JOURNAL OF ALGEBRA 114, 115–259 (1988)

Classification of the Restricted Simple Lie Algebras

RICHARD E. BLOCK

Department of Mathematics, University of California, Riverside, California 92521

AND

ROBERT LEE WILSON

Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903

Communicated by N. Jacobson

Received May 10, 1987

Contents. Introduction. 1. Preliminaries. 2. Simple Lie algebras of toral rank one. 3. Determination of rank one semisimples. 4. Determination of rank two semisimples. 5. Proper roots. 6. Distinguished maximal subalgebras. 7. Determination of G_0 if $\mathfrak{z}(G_0) \neq (0)$. 8. Determination of G_0 if $\mathfrak{z}(G_0) = (0)$. 9. Determination of all semisimples in which every two dimensional torus is maximal and standard. 10. All roots can be made proper. 11. Sections of a simple algebra. 12. Conclusion.

INTRODUCTION

This paper gives the proof of the following main theorem (Theorem 12.5.1): Let L be a finite-dimensional restricted simple Lie algebra over an algebraically closed field F of characteristic $p > 7$. Then L is of classical or Cartan type (and thus is completely classified). This result was conjectured by Kostrikin and Šafarevič [KS66] and was announced in [BW84].

Work on simple Lie algebras of characteristic p goes back to the pioneering work of Jacobson and Zassenhaus in the 1930s. The notion of restricted

0021-8693/88 $3.00

Lie algebra (also called Lie p-algebra) was introduced by Jacobson [Jac37] in 1937. We recall that ([Jac37]; cf. [Jac62, p. 187; BW82, Sect. 1.1]) a restricted Lie algebra over F is a Lie algebra L together with a map $x \mapsto x^p$ satisfying certain conditions, in particular,

$$(\operatorname{ad} x)^p = \operatorname{ad}(x^p) \qquad \text{for all} \quad x \in L.$$

(Indeed, if $(\operatorname{ad} x)^p$ is an inner derivation for every x in some basis for L then L can be given the structure of a restricted Lie algebra and if L is centerless there is a unique such structure.)

The study of restricted Lie algebras (and, in particular, the study of the classification problem for restricted simple Lie algebras) has proved fruitful for several reasons. First, Lie algebras which arise "in nature" are generally restricted, e.g., the derivation algebra $\operatorname{Der} A$ of any algebra A, the Lie algebra of an algebraic group, the primitive elements of an irreducible cocommutative Hopf algebra (infinitesimal formal group), and the Lie algebras corresponding to subfields of purely inseparable field extensions of exponent one. Second, all known nonrestricted simple Lie algebras are closely related to restricted simple Lie algebras; more explicitly, each of the restricted simple Lie algebras of Cartan type generalizes to a family of simple Lie algebras, and these families, together with the algebras of classical type, give all the known simple L. Third, certain technical tools are available in a restricted Lie algebra L which are not available in an arbitrary Lie algebra. (Here and throughout this paper, except where otherwise stated, we assume that all Lie algebras are finite-dimensional over a field F as above.) For example, an element $x \in L$ has a Jordan–Chevalley–Seligman decomposition into its semisimple and nilpotent parts [Sel67, Theorem V.7.2].

We remark that there is an important distinction between restricted simple Lie algebras (simple Lie algebras in which $(\operatorname{ad} x)^p$ is inner for every x) and simple restricted Lie algebras (nonabelian restricted Lie algebras which are simple as restricted Lie algebras, i.e., which have no nonzero proper ideals closed under the pth power map). Every restricted simple Lie algebra is a simple restricted Lie algebra, but a simple restricted Lie algebra need not be a simple Lie algebra. Indeed, if L is any simple Lie algebra and $\bar{L}$ denotes the restricted subalgebra of $\operatorname{Der} L$ generated by $\operatorname{ad} L$, then $[\bar{L}, \bar{L}] = L$ and $\bar{L}$ is a simple restricted Lie algebra. If U is any simple restricted Lie algebra then $[U, U]$ is simple and $U = \overline{[U, U]}$. Thus classification of the simple restricted Lie algebras is equivalent to classification of simple Lie algebras, a task which is beyond the scope of this paper.

We now briefly describe the Lie algebras arising in the statement of our main theorem. For each finite-dimensional Lie algebra A over the complex numbers $\mathbf{C}$, let $A_{\mathbf{Z}}$ be the $\mathbf{Z}$-span of a Chevalley basis of A, and extend scalars to F: $A_F = A_{\mathbb{Z}} \otimes F$. Dividing by the center $\mathfrak{z}(A_F)$ (which is nonzero when $A \cong \mathfrak{sl}(n)$, $p \mid n$), we get a Lie algebra which is simple and restricted. The simple Lie algebras obtained in this way are known as algebras of classical type. (At prime characteristic, one includes the five exceptional types among the classical-type algebras.) They thus correspond to the irreducible root systems A_n $(n \geqslant 1)$, $B_n (n \geqslant 3)$, C_n $(n \geqslant 2)$, D_n $(n \geqslant 4)$, G_2, F_4, E_6, E_7, E_8. The Lie algebras of classical type have been characterized axiomatically by Mills and Seligman [MS57]. They are the simple algebras which can be classified by extending the methods of Killing and Cartan, and include all simple Lie algebras with nondegenerate Killing form. (More generally, a simple L is of classical type if and only if it has a projective representation with nondegenerate trace form [Blo62, Kap71].)

Aside from the classical algebras, four classes of restricted simple Lie algebras are known, all generalizing the original p-dimensional example W_1 of Witt (see [Zas39], [Cha41]), and designated by W_m (or $W(m:\mathbf{1})$) $(m \geqslant 1)$, $S_m^{(1)}$ (or $S(m:\mathbf{1})^{(1)}$) $(m \geqslant 3)$, $H_m^{(2)}$ (or $H(m:\mathbf{1})^{(2)}$) (m even, $\geqslant 2$), and $K_m^{(1)}$ (or $K(m:\mathbf{1})^{(1)}$) (m odd, $\geqslant 3$). These were discovered respectively by Jacobson [Jac43], Frank [Fra54], Albert and Frank [AF54] and Frank [Fra64].

