.4.2}$$

provided the right-hand side is defined. We will prove this by induction on $\ell = n - k$. Let us assume that formula (2.2) holds for $l = m$ and prove it for $\ell = m + 1$. Without loss of generality we can take $j = 1, i = k + 1$. By homological relations (Theorem 1.4.3)

$$|X|_{k+1,1} = -|X^{k+1,1}|_{s,k+1}^{-1} \cdot |X^{k+1,k+1}|_{s1}$$

for an appropriate $1 \leqslant s \leqslant k$. Here

$$X^{k+1,1} = \begin{pmatrix} A_{2\dots k+1} & A_{k+2\dots n} \\ 0 & E_{n-k-1} \end{pmatrix},$$

$$X^{k+1,k+1} = \begin{pmatrix} A_{1\dots k} & A_{k+2\dots n} \\ 0 & E_{n-k-1} \end{pmatrix}.$$

By the induction assumption

$$|X^{k+1,1}|_{s,k+1} = -|A_{23\dots kk+2}|_{s,k+2}^{-1} \cdot |A_{23\dots k+1}|_{s,k+1},$$

$$|X^{k+1,k+1}|_{s1} = -|A_{23\dots kk+2}|_{s,k+2}^{-1} \cdot |A_{1\dots k}|_{s1}$$

and $|X|_{k+1,1} = -p_{k+1,1}^{23\dots k}$. $\square$

To prove Theorem 4.4.2 we apply the second formula in Theorem 1.4.4 to the matrix

$$X = \begin{pmatrix} A_{1\dots k} & A_{k+1\dots n} \\ 0 & E_{n-k} \end{pmatrix}$$

for $M = (k+1, \dots, n)$ and any L such that $|L| = n-k-1$. By Lemma 4.4.3, $|X|_{m_i\ell} = -q^{1\dots\hat{\ell}\dots k}(A), |X^{M_i,L}|_{m_iq} = -p^{1\dots n\setminus L}_{m_iq}(A)$, and Theorem 4.4.2 follows from Theorem 1.4.4. $\square$

To prove Theorem 4.4.1 it is sufficient to take the matrix X for $n = k+1$ and use homological relations.

Theorem 4.4.4. *Let $A = (a_{ij})$, $i = 1, \dots, k$, $j = 1, \dots, n$, be a matrix with formal entries and $f(a_{ij})$ an element of a free skew-field F generated by a_{ij}. Let f be invariant under the transformations*

$$A \to gA$$

for all invertible $k \times k$-matrices g over F. Then f is a rational function of the quasi-Plücker coordinates.

Proof. Let $b_{ij} = a_{ij}$ for $i,j = 1, \dots, k$. Consider the matrix $B = (b_{ij})$. Then $B^{-1} = (|B|_{ji}^{-1})$. Set $C = (c_{ij}) = B^{-1}A$. Then

$$c_{ij} = \begin{cases} \delta_{ij} & \text{for } j \leqslant k, \\ q_{ij}^{1\dots\hat{i}\dots k}(A) & \text{for } j > k. \end{cases}$$

By invariance, f is a rational expression of c_{ij} with $j > k$.

4.5. Right quasi-Plücker coordinates

Consider a matrix $B = (b_{pq})$, $p = 1, \dots, n$; $q = 1, \dots, k, k < n$ over a division ring F. Choose $1 \leqslant i, j, i_1, \dots, i_{k-1} \leqslant n$ such that $j \notin I = (i_1, \dots, i_{k-1})$. Let $B(i,j,i_1, \dots, i_{k-1})$ be the $(k+1) \times k$-submatrix of B with rows labeled by $i, j, i_1, \dots, i_{k-1}$.

Definition 4.5.1. Define right quasi-Plücker coordinates $r^I_{ij}(B)$ of the matrix B by the formula

$$r^I_{ij}(B) = r_{ij}(B(i,j,i_1, \dots, i_{k-1})).$$

By Proposition 4.2.3, right quasi-Plücker coordinates are given by the formula

$$r_{ij}^I(B) = \begin{vmatrix} b_{i1} & \dots & b_{ik} \\ b_{i_1 1} & \dots & b_{i_1 k} \\ & \dots & \\ b_{i_{k-1}1} & \dots & b_{i_{k-1}k} \end{vmatrix}_{it} \cdot \begin{vmatrix} b_{j1} & \dots & b_{jk} \\ b_{i_1 1} & \dots & b_{i_1 k} \\ & \dots & \\ b_{i_{k-1}1} & \dots & b_{i_{k-1}k} \end{vmatrix}_{jt}^{-1}$$

for an arbitrary $t, 1 \leqslant t \leqslant k$.

Proposition 4.5.2. *$r_{ij}^I(Bg) = r_{ij}^I(B)$ for each invertible $k \times k$-matrix g over F.*

4.6. *Identities for the right quasi-Plücker coordinates*

Identities for r_{ij}^I are dual to corresponding identities for the left quasi-Plücker coordinates q_{ij}^I. Namely,

(i) r_{ij}^I does not depend on the ordering on elements of I;
(ii) $r_{ij}^I = 0$ for $i \in I$;
(iii) $r_{ii}^I = 1$ and $r_{ij}^I \cdot r_{jk}^I = r_{ik}^I$.

Theorem 4.6.1 (Skew-symmetry). *Let $N, |N| = k+1$, be a set of indices, $i, j, m \in N$. Then*

$$r_{ij}^{N \backslash \{i,j\}} \cdot r_{jm}^{N \backslash \{j,m\}} \cdot r_{mi}^{N \backslash \{m,i\}} = -1.$$

Theorem 4.6.2 (Plücker relations). *Fix $M = (m_1, \dots, m_{k-1}), L = (\ell_1, \dots, \ell_k)$. Let $i \notin M$. Then*

$$\sum_{j \in L} r_{ij}^{L \backslash \{j\}} r_{ij}^M = 1.$$

4.7. *Duality between quasi-Plücker coordinates*

Let $A = (a_{ij})$, $i = 1, \dots, k$, $j = 1, \dots, n$; and $B = (b_{rs})$, $r = 1, \dots, n$, $s = 1, \dots, n-k$. Suppose that $AB = 0$. (This is equivalent to the statement that the subspace generated by the rows of A in the left linear space F^n is dual to the subspace generated by the columns of B in the dual right linear space.) Choose indices $1 \leqslant i, j \leqslant n$ and a subset $I \subset [1, n], |I| = k-1$, such that $i \notin I$. Set $J = ([1, n] \backslash I) \backslash \{i, j\}$.

Theorem 4.7.1. *We have*

$$q^I_{ij}(A) + r^J_{ij}(B) = 0.$$

4.8. Quasi-Pücker coordinates for $k \times n$-matrices for different k

Let $A = (a_{\alpha\beta}),\ \alpha = 1, \dots, k,\ \beta = 1, \dots, n$, be a $k \times n$-matrix over a noncommutative division ring R and A' a $(k-1) \times n$-submatrix of A. Choose $1 \leqslant i, j, m, j_1, \dots, j_{k-2} \leqslant n$ such that $i \neq m$ and $i, m \notin J = \{j_1, \dots, j_{k-2}\}$.

Proposition 4.8.1. *We have*

$$q^J_{ij}(A') = q^{J\cup\{m\}}_{ij}(A) + q^J_{im}(A') \cdot q^{J\cup\{i\}}_{mj}(A).$$

4.9. Applications of quasi-Plücker coordinates

Row and column expansion of a quasideterminant: Some of the results obtained in [GR,GR1,GR2] and partially presented in Section I can be rewritten in terms of quasi-Plücker coordinates.

Let $A = (a_{ij}),\ i, j = 1, \dots, n$, be a matrix over a division ring R. Choose $1 \leqslant \alpha, \beta \leqslant n$. Using the notation of section I let $B = A^{\{\alpha\},\emptyset}, C = A^{\emptyset,\{\beta\}}$ be the $(n-1) \times n$ and $n \times (n-1)$ submatrices of A obtained by deleting the αth row and the βth column respectively. For $j \neq \beta$ and $i \neq \alpha$ set

$$q_{j\beta} = q^{1\dots\hat{j}\dots\hat{\beta}\dots n}_{j\beta}(B),$$

$$r_{\alpha i} = r^{1\dots\hat{\alpha}\dots\hat{i}\dots n}_{\alpha i}(C).$$

Proposition 4.9.1. (i) $|A|_{\alpha\beta} = a_{\alpha\beta} - \sum_{j\neq\beta} a_{\alpha j} q_{j\beta}$,

(ii) $|A|_{\alpha\beta} = a_{\alpha\beta} - \sum_{i\neq\alpha} r_{\alpha i} a_{i\beta}$
provided the terms in the right-hand side of these formulas are defined.

Homological relations:

Proposition 4.9.2. *In the previous notation,*

(i) $|A|^{-1}_{ij} \cdot |A|_{i\ell} = -q_{j\ell}$ *(row relations)*,
(ii) $|A|_{ij} \cdot |A|^{-1}_{kj} = -r_{ik}$ *(column relations)*.

Corollary 4.9.3. *In the previous notation, let* $(i_1, \dots, i_s), (j_1, \dots, j_t)$ *be sequences of indices such that* $i \neq i_1, i_1 \neq i_2, \dots, i_{s-1} \neq i_s; j \neq j_1, j_1 \neq j_2, \dots, j_{t-1} \neq j_t$. *Then*

$$|A|_{i_s j_t} = q_{i_s i_{s-1}} \dots q_{i_2 i_1} q_{i_1 i} \cdot |A|_{ij} \cdot r_{jj_1} r_{j_1 j_2} \dots r_{j_{t-1} j_t}.$$

Example. For a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ we have

$$|A|_{22} = a_{21} \cdot a_{11}^{-1} \cdot |A|_{11} \cdot a_{22}^{-1} \cdot a_{22},$$

$$|A|_{11} = a_{12} \cdot a_{22}^{-1} \cdot a_{21} \cdot a_{11}^{-1} \cdot |A|_{11} \cdot a_{21}^{-1} \cdot a_{22} \cdot a_{12}^{-1} \cdot a_{11}.$$

Matrix multiplication: The following formula was already used in the proof of Theorem 4.4.4. Let $A = (a_{ij}),\ i = 1, \dots, n,\ j = 1, \dots, m,\ n < m,\ B = (a_{ij}),\ i = 1, \dots, n,\ j = 1, \dots, n,\ C = (a_{ik}),\ i = 1, \dots, n,\ k = n+1, \dots, m$.

Proposition 4.9.4. *Let the matrix B be invertible. Then* $q_{ik}^{1 \dots \hat{i} \dots n}(A)$ *are defined for* $i = 1, \dots n,\ k = n+1, \dots m$, *and*

$$B^{-1}C = (q_{ik}^{1 \dots \hat{i} \dots n}(A)), \quad i = 1, \dots, n,\ k = n+1, \dots, m.$$

Quasideterminant of the product: Let $A = (a_{ij}), B = (b_{ij}),\ i, j = 1, \dots n$ be matrices over a division ring R. Choose $1 \leqslant k \leqslant n$. Consider the $(n-1) \times n$-matrix $A' = (a_{ij}),\ i \neq k$, and the $n \times (n-1)$-matrix $B'' = (b_{ij}),\ j \neq k$.

Proposition 4.9.5. *We have*

$$|B|_{kk} \cdot |AB|_{kk}^{-1} \cdot |A|_{kk} = 1 + \sum_{\alpha \neq k} r_{k\alpha} \cdot q_{\alpha k},$$

where $r_{k\alpha} = r_{k\alpha}^{1 \dots \hat{\alpha} \dots n}(B'')$ *are right quasi-Plücker coordinates and* $q_{\alpha k} = q_{\alpha k}^{1 \dots \hat{\alpha} \dots n}(A')$ *are left quasi-Plücker coordinates, provided all expressions are defined.*

The proof follows from the multiplicative property of quasideterminants and Proposition 4.9.2.

Gauss decomposition: Consider a matrix $A = (a_{ij}),\ i, j = 1, \dots, n$, over a division ring R. Let $A_k = (a_{ij}),\ i, j = k, \dots n,\ B_k = (a_{ij}),\ i = 1, \dots n,\ j = k, \dots n$, and $C_k = (a_{ij}),\ i = k, \dots n,\ j = 1, \dots n$. These are submatrices of sizes $(n-k+1) \times (n-k+1), n \times (n-k+1)$, and $(n-k+1) \times n$ respectively. Suppose that the quasideterminants

$$y_k = |A_k|_{kk}, \quad k = 1, \dots, n,$$

are defined and invertible in R.

Theorem 4.9.6 (see Gelfand et al. [GR1,GR2]).

$$A = \begin{pmatrix} 1 & & x_{\alpha\beta} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} y_1 & & 0 \\ & \ddots & \\ 0 & & y_n \end{pmatrix} \begin{pmatrix} & & 0 \\ & \ddots & \\ z_{\beta\alpha} & & 1 \end{pmatrix},$$

where

$$x_{\alpha\beta} = r_{\alpha\beta}^{\beta+1\ldots n}(B_\beta), \quad 1 \leqslant \alpha < \beta \leqslant n,$$

$$z_{\beta\alpha} = q_{\beta\alpha}^{\beta+1\ldots n}(C_\beta), \quad 1 \leqslant \alpha < \beta \leqslant n.$$

Similarly, let $A^{(k)} = (a_{ij}),\ i,j = 1, \ldots, k,\ B^{(k)} = (a_{ij}),\ i = 1, \ldots, n,\ j = 1, \ldots, k,$ $C^{(k)} = (a_{ij}),\ i = 1, \ldots, k,\ j = 1, \ldots, n$. Suppose that the quasideterminants

$$y'_k = |A^{(k)}|_{kk}, \quad k = 1, \ldots, n,$$

are defined and invertible in R.

Theorem 4.9.7. *We have*

$$A = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ x'_{\beta\alpha} & & 1 \end{pmatrix} \begin{pmatrix} {y_1}' & & 0 \\ & \ddots & \\ 0 & & {y_n}' \end{pmatrix} \begin{pmatrix} 1 & & {z_{\alpha\beta}}' \\ & \ddots & \\ 0 & & 1 \end{pmatrix},$$

where

$$x'_{\beta\alpha} = r_{\beta\alpha}^{1\ldots\alpha-1}(B^{(\alpha)}), \quad 1 \leqslant \alpha < \beta \leqslant n,$$

$${z_{\alpha\beta}}' = q_{\alpha\beta}^{1\ldots\alpha-1}(C^{(\alpha)}), \quad 1 \leqslant \alpha < \beta \leqslant n.$$

Bruhat decompositions: A generalization of Theorem 4.9.6 is given by the following noncommutative analog of the Bruhat decomposition.

Definition: A square matrix P with entries 0 and 1 is called a permutation matrix if in each row of P and in each column of P there is exactly one entry 1.

Theorem 4.9.8 (Bruhat decomposition). *For an invertible matrix A over a division ring there exist an upper-unipotent matrix X, a low-unipotent matrix Y, a diagonal matrix D and a permutation matrix P such that*

$$A = XPDY.$$

Under the additional condition that $P^{-1}XP$ is an upper-unipotent matrix, the matrices X, P, D, Y are uniquely determined by A.

Note that one can always find a decomposition $A = XPDY$ that satisfies the additional condition.

