Calculus in positive characteristic \( p \)

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Abstract

We revisit hyperderivatives to build on the integral theory of calculus in positive characteristic \( p \). In particular, we give necessary and sufficient conditions for the exactness of a hyperdifferential form associated with hyperderivatives and then propose a closed formula for finding the hyperantiderivative of an exact form.

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1. Introduction

Calculus in positive characteristic \( p \) faces many obstacles. First, having an identically vanishing derivative does not guarantee that the function is a constant. Second, the \( p \)-th derivative of a function will always vanish identically. To compensate for the latter, Hasse first defined the sequence of hyperderivatives \( D_n f(x) \) for a power series \( f(x) \) over a field of arbitrary characteristic as follows:

\[
f(x + y) = \sum_{n=0}^{\infty} D_n f(x) y^n.
\]

Thus, Hasse shifted the focus from the usual \( n \)-th derivatives to the operators \( D_n \), which are indeed nontrivial even over a field of positive characteristic \( p \). Among others, hyperderivatives satisfy many formal properties that may be obtained from the classical theory as well as special properties.

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arising in finite characteristic. Since the pioneering work of Hasse [15], Schmidt [16], and Teichmüller [25], Dieudonné [7–10] carried out work on differential calculus in positive characteristic $p$. Let $R$ be the integer ring of a local field of characteristic $p$. Anderson and Goss [12] showed that the algebra of $R$-valued measures on $R$ is isomorphic to the algebra of divided power series over $R$, hence that of hyperderivatives on $R$. Brownawell [3] and Denis [4] in turn investigated the linear independence of values of the hyperderivatives of exponential and quasi-periodic functions related to a Drinfeld module. Recently, Kochubei [22] gave a thorough presentation of analysis over a local field of positive characteristic. In addition, Bosser and Pellarin [1,2,23] made heavy use of hyperderivatives to investigate differential properties of Drinfeld quasi-modular forms.

The purpose of the paper is to develop the integral theory of calculus in positive characteristic $p$; which has been extensively illustrated in a recent work by the author [18], and also motivated by the works of Dieudonné. To this end, in Section 2 we organize all of the properties on hyperderivatives, which are generally scattered in the literature, and provide simple proofs for differentiation rules. Section 3 recalls two known formulas for hyperderivatives and shows that they are essentially equivalent. Section 4 addresses higher logarithm derivatives, which are the analogue in positive characteristic $p$ for the classical logarithm derivative. In Section 5 we introduce the hyperantiderivative of an exact hyperdifferential form that is associated with hyperderivatives $D_{pi}$ and then provide necessary and sufficient conditions for a hyperdifferential form to be exact. We also propose a formula for identifying the hyperantiderivative of an exact form.

2. Hyperderivatives revisited

We deal extensively with hyperderivatives throughout the paper, and their well-known properties are reviewed in the two subsections below.

2.1. Definitions and properties

The basics of hyperderivatives are summarized following [10,5].

Let $k$ be a field of arbitrary characteristic and $k[x]$ be the ring of formal power series in one variable $x$ over $k$. As a natural substitute for the usual $n$-th derivative divided by $n!$ Hasse defined the hyperderivative $D_n$ on $k[x]$ given by Eq. (1).

One of basic properties that hyperderivatives satisfy is the Leibniz product formula:

$$D_n(fg) = \sum_{i+j=n} D_i(f)D_j(g), \quad (2)$$

which follows immediately from the multiplication of Eq. (1) for $f$ and $g$. Another property, which we call the composition rule, is

$$D_mD_n = \binom{m+n}{n}D_{m+n} = D_nD_m, \quad (3)$$

which follows by expressing $f(x + y + z)$ in two ways, using (1).

We now address hyperderivatives $D_n$ in a more algebraic way. The Leibniz formula (2) gives the $k$-algebra homomorphism on $k[x]$: $D : k[x] \to k[x][y]$, defined by $f \mapsto \sum_{n \geq 0} D_n(f)y^n$. \quad (4)

Then we see that a sequence of $D_n$ can be viewed as coefficient functions of the image of $D$. By comparing Eqs. (1) and (4), we have

$$f(x + y) = D(f). \quad (5)$$
We can also give a direct, alternate approach to the definition of hyperderivatives $D_n$ on the polynomial ring $k[x]$. For an integer $n \geq 0$, the hyperderivative $D_n = D_n,x$ of order $n$, defined by $D_n(x^m) = \binom{m}{n} x^{m-n}$ for $m \geq 0$, extends to the polynomial ring $k[x]$ by $k$-linearity. It is then well-known that a sequence of $k$-linear operators $D_n$ extends uniquely to $k$-linear operators on the extensions of $k[x]$ as specified in the following.

**Theorem 2.1.** Let $D_n$ be a sequence of $k$-linear operators on $k[x]$ defined above. Then

1. The maps $D_n$ extend uniquely to $k$-linear operators on the quotient field $k(x)$ of $k[x]$.
2. The maps $D_n$ extend uniquely to $k$-linear operators on any separable algebraic extension over $k(x)$.
3. The maps $D_n$ extend continuously to $k$-linear operators on the completion of $k(x)$ at any place.