Let B_m denote the commutative associative algebra of p-truncated polynomials, with generators $x_1, ..., x_m$ and relations $x_i^p = 0$ for $i = 1, ..., m$. Then $W_m = \operatorname{Der} B_m = \{\sum f_i D_i \mid f_i \in B_m\}$, where $D_i \in W_m$ is defined by $D_i x_j = \delta_{ij}$ for $i, j = 1, ..., m$ and S_m, H_m, K_m are certain subalgebras of W_m which may be described briefly as follows. (More details are given in Section 1.1). Consider the exterior algebra over B_m of differential forms in $dx_1, ..., dx_m$. Let

$$\omega_S = dx_1 \wedge \cdots \wedge dx_m,$$

$$\omega_H = \sum_{i=1}^{r} dx_i \wedge dx_{i+r}, \qquad m = 2r,$$

$$\omega_K = dx_{2r+1} + \sum_{i=1}^{r} (x_{i+r}\, dx_i - x_i\, dx_{i+r}), \qquad m = 2r+1.$$

Define

$$S_m = \{D \in W_m \mid D\omega_S = 0\},$$

$$H_m = \{D \in W_m \mid D\omega_H = 0\}, \qquad m = 2r,$$

$$K_m = \{D \in W_m \mid D\omega_K \in B_m \omega_K\}, \qquad m = 2r+1.$$

(Here the action of D on differential forms is extended from its action on B_m by requiring that D be a derivation of the exterior algebra satisfying $D(df) = d(Df)$, where $df = \sum (D_i f)\, dx_i$, $f \in B_m$.) Thus, for example, $S_m = \{\sum_{i=1}^m f_i D_i \mid \sum_{i=1}^m D_i f_i = 0\}$. For any Lie algebra A, let $A^{(m)}$ denote the mth derived algebra of A. Then $S_m^{(1)}$, $m \geqslant 3$, $H_m^{(2)}$, $m = 2r \geqslant 2$, and $K_m^{(1)}$, $m = 2r+1 \geqslant 3$, are restricted simple, with $\dim S_m^{(1)} = (m-1)(p^m - 1)$, $\dim H_m^{(2)} = p^m - 2$, and $\dim K_m^{(1)} = p^m$ if $p \nmid m+3$ and $\dim K_m^{(1)} = p^m - 1$ if $p \mid m+3$. The restricted simple Lie algebras just described, of types W, S, H, K, are called the restricted simple Lie algebras of *Cartan type*. The designation "Cartan type" refers to the fact that they are analogues over F, obtained by replacing power series by p-truncated polynomials, of the four classes of infinite simple Lie algebras over **C** used in the Cartan classification of Lie pseudogroups [Car09]. For a useful description of the Cartan-type algebras in terms of gradings and Cartan prolongations, see the brief accounts in [Blo82, BW86] or the fuller accounts in [KS69, Kac70, Kac74, Wil76].

We shall also need to give the notation for the known nonrestricted simple Lie algebras, since these will play a role in our proof even in the restricted case. Replacing B_m above by the completed algebra $\mathfrak{A}(m)$ of divided powers (in m generators) (see Section 1.1 for a more precise description), we obtain infinite-dimensional Lie algebras $W(m)$, $S(m)$, $H(m)$, $K(m)$. For each m-tuple $\mathbf{n} = (n_1, ..., n_m)$ of positive integers, we obtain a "$(p^{n_1}, ..., p^{n_m})$-truncated" Lie subalgebra $W(m : \mathbf{n})$ of $W(m)$ (of dimension $p^{|n|}$, where $|n| = n_1 + \cdots + n_m$). Intersecting $W(m : \mathbf{n})$ with $S(m)$, $H(m)$, $K(m)$, respectively, we obtain Lie algebras $S(m : \mathbf{n})$, $H(m : \mathbf{n})$, $K(m : \mathbf{n})$. (If $\mathbf{n} = \mathbf{1} = (1, ..., 1)$ these are the restricted algebras described above.) Finally, for an automorphism Φ of $W(m)$, and for $X = S$, H, or K, we write $X(m : \mathbf{n} : \Phi) = \Phi X(m) \cap W(m : \mathbf{n})$. Then under suitable conditions on Φ, the Lie algebras $S(m : \mathbf{n} : \Phi)^{(1)}$ $(m \geqslant 3)$, $H(m : \mathbf{n} : \Phi)^{(2)}$ $(m = 2r \geqslant 2)$, $K(m : \mathbf{n} : \Phi)^{(1)}$ $(m = 2r+1 \geqslant 3)$ are simple. These simple algebras, together with the algebras $W(m : \mathbf{n})$ (which are simple), are called the algebras of Cartan type (also known as algebras of generalized Cartan type). All known nonclassical simple Lie algebras over F (which have been constructed by a number of authors using a variety of techniques (see [Wil-pre] for references)) are isomorphic to algebras of Cartan type, and it is conjectured that every nonclassical simple Lie algebra over F is of Cartan type. We remark that $X(m : \mathbf{n} : \mathrm{id}) = X(m : \mathbf{n})$ (where id denotes the identity automorphism and $X = S$, H, or K).

We now describe the program begun by Kostrikin and Šafarevič [KS66, KS69] to classify the simple L, and then give an outline of our proof. We take the filtration of B_m obtained by giving each x_i degree one, and take the corresponding filtration of $W(m : \mathbf{1})$, with $\deg D_i = -1$ for all i. Intersecting this filtration with $X(m : \mathbf{1})^{(2)}$ $(X = S, H, K)$ except for type K

(where we give x_m degree 2 and D_m degree -2), we get a filtration of $X(m:\mathbf{1})^{(2)}$. All of these filtrations are Lie algebra filtrations

$$L = L_{-r} \supseteq L_{-r+1} \supseteq \cdots \supseteq L_0 \supseteq \cdots L_s \supseteq L_{s+1} = (0)$$

for some r, s, for which L_0 is a maximal subalgebra and for which the associated graded Lie algebra $G = \sum_{i=-r}^{s} G_i$ ($G_i = L_i/L_{i+1}$) satisfies (in addition to $[G_i, G_j] \subseteq G_{i+j}$ for all i, j):

(1) G_0 is a restricted Lie algebra and is a direct sum of restricted ideals each of which is classical simple, $\mathfrak{gl}(n)$, $\mathfrak{sl}(n)$, or $\mathfrak{pgl}(n)$ with $p \mid n$, or abelian.

(2) The action of G_0 on G_{-1} is restricted.

(3) If $i \leqslant 0$, $x \in G_i$, and $[x, G_1] = (0)$, then $x = 0$.

(4) If $i \geqslant 0$, $x \in G_i$, and $[x, G_{-1}] = (0)$, then $x = 0$.

(5) The action of G_0 on G_{-1} is irreducible.