The entries of matrices X and Y can be written in terms of quasi-Plücker coordinates of submatrices of A. The entries of D can be expressed as quasiminors of A.

Examples. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. If $a_{22} \neq 0$, then

$$A = \begin{pmatrix} 1 & a_{12}a_{22}^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} |A|_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_{22}^{-1}a_{21} & 1 \end{pmatrix}.$$

If $a_{22} = 0$ and the matrix A is invertible, then $a_{12} \neq 0$. In this case,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{21} & 0 \\ 0 & a_{12} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_{12}^{-1}a_{11} & 1 \end{pmatrix}.$$

An important example of quasi-Plücker coordinates for the Vandermonde matrix will be considered later.

4.10. Flag coordinates

Noncommutative flag coordinates were introduced in [GR1,GR2].

Let $A = (a_{ij}),\ i = 1, \ldots, k,\ j = 1, \ldots, n$, be a matrix over a division ring R. Let F_p be the subspace of the left vector space R^n generated by the first p rows of A. Then $\mathcal{F} = (F_1 \subset F_2 \subset \cdots \subset F_k)$ is a flag in R^n. Put

$$f_{j_1 \ldots j_k}(\mathcal{F}) = \begin{vmatrix} a_{1j_1} & \ldots & a_{1j_k} \\ & \ldots & \\ a_{kj_1} & \ldots & a_{kj_k} \end{vmatrix}_{kj_1}.$$

In [GR1,GR2] the functions $f_{j_1 \ldots j_k}(\mathcal{F})$ were called the *flag coordinates* of $\mathcal{F}$. Transformations properties of quasideterminants imply that $f_{j_1 \ldots j_k}(\mathcal{F})$ does not depend on the order of the indices $j_2, \ldots, j_k$.

Proposition 4.10.1 (see Gelfand et al. [GR1,GR2]). *The functions $f_{j_1 \ldots j_m}(\mathcal{F})$ do not change under left multiplication of A by an upper unipotent matrix.*

Theorem 4.10.2 (see Gelfand et al. [GR1,GR2]). *The functions* $f_{j_1\ldots j_k}(\mathcal{F})$ *possess the following relations*:

$$f_{j_1j_2j_3\ldots j_k}(\mathcal{F})f_{j_1j_3\ldots j_k}(\mathcal{F})^{-1} = -f_{j_2j_1\ldots j_k}(\mathcal{F})f_{j_2j_3\ldots j_k}(\mathcal{F})^{-1},$$

$$f_{j_1\ldots j_k}(\mathcal{F})f_{j_1\ldots j_{k-1}}(\mathcal{F})^{-1} + f_{j_2\ldots j_kj_1}(\mathcal{F})f_{j_2\ldots j_k}(\mathcal{F})^{-1}$$
$$+ \cdots + f_{j_kj_1\ldots j_{k-1}}(\mathcal{F})f_{j_kj_1\ldots j_{k-2}}(\mathcal{F})^{-1} = 0.$$

Example. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$. Then $f_{12}(\mathcal{F})a_{11}^{-1} = -f_{21}(\mathcal{F})a_{12}^{-1}$ and $f_{12}(\mathcal{F})a_{11}^{-1} + f_{23}(\mathcal{F})a_{12}^{-1} + f_{31}(\mathcal{F})a_{13}^{-1} = 0$.

It is easy to see that

$$q_{ij}^{i_1\ldots i_{k-1}}(A) = (f_{ii_1\ldots i_{k-1}}(\mathcal{F}))^{-1} \cdot f_{ji_1\ldots i_{k-1}}(\mathcal{F}).$$

Theorems 4.4.1 and 4.4.2 can be deduced from Theorem 4.10.2.

5. Factorization of Vandermonde quasideterminants and the Viète theorem

In this section we study factorizations of quasideterminants of Vandermonde matrices. It is well known that factorizations of Vandermonde determinants over commutative rings play a fundamental role in mathematics. Factorizations of noncommutative Vandermonde quasideterminants turn out to be equally important. This is why we devote a separate section to these results. We also use these factorizations to prove the noncommutative Vieté theorem, which was formulated in [GR3,GR4] using our noncommutative form of the Sylvester identity. In Sections 6 and 7 we will give other applications of factorizations of quasideterminants of Vandermonde matrices. A good exposition of decompositions of Vandermonde quasideterminants is given in [Os].

5.1. *Vandermonde quasideterminants*

Let $x_1, x_2, \ldots, x_k$ be a set of elements of a division ring R. For $k>1$ the quasideterminant

$$V(x_1, \ldots, x_k) = \begin{vmatrix} x_1^{k-1} & \cdots & x_k^{k-1} \\ & \cdots & \\ x_1 & \cdots & x_k \\ 1 & \cdots & 1 \end{vmatrix}_{1k}$$

is called the *Vandermonde* quasideterminant.

We say that a sequence of elements $x_1, \dots, x_n \in R$ is *independent* if all quasideterminants $V(x_1, \dots, x_k)$, $k = 2, \dots, n$, are defined and invertible. For independent sequences $x_1, \dots, x_n$ and $x_1, \dots, x_{n-1}, z$ set

$$y_1 = x_1, \quad z_1 = z,$$

$$y_k = V(x_1, \dots, x_k) x_k V(x_1, \dots, x_k)^{-1}, \quad k \geqslant 2,$$

$$z_k = V(x_1, \dots, x_{k-1}, z) z V(x_1, \dots, x_{k-1}, z)^{-1}, \quad k \geqslant 2.$$

In the commutative case $y_k = x_k$ and $z_k = z$ for $k = 1, \dots, n$.

5.2. Bezout and Viète decompositions of the Vandermonde quasideterminants

Theorem 5.2.1 (Bezout decomposition of the Vandermonde quasideterminant). *Suppose that sequences $x_1, \dots, x_n$ and $x_1, \dots, x_{n-1}, z$ are independent. Then*

$$V(x_1, \dots, x_n, z) = (z_n - y_n)(z_{n-1} - y_{n-1}) \cdots (z_1 - y_1). \tag{5.2.1}$$

Note that if z commutes with x_i, $i = 1, \dots, n$, then

$$V(x_1, \dots, x_n, z) = (z - y_n)(z - y_{n-1}) \cdots (z - y_1).$$

Theorem 5.2.2 (Viète decomposition of the Vandermonde quasideterminant). *For an independent sequence $x_1, \dots, x_n, z$ we have*

$$V(x_1, \dots, x_n, z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n, \tag{5.2.2}$$

where

$$a_k = (-1)^k \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} y_{i_k} y_{i_{k-1}} \cdots y_1. \tag{5.2.3}$$

In particular

$$a_1 = -(y_1 + \cdots + y_n),$$

$$a_2 = \sum_{1 \leqslant i < j \leqslant n} y_j y_i,$$

$$\dots$$

$$a_n = (-1)^n y_n \cdots y_1.$$

5.3. Proof of Theorem 5.2.2

By induction on n we show that Theorem 5.2.2 follows from Theorem 5.2.1. For $n = 1$ one has $V(x_1, z) = z - x_1$ and formulas (5.2.1) and (5.2.2) hold. Suppose that these formulas hold for $m = n - 1$. By Theorem 5.2.1

$$\begin{aligned} V(x_1, \dots, x_n, z) &= (z_n - y_n) V(x_1, \dots, x_{n-1}, z) \\ &= (V(x_1, \dots, x_{n-1}, z) \cdot z) - (y_n \cdot V(x_1, \dots, x_{n-1}, z)). \end{aligned}$$

By induction,

$$V(x_1, \dots, x_{n-1}, z) = z^{n-1} + b_1 z^{n-2} + \cdots + b_{n-1},$$

where

$$b_1 = -(y_1 + \cdots + y_{n-1}),$$

$$\dots$$

$$b_{n-1} = (-1)^n y_{n-1} \cdot \dots \cdot y_1.$$

Therefore,

$$\begin{aligned} V(x_1, \dots, x_n, z) &= z^n + (b_1 - y_n) z^{n-1} + (b_2 - y_n b_1) z^{n-2} + \cdots - y_n b_n \\ &= z^n + a_1 z^{n-1} + \cdots + a_n, \end{aligned}$$

where $a_1, \dots, a_n$ are given by (5.2.3). $\square$

5.4. Division lemma

To prove Theorem 5.2.1 we need the following result.

Lemma 5.4.1. *We have*

$$V(x_1, \dots, x_n, z) = V(\hat{x}_2, \dots, \hat{x}_n, \hat{z})(z - x_1),$$

where

$$\widehat{x}_k = (x_k - x_1)x_k(x_k - x_1)^{-1}, \quad k = 2, \dots, n$$

$$\widehat{z} = (z - x_1)z(z - x_1)^{-1}.$$

Proof. By definition,

$$V(x_1, \dots, x_n, z) = \left| \begin{matrix} x_1^n & x_2^n & \cdots & z^n \\ x_1^{n-1} & x_2^{n-1} & \cdots & z^{n-1} \\ & & \cdots & \\ x_1 & x_2 & \cdots & z \\ 1 & 1 & \cdots & 1 \end{matrix} \right|_{1,n+1}.$$

Multiply the kth row by x_1 from the left and subtract it from the $(k-1)$th row for $k = 2, \dots, n$. Since the quasideterminant does not change, we have

$$V(x_1, \dots, x_n, z) = \left| \begin{matrix} 0 & x_2^n - x_1 x_2^{n-1} & \cdots & z^n - x_z^{n-1} \\ 0 & x_2^{n-1} - x_1 x_2^{n-2} & & z^{n-1} - x_1 z^{n-2} \\ \vdots & \vdots & & \vdots \\ 0 & x_2 - x_1 & & z - x_1 \\ 1 & 1 & & 1 \end{matrix} \right|_{1,n+1}$$

$$= \left| \begin{matrix} 0 & (x_2 - x_1)x_2^{n-1} & \cdots & (z - x_1)z^{n-1} \\ \vdots & \vdots & & \vdots \\ 0 & x_2 - x_1 & & z - x_1 \\ 1 & 1 & & 1 \end{matrix} \right|_{1,n+1}.$$

Applying to the last quasideterminant Sylvester's theorem with the element of index $(n+1, 1)$ as the pivot we obtain

$$V(x_1, \dots, x_n, z) = \left| \begin{matrix} (x_2 - x_1)x_2^{n-1} & \cdots & (z - x_1)z^{n-1} \\ \cdots & & \cdots \\ x_2 - x_1 & & z - x_1 \end{matrix} \right|_{1n}.$$

According to elementary properties of quasideterminants, multiplying the kth column on the right by $(x_{k+1} - x_1)^{-1}$ for $k = 1, \dots, n-1$ and the last column by $(z - x_1)^{-1}$ results in the multiplication of the value of the quasideterminant on the right by $(z-x_1)^{-1}$. Therefore,

$$\begin{aligned}
&V(x_1,\dots,x_n,z)\\
&=\begin{vmatrix} (x_2-x_1)x_2^{n-1}(x_2-x_1)^{-1} & \cdots & (z-x_1)z^{n-1}(z-x_1)^{-1}\\ \vdots & & \vdots\\ 1 & & 1\end{vmatrix}_{1n}(z-x_1)\\
&=\begin{vmatrix} \widehat{x}_2^{n-1} & \cdots & \widehat{z}^{n-1}\\ \vdots & & \vdots\\ 1 & \cdots & 1\end{vmatrix}_{1n}\cdot(z-x_1)=V(\widehat{x}_2,\dots,\widehat{x}_n,\widehat{z})\cdot(z-x_1).
\end{aligned}$$

5.5. *Proof of Theorem 5.2.1*

We proceed by induction on n. By Lemma 5.4.1, Theorem 5.2.1 is valid for $n=2$. Also by Lemma 5.4.1,

$$V(x_1,\dots,x_n,z)=V(\widehat{x}_2,\dots,\widehat{x}_n,\widehat{z})(z-x_1). \tag{5.5.1}$$

Suppose that our theorem is valid for $m=n-1$. Then

$$V(\widehat{x}_2,\dots,\widehat{x}_n,\widehat{z})=(z_n{}'-y_n{}')\dots(z_2'-y_2'),$$

where

$$z_2'=\widehat{z},$$

$$y_2'=\widehat{x}_2,$$

$$z_k'=V(\widehat{x}_2,\dots,\widehat{x}_{k-1},\widehat{z})\widehat{z}V^{-1}(\widehat{x}_2,\dots,\widehat{x}_{k-1},\widehat{z}),$$

$$y_k'=V(\widehat{x}_2,\dots,\widehat{x}_k)\widehat{x}_kV^{-1}(\widehat{x}_2,\dots,\widehat{x}_k)\quad\text{for } k=3,\dots,n.$$

It suffices to show that $z_k'=z_k$ and $y_k'=y_k$ for $k=2,\dots,n$. For $k=2$ this is obvious. By Lemma 5.4.1,

$$V(\widehat{x}_2,\dots,\widehat{x}_{k-1},\widehat{z})=V(x_1,\dots,x_{k-1},z)(z-x_1)^{-1},$$

and by definition $\widehat{z}=(z-x_1)z(z-x_1)^{-1}$. So,

$$\begin{aligned}
z_k'=&\{V(x_1,\dots,x_{k-1},z)(z-x_1)^{-1}\}(z-x_1)z(z-x_1)^{-1}\\
&\times\{(z-x_1)V^{-1}(x_1,\dots,x_{k-1},z)\}=z_k\quad\text{for } k=3,\dots,n.
\end{aligned}$$

Similarly, $y'_k = y_k$ for $k = 3, \dots, n$ and from (5.5.1) we have

$$V(x_1, \dots, x_n, z) = (z_n - y_n) \cdots (z_2 - y_2)(z_1 - y_1). \qquad \square$$

5.6. *Another expression for the coefficients in Viète decomposition*

Another expression for the coefficients $a_1, \dots, a_n$ in Viète decomposition of $V(x_1, \dots, x_n, z)$ can be obtained from Proposition 1.5.1.

Theorem 5.6.1 (Gelfand et al. [GKLLRT]). *We have*

$$V(x_1, \dots, x_n, z) = z^n + a_1 z^{n-1} + \cdots + a_n,$$

where for $k = 1, \dots, n$

$$a_k = -\begin{vmatrix} x_1^n & \cdots & x_n^n \\ & \cdots & \\ x_1^{n-k+1} & \cdots & x_n^{n-k+1} \\ x_1^{n-k-1} & \cdots & x_n^{n-k-1} \\ & \cdots & \\ 1 & \cdots & 1 \end{vmatrix}_{1n} \cdot \begin{vmatrix} x_1^{n-1} & \cdots & x_n^{n-1} \\ & \cdots & \\ x_1^{n-k} & \cdots & x_n^{n-k} \\ & \cdots & \\ 1 & \cdots & 1 \end{vmatrix}_{kn}^{-1}. \tag{5.6.1}$$

From Theorem 5.6.1 we will get the Bezout and Viète formulas expressing the coefficients of the equation

$$z^n + a_1 z^{n-1} + \cdots + a_n = 0 \tag{5.6.2}$$

as polynomials in $x_1, \dots, x_n$ conjugated by Vandermonde determinants.

5.7. *The Bezout and Viète theorems*

Recall that the set of elements $x_1, \dots, x_n$ of a ring with unit is independent if all Vandermonde quasideterminants $V(x_{i_1}, \dots, x_{i_k})$ for $k \geqslant 2$ are defined and invertible.