**Proof.** See [5].

Thanks to Theorem 2.1(1) and (3) a sequence of $D_n$ extends continuously to $k((x))$, the field of formal Laurent series in the variable $x$ over a field $k$, so that we have the $k$-algebra homomorphism $\mathcal{D}$ on $k((x))$ extending Eq. (4). From this homomorphism we deduce directly that $D_n$ on $k((x))$ are given by

$$D_n \left( \sum_{m=m_0 \in \mathbb{Z}}^{\infty} c_m x^m \right) = \sum_{m=m_0 \in \mathbb{Z}}^{\infty} c_m \binom{m}{n} x^{m-n}. \quad (6)$$

Conversely, it is easily shown that the maps defined in Eq. (6) are $k$-linear continuous operators with respect to the usual absolute value on $k((x))$. By the Leibniz rule over $k((x))$ we have the $k$-algebra homomorphism $\mathcal{D} : k((x)) \to k((x))[y]$.

### 2.2. Differentiation rules

This subsection organizes various differentiation rules, which are scattered in the literature (hence complementing the formulas in [19]), and provides simple proofs of these rules. To do so, we make a heavy use of two expansion formulas for any powers of a formal power series with no constant term:

\[(I) \quad \left( \sum_{i=1}^{\infty} a_i y^i \right)^m = \sum_{n=m}^{\infty} \left( \sum_{i_1, \ldots, i_m \geq 1, i_1 + \cdots + i_m = n} a_{i_1} \cdots a_{i_m} \right) y^n,
\]

\[(II) \quad \left( \sum_{i=1}^{\infty} a_i y^i \right)^m = \sum_{n=m}^{\infty} \left( \sum_{i_1, \ldots, i_n \geq 0, i_1 + \cdots + i_n = m, i_1 + 2i_2 + \cdots + ni_n = n} \binom{m}{i_1, i_2, \ldots, i_n} a_{i_1}^1 \cdots a_{i_n}^n \right) y^n.
\]

Expansion (II) is derived from the repeated use of the binomial expansion formula (see [15]).

Let $f, f_1, \ldots, f_m$ be any functions in $k((x))$. The Leibniz rule extends to more than two factors, so we have the general product formula by equating the term $y^n$ in the expansion of two formal series:

$$\mathcal{D}(f_1 \cdots f_m) = \mathcal{D}(f_1) \cdots \mathcal{D}(f_m).$$

**General product rule.** For $n \geq 1$ and $m \geq 2$,

$$D_n(f_1 \cdots f_m) = \sum_{i_1, \ldots, i_n \geq 0, i_1 + \cdots + i_m = n} D_{i_1}(f_1) \cdots D_{i_m}(f_m).$$
As a special case of the general product rule, we have two types of power rules. To see this, we have

$$D(f^m) = (D(f))^m = (f + (D(f) - f))^m = \sum_{j=0}^{m} \binom{m}{j} f^{m-j} (D(f) - f)^j,$$

(7)

where $D(f) - f$ is a formal power series in $y$ without a constant term. Then Power rule I follows by applying Expansion (I) to Eq. (7).

**Power rule I.** For $n \geq 1$ and $m \geq 2$,

$$D_n(f^m) = \sum_{j=1}^{m} \binom{m}{j} f^{m-j} \sum_{\substack{i_1, \ldots, i_n \geq 1 \atop i_1 + \cdots + i_n = n}} (D_{i_1}(f)) \cdots (D_{i_n}(f)).$$

This formula can also be derived from the general product rule by collecting together all tuples $(i_1, \ldots, i_k)$ having the same number $j$ of positive coordinates (see [5]).

By applying Expansion (II) to Eq. (7) we have Power rule II below, which was first observed by Teichmüller [25].

**Power rule II.** For $n \geq 1$ and $m \geq 2$,

$$D_n(f^m) = \sum_{j=1}^{m} f^{m-j} \sum_{\substack{i_1, \ldots, i_n \geq 1 \atop i_1 + \cdots + i_n = n}} \frac{m(m-1) \cdots (m-j+1)}{i_1! \cdots i_n!} (D_{i_1}(f))^{i_1} \cdots (D_{i_n}(f))^{i_n}.$$

We now introduce power rules that work only in characteristic $p > 0$. To do that we recall the Lucas congruence theorem, which will be used throughout the paper, particularly in Section 5.

**Lucas’s congruence theorem.** Let $n$ and $m$ be positive integers so that $n = \sum_{i=0}^{r} a_i p^i$ with $0 \leq a_i < p$, and $m = \sum_{i=0}^{\ell} b_i p^i$ with $0 \leq b_i < p$. Then

$$\binom{n}{m} \equiv \binom{a_0}{b_0} \binom{a_1}{b_1} \cdots \binom{a_r}{b_r} \pmod{p}.$$

As a special case of Lucas’s congruence theorem we have the following congruence

$$\binom{p^m}{i} \equiv \binom{i}{j} \pmod{p}.$$

(8)

The $p$-th Power rule I follows from the first equality in Eq. (7) by equating the term $y^n$ with $n$ non-zero multiples of a $p$-th power. For a direct, alternate proof we refer to [3, Section 7, Lemma (a)].