(Actually, $G_0 = \mathfrak{gl}(m)$, $\mathfrak{sl}(m)$, $\mathfrak{sp}(2r)$, $\mathfrak{sp}(2r) \oplus F$ for $L = W(m:\mathbf{1})$, $S(m:\mathbf{1})^{(1)}$, $H(2r:\mathbf{1})^{(2)}$, $K(2r+1:\mathbf{1})^{(1)}$, respectively.) Filtrations for the nonrestricted simple Lie algebras of Cartan type for which the corresponding G has the above properties (1)–(5) are obtained similarly.

Now suppose L is any (finite-dimensional) simple Lie algebra over F and let L_0 be a maximal subalgebra and L_{-1} an ad L_0-invariant subspace containing L_0 such that L_{-1}/L_0 is ad L_0-irreducible. Define L_i inductively (following Cartan [Car09] and Weisfeiler [Wei68] by

$$L_{-i-1} = [L_{-i}, L_{-1}] + L_{-i}, \qquad i \geqslant 1,$$
$$L_{i+1} = \{x \in L_i \mid [x, L_{-1}] \subseteq L_i\}, \qquad i \geqslant 0.$$

Then we get a filtration of L, and the corresponding G satisfies properties (4) and (5) above. A major result on simple Lie algebras, the main stages of which were due to Kostrikin and Šafarevič [KS69], Kac [Kac70, Kac74], and Wilson [Wil76], is the Recognition Theorem for classical and Cartan-type algebras, which states that if L has a maximal subalgebra L_0 for which a corresponding graded G satisfies properties (1)–(3) above then L is of classical or Cartan type.

The problem then, for L restricted, is how to find a maximal subalgebra L_0 such that the pair (L, L_0) satisfies the hypotheses of the Recognition Theorem. We will actually give two ways of finding such an L_0. The first way will be used in certain low rank algebras; this classification in turn will enable us to find a suitable L_0 in the general rank case. To describe the approach briefly, we shall first need to mention some concepts which will be discussed more fully in the body of the paper. We recall that a *torus* in a restricted Lie algebra A is a subalgebra T, necessarily abelian, such that

$\mathrm{ad}_A\, t$ is semisimple for all $t \in T$; equivalently, T contains no nonzero nilpotent element t (where t is nilpotent if $t^{[p^e]} = 0$ for some e). The Cartan subalgebras of A are the centralizers $\mathfrak{z}_A(T)$ of the maximal tori T. We call a torus T *standard* if $[\mathfrak{z}_A(T), \mathfrak{z}_A(T)]$ is nil or, equivalently, if $\mathfrak{z}_A(T) = T + I$, where I is the largest nil ideal of $\mathfrak{z}_A(T)$ (the nil radical of $\mathfrak{z}_A(T)$). For a maximal torus T, we have the Cartan decomposition $A = \mathfrak{z}_A(T) + \sum_{\gamma \in \Gamma} A_\gamma$ of A with respect to T (or $\mathfrak{z}_A(T)$). If $\alpha \in \Gamma$ (respectively, α, $\beta \in \Gamma$ with α, β independent) we write $A^{(\alpha)} = \sum_{i \in \mathbf{Z}} A_{i\alpha}$ (respectively, $A^{(\alpha,\beta)} = \sum_{i,j \in \mathbf{Z}} A_{i\alpha + j\beta}$) and $A[\alpha] = A^{(\alpha)}/\mathrm{solv}\, A^{(\alpha)}$ (respectively, $A[\alpha, \beta] = A^{(\alpha,\beta)}/\mathrm{solv}\, A^{(\alpha,\beta)}$). The algebra $A[\alpha]$ (respectively, $A[\alpha, \beta]$) is called a *rank one section* (respectively, *rank two section*) of A (with respect to T).

Suppose that L is restricted simple. Then every torus in L and in every section of L is standard. Our proof involves determining the possible semisimple algebras which can occur as a rank one or rank two section of L. To do this we are led to determine the restricted semisimple A with the following property: every maximal torus of A is 1-dimensional (respectively, 2-dimensional) and standard. The proof of the case with a 2-dimensional maximal torus is particularly long, and parallels the proof in [BW82] but with the considerable added complication that here the nil radical I of $\mathfrak{z}_A(T)$ may be nonzero. Finally, with a list of the possible rank two sections in hand, we will construct the desired maximal subalgebra L_0 of L; this construction of L_0 extends that given for the case $I = (0)$ in [Wil83].

We now briefly indicate the contents of the twelve sections of the paper. Section 1 contains preliminaries, including details on Cartan-type algebras; conjugacy of tori in these algebras; the Recognition Theorem; lifting of maximal and standard tori; and Winter conjugates $e^x(T)$ of maximal tori T, including discussion of the relation between the root spaces for T and for $e^x(T)$. Section 2 gives some detailed results about certain (not necessarily restricted) simple Lie algebras S of toral rank 1, i.e., which have a Cartan decomposition with a root α such that all roots are in $\mathbf{Z}_p\alpha$. In particular, a list is given of those S which are such that $\bar{S}$ contains no torus of dimension $\geqslant 2$.

Section 3 determines the restricted semisimple A containing a 1-dimensional standard maximal torus but no tori of bigger dimension. These are of three types: $\mathfrak{sl}(2)$, $W(1:\mathbf{1})$, and certain algebras between $H(2:\mathbf{1})$ and $H(2:\mathbf{1})^{(2)}$. These are the possible rank one sections of a restricted simple L (relative to a torus of maximal dimension) which we will need to consider in the rest of the paper. Section 4 studies the analogous problem for the rank 2 sections. It examines the restricted semisimple A for which there is a 2-dimensional torus T, and all 2-dimensional tori are maximal and standard. A list of possible cases for such A is given, and in most cases A is closely related to certain specific small simple S, but in two cases only rather general structural properties are given; these are case (g), in which for some simple S we have $\bar{S} \subseteq A \subseteq \operatorname{Der} S$, $T \subseteq \bar{S}$, and $T \cap (S+I)$ is 1-dimensional and nonrestricted (where I is the nilradical of $\mathfrak{z}_A(T)$), and case (h), in which A is "nearly simple," i.e., for some simple S we have $\bar{S} \subseteq A \subseteq \operatorname{Der} S$ and $A = S + I$.