Lemma 5.7.1. *Suppose that* $x_1, \dots, x_n$ *is an independent set of roots of Eq.* (5.6.2). *Then the coefficients* $a_1, \dots, a_n$ *can be written in the form* (5.6.1).

Proof. Consider the system of right linear equations

$$x_i^n + a_1 x_i^{n-1} + \cdots + a_{n-1} x_i + a_n = 0, \quad i = 1, \dots, n$$

in variables $a_1, \dots, a_n$ and use Cramer's rule. $\square$

Theorem 5.7.2 (Noncommutative Bezout theorem). *Let $x_1, \ldots, x_n$ be an independent set of roots of equation* (5.6.2). *In notations of Theorem* 5.2.1,

$$z^n + a_1 z^{n-1} + \cdots + a_n = (z_n - y_n)\cdots(z_1 - y_1).$$

Proof. Use Lemma 5.7.1, Theorems 5.6.1 and 5.2.1. □

Theorem 5.7.2 (Noncommutative Viète theorem, see Gelfand and Retakh [GR3]). *Let $x_1, \ldots, x_n$ be an independent set of roots of Eq.* (5.6.2). *Then the coefficients $a_1, \ldots, a_n$ of the equation are given by formulas* (5.2.3).

Proof. Use Lemma 3.2.1, Theorems 3.1.4 and 3.1.2. □

A different proof of this theorem, using differential operators, appeared in [EGR]. Another noncommutative version of the Viète Theorem, based on notions of traces and determinants, was given by Connes and Schwarz in [CS].

6. Noncommutative symmetric functions

General theory of noncommutative symmetric functions was developed in the paper [GKLLRT]. In fact, [GKLLRT] was devoted to the study of different systems of multiplicative and linear generators in a free algebra **Sym** generated by a system of noncommuting variables Λ_i, $i = 1, 2, \ldots$. In [GKLLRT] these variables were called elementary symmetric functions, but the theory was developed independently of the origin of Λ_i. Thus, in [GKLLRT] only a formal theory of noncommutative symmetric functions "without variables" was introduced. The real theory of noncommutative symmetric functions got "the right to exist" only after the corresponding variables were introduced in [GR3,GR4] following the Vieté theorem and the basic theorem in the theory of noncommutative symmetric functions has been proved in [Wi].

In this section we apply the general theory to noncommutative symmetric functions generated by specific Λ_i. As in the commutative case, they depend of a set of roots of a polynomial equation.

6.1. Formal noncommutative symmetric functions

This theory was started in [GKLLRT] and developed in several papers (see, for example, [KLT,LST]). An extensive review was given in [Thi]. Here we just recall some basic constructions.

The algebra **Sym** is a free graded associative algebra over a field F generated by an infinite sequence of variables (Λ_k), $\deg \Lambda_k = k$, $k \geqslant 1$. The homogeneous component of degree n is denoted by $\mathbf{Sym}_n$. The direct sum $\oplus_{n\geqslant 1}\mathbf{Sym}_n$ is denoted by $\mathbf{Sym}_+$. Initially the Λ_k's were regarded as the elementary symmetric functions of some virtual set of arguments. A natural set of arguments was found later, see [GR3,GR4].

Recall some properties of the algebra Sym_N of symmetric commutative polynomials in variables $t_1, \ldots, t_N$. The algebra Sym_N has a natural grading, $\deg t_i = 1$, $i = 1, \ldots, N$, and is freely generated by the elementary symmetric functions $e_1(N) = \sum_i t_i$, $e_2(N) = \sum_{i<j} t_i t_j, \ldots, e_N(N) = t_1 t_2 \ldots t_N$. (There are other natural sets of generators in Sym_N). Setting $t_N = 0$ one gets a canonical epimorphism of graded algebras $p_N : \mathrm{Sym}_N \to \mathrm{Sym}_{N-1}$.

The projective limit of graded algebras Sym_N with respect to the system $\{p_N\}$ is called the algebra Sym of symmetric functions in infinite set of variables $t_1, t_2, \ldots$ (see [Mac]). One can view the algebra Sym as a free commutative algebra generated by formal series $e_1 = \sum_i t_i$, $e_2 = \sum_{i<j} t_i t_j, \ldots, e_k = \sum i_1 < \cdots < i_k t_{i_1} \ldots t_{i_k}, \ldots$. The series e_k is called the kth elementary symmetric function in $t_1, t_2, \ldots$.

In Sym, there are also other standard sets of generators (see, e.g., [Mac]). The most common among them are the complete symmetric functions $(h_k)_{k \geqslant 1}$ and the power symmetric functions $(p_k)_{k \geqslant 1}$. To express them in terms of (e_k) one can use generating functions. Namely, set $e_0 = h_0 = 1$. Let τ be a formal variable. Set $\lambda(\tau) = \sum_{k \geqslant 0} h_k \tau_k$, $\sigma(\tau) = \sum_{k \geqslant 0}$, $\psi(\tau) = \sum_{k \geqslant 1} p_k t^{k-1}$. Then

$$\lambda(\tau) = \sigma(-\tau) = 1,$$

$$\psi(\tau) = \frac{d}{d\tau} \log \sigma(\tau).$$

Define the canonical epimorphism $\pi : \mathbf{Sym} \to \mathrm{Sym}$ by setting $\pi(\Lambda_k) = e_k$, $k \geqslant 1$. Let $\mathcal{I}_N$ be an ideal **Sym** generated by all Λ_k, $k > N$. The epimorphism π induces the canonical epimorphism π_N of **Sym** onto the algebra Sym_N of symmetric polynomials in commuting variables $t_1, \ldots, t_N$, $N \geqslant 1$. Note that $\pi_N(\mathcal{I}_N) = 0$.

Noncommutative analogs of functions (h_k) and (p_k) can be constructed in the following way. Let τ be a formal variable commuting with all Λ_k. Set $\Lambda_0 = 1$ and define the generating series

$$\lambda(\tau) \coloneqq \sum_{k \geqslant 0} \Lambda_k \tau^k.$$

Definition 6.1.1. The complete homogeneous symmetric functions are the coefficients S_k in the generating series

$$\sigma(\tau) \coloneqq \sum_{k \geqslant 0} S_k \tau^k = \lambda(-t)^{-1}. \tag{6.1.1a}$$

The power sums symmetric functions of the first kind Ψ_k are the coefficients Ψ_k in the generating series

$$\sum_{k \geqslant 1} \Psi_k \tau^{k-1} \coloneqq \sigma(\tau)^{-1} \frac{d}{dt} \sigma(\tau). \tag{6.1.1b}$$

The power sums symmetric functions of the second kind Φ_k are defined by

$$\sum_{k\geqslant 1} \Phi_k \tau^{k-1} := \frac{d}{d\tau} \log \sigma(\tau). \tag{6.1.1c}$$

By using formulas (6.1.1a)–(6.1.1c) one can prove that $\pi_N(S_k)$ is the kth complete symmetric function and $\pi_N(\Psi_k) = \pi_N(\Phi_k)$ is the kth power symmetric function in N commuting variables. Note that in the right-hand sides of (6.1.1b) and (6.1.1c) different noncommutative analogs of the logarithmic derivative of $\sigma(t)$ are used.

Definition 6.1.1 leads to the following quasideterminantal formulas.

Proposition 6.1.2. *For every $k \geqslant 1$, one has*

$$S_k = (-1)^{k-1} \begin{vmatrix} \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{k-1} & \boxed{\Lambda_k} \\ 1 & \Lambda_1 & \cdots & \Lambda_{k-2} & \Lambda_{k-1} \\ 0 & 1 & \cdots & \Lambda_{k-3} & \Lambda_{k-2} \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & \Lambda_1 \end{vmatrix},$$

$$\Lambda_k = (-1)^{k-1} \begin{vmatrix} S_1 & 1 & 0 & \cdots & 0 \\ S_2 & S_1 & 1 & \cdots & 0 \\ S_3 & S_2 & S_1 & \ldots & 0 \\ & & \cdots & & \\ \boxed{S_k} & S_{k-1} & S_{k-2} & \cdots & S_1 \end{vmatrix},$$

$$kS_k = \begin{vmatrix} \Psi_1 & \Psi_2 & \cdots & \Psi_{k-1} & \boxed{\Psi_k} \\ -1 & \Psi_1 & \cdots & \Psi_{k-2} & \Psi_{k-1} \\ 0 & -2 & \cdots & \Psi_{k-3} & \Psi_{k-2} \\ & & \cdots & & \\ 0 & 0 & \cdots & -n+1 & \Psi_1 \end{vmatrix},$$

$$k\Lambda_k = \begin{vmatrix} \Psi_1 & 1 & 0 & \cdots & 0 \\ \Psi_2 & \Psi_1 & 2 & \cdots & 0 \\ & & \cdots & & \\ \boxed{\Psi_k} & \Psi_{k-1} & \Psi_{k-2} & \cdots & \Psi_1 \end{vmatrix},$$

$$\Psi_k = (-1)^{k-1} \begin{vmatrix} \Lambda_1 & 2\Lambda_2 & \cdots & (k-1)\Lambda_{k-1} & \boxed{k\Lambda_k} \\ 1 & \Lambda_1 & \cdots & \Lambda_{k-2} & \Lambda_{k-1} \\ 0 & 1 & \cdots & \Lambda_{k-3} & \Lambda_{k-2} \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & \Lambda_1 \end{vmatrix},$$

$$\Psi_k = \begin{vmatrix} S_1 & 1 & 0 & \cdots & 0 \\ 2S_2 & S_1 & 1 & \cdots & 0 \\ & & \cdots & & \\ \boxed{kS_k} & S_{k-1} & S_{k-2} & \cdots & S_1 \end{vmatrix}.$$

Each of the four sequences (Λ_k), (S_k), (Ψ_k), and (Φ_k) is a set of generators in **Sym**. Therefore, each of the four sets of products $F_{i_1} \dots F_{i_N}$, $i_1, \dots, i_N \geqslant 1$, where F_{i_k} equals to Λ_{i_k}, S_{i_k}, Ψ_{i_k}, or Φi_k, is a linear basis in $\mathbf{Sym}_+$. Linear relations between these bases were given in [GKLLRT].

Another important example of a linear basis in $\mathbf{Sym}_+$ is given by *ribbon Schur functions*.

6.2. Ribbon Schur functions

Commutative ribbon Schur functions were defined by MacMahon [M]. Here we follow his ideas.

Let $I = (i_1, \dots, i_k)$, $i_1, \dots, i_k \geqslant 1$, be an ordered set.

Definition 6.2.1 (Gelfand et al. [GKLLRT]). The ribbon Schur function R_I is defined by the formula

$$R_I = (-1)^{k-1} \begin{vmatrix} S_{i_1} & S_{i_1+i_2} & S_{i_1+i_2+i_3} & \cdots & \boxed{S_{i_1+\cdots+i_k}} \\ 1 & S_{i_2} & S_{i_2+i_3} & \cdots & S_{i_2+\cdots+i_k} \\ 0 & 1 & S_{i_3} & \cdots & S_{i_3+\cdots+i_k} \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & S_{i_k} \end{vmatrix}.$$

Definition 6.2.1 allows us to express R_I's as polynomials in S_k's. To do this we need the following ordering of sets of integers.

Let $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_s)$. We say that $I \leqslant J$ if $i_1 = j_1 + j_2 + \cdots + j_{t_1}$, $i_2 = j_{t_1+1} + \cdots + j_{t_2}, \dots, i_s = j_{t_{s-1}+1} + \cdots + j_s$. For example, if $I \leqslant (12)$, then $I = (12)$ or $I = (3)$. If $I \leqslant (321)$, then I is equal to one of the sets (321), (51), (33), or (6).

For $I = (i_1, \dots, i_r)$ set $l(I) = r$ and $S^I = S_{i_1} S_{i_2} \dots S_{i_r}$.

Proposition 6.2.2 (Gelfand et al. [GKLLRT, p. 254]).

$$R_J = \sum_{I \leqslant J} (-1)^{l(J)-l(I)} S^I.$$

Example. $R_{123} = S_6 - S_3^2 - S_1 S_5 + S_1 S_2 S_3$.

Definition 6.1.2 implies that $R_I = S_m$ for $I = \{m\}$ and $R_I = \Lambda_k$ for $i_1 = \cdots = i_k = 1$. For each N the homomorphism π_N maps R_I to the corresponding MacMahon ribbon Schur function.

In [GKLLRT] similar formulas expressing R_I as quasideterminants of matrices with entries Λ_k, as well as linear relations with different bases in $\mathbf{Sym}_+$ defined in Section 6.1, are given.

Natural bases in algebra Sym of commutative symmetric functions are indexed by weakly decreasing (or, weakly increasing) finite sequences of integers. Examples are products of elementary symmetric functions $e_{i_1} \ldots e_{i_k}$ where $i_1 \geqslant i_2 \cdots \geqslant i_k$ and Schur functions s_λ where $\lambda = (i_1, \ldots, i_k)$. The following theorem gives a natural basis in the algebra of noncommutative symmetric functions. Elements of this basis are indexed by all finite sequences of integers.

Theorem 6.2.3 (Gelfand et al. [GKLLRT]). *The ribbon Schur functions R_I form a linear basis in* **Sym**.

Let π : **Sym** $\rightarrow$ Sym be the canonical morphism. Then it is known (see [M]) that the commutative ribbon Schur functions $\pi(R_I)$ are not linearly independent. For example, commutative ribbon Schur functions defined by sets (ij) and (ji) coincide. This means that the kernel $\operatorname{Ker} \pi$ is nontrivial.

Remark. In the commutative case, ribbon Schur functions $\pi(R_I)$ with weakly decreasing I constitute a basis in the space of symmetric functions. However, this basis is not frequently used.

The description of the kernel $\operatorname{Ker} \pi$ in terms of ribbon Schur functions is given by the following theorem.

For an ordered set I is denote by $u(I)$ the corresponding unordered set.

Theorem 6.2.4. *The kernel of π is linearly generated by the elements*

$$\Delta_{J,J'} = \sum_{I \leqslant J} R_I - \sum_{I' \leqslant J'} R_{I'}$$

for all J, J' such that $u(J) = u(J')$.

Example. 1. Let $J = (12)$, $J' = (21)$. Then $\Delta_{J,J'} = (R_{12} + R_3) - (R_{21} + R_3) = R_{12} - R_{21}$ and $\pi(R_{12}) = \pi(R_{21})$.

2. Let $J = (123)$, $J' = (213)$. Then $\Delta_{J,J'} = (R_{123} + R_{33} + R_{15} + R_6) - (R_{213} + R_{33} + R_{24} + R_6) = R_{123} + R_{15} - R_{213} - R_{24}$. This shows, in particular, that $\pi(R_{123}) - \pi(R_{213}) = \pi(R_{24}) - \pi(R_{15}) \neq 0$.

The homological relations for quasideterminants imply the multiplication rule for the ribbon Schur functions. Let $I = (i_1, \dots, i_r)$, $J = (j_1, \dots, j_s)$, $i_p \geqslant 1, j_q \geqslant 1$ for all p, q. Set $I + J = (i_1, \dots, i_{r-1}, i_r + j_1, j_2, \dots, j_s)$ and $I \cdot J = (i_1, \dots, i_r, j_1, \dots, j_s)$.