**$p$-th Power rule I.** Let $p > 0$ be the characteristic of $k$. We have, for $n \geq 1$ and $f \in k((x))$,

$$D_n(f^{p^m}) = \begin{cases} (D_{j}(f))^{p^n} & \text{if } n = j p^m, \\ 0 & \text{if } n \not\equiv 0 \pmod{p^m}. \end{cases}$$

The $p$-th Power rule II in [17] follows from Eqs. (6) and (8).
**p-th Power rule II.** Let $p > 0$ be the characteristic of $k$ and $u$ be another variable in $k((x))$. We have, for $n \geq 1$ and $f \in k((x))$,

$$D_{n,u} f(u^p) = \begin{cases} (D_{j,x} f)(x^{p^m}) & \text{if } n = jp^m, \\ 0 & \text{if } n \not\equiv 0 \pmod{p^m} \end{cases}$$

where $D_{n,u}$ is the hyperderivative of order $n$ with respect to $u$ in $k((u))$ and $D_{j,x}$ is the hyperderivative for the variable $x$.

We now turn to the quotient rules. The Quotient rules I and II can be derived from the $k$-algebra homomorphism $D$: For $f \neq 0 \in k((x))$,

$$1 = D(f) D(1/f).$$

Hence we yield

$$D(1/f) = \frac{1}{D(f)} = \frac{1}{f + (D(f) - f) f} = \frac{1}{f} \sum_{n=0}^{\infty} r^n$$

where $r = -(D(f)/f - 1)$, which is a formal power series in $y$ with no constant term. Quotient rules I and II follow respectively by applying Expansions (I) and (II) to Eq. (9).

**Quotient rule I.** For $n \geq 1$ and $0 \neq f \in k((x))$,

$$D_n \left( \frac{1}{f} \right) = \sum_{j=1}^{n} (-1)^j \sum_{\begin{subarray}{c} i_1, \ldots, i_j \geq 1 \\ i_1 + \cdots + i_j = n \end{subarray}} D_{i_1}(f) \cdots D_{i_j}(f).$$

**Quotient rule II.** For $n \geq 1$ and $0 \neq f \in k((x))$,

$$D_n \left( \frac{1}{f} \right) = \sum_{j=1}^{n} (-1)^j \sum_{\begin{subarray}{c} i_1, \ldots, i_n \geq 0 \\ i_1 + \cdots + i_n = j \\ i_1 + 2i_2 + \cdots + ni_n = n \end{subarray}} \left( \begin{array}{c} j \\ i_1, i_2, \ldots, i_n \end{array} \right) (D_1(f))^{i_1} \cdots (D_n(f))^{i_n}.$$

Quotient rule I can be also proved by induction on $n$ as Göttfert [13, Theorem 1] did in the completion of $F(x)$ at the infinite prime.

There is another less familiar quotient rule (see [14, Eq. (10), p. 47]).

**Quotient rule III.** For $n \geq 1$ and $0 \neq f \in k((x))$,

$$D_n \left( \frac{1}{f} \right) = \sum_{j=1}^{n} \frac{n+1}{j+1} \frac{(-1)^j}{f^{j+1}} D_n (f^j)$$

$$= \sum_{j=1}^{n} \frac{n+1}{j+1} \frac{(-1)^j}{f^{j+1}} \sum_{\begin{subarray}{c} i_1, \ldots, i_j \geq 0 \\ i_1 + \cdots + i_j = n \end{subarray}} D_{i_1}(f) \cdots D_{i_j}(f).$$
The first equality in the statement follows from Quotient rule I, together with Power rule I and the binomial coefficient identity: for all integers \( m > 0, \)
\[
\sum_{j=m}^{n} (-1)^j \binom{n+1}{j+1} \binom{j}{m} = (-1)^m,
\]
which is a special case of the well-known binomial identity in [21, Exercise 63, p. 74]. The second follows from the general product formula.

As for the chain rule, let \( u = f(x) \) be a function in \( k((x)) \) as another variable and denote by \( D_{n,u}(g) \) the \( n \)-th hyperderivative of a function \( g(u) \) in \( F((u)) \) with respect to \( u \). From Eq. (5), we then have
\[
g \circ f(x + y) = D(g \circ f) = \sum_{n \geq 0} D_{n,x}(g \circ f)y^n.
\]
Using Eq. (5), we also obtain
\[
g \circ f(x + y) = g(D(f)) = g(f + v) \quad \text{with } v = D(f) - f = \sum_{n \geq 0} D_{n,u}(g(u))v^n \in k((x))[[y]].
\]
Thus we have
\[
\sum_{n \geq 0} D_{n,x}(g \circ f)y^n = \sum_{n \geq 0} D_{n,u}(g(u))(D(f) - f)^n
\]
(10)

Now Chain rules I and II follow immediately from applying two expansions, (I) and (II) to the right-hand side of Eq. (10).