Section 5 examines restricted algebras A with a standard maximal torus T. For α a root of A with respect to T, the spaces $R_\alpha = \{x \in A_\alpha \mid [x, A_{-\alpha}] \subseteq I\}$ and $K_\alpha = \{x \in A_\alpha \mid \alpha([x, A_{-\alpha}]) = (0)\}$ are defined. When T is 2-dimensional, a tight bound on $\dim A_\alpha / R_\alpha$ is obtained. The important notion of the root α being *proper* is introduced. This says that $K_{i\alpha} = A_{i\alpha}$ for some $i \in \mathbf{Z}_p - \{0\}$. The distinction between proper and improper roots may be seen in the p-dimensional Witt algebra W_1. Here there are two conjugacy classes (under $\operatorname{Aut} W_1$) of maximal tori; one is spanned by $x_1 D_1$ and the other is spanned by $(x_1 + 1) D_1$. Any root with respect to the first of these tori is proper, and any root with respect to the second is improper. As our choice of name indicates, the first type of torus is better for our purposes (essentially because it is contained in the maximal subalgebra $(W_1)_0$). A related characterization of α being proper is given, namely, that α is proper if and only if T is contained in a compositionally classical subalgebra of codimension $\leqslant 2$ in $A^{(\alpha)}$ (or $A[\alpha]$), where compositionally classical means that all composition factors are abelian or simple of classical type.

Sections 6–9 again study restricted semisimple A which contain a 2-dimensional torus T and in which all 2-dimensional tori are maximal and standard. Section 6 examines the effect of switching tori by a Winter conjugacy and introduces the notion of optimal torus, this being a T with the maximal number of lines of proper roots, and of distinguished maximal subalgebra, this being a maximal subalgebra containing the centralizer of an optimal torus T and the spaces R_α for all roots with respect to T. Sections 7–9 examine (for A as in Section 6) the graded algebra G associated to a filtration coming from a distinguished maximal subalgebra. In Section 7 (the longest section in the paper), under the hypothesis that the center $\mathfrak{z}(G_0) \neq (0)$, and hence $\mathfrak{z}(G_0) = Fz$ for some z, it is proved that either $G_1 = (0)$ or $G_0 = \mathfrak{sl}(2) \oplus Fz$, and in the latter case that property (3) (of the hypotheses of the Recognition Theorem) above is satisfied. In Section 8, under the hypothesis that $\mathfrak{z}(G_0) = (0)$, it is proved that either (a) A is one of the algebras in case (h) (the "nearly simple" case) of Section 4 or (b) G_0 is $S_1 \oplus S_2$, where $S_i \cong \mathfrak{sl}(2)$ or $W(1:\mathbf{1})$; furthermore, if (b) holds and $G_1 \neq (0)$ then $G_0 \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ and property (3) above holds. Section 9 uses Sections 7 and 8 to complete the analysis of cases (g) and (h) of Section 4, thus giving in Theorem 9.1.1 a more explicit list of the restricted semisimple A with all tori of maximal dimension 2-dimensional and standard.

Section 10 shows that if L is restricted simple then it contains a torus T, of maximal dimension, such that all roots with respect to T are proper. This uses switching of tori by Winter conjugacy to send an improper root to a proper root while not simultaneously sending any proper root to an improper root. It suffices to prove the corresponding property in the rank two sections of L, these being among the semisimple algebras listed in Theorem 9.1.1. The desired property is shown to hold except for certain algebras in case (b) of that theorem, and a separate argument shows that those algebras cannot occur as a rank 2 section of L.

Section 11 shows that if L is restricted simple, T is a torus of maximal dimension, and all roots with respect to T are proper then more cases of the semisimple algebras listed in Theorem 9.1.1 cannot occur as a rank 2 section of L.

The concluding Section, Section 12, shows how, for L restricted simple but not classical, to construct a maximal subalgebra L_0 such that the hypotheses of the Recognition Theorem are satisfied by the pair (L, L_0). To construct L_0, we take a torus T, necessarily standard, of maximal dimension, with the property that all roots with respect to T are proper. For each root α, $L^{(\alpha)}$ has a unique compositionally classical subalgebra $U^{(\alpha)}$ of maximal dimension, and we define $Q(L) = \sum_\alpha U^{(\alpha)}$. Using the list of possible rank 2 sections from Sections 11, we prove that Q is a subalgebra which is large in a certain sense involving the number of roots β in a root string for which $L_\beta \neq Q_\beta$. It is shown that $Q \neq L$ (otherwise L would be classical). Then L_0 is taken to be a maximal subalgebra of L containing Q, and we prove that the hypotheses of the Recognition Theorem hold, giving our main result.

We are indebted to G. Seligman and H. Strade for many helpful comments on an earlier version of this paper.

1. Preliminaries

This section contains a number of definitions and results used throughout the paper. Most of the main results of this section are proved elsewhere and the proofs are not repeated here. Topics treated are: definition of algebras of Cartan type (Section 1.1); the Recognition Theorem for algebras of classical or Cartan type (Section 1.2); Demuškin's conjugacy theorems for toral elements and for maximal tori in the restricted Lie algebras of Cartan type (Section 1.3); the notations $A^{(X)}$, $A[X]$, and Ψ_X (Section 1.4); definition and properties of standard tori (Sections 1.5 and 1.6); properties of tori of maximal dimension (Section 1.7); properties of solvable algebras (Section 1.8); Winter's exponential map (Section 1.9); the Blattner–Dixmier theorem (Section 1.10); subalgebras of maximal dimension in $W(1:\mathbf{1})$ and $H(2:\mathbf{1})^{(2)}$ (Section 1.11); and Schue's lemma on Cartan decompositions of simple Lie algebras (Section 1.12).

Throughout the entire paper F will denote an algebraically closed field of characteristic $p>7$. We use the following notations: $\mathfrak{z}_A(B)$ denotes the centralizer of B in A, $\mathfrak{z}(A)$ denotes the center of A, $N_A(B)$ denotes the normalizer of B in A, $\mathbf{Z}$ denotes the ring of integers, $\mathbf{N}$ denotes the set of natural numbers, $\mathbf{Z}_p$ denotes $\mathbf{Z}/p\mathbf{Z}$, the prime field of F, and $\mathbf{Z}_p^*$ denotes the set of nonzero elements of $\mathbf{Z}_p$. Also, if B is a subalgebra of a restricted Lie algebra A, $\bar{B}$ denotes the restricted subalgebra of A generated by B.