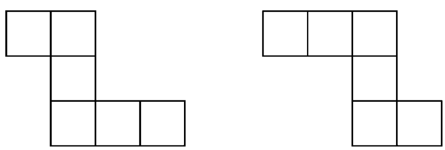
The following picture illustrates this definition (and explains the origin of the name "ribbon Schur functions"). To each ordered set $I = \{i_1, i_2, \dots, i_k\}$ we can associate a ribbon, i.e., a sequence of square cells on the square rules paper starting at the square $(0, 0)$ and going right and down, with i_1 squares in the first column, i_2 squares in the second column, and so on, see Fig. 1 for the ribbons corresponding to $I = (2, 1, 3)$ and $J = (3, 1, 2)$. Then the construction of ribbons $I + J$ and $I \cdot J$ has a simple geometric meaning as shown in Fig. 2 for $I = (2, 1, 3)$ and $J = (3, 1, 2)$.

Theorem 6.2.5 (Gelfand et al. [GKLLRT]). *We have*

$$R_I R_J = R_{I+J} + R_{I.J}.$$

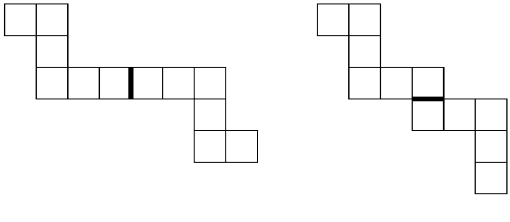
The commutative version of this multiplication rule is due to MacMahon.

Naturality of ribbon Schur functions R_I can be explained in terms of the following construction.



I=(2,1,3) J=(3,1,2)

Fig. 1.



I+J=(2,1,6,1,2) I·J=(2,1,3,3,1,2)

Fig. 2.

6.3. Algebras with two multiplications

The relations between the functions Λ_k and the functions S_k can be illuminated by noting that the ideal $\mathbf{Sym}_+$ has two natural associative multiplications $*_1$ and $*_2$. In terms of ribbon Schur functions it can be given as $R_I *_1 R_J = R_{I.J}$ and $R_I *_2 R_J = R_{I+J}$. We formalize this notion as follows.

Definition 6.3.1. A linear space A with two bilinear products $\circ_1$ and $\circ_2$ is called a biassociative algebra with products $\circ_1$ and $\circ_2$ if

$$(a\circ_i b)\circ_j c = a\circ_i(b\circ_j c)$$

for all $a, b, c\in A$ and all $i, j\in\{1, 2\}$.

Note that if the products $\circ_1$ and $\circ_2$ in a biassociative algebra A have a common identity element 1 (i.e., if $1\circ_i a = a\circ_i 1 = a$ for all $a\in A$ and $i = 1, 2$, then

$$a\circ_1 b = (a\circ_2 1)\circ_1 b = a\circ_2(1\circ_1 b) = a\circ_2 b$$

for all $a, b\in A$ and so $\circ_1 = \circ_2$.

Note also that if A is a biassociative algebra with two products $\circ_1$ and $\circ_2$, then for $r, s\in F$ one can define the linear combination $\circ_{r,s} = r\circ_1 + s\circ_2$ by the formula

$$a\circ_{r,s} b = r(a\circ_1 b) + s(a\circ_2 b), \quad a, b\in A.$$

Then A is a biassociative algebra with the products $\circ_{r,s}$ and $\circ_{t,u}$ for each $r, s, t, u\in F$.

Jacobson's discussion of isotopy and homotopy of Jordan algebras (see [Ja2, p. 56,ff]) shows that if A is an associative algebra with the product $\circ$ and $\circ_a$ for $a\in A$ is defined by the formula

$$b\circ_a c = b\circ a\circ c,$$

then A is a biassociative algebra with the products $\circ$ and $\circ_a$.

We now endow the ideal $\mathbf{Sym}_+\subset\mathbf{Sym}$ with the structure of a biassociative algebra in two different ways. Recall that the nontrivial monomials $(\Lambda_{i_1}\dots\Lambda_{i_r})$ as well as the nontrivial monomials $(S_{i_1}\dots S_{i_r})$ form linear bases in $\mathbf{Sym}+$.

Definition 6.3.2. Define the linear map $*_1 : \mathbf{Sym}_+\otimes\mathbf{Sym}_+\to\mathbf{Sym}_+$ by

$$(\Lambda_{i_1}\dots\Lambda_{i_r}) *_1 (\Lambda_{j_1}\dots\Lambda_{j_s}) = \Lambda_{i_1}\dots\Lambda_{i_{r-1}}\Lambda_{i_r+j_1}\Lambda_{j_2}\dots\Lambda_{j_s}$$

and the linear map $*_2 : \mathbf{Sym}_+ \otimes \mathbf{Sym}_+ \to \mathbf{Sym}_+$ by

$$(S_{i_1} \dots S_{i_r}) *_2 (S_{j_1} \dots S_{j_s}) = S_{i_1} \dots S_{i_{r-1}} S_{i_r+j_1} S_{j_2} \dots S_{j_s}.$$

Write $ab = a *_0 b$ for $a, b \in \mathbf{Sym}_+$. Then it is clear that $a *_i (b *_j c) = (a *_i b) *_j c$ for all $a, b, c \in \mathbf{Sym}_+$ and $i, j = 0, 1$ or $i, j = 0, 2$. Thus we have the following result.

Lemma 6.3.3. $\mathbf{Sym}_+$ *is a biassociative algebra with products* $*_0$ *and* $*_1$ *and also a biassociative algebra with products* $*_0$ *and* $*_2$.

In fact, $*_0$, $*_1$ and $*_2$ are closely related.

The following Lemma is just a restatement of Theorem 6.2.5.

Lemma 6.3.4. $*_0 = *_1 + *_2$.

Proof. We have

$$\lambda(-t)^{-1} = \left(1 + \sum_{i>0} (-1)^i \Lambda_i t^i\right)^{-1} = 1 + \sum_{j>0} \sum_{i_1+\dots+i_l=j} (-1)^{l+j} \Lambda_{i_1} \dots \Lambda_{i_l} t^j.$$

Since $\Lambda_i = \Lambda_1 *_1 \Lambda_1 *_1 \cdots *_1 \Lambda_1$, where there are $i-1$ occurrences of $*_1$, the coefficient at t^j in $\lambda(-t)^{-1}$ is

$$\sum_{u_1,\dots,u_{j-1} \in \{0,1\}} (-1)^k \Lambda_1 *_{u_1} \Lambda_1 *_{u_2} \cdots *_{u_{j-1}} \Lambda_1,$$

where k is the number of u_t equal to 1.

Since $1 + \sum S_i t^i = \lambda(-t)^{-1}$ we have

$$S_j = \sum_{u_1,\dots,u_{j-1} \in \{0,1\}} (-1)^k \Lambda_1 *_{u_1} \Lambda_1 *_{u_2} \cdots *_{u_{j-1}} \Lambda_1.$$

Therefore $S_i *_0 S_j - S_i *_1 S_j = S_i(*_0 - *_1) S_j = S_{i+j} = S_i *_2 S_j$ and $*_2 = *_0 - *_1$, as required. $\square$

Now let U be the two-dimensional vector space with basis $\{u_0, u_1\}$ and

$$F\langle U \rangle = \sum_{k \geqslant 0} F\langle U \rangle_k$$

the (graded) free associative algebra on U, with the homogeneous components

$$F\langle U \rangle_k = U^{\otimes k}.$$

We use the products $*_1$ and $*_2$ to define two isomorphisms, ϕ_1 and ϕ_2, of $F\langle U\rangle$ to $\mathbf{Sym}_+$. Namely, for a basis element $u_{i_1}\dots u_{i_l}\in F\langle U\rangle_k$ set

$$\phi_1 : u_{i_1}\dots u_{i_l} \mapsto \Lambda_1 *_{i_1} \Lambda_1 *_{i_2} \cdots *_{i_l} \Lambda_1 \in (\mathbf{Sym}_+)_{l+1}$$

and

$$\phi_2 : u_{i_1}\dots u_{i_l} \mapsto \Lambda_1 *_{j_1} \Lambda_1 *_{j_2} \cdots *_{j_l} \Lambda_1 \in (\mathbf{Sym}_+)_{l+1},$$

where $j_t = 0$ if $i_t = 0$ and $j_t = 2$ if $i_t = 1$. Note that ϕ_1 and ϕ_2 shift degree.

Define the involution θ of U by $\theta(u_0) = u_0$ and $\theta(u_1) = u_0 - u_1$. Then θ extends to an automorphism Θ of $F\langle U\rangle$ and the restriction Θ_k of this automorphism to $F\langle U\rangle_k$ is the kth tensor power of θ. Clearly $\phi_1\Theta = \phi_2$ and so we recover Proposition 4.13 in [GKLLRT], which describes, in terms of tensor powers, the relation between the bases of $\mathbf{Sym}_+$ consisting of nontrivial monomials in Λ_i and of nontrivial monomials in S_i.

Similarly, taking the identity

$$a^{n-1} + (-1)^n b^{n-1} + \sum_{k=1}^{n-1} (-1)^{n-k} a^{k-1}(a+b)b^{n-k-1} = 0,$$

valid in any associative algebra, setting $a = u_0 - u_1$, $b = u_1$, and applying ϕ_1, we obtain the identity

$$0 = \sum_{k=0}^{n} (-1)^{n-k} \Lambda_k S_{n-k}$$

between the elementary and complete symmetric functions (Proposition 3.3 in [GKLLRT]). Using Proposition 6.1.2, one can express these identities in terms of quasideterminants.

6.4. *Quasi-Schur functions*

Quasi-Schur functions were defined in [GKLLRT]. They are elements not of $\mathbf{Sym}$ but of the free-skew field generated by $S_1, S_2\dots$. Let $I = (i_1, \dots, i_k)$.

Definition 6.4.1. Define $\check{S}_I$ by the formula

$$\check{S}_I = (-1)^{k-1} \begin{vmatrix} S_{i_1} & S_{i_2+1} & \cdots & \boxed{S_{i_k+k-1}} \\ S_{i_1-1} & S_{i_2} & \cdots & S_{i_k+k-2} \\ & & \cdots & \\ S_{i_1-k+1} & S_{i_2-k+2} & \cdots & S_{i_k} \end{vmatrix}. \tag{6.4.1}$$

If $I = (i_1, \dots, i_k)$ is a partition, i.e., a weakly increasing sequence of nonnegative sequences, the element $\check{S}_I$ is called a quasi-Schur function. For an arbitrary set I, $\check{S}_I$ is called a generalized quasi-Schur function.

In particular, $\check{S}_{\{k\}} = S_k$ and $\check{S}_{\{1^k\}} = \Lambda_k$, where $1^k = (1\dots1)$ with k occurrences of 1.

Definition 6.4.1 is a noncommutative analog of Jacobi–Trudi formula. In the commutative case, $\check{S}_I$ for a partition I is the *ratio* of two Schur functions S_I/S_J, where $J = (i_1 - 1, \dots, i_{k-1} - 1)$. It shows that in general $\check{S}$ cannot be represented as a polynomial in S_k.

Remark. The homological relations and the transformation properties of quasideterminants imply that any generalized quasi-Schur function $\check{S}_I$ can be expressed as a rational function in the quasi-Schur functions. For example,

$$\check{S}_{42} = -\begin{vmatrix} S_4 & \boxed{S_3} \\ S_3 & S_2 \end{vmatrix} = -\begin{vmatrix} \boxed{S_3} & S_4 \\ S_2 & S_3 \end{vmatrix} = \begin{vmatrix} S_3 & \boxed{S_4} \\ S_2 & S_3 \end{vmatrix} S_3^{-1} S_2 = \check{S}_{33} S_3^{-1} S_2.$$

6.5. *Symmetric functions in noncommutative variables*

We fix n independent indeterminants $x_1, x_2, \dots, x_n$ and construct new variables $y_1, \dots, y_n$ which are *rational* functions in $x_1, \dots, x_n$ as follows. Recall that in 5.1 we defined the Vandermonde quasideterminant

$$V(x_1, \dots, x_k) = \begin{vmatrix} x_1^{k-1} & \cdots & \boxed{x_k^{k-1}} \\ x_1^{k-2} & \cdots & x_k^{k-2} \\ & \cdots & \\ 1 & \cdots & 1 \end{vmatrix}.$$

Set

$$y_1 = x_1,$$

$$y_2 = V(x_1, x_2) x_2 V(x_1, x_2)^{-1} = (x_2 - x_1) x_2 (x_2 - x_1)^{-1},$$

$$\dots$$

$$y_n = V(x_1, \dots, x_n) x_n V(x_1, \dots, x_n)^{-1}.$$

In the commutative case $x_i = y_i$, $i = 1, \dots, n$. In the noncommutative case x_i and y_i are obviously different.

Remark. Consider the free skew-field R generated by $x_1, \dots, x_n$. Define on R differential operators ∂_i by formula $\partial_i x_j = \delta_{ij}$ and the Leibniz rule $\partial_i(fg) = \partial_i(f)g + f\partial_i(g)$, for $i = 1, \dots, n$. It easy to see that $\partial_i y_j \neq \delta_{ij}$. However, denote $\partial = \partial_1 + \cdots + \partial_n$. Then

$$\partial(V(x_1, \dots, x_k)) = 0, \quad k = 2, \dots, n$$

and

$$\partial(y_i) = \partial(x_i) = 1, \quad i = 1, \dots, n.$$

Elementary symmetric functions:

Definition 6.5.1. The functions

$$\Lambda_1(x_1, \dots, x_n) = y_1 + y_2 + \cdots + y_n,$$

$$\Lambda_2(x_1, \dots, x_n) = \sum_{i<j} y_j y_i,$$

$$\dots$$

$$\Lambda_n(x_1, \dots, x_n) = y_n \dots y_1$$

are called elementary symmetric functions in $x_1, \dots, x_n$.

In the commutative case these functions are the standard elementary symmetric functions of $x_1, \dots, x_n$. By the noncommutative Viète theorem (Theorem 5.7.1), $\Lambda_i(x_1, \dots, x_n) = (-1)^i a_i,\ i = 1, \dots, n$, where $x_1, \dots, x_n$ are the roots of the equation

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$$

This implies

Proposition 6.5.2. *The functions $\Lambda_i(x_1, \dots, x_n)$ are symmetric in $x_1, \dots, x_n$.*

Denote by $\overline{\mathbf{Sym}}_n$ the subalgebra of the algebra of rational functions in $x_1, \dots, x_n$ generated by $\Lambda_k(x_1, \dots, x_n),\ k = 1, \dots, n$. Define the surjective homomorphism

$$\phi : \mathbf{Sym} \to \overline{\mathbf{Sym}}_n \qquad (6.5.1)$$

by setting $\phi(\Lambda_k) = \Lambda_k(x_1, \dots, x_n)$.

Theorem 6.5.3. *The kernel of ϕ is generated by Λ_k for $k > n$.*

Remark. The order of $y_1, \dots, y_n$ is essential in the definition of $\Lambda_i(x_1, \dots, x_n)$, $i = 1, \dots, n$. For example, $\Lambda_2(x_1, x_2) = y_2 y_1$ is symmetric in x_1, x_2, whereas the product $y_1 y_2$ is not symmetric. To see this set $\Delta = x_2 - x_1$. The symmetricity in x_1, x_2 of the product $y_1 y_2$ would imply that $x_1 \Delta^2 = \Delta^2 x_1$.