**Chain rule I.** For \( n \geq 1 \) and \( g \circ f(x) \in k((x)), \)
\[
D_{n,x}(g \circ f(x)) = \sum_{j=1}^{n} D_{j,u}(g(u)) \sum_{i_1 + \cdots + i_j = n} \prod_{i=1}^{j} D_{i_1,x}(f) \cdots D_{i_j,x}(f).
\]

**Chain rule II.** For \( n \geq 1 \) and \( g \circ f(x) \in k((x)), \)
\[
D_{n,x}(g \circ f(x)) = \sum_{j=1}^{n} D_{j,u}(g(u)) \sum_{i_1 + \cdots + i_j = n} \binom{j}{i_1, i_2, \ldots, i_n} (D_{1,x}(f))^{i_1} \cdots (D_{n,x}(f))^{i_n}.
\]

Chain rules I and II in characteristic \( p > 0 \) were respectively first observed by Teichmüller [25] and Hasse [15]. On the other hand, Chain rule II for the ordinary higher derivatives was observed by L.F.A. Arbogast around 1800. Since then, this rule has been known as Faà di Bruno’s formula in the literature. Interested readers can consult [6] or [20] for the curious history on Chain rule II as well as for different derivations of this rule.

Finally, we present another chain rule which was written in terms of the ordinary higher derivatives in [24]. Interested readers are encouraged to derive this lesser-known formula. Indeed, by using Power rule I and some well-known binomial identities, we can expand the right-hand side of Chain rule III, resulting in Chain rule I.
Chain rule III. For \( n \geq 1 \) and \( g \circ f(x) \in k((x)) \),

\[
D_{n,x}(g \circ f(x)) = \sum_{j=1}^{n} (-1)^{j} D_{j,u}(g(u)) \sum_{k=1}^{j} (-1)^{k} D_{k,u}(u^{k}) D_{n,x}(u^{k}),
\]

where \( u = f(x) \).

### 3. Two equivalent formulas for hyperderivatives

From this section onward we work over a field \( k \) of positive characteristic \( p \). For simplicity, we put

\[
D_{m} = D_{p^{m}} (m \geq 0),
\]

which will be extensively used in the rest of the paper. We first recall Dieudonné's formula [8], which plays a role in building on the integral theory for \( D_{m} \) in Section 5.

**Theorem 3.1.** For an integer \( n \geq 0 \) write \( n \) in the \( p \)-adic form as \( n = \sum_{i=0}^{r} a_{i} p^{i} \) with \( 0 \leq a_{i} < p \), and \( a_{r} \neq 0 \). Then

\[
D_{n} = \frac{1}{a_{0}! \cdots a_{r}!} D_{0}^{a_{0}} \cdots D_{r}^{a_{r}},
\]

where the product on the right is a composite, not a product of operators.

**Proof.** See [8]. Also it follows from the composition rule (3) and Lucas's congruence theorem. \( \Box \)

This theorem says that the hyperderivatives \( D_{n} \), \( n \geq 0 \) are completely determined by the operators \( D_{m} \) for \( m \geq 0 \). In parallel with Theorem 3.1 there is another formula by Conrad [5].

**Theorem 3.2.** Let \( n \) be the same as in Theorem 3.1. Then

\[
D_{n} = \frac{1}{n!} \left( p! \right)^{a_{r}} D_{0}^{a_{0}} \cdots D_{r}^{a_{r}}.
\]

**Proof.** It follows from the composition rule (3). \( \Box \)

We now show that the two formulas are equivalent by the following.

**Lemma 3.3.** For an integer \( n \geq 0 \) write \( n \) in the \( p \)-adic form as \( n = \sum_{i=0}^{r} a_{i} p^{i} \) with \( 0 \leq a_{i} < p \), and \( a_{r} \neq 0 \). Then we have the following congruence:

\[
\frac{n!}{p! a_{1}! \cdots p^{r} a_{r}!} = a_{0}! \cdots a_{r}! \pmod{p}.
\]

**Proof.** We invoke the well-known formula:

\[
v_{p}(n!) = \frac{n - s(n)}{p - 1},
\]

where \( v_{p} \) is a \( p \)-adic valuation on \( \mathbb{Z} \) and \( s(n) = a_{0} + a_{1} + \cdots + a_{r} \). By this formula, we have

\[
v_{p} \left( p! a_{1}! \cdots p^{r} a_{r}! \right) = \frac{n - s(n)}{p - 1} = v_{p} \left( \prod_{i=0}^{r} a_{i} p^{i}! \right).
\]
By the formula in Eq. (8), we see, for each \(i\),
\[
p! a_i! \equiv p! \left( \frac{2p^i}{p^i} \right) \cdots \left( \frac{a_i p^i}{(a_i - 1) p^i} \right) = a_i p^i! \pmod{p}.
\]
By Eq. (13), we see that the congruence in the statement is equivalent to
\[
n! \equiv \prod_{i=0}^{r} a_i p^i! \pmod{p^{e+1}},
\]
where \(e = v_p(n!)\). By Eq. (13) again, the preceding congruence is also equivalent to
\[
n! a_0! a_1 p! \cdots a_r p^r! \equiv 1 \pmod{p},
\]
which follows by a direct way or the Lucas’s congruence formula for multinomials.