1.1. We begin by describing the simple Lie algebras of Cartan type over F.

Let $A(m)$ denote the monoid (under addition) of all m-tuples of non-negative integers. For $1 \leqslant i \leqslant m$ let ε_i denote the m-tuple $(\delta_{i1}, ..., \delta_{im})$. For α, $\beta \in A(m)$ define $\binom{\alpha}{\beta} = \binom{\alpha(1)}{\beta(1)} \cdots \binom{\alpha(m)}{\beta(m)}$ and $\alpha! = \prod_{i=1}^{m} \alpha(i)!$. For $\mathbf{n} = (n_1, ..., n_m)$, an m-tuple of positive integers, let $A(m:\mathbf{n})$ denote $\{\alpha \in A(m) \mid \alpha(i) < p^{n_i}$ for $1 \leqslant i \leqslant m\}$ (where p is the characteristic of the field F under consideration).

We now define the completed free divided power algebra $\mathfrak{A}(m)$. Give the polynomial algebra $F[X_1, ..., X_m]$ its usual coalgebra structure with each X_i primitive. Then the dual space $\mathfrak{A}(m) = F[X_1, ..., X_m]^*$ is an infinite-dimensional commutative associative algebra consisting of all formal sums $\sum a_\alpha x^\alpha$, where α ranges over $A(m)$, $a_\alpha \in F$ and $x^\alpha(X^{\beta(1)} \cdots X^{\beta(m)}) = \delta_{\alpha\beta}$, so that multiplication is determined by

$$x^\alpha x^\beta = \binom{\alpha+\beta}{\alpha} x^{\alpha+\beta}.$$

For $\mathbf{n} = (n_1, ..., n_m)$ an m-tuple of positive integers, we let $\mathfrak{A}(m:\mathbf{n})$ denote the span of the x^α with $\alpha \in A(m:\mathbf{n})$. Then $\mathfrak{A}(m:\mathbf{n})$ is a subalgebra of $\mathfrak{A}(m)$. Write $\mathbf{1}$ for the m-tuple $(1, ..., 1)$. Note that $\mathfrak{A}(m:\mathbf{1}) \cong F[x_1, ..., x_m]/(x_1^p, ..., x_m^p)$. We will also denote this algebra by B_m.

For any m-tuple $\mathbf{n}$ of positive integers we may define a filtration of $\mathfrak{A}(m)$ by taking $\mathfrak{A}(m)_j$ to consist of all (possibly infinite) sums of elements $a_\alpha x^\alpha$ with $\alpha \in A(m)$ and $n_1\alpha(1) + \cdots + n_m\alpha(m) \geqslant j$. Furthermore, $\mathfrak{A}(m)$ has a topological (i.e., allowing infinite sums) grading $\mathfrak{A}(m) = \sum \mathfrak{A}(m)_{[j]}$, where $\mathfrak{A}(m)_{[j]}$ is the span of all x^α with $\alpha \in A(m)$ and $n_1\alpha(1) + \cdots + n_m\alpha(m) = j$. Such a filtration gives $\mathfrak{A}(m)$ the structure of a topological algebra. Note that any choice of the m-tuple $\mathbf{n}$ of positive integers gives the same topology and also gives the same subspace $\mathfrak{A}(m)_1$. A continuous linear transformation from $\mathfrak{A}(m)$ to $\mathfrak{A}(m)$ is determined by its effect on the x^α, $\alpha \in A(m)$.

It is easy to see (cf. Lemma 1 of [Wil71a]) that there is a unique sequence of continuous mappings $y \mapsto y^{(r)}$, $r \in \mathbf{N}$, of $\mathfrak{A}(m)_1$ into $\mathfrak{A}(m)$ satisfying

$$x^{(0)} = 1 \qquad \text{for all} \quad x \in \mathfrak{A}(m)_1;$$

$$(x^\alpha)^{(r)} = ((r\alpha)!/(\alpha!)^r\, r!)\, x^{r\alpha} \qquad \text{for all } 0 \neq \alpha \in A(m) \text{ and all } r \in \mathbf{N};$$

$$(ax)^{(r)} = a^r x^{(r)} \qquad \text{for all} \quad a \in F, x \in \mathfrak{A}(m)_1, r \in \mathbf{N};$$

$$(x+y)^{(r)} = \sum_{i=0}^{r} x^{(i)} y^{(r-i)} \qquad \text{for all} \quad x, y \in \mathfrak{A}(m)_1, r \in \mathbf{N}.$$

We call the mappings $y \mapsto y^{(r)}$ *divided power mappings*. This makes $\mathfrak{A}(m)$ a divided power algebra (cf. [Blo85]).

A continuous automorphism ϕ of the associative algebra $\mathfrak{A}(m)$ is said to be a *divided power automorphism* (also called an *admissible automorphism*) if it preserves the divided power structure, i.e., if

$$(\phi x)^{(r)} = \phi(x^{(r)}) \qquad \text{for all} \quad x \in \mathfrak{A}(m)_1, r \in \mathbf{N}.$$

For each i there is a continuous derivation D_i of $\mathfrak{A}(m)$ with

$$D_i(x^\alpha) = x^{\alpha - \varepsilon_i} \qquad \text{if} \quad \alpha \in A(m) \text{ and } \alpha(i) > 0$$

and

$$D_i(x^\alpha) = 0 \qquad \text{if} \quad \alpha \in A(m) \text{ and } \alpha(i) = 0.$$

We will frequently write x_i for x^{ε_i}. (Consequently $x^{j\varepsilon_i} = x_i^j/j!$ for $1 \leqslant j \leqslant p-1$.) The set $\{u_1 D_1 + \cdots + u_m D_m \mid u_i \in \mathfrak{A}(m)$ (respectively, $u_i \in \mathfrak{A}(m:\mathbf{n}))\}$ (where $(uD)v = u(Dv)$) is a subalgebra of Der $\mathfrak{A}(m)$, which is denoted by $W(m)$ (respectively, $W(m:\mathbf{n})$). The algebra $W(m:\mathbf{n})$ is simple and of dimension mp^n, where $n = n_1 + \cdots + n_m$. It is restricted if and only if $\mathbf{n} = \mathbf{1}$. $W(m)$ is filtered by

$$W(m)_j = \sum_{k=1}^{m} \mathfrak{A}(m)_{j+n_k} D_k$$

and has a topological grading $W(m) = \sum W(m)_{[j]}$, where

$$W(m)_{[j]} = \sum_{k=1}^{m} \mathfrak{A}(m)_{[j+n_k]} D_k.$$

Any subalgebra $M \subseteq W(m)$ is filtered by $M_j = W(m)_j \cap M$. Such a filtration gives $W(m)$ the structure of a topological algebra (and any m-tuple $\mathbf{n}$ of positive integers gives the same topology).