Complete symmetric functions:

Definition 6.5.4. The functions

$$S_k(x_1, \dots, x_n) = \sum_{i_1 \leqslant i_2 \leqslant \dots \leqslant i_k} y_{i_1} \dots y_{i_k}, \quad k = 1, 2, 3, \dots$$

are called complete symmetric functions in $x_1, \dots, x_n$.

In the commutative case these functions are the standard complete symmetric functions in $x_1, \dots, x_n$.

Let t be a formal variable commuting with x_i, $i = 1, \dots, n$. Define the generating functions

$$\lambda(t) = 1 + \Lambda(x_1, \dots, x_n)t + \dots + \Lambda_n(x_1, \dots, x_n)t^n$$

$$\sigma(y) = 1 + \sum_i S_i(x_1, \dots, x_n)t^i = \lambda(-t)^{-1}.$$

Proposition 6.5.5. *We have*

$$\sigma(t)\lambda(-t) = 1.$$

In the commutative case, $S_k(x_1, \dots, x_n)$ are the standard complete symmetric functions.

Proposition 6.5.6. *The functions $S_k(x_1, \dots, x_n)$ are symmetric in $x_1, \dots, x_n$.*

Proof. Use Proposition 6.5.5, Theorem 6.5.3 and Proposition 6.1.2. $\square$

Remark. The order of elements y_s in the definition of S_k is essential: $S_2(x_1, x_2) = y_1^2 + y_1 y_2 + y_2^2$ is symmetric in $x_1, \dots, x_n$ whereas $y_1^2 + y_2 y_1 + y_2^2$ is not symmetric (cf. the remark after Theorem 6.5.3).

Ribbon Schur functions: We define "ribbon Schur functions with arguments" $R_I(x_1, \dots, x_n)$ similarly to Definition 6.2.1, replacing R_I with $R_I(x_1, \dots, x_n)$ and S_k with $S_k(x_1, \dots, x_n)$. Evidently, ribbon Schur functions $R_I(x_1, \dots, x_n)$ are symmetric in $x_1, \dots, x_n$ and form a linear basis in $\overline{\mathbf{Sym}}_n$.

Proposition 6.5.7. *We have* $R_I(x_1, \ldots, x_n) = \phi(R_I)$, *where* ϕ *is defined by formula* (6.5.1).

To express $R_I(x_1, \ldots, x_n)$ as a sum of monomials in $y_1, \ldots, y_n$ we need some notation. Let $w = a_{i_1} \ldots a_{i_k}$ be a word in ordered letters $a_1 < \cdots < a_n$. An integer m is called a *descent* of w if $1 \leqslant m \leqslant k-1$ and $i_m > i_{m+1}$. Let $M(w)$ be the set of all descents of w.

Let $J = (j_i, \ldots, j_k)$ be a set of positive integers.

Theorem 6.5.8.

$$R_J(x_1, \ldots, x_n) = \sum y_{i_1} \ldots y_{i_m}, \tag{6.5.2}$$

where the sum is taken over all words $w = y_{i_1} \ldots y_{i_m}$ *such that* $M(w) = \{j_1, j_1 + j_2, \ldots, j_1 + j_2 + \cdots + j_{k-1}\}$.

The proof of the theorem was essentially given in [GKLLRT, Section VII].

6.6. *Main theorem for noncommutative symmetric functions*

In the commutative case the classical main theorem of the theory of symmetric functions says that every symmetric polynomial of n variables is a polynomial of (elementary) symmetric functions of these variables. Its analogue for a noncommutative case is given by the following theorem. Denote $\Lambda_k(x_1, \ldots, x_n)$ as $\Lambda_k(X)$, $k = 1, \ldots, n$.

Recall that in the previous section we defined the elements y_k by the formulas $y_1 = x_1$, $y_k = V(x_1, \ldots, x_k) x_k V(x_1, \ldots, x_k)^{-1}$ for $k = 2, \ldots, n$.

Theorem 6.6.1 (Wilson [Wi]). *Let a polynomial* $P(y_1, \ldots, y_n)$ *over* $\mathbb{Q}$ *be symmetric in* $x_1, \ldots, x_n$. *Then* $P(y_1, \ldots, y_n) = Q(\Lambda_1(X), \ldots, \Lambda_n(X))$, *where* Q *is a noncommutative polynomial over* $\mathbb{Q}$.

Remark. Recall that $P(y_1, \ldots, y_n)$ is a polynomial in y_i and not in x_i. We can express this by saying that in the natural variables x_i, noncommutative symmetric polynomials are not polynomials but rational functions.

Corollary 6.6.2. *A polynomial* $P(y_1, \ldots, y_n)$ *with coefficients in* $\mathbb{Q}$ *is symmetric in* $x_1, \ldots, x_n$ *if and only if* $P(y_1, \ldots, y_n)$, *viewed as a rational function of* $x_1, \ldots, x_n$, *is a linear combination of* $R_J(x_1, \ldots, x_n)$.

6.7. *Quasi-Plücker coordinates of Vandermonde matrices and symmetric functions*

Here we study right quasi-Plücker coordinates $r_{ij}^{i_1, \ldots, i_{n-1}}(V_n)$, where $V_n = (x_j^i)$, $i \in \mathbb{Z}$, $j = 1, \ldots, n$, is the Vandermonde matrix, x_j, $j = 1, \ldots, n$ are noncommuting variables. The matrix V_n has n columns and infinitely many rows. The following proposition shows the importance of such coordinates.

Proposition 6.7.1. *We have*

$$r^{0,1,\dots,n-2}_{n+k-1,n-1}(V_n) = S_k(x_1,\dots,x_n), \quad k = 0,1,2,\dots,$$

$$r^{0,1,\dots,n-k-1,n-k+1,\dots,n-1}_{n,n-k}(V_n) = (-1)^{k-1}\Lambda_k(x_1,\dots,x_n), \quad k = 0,1,\dots,n.$$

Examples.

$$S_k(x_1,x_2) = \begin{vmatrix} x_1^{k+1} & \boxed{x_2^{k+1}} \\ 1 & 1 \end{vmatrix} \begin{vmatrix} x_1 & \boxed{x_2} \\ 1 & 1 \end{vmatrix}^{-1} = (x_2^{k+1} - x_1^{k+1})(x_2 - x_1)^{-1}$$
$$= y_2^k + y_2^{k-1}y_1 + \cdots + y_2 y_1^{k-1} + y_1^k,$$

where $y_1 = x_1$, $y_2 = (x_2 - x_1)x_2(x_2 - x_1)^{-1}$;

$$\Lambda_1(x_1,x_2) = \begin{vmatrix} x_1^2 & \boxed{x_2^2} \\ 1 & 1 \end{vmatrix} \begin{vmatrix} x_1 & \boxed{x_2} \\ 1 & 1 \end{vmatrix}^{-1} = (x_2^2 - x_1^2)(x_2 - x_1)^{-1} = y_1 + y_2,$$

$$\Lambda_2(x_1,x_2) = -\begin{vmatrix} x_1^2 & \boxed{x_2^2} \\ x_1 & x_2 \end{vmatrix}^{-1} \begin{vmatrix} x_1 & x_2 \\ 1 & \boxed{1} \end{vmatrix}^{-1} = (x_2^2 - x_1x_2)(1 - x_1^{-1}x_2)^{-1} = y_2y_1.$$

Remark. Formulas for S_k in Proposition 6.7.1 are valid for all $k \in \mathbb{Z}$.

An important "periodicity" property of quasi-Plücker coordinates of Vandermonde matrices is given by the following proposition.

Proposition 6.7.2. *For any $k \in \mathbb{Z}$ we have*

$$r^{i_1,\dots,i_{n-1}}_{ij}(V_n) = r^{i_1+k,\dots,i_{n-1}+k}_{i+k,\,j+k}(V_n).$$

Proposition 6.7.1 can be generalized as follows. Recall that in 6.4 we defined generalized quasi-Schur functions. Let $I = \{i_1, i_2, \dots, i_m\}$.

Proposition 6.7.3. *Let* $I=\{i_1,i_2,\dots,i_m\}$ *and* $i_1\geqslant i_2\geqslant\dots\geqslant i_m$. *Set* $J=\{0,1,\dots,n-m-1,n-m+i_1,\dots,n-2+i_{m-1}\}$. *Then*

$$r^J_{n-1+i_1,n-m}(V_n)=\check{S}_I.$$

7. Universal quadratic algebras associated with pseudo-roots of noncommutative polynomials and noncommutative differential polynomials

7.1. Pseudo-roots of noncommutative polynomials

During the last years the authors introduced and studied universal algebras associated with pseudo-roots of noncommutative polynomials. The results appeared in [GRW,GGR,GGRSW].

Let C be an algebra with unit and $P(t)\in C[t]$ be a polynomial (where t is a formal variable commuting with elements of C). We say that an element $c\in C$ is a *pseudo-root* of $P(t)$ if there exist polynomials $L_c(t),R_c(t)\in C[t]$ such that $P(t)=L_c(t)(t-c)R_c(t)$. If $P(t)=a_0t^n+a_1t^{n-1}+\dots+a_{n-1}t+a_n$ and c is a pseudo-root of $P(t)$ with $R_c(t)=1$, then

$$a_0c^n+a_1c^{n-1}+\dots+a_{n-1}c+a_n=0,$$

i.e., c is a root of the polynomial $P(x)=a_0x^n+a_1x^{n-1}+\dots+a_{n-1}x+a_n$ (where x is a noncommuting variable). Our theory shows that the analysis of noncommutative polynomials is impossible without studying pseudo-roots.

Let $x_1,\dots,x_n$ be roots of a *generic* monic polynomial $P(x)=x^n+a_1x^{n-1}+\dots+a_n$ over an algebra C. There are two important classical problems: (a) to express the coefficients $a_1,\dots,a_n$ via the roots, (b) to determine all factorizations of $P(x)$, or $P(t)$.

When C is a division ring, the first problem was solved in [GR3,GR4] using the theory of quasideterminants; the solution is presented in Section 5. Let $V(x_{i_1},\dots,x_{i_k})$ be the Vandermonde quasideterminant corresponding to the sequence $x_{i_1},\dots,x_{i_k}$. For an ordering $\{i_1,\dots,i_n\}$ of $\{1,\dots,n\}$, we constructed the elements

$$x_{\emptyset,i_1}=x_{i_1},$$

$$x_{\{i_1,i_2,\dots,i_{k-1}\},i_k}=V(x_{i_1},\dots,x_{i_k})x_{i_k}V(x_{i_1},\dots,x_{i_k})^{-1},\quad k=2,\dots,n,$$

in C such that for every $m=1,\dots,n$,

$$(-1)^m a_m=\sum_{j_1>j_2>\cdots j_m} y_{j_1}y_{j_2}\dots y_{j_m},\tag{7.1.1}$$

where $y_1=x_{i_1}$, $y_k=x_{\{i_1,\dots,i_{k-1}\},i_k}$, $k=2,\dots,n$.

It is surprising that the left-hand side in formula (7.1.1) does not depend on the ordering of $\{1, \ldots, n\}$ whereas the right-hand side a priori does depend on it. The independence of the right-hand side in (7.1.1) on the ordering of $\{1, \ldots, n\}$ was a key point in the theory of noncommutative symmetric functions developed in [GR3,GR4], see Section 6 of the present article.

The element $x_{\{i_1,\ldots,i_{k-1}\},i_k}$ has an interesting structure. As we have already mentioned, it is symmetric in $x_{i_1}, \ldots, x_{i_{k-1}}$. Next, it is a rational function in $x_{i_1}, \ldots, x_{i_k}$ containing, in the general case, $k-1$ inversions. In other words, $x_{\{i_1,\ldots,i_{k-1}\},i_k}$ is a rational expression of height $k-1$.

Set $A_k = \{i_1, \ldots, i_{k-1}\}$ for $k = 2, \ldots, n$. Recall that formulas (7.1.1) are equivalent to the decomposition

$$P(t) = (t - x_{A_n,i_n})(t - x_{A_{n-1},i_{n-1}})\ldots(t - x_{i_1}). \tag{7.1.2}$$

This formula shows that the elements $x_{A,i}$ are pseudo-roots of $P(t)$, and in the general case the polynomial $P(t)$ has at least $n2^{n-1}$ different pseudo-roots. We study all these pseudo-roots together by constructing the universal algebra of pseudo-roots Q_n.

7.2. *Universal algebra of pseudo-roots*

It is easy to see that the elements $x_{A,i}$, $i \notin A$, satisfy the following simple relations:

$$x_{A\cup\{i\},j} + x_{A,i} = x_{A\cup\{j\},i} + x_{A,j}, \tag{7.2.1a}$$

$$x_{A\cup\{i\},j} \cdot x_{A,i} = x_{A\cup\{j\},i} \cdot x_{A,j}, \tag{7.2.1b}$$

for all $A \subseteq \{1, \ldots, n\}$, $i, j \notin A$.

In order to avoid inversions and to make our construction independent of the algebra C, we define the universal algebra of pseudo-roots Q_n over a field F to be the algebra with generators $z_{A,i}$, $A \subseteq \{1, \ldots, n\}$, $i \notin A$, and relations corresponding to 7.2.1 (with x replaced by z).

Each algebra Q_n has a natural derivation $\partial : Q_n \to Q_n$, $\partial(z_{A,i}) = 1$ and a natural anti-involution $\theta : Q_n \to Q_n$, $\theta(z_{A,i}) = z_{C,i}$ where $C = \{1, \ldots, n\} \backslash A \backslash \{i\}$.

The algebra contains, as a subalgebra, the free associative algebra generated by the $z_i = z_{\emptyset,i}$, $i = 1, \ldots, n$. The algebra Q_n also admits a natural homomorphism α_n to the skew-field generated by elements $z_1, \ldots, z_n$. Namely, let $A = \{i_1, \ldots, i_k\}$. Set

$$\alpha_n(z_{A,i}) = \begin{vmatrix} z_{i_1}^k & \cdots & z_{i_k}^k & \boxed{z_i^k} \\ z_{i_1}^{k-1} & \cdots & z_{i_k}^{k-1} & z_i^{k-1} \\ & \cdots & & \\ 1 & \cdots & 1 & 1 \end{vmatrix} z_i \begin{vmatrix} z_{i_1}^k & \cdots & z_{i_k}^k & \boxed{z_i^k} \\ z_{i_1}^{k-1} & \cdots & z_{i_k}^{k-1} & z_i^{k-1} \\ & \cdots & & \\ 1 & \cdots & 1 & 1 \end{vmatrix}^{-1}.$$

Conjecture 7.2.1. *The homomorphism α_n is a monomorphism.*

We can prove this conjecture for $n = 2, 3$.

The algebra Q_n admits a natural commutative specialization $\pi_n : Q_n \to F[t_1, \dots, t_n]$ given by

$$\pi_n(z_{A,i}) = t_i, \quad i = 1, \dots, n. \tag{7.2.2}$$

In particular, for each $i \notin A$, the image of $z_{A,i} - z_i$ under π_n equals zero.