In calculus there is a well-known result which says that a function \(f\) is identically constant if and only if \(f' = 0\). We now give the characteristic \(p\) analogue of this result.

**Proposition 3.4.** A function \(f\) is identically constant in \(k((x))\) if and only if \(D_m f\) is identically zero for all \(m = 0, \ldots\).

**Proof.** The result follows at once from Dieudonné’s Theorem 3.1. Or see [11, Prop. 1] for an alternate, direct proof. \(\square\)

**4. Higher logarithm derivatives**

It is well-known in calculus that two functions having the same logarithmic derivatives are equal up to the multiplication of a constant; indeed, for differentiable functions \(f\) and \(g\),
\[
\frac{f'}{f} = \frac{g'}{g} \implies f(x) = \lambda g(x),
\]
where \(\lambda\) is a non-zero constant. In this section, we give the analogue in positive characteristic \(p\) for this classical result. For a fixed integer \(m > 0\), let \(k((x^{pm}))\) be the field of formal Laurent series in the variable \(x^{pm}\) over \(k\). Then it is shown in [7] that \(D_m\) is a semi-derivation of height \(m\) in \(k((x))\), which means that its restriction to the subfield \(k((x^{pm}))\) is a derivation. More generally, for \(f \in k((x^{pm}))\) and \(g \in k((x))\), we have the following identity in [7]:
\[
D_m (fg) = (D_m f) g + f (D_m g).
\]

**Theorem 4.1.** Let \(f(x)\) be a non-zero function in \(k((x))\). Then higher logarithm derivatives \(D_m f / f\) \((m = 0, 1, \ldots)\) determine \(f\) up to the multiplication of a constant.

**Proof.** It is straightforward to see that the aforementioned statement can be reformulated into the following:
\[
\frac{D_m f}{f} = \frac{D_m g}{g}\quad \text{for all } m = 0, 1, \ldots\quad \text{imply that}\quad f(x) = \lambda g(x),
\]
where \(\lambda\) is a non-zero constant in \(k\). We here use induction on \(m\) to show that \(f(x) = g(x) h_m(x^{pm+1})\) for \(h_m(x) \in k((x))\) for all \(m\). The relation \(\frac{D_0 f}{f} = \frac{D_0 g}{g}\) implies \(D_0 (f / g) = 0\), so that \(f / g = h_0(x^p)\) for
some $h_0(x) \in k((x))$. Assuming that $f(x) = g(x)h_m(x^{p^{m+1}})$, we use Eq. (14) to obtain
\[
\frac{D_{m+1}f}{f} = \frac{D_{m+1}g}{g} + \frac{D_{m+1}h_m(x^{p^{m+1}})}{h_m(x^{p^{m+1}})}.
\]
By the hypothesis, we see $D_{m+1}h_m(x^{p^{m+1}})$ is identically zero. On the other hand, the $p$-th Power rule I applied to $h_m(x^{p^{m+1}})$ gives
\[
D_{m+1}(h_m(x^{p^{m+1}})) = (D_0h_m)(x^{p^{m+1}}).
\]
Hence $D_0h_m(x) = 0$, thus $h_m(x) = h_{m+1}(x^p)$ for some $h_{m+1} \in k((x))$. This completes the proof for the case $m + 1$. Finally, the result follows from Proposition 3.4 as $f/g$ belongs to the kernel of $D_m$ for all $m = 0, 1, \ldots$

Theorem 4.1 assists in the determination whether there is a relation between the usual logarithm derivative and higher logarithm derivatives. This question is addressed by the following formula which we call the reciprocity formula.

**Theorem 4.2.** Let $m$ be any nonnegative integer, $f$ be a non-zero function in $k((x))$, and put $h = 1/f^{p^{m-1}}$. Then
\[
\frac{D_m f}{f} = \left(\frac{D_0f}{f}\right)^{p^m} + \frac{D_m h}{h}.
\]