A continuous derivation D of $\mathfrak{A}(m)$ is said to be a *divided power derivation* (also called a *special derivation*) if

$$Dx^{(r)} = x^{(r-1)}\,Dx \qquad \text{for all} \quad x \in \mathfrak{A}(m)_1,\ r \in \mathbf{N}.$$

It is easily seen that $W(m)$ is the set of all divided power derivations of $\mathfrak{A}(m)$. Furthermore (cf. [Kac74, Sect. 5.2]), if Φ is a continuous automorphism of $W(m)$ then there is a divided power automophism ϕ of $\mathfrak{A}(m)$ such that

$$\Phi D = \phi\, D \phi^{-1} \qquad \text{for all} \quad D \in W(m).$$

We now define the algebra $\Omega(m)$ of differential forms on the divided power algebra $\mathfrak{A}(m)$, following [Kac74, Sect. 1.4]. Define

$$d\colon \mathfrak{A}(m) \to \operatorname{Hom}_{\mathfrak{A}(m)}(W(m), \mathfrak{A}(m))$$

by

$$du\colon D \mapsto Du \qquad \text{for} \quad u \in \mathfrak{A}(m),\ D \in W(m).$$

Then $\operatorname{Hom}_{\mathfrak{A}(m)}(W(m), \mathfrak{A}(m))$ is a free $\mathfrak{A}(m)$-module with base $\{dx_1, ..., dx_m\}$ and for $u \in \mathfrak{A}(m)$

$$du = \sum_{i=1}^{m} (D_i u)\, dx_i.$$

Define $\Omega(m)$ to be the exterior algebra over $\mathfrak{A}(m)$ on the $\mathfrak{A}(m)$-module $\operatorname{Hom}_{\mathfrak{A}(m)}(W(m), \mathfrak{A}(m))$.

Now $W(m)$ acts on $\operatorname{Hom}_{\mathfrak{A}(m)}(W(m), \mathfrak{A}(m))$ by

$$D\lambda = D \circ \lambda - \lambda \circ (\operatorname{ad} D) \qquad \text{for} \quad \lambda \in \operatorname{Hom}_{\mathfrak{A}(m)}(W(m), \mathfrak{A}(m)),\ D \in W(m).$$

In particular,

$$D(df) = d(Df) \qquad \text{for} \quad D \in W(m),\ f \in \mathfrak{A}(m).$$

It is easily seen that

$$D(u\lambda) = (Du)\lambda + u(D\lambda)$$
$$\text{for} \quad u \in \mathfrak{A}(m),\ \lambda \in \operatorname{Hom}_{\mathfrak{A}(m)}(W(m), \mathfrak{A}(m)),\ D \in W(m).$$

Then D extends to a derivation of $\Omega(m)$ (so $D(\alpha \wedge \beta) = D\alpha \wedge \beta + \alpha \wedge D\beta$ and $D(f\alpha) = (Df)\alpha + f(D\alpha)$ for all $\alpha, \beta \in \Omega(m), f \in \mathfrak{A}(m)$). Furthermore, if

ϕ is a divided power automorphism of $\mathfrak{A}(m)$ then ϕ acts on $\mathrm{Hom}_{\mathfrak{A}(m)}(W(m), \mathfrak{A}(m))$ by

$$\phi(\lambda) = \phi \circ \lambda \circ \Phi^{-1}$$

for $\lambda \in \mathrm{Hom}_{\mathfrak{A}(m)}(W(m), \mathfrak{A}(m))$ (recall $\Phi^{-1}E = \phi^{-1}E\phi$). Clearly

$$\phi(u\lambda) = \phi(u)\,\phi(\lambda) \qquad \text{for} \quad u \in \mathfrak{A}(m), \lambda \in \mathrm{Hom}_{\mathfrak{A}(m)}(W(m), \mathfrak{A}(m)).$$

Then ϕ extends to an automorphism of $\Omega(m)$. Note that, in particular,

$$\phi(df) = d(\phi f) \qquad \text{for} \quad f \in \mathfrak{A}(m). \tag{1.1.1}$$

We claim that if $D \in W(m)$ and if ϕ is a divided power automorphism of $\mathfrak{A}(m)$ then

$$(\Phi D)(\lambda) = \phi(D(\phi^{-1}(\lambda))) \qquad \text{for all} \quad \lambda \in \Omega(m). \tag{1.1.2}$$

Since both ΦD and $\phi \circ D \circ \phi^{-1}$ are derivations of $\Omega(m)$ and since they agree (by definition) on $\mathfrak{A}(m)$, it suffices to show that they agree on $\mathrm{Hom}_{\mathfrak{A}(m)}(W(m), \mathfrak{A}(m))$. Let $\lambda \in \mathrm{Hom}_{\mathfrak{A}(m)}(W(m), \mathfrak{A}(m))$. Then

$$\begin{aligned}
\phi(D(\phi^{-1}\lambda)) &= \phi \circ (D(\phi^{-1}\lambda)) \circ \Phi^{-1} \\
&= \phi \circ (D \circ (\phi^{-1}\lambda) - (\phi^{-1}\lambda) \circ \mathrm{ad}\, D) \circ \Phi^{-1} \\
&= \phi \circ (D \circ \phi^{-1} \circ \lambda \circ \Phi - \phi^{-1} \circ \lambda \circ \Phi \circ \mathrm{ad}\, D) \circ \Phi^{-1} \\
&= \Phi D \circ \lambda - \lambda \circ (\mathrm{ad}\, \Phi D) = (\Phi D)(\lambda),
\end{aligned}$$

proving (1.1.2).