To study the kernel of π_n further, it is convenient to define, for each pair $A, B \subseteq \{1, \dots, n\}$, with $A \cap B = \emptyset$ the element $z_{A,B} \in Q_n$ by the recurrence formula

$$z_{\emptyset,\emptyset} = 0,$$

$$z_{A\cup\{i\},B} - z_{A,B\cup\{i\}} = z_{A,B} \quad \text{for } i \notin A, B.$$

One can easily see that when B contains more than one element, the element $z_{A,B}$ is "invisible" in the commutative case, i.e., $\pi_n(z_{A,B}) = 0$.

In [GRW] it was proved that

$$z_{A,i} - z_i = \sum_{\emptyset \neq C \subseteq A} z_{\emptyset, C\cup\{i\}}.$$

The terms on the right-hand side in (7.2.2) measure the "noncommutativity" of $z_{A,i}$. Moreover, in a sense, the "degree of noncommutativity" carried by $z_{\emptyset,B}$ depends on the size of B: the greater $|B|$, the more "noncommutative" the element by $z_{\emptyset,B}$ is.

7.3. Bases in the algebra Q_n

The algebra Q_n has a natural graded structure $Q_n = \sum_{l \geqslant 0} Q_{n,l}$ where $Q_{n,l}$ is the span of all products of l generators $z_{A,i}$.

One can see that elements $z_{A,\emptyset}$ and, similarly, the elements $z_{\emptyset,A}$, for all $A \subseteq \{1, \dots, n\}$, $A \neq \emptyset$, constitute a basis in the subspace of $Q_{n,1}$. These elements satisfy simple quadratic relations.

Our study of Q_n relies on the construction of a basis in Q_n, which is a hard combinatorial problem.

For $A \subseteq \{1, \dots, n\}$ let $\min(A)$ denote the smallest element of A. Then set $A^{(0)} = A$, $A^{(1)} = A \backslash \{\min(A)\}$, $A^{(i+1)} = (A^{(i)})^{(1)}$. Set $r_A = z_{\emptyset,A}$.

Theorem 7.3.1 (see Gelfand et al. [GRW]). *The set of all monomials*

$$r_{A_1^{(0)}} r_{A_1^{(1)}} \dots r_{A_1^{(j_1)}} r_{A_2^{(0)}} r_{A_2^{(1)}} \dots r_{A_2^{(j_2)}} \dots r_{A_l^{(0)}} r_{A_l^{(1)}} \dots r_{A_l^{(j_l)}},$$

where $A_1, \dots, A_l \subseteq \{1, \dots, n\}$ *and for each* $1 \leqslant i \leqslant l-1$, *either* $A_{i+1} \nsubseteq A_i$ *or* $|A_{i+1}| \neq |A_i| - j_i - 1$, *is a basis in* Q_n.

Remark. It would be interesting to study in details combinatorial properties of the basis and to give constructions of similar bases in Q_n using different techniques including noncommutative Gröbner bases and Bergman's Diamond lemma.

7.4. *Algebra of noncommutative symmetric polynomials as a subalgebra in Q_n*

For each ordering $I = (i_1, \ldots, i_n)$ of $\{1, \ldots, n\}$, there is a natural free subalgebra $Q_{n,I} \subset Q_n$ generated by $\{z_{\{i_1,\ldots,i_k\},\emptyset} \mid 1 \leqslant k \leqslant n\}$. Using the basis theorem in [GRW], we can describe arbitrary intersections of subalgebras $Q_{n,I}$. In particular, we can prove the following theorem. Let $\mathbf{S}_n$ be a subalgebra in Q_n generated by all coefficients a_m (see formula (7.1.1)). One can identify $\mathbf{S}_n$ with the algebra $\overline{\mathbf{Sym}}_n$ of noncommutative symmetric functions in $x_1, \ldots, x_n$.

Theorem 7.4.1 (see Gelfand et al. [GRW]). *The intersection of all subalgebras $Q_{n,I}$ coincides with algebra $\mathbf{S}_n$.*

This is a purely noncommutative phenomenon: under the commutative specialization π_n, all algebras $Q_{n,I}$ map to the algebra of all polynomials and algebra $\mathbf{S}_n$ maps to the algebra of symmetric functions.

7.5. *The dual algebra $Q_n^!$*

The definition of the dual quadratic algebra and of Koszul quadratic algebras can be found, e.g., in [Lö].

Recall, that the quadratic algebra Q_n has a natural graded structure $Q_n = \sum_{i \geqslant 0} Q_{n,i}$ where $Q_{n,i}$ is the span of all products of i generators. As usual, we denote the Hilbert series of Q_n by $H(Q_n, \tau) = \sum_{i \geqslant 0} \dim(Q_{n,i})\tau^i$.

In [GGRSW] we computed the Hilbert series of Q_n and of its dual quadratic algebra $Q_n^!$. In particular, the following result was proved.

Theorem 7.5.1. *We have*

$$H(Q_n, \tau) = \frac{1 - \tau}{1 - \tau(2 - \tau)^n},$$

$$H(Q_n^!, \tau) = \frac{1 + \tau(2 + \tau)^n}{1 + \tau}.$$

In particular, since $H(Q_n^!, \tau)$ is a polynomial in τ, the dual algebra $Q_n^!$ is finite dimensional. Similarly to Q_n, it also has a rich and interesting structure.

Theorem 7.5.1 shows that

$$H(Q_n, \tau) \cdot H(Q_n^!, -\tau) = 1.$$

This also follows from the koszulity of Q_n, which was recently proved by Serconek and Wilson [SW].

7.6. Quotient algebras of Q_n

There are at least two reasons to study quotient algebras of Q_n. The noncommutative nature of Q_n can be studied by looking at quotients of Q_n by ideals generated by some $z_{\emptyset,A}$. These quotients are "more commutative" than Q_n. For example, the quotient of Q_n by the ideal generated by all $z_{\emptyset,A}$ with $|A|\geqslant 2$ is isomorphic to the algebra of commutative polynomials in n variables.

To consider more refined cases, we need to turn to a "noncommutative combinatorial topology". In this approach the algebra Q_n corresponds to an n-simplex Δ_n, and we consider quotients of Q_n by ideals generated by some $z_{\emptyset,A}$ corresponding to subcomplexes of Δ_n. In [GGR] we described generators and relations for those quotients. Special attention was given to the quotients of Q_n corresponding to one-dimensional subcomplexes of Δ_n (they are close to algebras of commutative polynomials). However, we need to study other quotient algebras of this type and to be able to "glue" together such quotient algebras. This will lead to a construction of a "noncommutative combinatorial topology".

Another interesting class of quotients of Q_n consists of algebras corresponding to special types of polynomials (such as $x^n = 0$ or polynomials with multiple roots). Here is an example.

Example. Let F be a field. We consider quotients of the F-algebra Q_2. The algebra Q_2 itself is generated by z_1, z_2, $z_{1,2}$, $z_{2,1}$. It corresponds to a polynomial $P(t) = t^2 - pt + q$ with $p = z_1 + z_{1,2} = z_2 + z_{2,1}$, $q = z_{1,2}z_1 = z_{2,1}z_2$.

There is an "invisible element" $z_{\emptyset,12} = z_{1,2} - z_2 = z_{2,1} - z_1$. This element satisfies the relation $z_{\emptyset,12}(z_1 - z_2) = z_1z_2 - z_2z_1$.

There are three quotient algebras corresponding to special cases of $P(t)$.

(i) The algebra corresponding to the polynomial t^2 is $Q_2/(p,q)$. This algebra is isomorphic to $F\langle z_1, z_2\rangle/(z_1^2, z_2^2)$.

(ii) The algebra corresponding to a polynomial with multiple roots is $Q_2/(z_1 - z_2)$. It is isomorphic to the free algebra with generators z_1 and $z_{1,2}$.

(iii) The algebra $Q_2/(z_{\emptyset,12})$ is isomorphic to the algebra of commutative polynomials in two variables.

Another class of quotient algebras of Q_n was introduced in [GGR].

7.7. Quadratic algebras associated with differential noncommutative polynomials

In [GRW] we also constructed, similarly to algebras Q_n, the universal algebras of pseudo-roots of noncommutative differential polynomials.

Let C be an algebra over a field F. Recall that a derivation D of C is an F-linear map $D: C \to C$ such that $D(ab) = D(a)b + aD(b)$. Any element $a \in C$ acts on C by the multiplication from the left. It is obvious that the commutator $[D, a] = Da - aD$ acts on C as the left multiplication by $D(a)$. Also, any polynomial $P = P(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_n$, $a_i \in C$ for $i = 0, 1, \ldots, n$ acts on C by the formula

$$P(D)(\phi) = a_0 D^n(\phi) + a_1 D^{n-1}(\phi) + \cdots + a_{n-1} D(\phi) + a_n \phi.$$

We say that an element $c \in C$ is a pseudo-root of $P(D)$ if there exist polynomials $L_c(D)$ and $R_c(D)$ with coefficients in C such that $P(D) = L_c(D)(D - c)R_c(D)$ (taking into account the commutation rule 7.7.1). We say that c is a root of $P(D)$ if $R_c(D) = 1$.

Suppose that $a_0 = 1$ and the differential polynomial $P(D)$ has n different roots $f_1, \ldots, f_n \in C$. Following [EGR], for any ordering $(i_1, \ldots, i_n)$ of $\{1, \ldots, n\}$, in [GRW] we constructed, for a generic P, pseudo-roots $f_{i_1,i_2}, \ldots, f_{i_1,\ldots,i_{n-1},i_n}$ such that

$$P(D) = (D - f_{i_1,\ldots,i_{n-1},i_n}) \ldots (D - f_{i_1,i_2})(D - f_{i_1}).$$

For $k = 2, \ldots, n$ the element $f_{i_1,\ldots,i_{k-1},i_k}$ does not depend on the order of elements $(i_1, \ldots, i_{k-1})$.

Set $f_{\emptyset,i} = f_i$. It was proved in [GRW] that for any $A \subset \{1, \ldots, n\}$ such that $|A| < n - 1$, and for any $i, j \notin A$ we have

$$f_{A \cup i, j} + f_{A,i} = f_{A \cup j, i} + f_{A,j}, \tag{7.7.1a}$$

$$f_{A \cup i, j} f_{A,i} - D(f_{A,i}) = f_{A \cup j, i} f_{A,j} - D(f_{A,j}). \tag{7.7.1b}$$

Based on these formulas one can define universal algebras DQ_n of pseudo-roots of noncommutative differential polynomials. They are defined by elements $f_{A,i}$ for $i \notin A$ satisfying relations 7.7.1. The theory of algebras DQ_n seems to be useful in the study of noncommutative integrable systems.

8. Noncommutative traces, determinants and eigenvalues

In this section we discuss noncommutative traces, determinants and eigenvalues. Our approach to noncommutative determinants in this section is different from our approach described in Section 3.

Classical (commutative) determinants play a key role in representation theory. Frobenius developed his theory of group characters by studying factorizations of group determinants (see [L]). Therefore, one cannot start a noncommutative representation theory without looking at possible definition of noncommutative determinants and traces. The definition of a noncommutative determinant given in this section is different from the definition given in Section 3. However, for matrices over commutative algebras, quantum and Capelli matrices both approach give the same results.

8.1. Determinants and cyclic vectors

Let R be an algebra with unit and $A: R^m \to R^m$ a linear map of right vector spaces, A vector $v \in R^m$ is an *A-cyclic vector* if $v, Av, \ldots, A^{m-1}v$ is a basis in R^m regarded as a right R-module. In this case there exist $\Lambda_i(v, A) \in R$, $i = 1, \ldots, m$, such that

$$(-1)^m v\Lambda_m(v,A) + (-1)^{m-1}(Av)\Lambda_{m-1}(v,A) + \cdots - (A^{m-1}v)\Lambda_1(v,A) + A^m v = 0.$$

Definition 8.1.1. We call $\Lambda_m(v, A)$ the *determinant* of (v, A) and $\Lambda_1(v, A)$ the *trace* of (v, A).

We may express $\Lambda_i(v, A) \in R$, $i = 1, \ldots, m$, as quasi-Plücker coordinates of the $m \times (m+1)$ matrix with columns $v, Av, \ldots, A^n v$ (following [GR4]).

In the basis $v, Av, \ldots, A^{m-1}$ the map A is represented by the Frobenius matrix A_v with the last column equal to $((-1)^m \Lambda_m(v, A), \ldots, -\Lambda_1(v, A))^T$. From Theorem 3.1.3 it follows that if determinants of A_v are defined, then they coincide up to a sign with $\Lambda_m(V, A)$. This justifies our definition.

Also, when R is a commutative algebra, $\Lambda_m(v, A)$ is the determinant of A and $\Lambda_1(v, A)$ is the trace of A.

When R is noncommutative, the expressions $\Lambda_i(v, A) \in R$, $i = 1, \ldots, m$, depend on vector v. However, they provide some information about A. For example, the following statement is true.

Proposition 8.1.2. *If the determinant* $\Lambda_m(v, A)$ *equals zero, then the map* A *is not invertible.*

Definition 8.1.1 of noncommutative determinants and traces was essentially used in [GKLLRT] for linear maps given by matrices $A = (a_{ij})$, $i, j = 1, \ldots, m$ and unit vectors e_s, $s = 1, \ldots, m$. In this case $\Lambda_i(e_s, A)$ are quasi-Plücker coordinates of the corresponding Krylov matrix $K_s(A)$. Here (see [G]) $K_s(A)$ is the matrix (b_{ij}), $i = m, m-1, \ldots, 1, 0$, $j = 1, \ldots, m$, where b_{ij} is the (sj)-entry of A^i.

Example. Let $A = (a_{ij})$ be an $m \times m$-matrix and $v = e_1 = (1, 0, \ldots, 0)^T$. Denote by $a_{ij}^{(k)}$ the corresponding entries of A^k. Then

$$\Lambda_m(v,A) = (-1)^{m-1} \begin{vmatrix} \boxed{a_{11}^{(m)}} & a_{12}^{(m)} & \cdots & a_{1m}^{(m)} \\ a_{11}^{(m-1)} & a_{12}^{(m-1)} & \cdots & a_{1m}^{(m-1)} \\ & \cdots & \cdots & \\ a_{11} & a_{12} & \cdots & a_{1m} \end{vmatrix}$$

For $m = 2$ the "noncommutative trace" Λ_1 equals $a_{11} + a_{12}a_{22}a_{12}^{-1}$ and the "noncommutative determinant" Λ_2 equals $a_{12}a_{22}a_{12}^{-1}a_{11} - a_{12}a_{21}$.

It was shown in [GKLLRT] that if A is a quantum matrix, then Λ_m equals $\det_q A$ and A is a Capelli matrix, then Λ_m equals the Capelli determinant.

A construction of a noncommutative determinant and a noncommutative trace in terms of cyclic vectors in a special case was used in [Ki].

One can view the elements $\Lambda_i(v,A)$ as elementary symmetric functions of "eigenvalues" of A.

Following Section 6 we introduce complete symmetric functions $S_i(v,A)$, $i = 1,2,\dots,$ of "eigenvalues" of A as follows. Let t be a formal commutative variable. Set $\lambda(t) = 1 + \Lambda_1(v,A)t + \cdots + \Lambda_m(v,A)t^m$ and define the elements $S_i(v,A)$ by the formulas

$$\sigma(t) := 1 + \sum_{k>0} S_k t^k = \lambda(-t)^{-1}.$$

Recall that in Section 6 we introduced ribbon Schur functions and that $R_{1^k l}$ is the ribbon Schur function corresponding to the hook with k vertical and l horizontal boxes. In particular, $\Lambda_k = R_{1^k}$, $S_l = R_l$.