**Proof.** We begin by invoking the Quotient rule II with $n = p^m$:
\[
D_{p^m}\left(\frac{1}{f}\right) = \sum_{j=1}^{p^m} \binom{p^m+1}{j+1} (-1)^j f^{j+1} D_{p^m}(f^j).
\]
By Lucas’ congruence theorem, we see that the nonvanishing terms in the preceding equation are those $j$’s that satisfy
\[
\binom{p^m+1}{j+1} \equiv 0 \pmod{p} \iff j = p^m - 1 \text{ or } p^m.
\]
Hence we have
\[
D_m\left(\frac{1}{f}\right) = \frac{1}{f^{p^m}} D_m(f^{p^m-1}) - \frac{1}{f^{p^m+1}} D_m(f^{p^m}).
\]
By the $p$-th Power rule I, rewrite it as
\[
D_m\left(\frac{1}{f}\right) = \frac{1}{f^{p^m}} D_m(f^{p^m-1}) - \frac{1}{f^{p^m+1}} \left(\frac{D_0f}{f}\right)^{p^m}.
\]
Replace $f$ with $1/f$, then a simple identity $\frac{D_0}{f} = -\frac{D_0f}{f}$ gives
\[
D_m(f) = f^{p^m} D_m\left(\frac{1}{f^{p^{m-1}}}\right) + f\left(\frac{D_0f}{f}\right)^{p^m}.
\]
Finally, on dividing through by $f$, we have the desired result. ☐

We remark that Theorem 4.2 also follows by applying Dieudonné’s formula (14) to $f$, along with the $p$-th Power rule I.
5. Hyperantiderivatives

Recall that \( k \) is a field of characteristic \( p > 0 \), that \( k[J^x] \) is the formal power series ring over \( k \) and that \( D_j = D_{p^j} \) as is defined in Eq. (11). For a sequence of power series \( \{f_j\}_{j \geq 0} \) in \( k[x] \) we say that the hyperdifferential form \( \omega \)

\[
\omega := \sum_{j=0}^{\infty} f_j(x) d_jx
\]  

is exact if and only if there exists a function \( F(x) \) in \( k[J^x] \) such that

\[
D_j F = f_j \quad \text{for all} \quad j = 0, \ldots
\]  

Then such a function \( F \) is unique up to a constant by Proposition 3.4. In this case, we call \( F \) the hyperantiderivative of the hyperdifferential form \( \omega \). For simplicity, the hyperdifferential form (or hyperantiderivative) will subsequently be referred to as the differential form (or antiderivative). In this section, we answer two crucial questions regarding the antiderivative of \( \omega \);

**Question 1.** What are the necessary and sufficient conditions for \( \omega \) to have an antiderivative?

**Question 2.** Can we give a formula for finding an antiderivative of the exact form \( \omega \)?

To this end, we introduce some necessary terminologies. We say that a power series \( f(x) \) in \( k[J^x] \) to have an antiderivative \( F(x) \) of level \( j \) if \( D_j F(x) = f(x) \) for some \( F(x) \in k[x] \). Then it is easily seen by Lemma 5.1 below that such a function \( F \) is not uniquely determined. We also define the coefficient support of a power series \( f(x) = \sum_{i \geq 0} c_i x^i \), denoted \( C_f \), by putting

\[
C_f = \{ i \mid c_i \neq 0 \}.
\]  

And, for each integer \( j \geq 0 \), put

\[
S_j = \left\{ i \mid \left( \frac{i}{p-1} p^j \right) \equiv 0 \pmod{p} \right\},
\]  

which is closely related with the coefficient support of a power series having an antiderivative of level \( j \).

First, we need a lemma regarding equivalent conditions for a power series \( f(x) \) in \( k[x] \) to have an antiderivative of level \( j \).

**Lemma 5.1.** The following are equivalent; For a power series \( f = \sum_{i \geq 0} c_i x^i \),

1. \( f \) has an antiderivative \( F \) of level \( j \), that is, \( D_j F = f \).
2. \( D_j^{p^j-1} f(x) = 0 \).
3. \( C_f \subset S_j \) for some \( j \).

In one (hence all) case, \( f \) has an antiderivative \( F(x) \) of level \( j \), given by

\[
F(x) = \sum_{i \in C_f} \frac{c_i}{i+p^j} x^{i+p^j} + G_j(x),
\]  

for any power series \( G_j(x) \) such that \( D_j G_j = 0 \).
Lemma 5.2. Let \( m \) and \( n \) be two distinct nonnegative integers and let \( f \) be a function in \( k[x] \) such that \( D_m D_n(f) = 0 \). Then there exist \( g(x) \) with \( D_m g = 0 \) and \( h(x) \) with \( D_n h = 0 \) in \( k[x] \) such that \( f(x) = g(x) + h(x) \).

Proof. We use \( D_m D_n = D_{p^m + p^n} \) (by Eq. (3)) to compute, for \( f(x) = \sum_{i=0}^{\infty} c_i x^i \),

\[
D_{p^m + p^n}(f(x)) = \sum_{i \geq p^m + p^n} c_i \left( \frac{i}{p^m + p^n} \right) x^{i-p^m-p^n} = 0.
\]

From this we see that if \( c_i \) is non-zero for each \( i \geq p^m + p^n \), then \( \left( \frac{i}{p^m + p^n} \right) \) is zero modulo \( p \), implying by Lucas’s theorem that \( d_m(i) = 0 \) or \( d_n(i) = 0 \), where \( d_m(i) \) denotes the \( m \)-th digit in the \( p \)-adic expansion of \( i \). We now regroup \( f \) by degrees as

\[
f(x) = \sum_{i, d_m(i) = 0} c_i x^i + \sum_{i, d_m(i) \neq 0} c_i x^i = \sum_{i, d_m(i) = 0} c_i x^i + \sum_{i, d_m(i) \neq 0, d_n(i) = 0} c_i x^i + \sum_{i, d_m(i) \neq 0, d_n(i) \neq 0} c_i x^i.
\]

Since the last sum is identically zero, we see that the two remaining sums are respectively killed by \( D_m \) and \( D_n \).