Define ω_S, ω_H, and $\omega_K \in \Omega(m)$ by

$$\omega_S = dx_1 \wedge \cdots \wedge dx_m,$$

$$\omega_H = \sum_{i=1}^{r} dx_i \wedge dx_{i+r} \qquad \text{for} \quad m = 2r,$$

and

$$\omega_K = dx_{2r+1} + \sum_{i=1}^{r} (x_{i+r}\, dx_i - x_i\, dx_{i+r}) \qquad \text{for} \quad m = 2r+1.$$

Define subalgebras $S(m)$, $CS(m)$, $H(m)$, $CH(m)$, $K(m) \subseteq W(m)$ by

$$\begin{aligned}
S(m) &= \{D \in W(m) \mid D(\omega_S) = 0\} \qquad (m \geqslant 3), \\
CS(m) &= \{D \in W(m) \mid D(\omega_S) \in F\omega_S\} \\
&= S(m) + F(x_1 D_1) \qquad (m \geqslant 3),
\end{aligned}$$

$$H(m) = \{D \in W(m) \mid D(\omega_H) = 0\} \qquad (m = 2r \geqslant 2),$$

$$CH(m) = \{D \in W(m) \mid D(\omega_H) \in F\omega_H\}$$
$$= H(m) + F(x_1 D_1 + \cdots + x_m D_m) \qquad (m = 2r \geqslant 2),$$

and

$$K(m) = \{D \in W(m) \mid D(\omega_K) \in \mathfrak{A}(m)\, \omega_K\} \qquad (m = 2r + 1 \geqslant 3).$$

For any automorphism Φ of $W(m)$ and for $X = W$, S, H, or K, define

$$X(m : \mathbf{n} : \Phi) = \Phi X(m) \cap W(m : \mathbf{n}).$$

(Of course $W(m : \mathbf{n} : \Phi) = W(m : \mathbf{n})$ for all Φ.) Write $X(m : \mathbf{n})$ for $X(m : \mathbf{n} : \mathrm{id})$. Direct computation (cf. [KS69, SF88]) shows that $X(m : \mathbf{n})^{(1)} = X(m : \mathbf{n})^{(2)}$ for $X = S, H, K$ except for $X = H$, $m = 2$ and that $H(2 : \mathbf{n})^{(2)} = H(2 : \mathbf{n})^{(3)}$.

Unless specified otherwise, we assume that the m-tuple used to define a filtration or gradation is $\mathbf{1}$ unless we are considering one of the algebras $K(m)$ or $K(m : \mathbf{n} : \Phi)$ and in that case we assume that the m-tuple is $(1, ..., 1, 2)$. Then for $X = S$, CS, H, CH, or K, $X(m : \mathbf{n})$ is graded by $X(m : \mathbf{n})_{[j]} = X(m : \mathbf{n}) \cap W(m)_{[j]}$. Note that the filtered algebra $X(m : \mathbf{n} : \Phi)$ is a filtered subalgebra of the filtered algebra $W(m : \mathbf{n})$ and so we may view the associated graded algebra $\mathrm{gr}(X(m : \mathbf{n} : \Phi))$ as a graded subalgebra of $\mathrm{gr}(W(m : \mathbf{n})) = W(m : \mathbf{n})$.

If $X = W$, S, H, or K, and

$$\mathrm{gr}(X(m : \mathbf{n} : \Phi) \supseteq X(m : \mathbf{n})^{(2)} \qquad (1.1.3)$$

then $X(m : \mathbf{n} : \Phi)^{(2)}$ is simple (cf. [Kac74; Wil76, Corollary 2.5]). Such algebras are called simple Lie algebras of *Cartan type* (or in some references *generalized Cartan type*). Any graded subalgebra of $X(m : \mathbf{n})$ ($X = W$, S, CS, H, CH, or K) containing $X(m : \mathbf{n})^{(2)}$ is called a *graded Lie algebra of Cartan type*.

PROPOSITION 1.1.1 [Kac74, Theorem 2]. *Let L be a restricted simple Lie algebra of Cartan type. Then L is isomorphic to one of $W(m : \mathbf{1})$ $(m \geqslant 1)$, $S(m : \mathbf{1})^{(1)}$ $(m \geqslant 3)$, $H(m : \mathbf{1})^{(2)}$ $(m = 2r \geqslant 2)$, $K(m : \mathbf{1})^{(1)}$ $(m = 2r + 1 \geqslant 3)$.*

1.2. In this section we will state two important "recognition theorems" which allow us to conclude that certain algebras are of classical or Cartan type.

Let L be a classical simple Lie algebra (cf. [Sel67, Chap. II; BW82, Sect. 1.2]) with Cartan subalgebra H, root system Γ, and base Φ. If $\alpha \in \Gamma \cup \{0\}$ write $\alpha = \sum_{\beta \in \Phi} n_\beta(\alpha)\beta$. Then for any $\beta \in \Phi$, L is graded by

$$L_{[i]} = \sum_{n_\beta(\alpha) = i} L_\alpha.$$

Call such a grading of L a *standard grading*. Define standard gradings of the algebras $\mathfrak{pgl}(n)$ with $p \mid n$ similarly.

THEOREM 1.2.1 [Kac70, Theorem 3]. *Let G be a finite-dimensional graded Lie algebra over F. Assume that*:

G_0 is a restricted Lie algebra and is a direct sum of restricted ideals each of which is classical simple, $\mathfrak{gl}(n)$, $\mathfrak{sl}(n)$, or $\mathfrak{pgl}(n)$ with $p \mid n$, or abelian. (1.2.1)

The action of G_0 on G_{-1} is restricted. (1.2.2)

If $i \leqslant 0$, $x \in G_i$, and $[x, G_1] = (0)$, then $x = 0$. (1.2.3)

If $i \geqslant 0$, $x \in G_i$, and $[x, G_{-1}] = (0)$, then $x = 0$. (1.2.4)

The action of G_0 on G_{-1} is irreducible. (1.2.5)

Then G is isomorphic, as a graded algebra, to a classical Lie algebra with a standard grading, to $\mathfrak{pgl}(n)$ with $p \mid n$ and with a standard grading, or to a graded Lie algebra of Cartan type.

Let L be a finite-dimensional simple Lie algebra over F and L_0 be a maximal subalgebra of L. Then the adjoint action of L on itself induces a representation of L_0 on L/L_0. Let $L \supseteq L_{-1} \supseteq L_0$ be such that L_{-1}/L_0 is an irreducible L_0-submodule of L/L_0. Following Cartan [Car09] and Weisfeiler [Wei68] we define a filtration of L by

$$L_{i-1} = [L_i, L_{-1}] + L_i \qquad \text{for} \quad i < 0,$$

and

$$L_{i+1} = \{x \in L_i \mid [x, L_{-1}] \subseteq L_i\} \qquad \text{for} \quad i \geqslant 0.$$

We call this a filtration *corresponding to* the maximal subalgebra L_0. Let $G = \sum G_i$, $G_i = L_i/L_{i+1}$, be the associated graded algebra.

The following theorem contains results of [Kac74, Wil76].