Let $A : R^m \to R^m$ be a linear map of right linear spaces.

Proposition 8.1.3. *For* $k \geqslant 0$:

$$A^{m+k}v = (-1)^{m-1} v R_{1^{m-1}(k+1)} + (-1)^{m-2}(Av(R_{1^{m-2}(k+1)}) + \cdots + (A^{m-1}v)R_{k+1}).$$

Let $A = \mathrm{diag}(x_1,\dots,x_m)$. In the general case for a cyclic vector one can take $v = (1,\dots,1)^T$. In this case, the following two results hold.

Proposition 8.1.4. *For* $k = 1,\dots,m$

$$\Lambda_k(v,A) = \begin{vmatrix} 1 & \cdots & \boxed{x_m^{m-k}} & \cdots & x_m^{m-1} \\ & & \cdots & & \\ 1 & \cdots & x_1 & \cdots & x_1^{m-1} \end{vmatrix}^{-1} \cdot \begin{vmatrix} 1 & \cdots & x_m^{m-k-1} & x_m^{m-k+1} & \cdots & \boxed{x_m^m} \\ & & \cdots & & & \\ 1 & \cdots & x_1^{m-k-1} & x_1^{m-k+1} & \cdots & x_1^m \end{vmatrix}.$$

Proposition 8.1.5. *For any* $k>0$

$$S_k(v,A) = \begin{vmatrix} 1 & \cdots & \boxed{x_m^{m-1}} \\ & \cdots & \\ 1 & \cdots & x_1^{m-1} \end{vmatrix}^{-1} \cdot \begin{vmatrix} 1 & \cdots & x_m^{m-2} & \boxed{x_m^{m+k-1}} \\ & & \cdots & \\ 1 & \cdots & x_1^{m-2} & x_1^{m+k-1} \end{vmatrix}.$$

Note that formulas for S_k look somewhat simpler than formulas for Λ_k.

8.2. Noncommutative determinants and noncommutative eigenvalues

One can also express $\Lambda_i(v, A) \in R$ in terms of left eigenvalues of A.

Let a linear map $A : R^m \to R^m$ of the right vector spaces is represented by the matrix (a_{ij}).

Definition 8.2.1. A nonzero row-vector $u = (u_1, \dots, u_m)$ is a left eigenvector of A if there exists $\lambda \in R$ such that $uA = \lambda u$.

We call λ a *left eigenvalue* of A corresponding to vector u. Note, that λ is the eigenvalue of A corresponding to a left eigenvector u then, for each $\alpha \in R$, $\alpha\lambda\alpha^{-1}$ is the eigenvalue corresponding to the left eigenvector αu. Indeed, $(\alpha u)A = \alpha\lambda\alpha^{-1}(\alpha u)$.

For a row vector $u = (u_1, \dots, u_m)$ and a column vector $v = (v_1, \dots, v_m)^T$ denote by $\langle u, v \rangle$ the inner product $\langle u, v \rangle = u_1 v_1 + \dots u_m v_m$.

Proposition 8.2.2. *Suppose that* $u = (u_1, \dots, u_m)$ *is a left eigenvector of* A *with the eigenvalue* λ, $v = (v_1, \dots, v_m)^T$ *is a cyclic vector of* A, *and* $\langle u, v \rangle = 1$. *Then the eigenvalue* λ *satisfies the equation*

$$(-1)^m \Lambda_m(v, A) + (-1)^{m-1} \lambda \Lambda_{m-1}(v, A) + \cdots - \lambda^{m-1} \Lambda_1(v, A) + \lambda^m = 0. \qquad (8.2.1)$$

Eq. (8.2.1) and the corresponding Viète theorem (see Section 3) show that if the map $A : R^m \to R^m$ has left eigenvectors $u^1, \dots, u^m$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_m$ such that $\langle u^i, v \rangle = 1$ for $i = 1, \dots, m$ and any submatrix of the Vandermonde matrix (λ_i^j) is invertible, then all $\Lambda_i(v, A)$ can be expressed in terms of $\lambda_1, \dots, \lambda_m$ as "noncommutative elementary symmetric functions" by formulas similar to those in Definition 6.5.1.

8.3. Multiplicativity of determinants

In the commutative case the multiplicativity of determinants and the additivity of traces are related to computations of determinants and traces with diagonal block-matrices. In the noncommutative case we suggest to consider the following construction.

Let R be an algebra with a unit. Let $A : R^m \to R^m$ and $D : R^n \to R^n$ be linear maps of right vector spaces, $v \in R^m$ an A-cyclic vector and $w \in R^n$ a D-cyclic vector.

There exist $\Lambda_i(w, D) \in R$, $i = 1, \dots, n$, such that

$$(-1)^n v\Lambda_n(w, D) + (-1)^{n-1}(Dw)\Lambda_{n-1}(w, D) + \cdots - (D^{m-1}v)\Lambda_1(w, D) + D^n w = 0.$$

Denote also by $S_i(w, D)$, $i = 1, 2, \dots$, the corresponding complete symmetric functions.

The matrix $C = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ acts on R^{m+n}. Suppose that the vector $u = \begin{pmatrix} v \\ w \end{pmatrix}$ is a cyclic vector for matrix C. We want to express $\Lambda_i(u, C)$, $i = 1, \ldots, m+n$ in terms of $\Lambda_j(v, A)$, $S_k(v, A)$, $\Lambda_p(w, D)$, and $S_q(w, D)$.

Denote, for brevity, $\Lambda_j(v, A) = \Lambda_j$, $S_k(v, A) = S_k$, $\Lambda_p(w, D) = \Lambda_p{}'$, $S_q(w, D) = S_q{}'$.

For two sets of variables $\alpha = \{a_1, a_2, \ldots,\}$ and $\beta = \{b_1, b_2, \ldots,\}$ introduce the following $(m+n) \times (m+n)$-matrix $M(m, n; \alpha, \beta)$:

$$\begin{pmatrix}
1 & a_1 & a_2 & \cdots & \cdots & \cdots & a_{m-1} & \cdots & a_{m+n-1} \\
0 & 1 & a_1 & a_2 & \cdots & \cdots & a_{m-1} & \cdots & a_{m+n-2} \\
 & & & & & \cdots & & & \\
0 & 0 & 0 & \cdots & \cdots & \cdots & 1 & \cdots & a_m \\
1 & b_1 & b_2 & \cdots & b_{n-1} & \cdots & \cdots & \cdots & b_{m+n-1} \\
0 & 1 & b_1 & b_2 & \cdots & \cdots & \cdots & \cdots & b_{m+n-2} \\
 & & & & & \cdots & & & \\
0 & 0 & 0 & \cdots & 1 & b_1 & \cdots & \cdots & b_n
\end{pmatrix}.$$

Proposition 8.3.1. *For any $j = 2, \ldots, m+n$ we have*

$$|M(m, n; \alpha, \beta)|_{1j} = -|M(m, n; \alpha, \beta)|_{m+1, j}.$$

The elements $S_i(u, C)$, $i = 1, 2, \ldots,$ can be computed as follows. Denote by $N_k(m, n; \alpha, \beta)$ the matrix obtained from M by replacing its last column by the following column:

$$(a_{m+n+k-1}, a_{m+n+k-2}, \ldots, a_{n+k-1}, b_{m+n+k-1}, b_{m+n+k-2}, \ldots, b_{m+k-1})^T.$$

Set $\alpha = \{-S_1, S_2, \ldots, (-1)^k S_k, \ldots\}$, $\alpha' = \{-S'_1, S'_2, \ldots, (-1)^k S'_k, \ldots\}$.

Theorem 8.3.2. *For $k = 1, 2, \ldots$ we have*

$$S_k(u, C) = |M(m, n; \alpha, \alpha')|^{-1}_{1m+n} \cdot |N_k(m, n; \alpha, \alpha')|_{1m+n}.$$

Example. For $m = 3$, $n = 2$ and $k = 1, 2, \ldots$. Then

$S_k(u, C) =$

$$(-1)^{k-1} \begin{vmatrix}
1 & -S_1 & S_2 & -S_3 & S_4 \\
0 & 1 & -S_1 & S_2 & -S_3 \\
0 & 0 & 1 & -S_1 & S_2 \\
1 & -S'_1 & S'_2 & -S_3{}' & S_4{}' \\
0 & 1 & -S'_1 & S'_2 & -S_3{}'
\end{vmatrix}^{-1}_{15}
\begin{vmatrix}
1 & -S_1 & S_2 & -S_3 & S_{4+k} \\
0 & 1 & -S_1 & S_2 & -S_{3+k} \\
0 & 0 & 1 & -S_1 & S_{2+k} \\
1 & -S'_1 & S'_2 & -S_3{}' & S'_{4+k} \\
0 & 1 & -S'_1 & S'_2 & -S'_{3+k}
\end{vmatrix}_{15}.$$

For $n = 1$ denote $\Lambda_1(D) = S_1(D)$ by λ'.

Proposition 8.3.3. *If $n = 1$, then for $k = 1, 2, \ldots$ we have*

$$S_k(u, C) = S_k(v, A) + S_{k-1}(v, A)|M(m, n; \alpha, \alpha')|^{-1}_{1m+n}\lambda'|M(m, n; \alpha, \alpha')|_{1m+n+1}.$$

Note that

$$\Lambda_{m+1}(u, C) = |M(m, n; \alpha, \alpha')|^{-1}_{1m+n}\lambda'|M(m, n; \alpha, \alpha')|_{1m+n+1}\Lambda_m(v, A),$$

i.e. the "determinant" of the diagonal matrix equals the product of two "determinants".

9. Some applications

In this section we mainly present some results from [GR1,GR2,GR4].

9.1. Continued fractions and almost triangular matrices

Consider an infinite matrix A over a skew-field:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n}\cdots \\ -1 & a_{22} & a_{23} & \cdots & a_{2n}\cdots \\ 0 & -1 & a_{33} & \cdots & a_{3n}\cdots \\ 0 & 0 & -1 & \cdots & \cdots \end{pmatrix}.$$

It was pointed out in [GR1,GR2] that the quasideterminant $|A|_{11}$ can be written as a generalized continued fraction

$$|A|_{11} = a_{11} + \sum_{j_1 \neq 1} a_{1j_1} \frac{1}{a_{2j_1} + \sum_{j_2 \neq 1, j_1} a_{2j_2} \frac{1}{a_{3j_2} + \cdots}}.$$

Let

$$A_n = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ -1 & a_{22} & \cdots & a_{2n} \\ 0 & -1 & \cdots & a_{3n} \\ & & \cdots & \\ \cdots & 0 & -1 & a_{nn} \end{pmatrix}.$$

The following proposition was formulated in [GR1,GR2].

Proposition 9.1.1. $|A_n|_{11} = P_n Q_n^{-1}$, *where*

$$P_n = \sum_{1 \leqslant j_1 < \cdots < j_k < n} a_{1j_1} a_{j_1+1,\,j_2} a_{j_2+1,\,j_3} \dots a_{j_k+1,n}, \tag{9.1.1}$$

$$Q_n = \sum_{2 \leqslant j_1 < \cdots < j_k < n} a_{2j_1} a_{j_1+1,\,j_2} a_{j_2+1,\,j_3} \cdots a_{j_k+1,n}. \tag{9.1.2}$$

Proof. From the homological relations one has

$$|A_n|_{11} |A_n^{1n}|_{21}^{-1} = -|A_n|_{1n} |A_n^{11}|_{2n}^{-1}.$$

We will apply formula (1.2.2) to compute $|A_n|_{1n}$, $|A_n^{11}|_{2n}$, and $|A_n^{1n}|_{21}$. It is easy to see that $|A_n^{1n}|_{21} = -1$. To compute the two other quasideterminants, we have to invert triangular matrices. Setting $P_n = |A_n|_{1n}$ and $Q_n = |A_n^{11}|_{2n}$ we arrive at formulas (9.1.1), (9.1.2). □

Remark. In the commutative case Proposition 9.1.1 is well known. In this case $P_n = |A_n|_{1n} = (-1)^n \det A_n$ and $Q_n = (-1)^{n-1} \det A_n^{11}$.

Formulas (9.1.1), (9.1.2) imply the following result (see [GR1,GR2]).

Corollary 9.1.2. *The polynomials P_k for $k \geqslant 0$ and Q_k for $k \geqslant 1$ are related by the formulas*

$$P_k = \sum_{s=0}^{k-1} P_s a_{s+1,k}, \quad P_0 = 1, \tag{9.1.3}$$

$$Q_k = \sum_{s=1}^{k-1} Q_s a_{s+1,k}, \quad Q_1 = 1. \tag{9.1.4}$$

Corollary 9.1.3. *Suppose that for any $i \neq j$ and any p, q the elements of the matrix A satisfy the conditions*

$$a_{ij} a_{pq} = a_{pq} a_{ij},$$

$$a_{jj} a_{ii} - a_{ii} a_{jj} = a_{ij}, \quad 1 \leqslant i < j \leqslant n.$$

Then

$$P_n = |A_n|_{1n} = a_{nn} a_{n-1n-1} \dots a_{11}. \tag{9.1.5}$$

The proof follows from (9.1.3).

Corollary 9.1.4 (Gelfand and Retakh [GR1,GR2]). *For the Jacoby matrix*

$$A = \begin{pmatrix} a_1 & 1 & 0 & \cdots \\ -1 & a_2 & 1 & \\ 0 & -1 & a_3 & \cdots \end{pmatrix}$$

we have

$$|A|_{11} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}},$$

and

$$P_0 = 1, \quad P_1 = a_1, \quad P_k = P_{k-1}a_k + P_{k-2}, \quad \textit{for } k \geqslant 2;$$

$$Q_1 = 1, \quad Q_2 = a_2, \quad Q_k = Q_{k+1}a_k + Q_{k-2}, \quad \textit{for } k \geqslant 3.$$

In this case P_k is a polynomial in $a_1, \dots, a_k$ and Q_k is a polynomial in $a_2, \dots, a_k$.

9.2. Continued fractions and formal series

In the notation of the previous subsection the infinite continued fraction $|A|_{11}$ may be written as a ratio of formal series in the letters a_{ij} and a_{ii}^{-1}. Namely, set

$$\begin{aligned} P_\infty &= \sum_{\substack{1 \leqslant j_1 < j_2 \cdots < j_k < r-1 \\ r = 1,2,3,\dots}} a_{1j_1} a_{j_1+1 j_2} \dots a_{j_k+1 r} a_{rr}^{-1} \cdot \dots \cdot a_{11}^{-1} \\ &= 1 + a_{12}a_{22}^{-1}a_{11}^{-1} + a_{13}a_{33}^{-1}a_{22}^{-1}a_{11}^{-1} + a_{11}a_{23}a_{33}^{-1}a_{22}^{-1}a_{11}^{-1} + \cdots, \end{aligned}$$

and

$$\begin{aligned} Q_\infty &= a_{11}^{-1} + \sum_{\substack{2 \leqslant j_1 < j_2 \cdots < j_k < r-1 \\ r = 2,3 \dots}} a_{2j_1} a_{j_1+1 j_2} \dots a_{j_k+1 r} a_{rr}^{-1} \cdot \dots \cdot a_{11}^{-1} \\ &= a_{11}^{-1} + a_{23}a_{33}^{-1}a_{22}^{-1}a_{11}^{-1} + a_{24}a_{44}^{-1}a_{33}^{-1}a_{22}^{-1}a_{11}^{-1} + \cdots. \end{aligned}$$

Since each monomial appears in these sums at most once, these are well-defined formal series.