Now we give necessary and sufficient conditions for \( \omega \) to be exact, which answers Question 1.
Proof. (⇒): Part (1) follows immediately from Lemma 5.1(1), and part (2) follows from the composition rule (3).

(⇐): From assumption (1), we see by Lemma 5.1 that for all \( j \) there exists an antiderivative \( F_j \) of level \( j \) such that \( D_j F_j = f_j \). Then we need to show that \( F_i + G_i = F_j + G_j \) for all \( i \) and \( j \) by taking an appropriate function \( G_j \) with \( D_j G_j = 0 \) for each \( j \). From assumption (2), we have \( D_j D_i F_i = D_i D_j F_j \) for all \( i \) and \( j \). Hence \( f_{i,j} = F_i - F_j \) is killed by \( D_i D_j = D_j D_i \). Using Lemma 5.2 we decompose \( f_{i,j}(x) \) as \( f_{i,j}(x) = g_i(x) + g_j(x) \) for some functions \( g_i \) with \( D_i(g_i) = 0 \) where \( i = i, j \). Thus, we have \( F_i - g_i = F_j + g_j \) for all \( i \) and \( j \). The proof is complete by taking \( G_i = -g_i \) and \( G_j = g_j \).

We are now in a position of deriving a formula for the antiderivative \( F(x) \) of the exact form \( \omega \). Before proceeding further we fix some additional notations. For each integer \( j \geq 0 \), we put

\[
K_j = \{ g \in k[[x]] \mid D_ig = 0 \text{ for all } i = 0, \ldots, j \} \tag{19}
\]

and

\[
K_\infty = \{ g \in k[x] \mid D_ig = 0 \text{ for all } i = 0, \ldots, j \}.
\]

By Proposition 3.4 we know \( K_\infty = k \) and then by the \( p \)-th Power rule that \( f(x) \) is in \( K_j \) if and only if \( f(x) = g(x^{p^{j+1}}) \) for some \( g(x) \in k[x] \).

We begin with the exact form \( \omega = \sum_{i=0}^{\infty} f_j(x) \, d_j x \) where \( f_j = \sum_{i \geq 0} c_{j,i} x^i \) with its coefficient support \( C_j \) as in Eq. (17). By Lemma 5.1, write the antiderivative \( F \) as

\[
F = F_j + G_j \quad (j \geq 0),
\]

where

\[
F_j(x) = \sum_{i \in C_j} \frac{c_{j,i}}{i^{i+p^j}} x^{i+p^j} \tag{21}
\]

and \( G_j(x) \) belongs to the kernel of \( D_j \). For a fixed integer \( r \) with \( 0 \leq r \leq p^{j+1} - 1 \), put

\[
F_{j,r(p^{j+1})} = \sum_{i \in C_j \text{ (mod } p^{j+1})} \frac{c_{j,i}}{(i+p^j)} x^{i+p^j}. \tag{22}
\]

Then \( F_{j,r(p^{j+1})} \) is nothing but a power series obtained from \( F_j \) by collecting all the terms of degrees \( i \) satisfying \( i \equiv r \pmod{p^{j+1}} \). The relation \( C_j \subset S_j \) (Lemma 5.1(3)) enables us to regroup \( F_j \) by degrees. Thus we have the decomposition of \( F_j \) in terms of \( F_{j,r(p^{j+1})} \) in Eq. (22):

\[
F_j = \sum_{r=p^j}^{p^{j+1}-1} F_{j,r(p^{j+1})} \quad (j \geq 0). \tag{23}
\]
In particular, we have the trivial decomposition of $F_0$: For each integer $j > 0$,

$$F_0 = \sum_{i_0=1}^{p-1} \sum_{i_1=0}^{p-1} \cdots \sum_{i_j=0}^{p-1} F_{0,i_0+i_1 p+\ldots+i_j p} (p^{j+1}).$$ \hspace{1cm} (24)

As a solution to Question 2 we have then the following result.

**Theorem 5.4.** Let notations be the same as above. If a differential form $\omega = \sum_{j=0}^{\infty} f_j(x) d_j x$ is exact, then the antiderivative $F(x)$ of $\omega$ is given by

$$F = F_0 + \sum_{j=1}^{\infty} \sum_{r=1}^{p-1} F_{j,r p^j (p^{j+1})} + u,$$

for some constant $u$ in a field $k$.

**Proof.** By Eq. (20) we have the equation:

$$F_0 + G_0 = F = F_j + G_j \quad (j > 0).$$ \hspace{1cm} (25)

Take a sequence of functions $\{G_{0,j}\}_{j \geq 1}$ with $D_0 G_{0,j} = 0$, recursively defined by

$$G_0 = \sum_{i=1}^{j} \sum_{r=1}^{p-1} F_{i,r p^i (p^{i+1})} + G_{0,j}.$$ \hspace{1cm} (26)

Then we claim that for each $j$ $G_{0,j}$ belongs to $K_j$ in Eq. (19), by which the result follows from Eqs. (25) and (26) since $G_{0,j}$ converges to some constant $u$ in $k$ as $j$ goes to $\infty$. It suffices now to show the claim by induction on $j$. To do so we need to take another sequence of functions $\{G_{j,1}\}_{j \geq 1}$ with $D_j G_{j,1} = 0$, given by

$$G_j = \sum_{i_0=1}^{p-1} \sum_{i_1=0}^{p-1} \cdots \sum_{i_{j-1}=0}^{p-1} F_{0,i_0+i_1 p+\ldots+i_{j-1} p^{j-1} (p^{j+1})} + \sum_{r=1}^{p-1} F_{j-1,r p^{j-1} (p^{j+1})} + G_{j,1}.$$ \hspace{1cm} (27)

We first plug Eqs. (24), (26), (23) and (27) with $j = 1$ into Eq. (25). Then we have

$$\sum_{i_0=1}^{p-1} \sum_{i_1=0}^{p-1} F_{0,i_0+i_1 p} (p^2) + \sum_{r=1}^{p-1} F_{1,r p^2} (p^2) + G_{0,1} = \sum_{r=p}^{p^2-1} F_{1,r p^2} + \sum_{r=1}^{p-1} F_{0,r p^2} + G_{1,1}.$$ \hspace{1cm} (28)

Canceling out Eq. (28) gives

$$\sum_{i_0=1}^{p-1} \sum_{i_1=1}^{p-1} F_{0,i_0+i_1 p} (p^2) + G_{0,1} = \sum_{r=p, p|r}^{p^2-1} F_{1,r p^2} + G_{1,1}.$$ \hspace{1cm} (29)

Since $G_{0,1}$ is a function of $x^p$ we take $D_0 D_1$ into Eq. (29) to deduce that $G_{0,1} = G_{1,1}$. Hence $G_{0,1}$ belongs to $K_1$.

Assume that $G_{0,j}$ lies in $K_j$ for $j > 1$. Then by Eq. (26) it implies $G_{0,j+1}$ also belongs to $K_j$. As is in case $j = 1$ we also plug Eqs. (24), (26), (23) and (27) with all $j$ replaced by $j + 1$ into Eq. (25). Then we have
Example 1. In characteristic 2, we want to find an antiderivative of the exact form
\[ \omega = f_0 d_0 x + f_1 d_1 x + f_2 d_2 x \]
where
\[ f_0 = 1 + x^2 + x^4 + x^6 + x^{10} + x^{12} + x^{14}, \]
\[ f_1 = 1 + x + x^4 + x^5 + x^8 + x^9 + x^{12} + x^{13}, \]
\[ f_2 = 1 + x + x^2 + x^3 + x^8 + x^9 + x^{10} + x^{11}. \]

If we use Eq. (21) to find antiderivatives \( F_j \) of level \( j \) where \( j = 0, 1, 2 \) we have
\[ F_0 = x + x^3 + x^5 + x^7 + x^{11} + x^{13} + x^{15}, \]
\[ F_1 = x^2 + x^3 + x^6 + x^7 + x^{10} + x^{11} + x^{14} + x^{15}, \]
\[ F_2 = x^4 + x^5 + x^6 + x^7 + x^{12} + x^{13} + x^{14} + x^{15}. \]

From the two polynomials \( F_1 \) and \( F_2 \), we find
\[ F_{1,2(4)} = x^2 + x^6 + x^{10} + x^{14} \quad \text{and} \quad F_{2,4(8)} = x^4 + x^{12}. \]
Thus, by Theorem 5.4, the antiderivative $F$ is of the form

$$F = (x + x^3 + x^5 + x^7 + x^{11} + x^{13} + x^{15}) + (x^2 + x^6 + x^{10} + x^{14}) + (x^4 + x^{12}) + u$$

for some $u \in k$.

**Example 2.** In characteristic 3, we want to find an antiderivative of the exact form $\omega = f_1 d_1 x + f_2 d_2 x$ where

$$f_1 = 1 + x^9 \quad \text{and} \quad f_2 = 1 + x^3 + 2x^9.$$

By the formula in Eq. (21), we find antiderivatives $F_j$ of level $j = 1, 2$:

$$F_1 = x^3 + x^{12},$$
$$F_2 = x^9 + x^{12} + x^{18},$$

from which we obtain that $F_{1,3(9)} = x^3 + x^{12}$, $F_{1,6(9)} = 0$, $F_{2,9(27)} = x^9$ and $F_{2,18(27)} = x^{18}$. By Theorem 5.4, the antiderivative $F$ is of the form

$$F = x^3 + x^{12} + x^9 + x^{18} + u$$

for some $u \in k$.

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**References**