The following theorem was proved in [PPR]. Another proof was given in [GR4].

Theorem 9.2.1. *We have*

$$|A|_{11} = P_\infty \cdot Q_\infty^{-1}.$$

Proof. Set $b_{ij} = a_{ij}a_{jj}^{-1}$ and consider matrix $B = (b_{ij}),\ i,j = 1,2,3,\ldots$. According to a property of quasideterminants $|A|_{11} = |B|_{11}a_{11}$. Applying the noncommutative Sylvester theorem to B with matrix $(b_{ij}),\ i,j \geqslant 3$, as the pivot, we have

$$|B|_{11} = 1 + |B^{21}|_{12}|B^{11}|_{22}^{-1}a_{11}^{-1}.$$

Therefore

$$|A|_{11} = (a_{11}|B^{11}|_{22}a_{11}^{-1} + |B^{21}|_{12}a_{11}^{-1})(|B^{11}|_{22}a_{11}^{-1})^{-1}. \tag{9.2.1}$$

By [GKLLRT], Proposition 2.4, the first factor in (9.2.1) equals P_∞, and the second equals Q_∞^{-1}. □

9.3. Noncommutative Rogers–Ramanujan continued fraction

The following application of Theorem 9.2.1 to Rogers–Ramanujan continued fraction was given in [PPR]. Consider a continued fraction with two formal variables x and y:

$$A(x,y) = \cfrac{1}{1 + x\cfrac{1}{1+x\frac{1}{1+\cdots}y}y}.$$

It is easy to see that

$$A(x,y) = \begin{vmatrix} 1 & x & & \cdot & & & 0 \\ -y & 1 & x & & \cdot & & \\ & -y & 1 & x & & \cdot & \\ & & & 1 & & & \cdot \\ & 0 & & \ddots & \ddots & & \cdot \end{vmatrix}_{11}^{-1} = \begin{vmatrix} 1 & x & 0 & & \\ -1 & y^{-1} & xy^{-1} & & \\ 0 & -1 & y^{-1} & xy^{-1} & \\ & & -1 & y^{-1} & \ddots \end{vmatrix}_{11}$$

Theorem 9.2.1 implies the following result.

Corollary 9.3.1. *$A(x,y) = P \cdot Q^{-1}$, where $Q = yPy^{-1}$ and*

$$P = 1 + \sum_{\substack{k \geqslant 1 \\ n_1,\ldots,n_k \geqslant 1}} y^{-n_1}xy^{-n_2}x\ldots y^{-n_k}xy^{k+n_1+n_2+\cdots+n_k}.$$

Following [PPR], let us assume that $xy = qyx$, where q commutes with x and y. Set $z = yx$. Then Corollary 9.3.1 implies Rogers–Ramanujan continued fraction identity

$$A(x,y) = \frac{1}{1+\frac{qz}{1+\frac{q^2z}{1+\cdots}}} = \frac{1+\sum_{k\geqslant 1}\frac{q^{k(k+1)}}{(1-q)\ldots(1-q^k)}z^k}{1+\sum_{k\geqslant 1}\frac{q^{k^2}}{(1-q)\ldots(1-q^k)}z^k}.$$

9.4. Quasideterminants and characteristic functions of graphs

Let $A=(a_{ij})$, $i,j=1,\ldots,n$, where a_{ij} are formal noncommuting variables. Fix $p,q\in\{1,\ldots,n\}$ and a set $J\subset\{1,\ldots,\hat{p},\ldots,n\}\times\{1,\ldots,\hat{q},\ldots,n\}$ such that $|J|=n-1$ and both projections of J onto $\{1,\ldots,\hat{p},\ldots,n\}$ and $\{1,\ldots,\hat{q},\ldots,n\}$ are surjective. Introduce new variables b_{kl}, $k,l=1,\ldots,n$, by the formulas $b_{kl}=a_{kl}$ for $(l,k)\notin J$, $b_{kl}=a_{lk}^{-1}$ for $(l,k)\in J$. Let F_J be a ring of formal series in variables b_{kl}.

Proposition 9.4.1. *The quasideterminant $|A|_{ij}$ is defined in the ring F_J and is given by the formula*

$$|A|_{ij} = b_{ij} - \sum(-1)^s b_{ii_1}b_{i_1i_2}\ldots b_{i_sj}. \tag{9.4.1}$$

The sum is taken over all sequences $i_1,\ldots,i_s$ such that $i_k\neq i,j$ for $k=1,\ldots,s$.

Proposition 9.4.2. *The inverse to $|A|_{ij}$ is also defined in the ring F_J and is given by the following formula*:

$$|A|_{ij} = b_{ij} - \sum(-1)^s b_{ii_1}b_{i_1i_2}\ldots b_{i_sj}. \tag{9.4.2}$$

The sum is taken over all sequences $i_1,\ldots,i_s$.

All relations between quasideterminants, including the Sylvester identity, can be deduced from formulas (9.4.1) and (9.4.2).

Formulas (9.4.1) and (9.4.2) can be interpreted in terms graph theory. Let Γ_n be a complete oriented graph with vertices $1,\ldots,n$ and edges e_{kl}, where $k,l=1,\ldots,n$. Introduce a bijective correspondence between edges of the graph and elements b_{kl} such that $e_{kl}\mapsto b_{kl}$.

Then there exist a bijective correspondence between the monomials $b_{ii_1}b_{i_1i_2}\ldots b_{i_sj}$ and the paths from the vertex i to the vertex j.

9.5. Factorizations of differential operators and noncommutative variation of constants

Let R be an algebra with a derivation $D: R\to R$. Denote Dg by g' and D^kg by $g^{(k)}$. Let $P(D)=D^n+a_1D^{n-1}+\cdots+a_n$ be a differential operator acting on R and ϕ_i, $i=1,\ldots,n$, be solutions of the homogeneous equation $P(D)\phi=0$, i.e., $P(D)\phi_i=0$ for all i.

For $k = 1, \dots, n$ consider the Wronski matrix

$$W_k = \begin{pmatrix} \phi_1^{(k-1)} & \cdots & \phi_k^{(k-1)} \\ & \cdots & \\ \phi_1 & \cdots & \phi_k \end{pmatrix}$$

and suppose that any square submatrix of W_n is invertible.

Set $w_k = |W|_{1k}$ and $b_k = w_k' w_k^{-1}$, $k = 1, \dots, n$.

Theorem 9.5.1 (Etingof et al. [EGR]).

$$P(D) = (D - b_n)(D - b_{n-1}) \dots (D - b_1).$$

Corollary 9.5.2. *Operator $P(D)$ can be factorized as*

$$P(D) = (w_n \cdot D \cdot w_n^{-1})(w_{n-1} \cdot D \cdot w_{n-1}^{-1}) \dots (w_1 \cdot D \cdot w_1^{-1}).$$

One can also construct solutions of the nonhomogeneous equation $P(D)\psi = f$, $f \in R$, starting with solutions $\phi_1, \dots, \phi_n$ of the homogeneous equation. Suppose that any square submatrix of W_n is invertible and that there exist elements $u_j \in R$, $j = 1, \dots, n$, such that

$$u_j' = |W|_{1j}^{-1} f. \tag{9.5.1}$$

Theorem 9.5.3. *The element $\psi = \sum_{j=1}^{j=n} \phi_j u_j$ satisfies the equation*

$$(D^n + a_1 D^{n-1} + \cdots + a_n)\psi = f.$$

In the case where R is the algebra of complex valued functions $g(x)$, $x \in \mathbb{R}$ the solution ψ of the nonhomogeneous equation is given by the classical formula

$$\psi(x) = \sum_{j=1}^{j=n} \phi_j \int \frac{\det W_j}{\det W} dx,$$

where matrix W_j is obtained from the Wronski matrix W by replacing the entries in the jth column of W by $f, 0, \dots, 0$. It is easy to see that formula (9.5.1) and Theorem 9.5.3 imply formula (9.5.2).

9.6. Iterated Darboux transformations

Let R be a differential algebra with a derivation $D: R \to R$ and $\phi \in R$ be an invertible element. Recall that we denote $D(g) = g'$ and $D^k(g) = g^{(k)}$. In particular $D^{(0)}(g) = g$.

For $f \in R$ define $\mathcal{D}(\phi;f) = f' - \phi'\phi^{-1}f$. Following [Mat] we call $\mathcal{D}(\phi;f)$ the *Darboux transformation* of f defined by ϕ. This definition was known for matrix functions $f(x)$ and $D = \partial_x$. Note that

$$\mathcal{D}(\phi;f) = \begin{vmatrix} \boxed{f'} & \phi' \\ f & \phi \end{vmatrix}.$$

Let $\phi_1, \dots, \phi_k$. Define the *iterated* Darboux transformation $\mathcal{D}(\phi_k, \dots \phi_1; f)$ by induction as follows. For $k = 1$, it coincides with the Darboux transformation defined above. Assume that $k > 1$. The expression $\mathcal{D}(\phi_k, \dots, \phi_1; f)$ is defined if $\mathcal{D}(\phi_k, \dots, \phi_2; f)$ is defined and invertible and $\mathcal{D}(\phi_k; f)$ is defined. In this case,

$$\mathcal{D}(\phi_k, \dots \phi_1; f) = \mathcal{D}(\mathcal{D}(\phi_k, \dots \phi_2; f); \mathcal{D}(\phi_1; f)).$$

Theorem 9.6.1. *If all square submatrices of matrix* $(\phi_i^{(j)})$, $i = 1, \dots, k$; $j = k - 1, \dots, 0$ *are invertible, then*

$$\mathcal{D}(\phi_k, \dots, \phi_1; f) = \begin{vmatrix} \boxed{f^{(k)}} & \phi_1^{(k)} & \cdots & \phi_k^{(k)} \\ \cdots & \cdots & \cdots & \cdots \\ f & \phi_1 & \cdots & \phi_k \end{vmatrix}.$$

The proof follows from the noncommutative Sylvester theorem (Theorem 1.5.2).

Corollary 9.6.2. *The iterated Darboux transformation* $\mathcal{D}(\phi_k, \dots \phi_1; f)$ *is symmetric in* $\phi_1, \dots, \phi_k$.

The proof follows from the symmetricity of quasideterminants.

Corollary 9.6.3 (Matveev [Mat]). *In commutative case, the iterated Darboux transformation is a ratio of two Wronskians,*

$$\mathcal{D}(\phi_k, \dots, \phi_1; f) = \frac{W(\phi_1, \dots \phi_k, f)}{W(\phi_1, \dots, \phi_k)}.$$

9.7. Noncommutative Sylvester–Toda lattices

Let R be a division ring with a derivation $D : R \to R$. Let $\phi \in R$ and the quasideterminants

$$T_n(\phi) = \begin{vmatrix} \phi & D\phi & \cdots & D^{n-1}\phi \\ D\phi & D^2\phi & \cdots & D^n\phi \\ \cdots & \cdots & \cdots & \cdots \\ D^{n-1}\phi & D^n\phi & \cdots & \boxed{D^{2n-2}\phi} \end{vmatrix} \tag{9.7.1}$$

are defined and invertible. Set $\phi_1 = \phi$ and $\phi_n = T_n(\phi)$, $n = 2, 3, \dots$.

Theorem 9.7.1. *Elements ϕ_n, $n = 1, 2, \dots$, satisfy the following system of equations*:

$$D((D\phi_1)\phi_1^{-1}) = \phi_2\phi_1^{-1},$$

$$D((D\phi_n)\phi_n^{-1}) = \phi_{n+1}\phi_n^{-1} - \phi_n\phi_{n-1}^{-1}, \quad n \geqslant 2.$$

If R is commutative, the determinants of matrices used in formulas (9.7.1) satisfy a nonlinear system of differential equations. In the modern literature this system is called the Toda lattice (see, for example, [Ok] but in fact it was discovered by Sylvester in 1862 [Syl]) and, probably, should be called the Sylvester–Toda lattice. Our system can be viewed as a noncommutative generalization of the Sylvester–Toda lattice. Theorem 9.7.1 appeared in [GR1,GR2] and was generalized in [RS] and [EGR].

The following theorem is a noncommutative analog of the famous Hirota identities.

Theorem 9.7.2. *For $n \geqslant 2$*

$$T_{n+1}(\phi) = T_n(D^2\phi) - T_n(D\phi) \cdot (T_{n-1}(D^2\phi)^{-1} - T_n(\phi)^{-1})^{-1} \cdot T_n(D\phi).$$

The proof follows from the Theorem 1.5.2.

9.8. Noncommutative orthogonal polynomials

The results described in this subsection were obtained in [GKLLRT]. Let $S_0, S_1, S_2, \dots$ be elements of a skew-field R and x be a commutative variable. Define a sequence of elements $P_i(x) \in R[x]$, $i = 0, 1, \ldots$, by setting $P_0 = S_0$ and

$$P_n(x) = \begin{vmatrix} S_n & \cdots & S_{2n-1} & \boxed{x^n} \\ S_{n-1} & \cdots & S_{2n-2} & x^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ S_0 & \cdots & S_{n-1} & 1 \end{vmatrix} \tag{9.8.1}$$

for $n \geqslant 1$. We suppose here that quasideterminants in (9.8.1) are defined. Proposition 1.5.1 implies that $P_n(x)$ is a polynomial of degree n. If R is commutative, then P_n, $n \geqslant 0$, are orthogonal polynomials defined by the moments S_n, $n \geqslant 0$. We are going to show that if R is a free division ring generated by S_n, $n \geqslant 0$, then polynomials P_n are indeed orthogonal with regard a natural noncommutative R-valued product on $R[x]$.

Let R be a free skew-field generated by c_n, $n \geqslant 0$. Define on R a natural anti-involution $a \mapsto a^*$ by setting $c_n^* = c_n$ for all n. Extend the involution to $R[x]$ by setting $(\sum a_i x^i)^* = \sum a_i x^i$. Define the R-valued inner product on $R[x]$ by setting

$$\left\langle \sum a_i x^i, \sum b_j x^j \right\rangle = \sum a_i c_{i+j} b_j^*.$$

Theorem 9.8.1. *For $n \neq m$ we have*

$$\langle P_n(x), P_m(x) \rangle = 0.$$

The three term relation for noncommutative orthogonal polynomials $P_n(x)$ can be expressed in terms of noncommutative quasi-Schur functions $\check{S}_{i_1,\ldots,i_N}$ defined in 6.4. We will use a notation $\check{S}_{i^{N-1}j}$ if $i_1 = \cdots = i_{N-1}$ and $i_N = j$ and write $\check{S}_{i^N}$ if $i_1 = \cdots = i_N$.

Theorem 9.8.2. *The noncommutative orthogonal polynomials $P_n(x)$ satisfy the three term recurrence relation*

$$P_{n+1}(x) - (x - \check{S}^*_{n^n(n+1)} \check{S}^{-1}_{n^{n+1}} + \check{S}^*_{(n-1)^{n-1}n} \check{S}^{-1}_{(n-1)^n}) P_n(x) + \check{S}^*_{n^{n+1}} \check{S}^{-1}_{(n-1)^n} P_{n-1}(x) = 0$$

for $n \geqslant 1$.

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