THEOREM 1.2.2 (The Recognition Theorem). *Let L be a finite-dimensional simple Lie algebra over F with maximal subalgebra L_0. Give L a corresponding filtration and let G be the associated graded algebra. Suppose G satisfies* (1.2.1)–(1.2.3). *Then L is of classical or of Cartan type.*

1.3. Let A be a restricted Lie algebra. We say an element $t \in A$ is *toral* if $t^p = t$.

Demuškin [Dem70, Dem72] has proved conjugacy theorems for toral elements and for maximal tori in the restricted simple Lie algebras of Cartan type. The following theorem combines some of these results and also includes a strengthening of Demuškin's result on conjugacy of toral elements in $K(m:\mathbf{1})^{(1)}$.

THEOREM 1.3.1 (Demuškin's Conjugacy Theorem). (a) *Let $t \in W(m:\mathbf{1})$, $t \notin W(m:\mathbf{1})_0$, be a toral element. Then t is conjugate to $(x_1+1)D_1$.*

(b) *Let $t \in W(m:\mathbf{1})_0$ be a toral element. Then t is conjugate to an element in the $\mathbf{Z}_p$-span of $\{x_1 D_1, ..., x_m D_m\}$.*

(c) *Let $T \subseteq W(m:\mathbf{1})$ be a maximal torus. Then T is conjugate to $T_i = \operatorname{span}\{(x_1+1)D_1, ..., (x_i+1)D_i, x_{i+1}D_{i+1}, ..., x_m D_m\}$ for some i, $0 \leqslant i \leqslant m$. For any i, $\mathfrak{z}_{W(m:\mathbf{1})}(T_i) = T_i$.*

(d) *Let $t \in S(m:\mathbf{1})^{(1)}$, $m \geqslant 3$, $t \notin (S(m:\mathbf{1})^{(1)})_0$, be a toral element. Then t is conjugate to $(x_1+1)D_1 - x_2 D_2$.*

(e) *Let $t \in (S(m:\mathbf{1})^{(1)})_0$, $m \geqslant 3$, be a toral element. Then t is conjugate to an element in the $\mathbf{Z}_p$-span of $\{x_i D_i - x_{i+1} D_{i+1} \mid 1 \leqslant i \leqslant m-1\}$.*

(f) *Let $T \subseteq S(m:\mathbf{1})^{(1)}$, $m \geqslant 3$, be a maximal torus. Then T is conjugate to $T_i \cap S(m:\mathbf{1})^{(1)}$ for some i, $0 \leqslant i \leqslant m-1$ (where T_i is as in* (c)*).*

(g) *Let $t \in H(m:\mathbf{1})^{(2)}$, $m = 2r \geqslant 2$, $t \notin (H(m:\mathbf{1})^{(2)})_0$, be a toral element. Then t is conjugate to $(x_1+1)D_1 - x_{r+1}D_{r+1}$.*

(h) *Let $t \in (H(m:\mathbf{1})^{(2)})_0$, $m = 2r \geqslant 2$, be a toral element. Then t is conjugate to an element in the $\mathbf{Z}_p$-span of $\{x_i D_i - x_{i+r} D_{i+r} \mid 1 \leqslant i \leqslant r\}$.*

(i) *Let $T \subseteq H(m:\mathbf{1})^{(2)}$, $m = 2r \geqslant 2$, be a maximal torus. Then T is conjugate to $T_i \cap H(m:\mathbf{1})^{(2)}$ for some i, $0 \leqslant i \leqslant r$ (where T_i is as in* (c)*).*

(j) *Let $t \in K(m:\mathbf{1})^{(1)}$, $t \notin (K(m:\mathbf{1})^{(1)})_{-1}$, $m = 2r+1 \geqslant 3$, be a toral element. Then t is conjugate to $2(x_m+1)D_m + \sum_{i=1}^{m-1} x_i D_i$.*

(k) *Let $t \in (K(m:\mathbf{1})^{(1)})_{-1}$, $t \notin (K(m:\mathbf{1}))^{(1)})_0$, $m = 2r+1 \geqslant 3$, be a toral element. Then t is conjugate to $(x_1+1)D_1 - x_{r+1}D_{r+1} + x_{r+1}D_m$.*

(l) *Let $t \in (K(m:\mathbf{1})^{(1)})_0$, $m = 2r+1 \geqslant 3$, be a toral element. Then t is conjugate to an element in the $\mathbf{Z}_p$-span of $\{x_i D_i - x_{i+r} D_{i+r} \mid 1 \leqslant i \leqslant r\} \cup \{2x_m D_m + \sum_{i=1}^{m-1} x_i D_i\}$.*

Proof. Reference [Dem70] contains proofs of (a) (Lemma 6), (c) (Theorem 1 and Corollary 2), (d) (Lemma 8), and (f) (Theorem 2). Reference [Dem72] contains proofs of (g) (Lemma 2) and (i) (Theorem 2). Furthermore, (b) is proved at the beginning of the proof of Theorem 1 of [Dem70], the proof of (e) is contained within the proof of Theorem 2 of

[Dem70] and the proof of (h) is contained within the proof of Theorem 2 of [Dem72]. Actually, using Kreknin's result [Kre71] that $(X(m:\mathbf{n})^{(2)})_0$ is invariant under $\mathrm{Aut}(X(m:\mathbf{n})^{(2)})$ (for $X=W, S, H, K$), it is easy to deduce (b) from (c), for the toral element $t\in W(m:\mathbf{1})_0$ can be imbedded in a maximal torus of $W(m:\mathbf{1})$ contained in $W(m:\mathbf{1})_0$). Part (e) follows similarly from (f) and part (h) follows similarly from (i).

The proofs of (j), (k), (l) require computations in $K(m:\mathbf{1})$ $(m=2r+1)$. Accordingly, we recall (cf. [KS69, Sect. 7.1; SF88, Sect. IV.5]) that $K(m:\mathbf{1})=\{\mathscr{D}_K(f)\mid f\in\mathfrak{A}(m:\mathbf{1})\}$, where

$$\mathscr{D}_K(f)=\sum_{i=1}^{r}(D_{i+r}f+x_iD_mf)D_i+\sum_{i=r+1}^{2r}(-D_{i-r}f+x_iD_mf)D_i + \left(2f-\sum_{i=1}^{2r}x_iD_if\right)D_m, \tag{1.3.1}$$

and that

$$[\mathscr{D}_K(f),\mathscr{D}_K(g)]=\mathscr{D}_K\Bigg((D_mg)\left(2f-\sum_{i=1}^{2r}x_iD_if\right) -(D_mf)\left(2g-\sum_{i=1}^{2r}x_iD_ig\right) -\sum_{i=1}^{r}((D_if)(D_{i+r}g)-(D_{i+r}f)(D_ig))\Bigg). \tag{1.3.2}$$

Now assume that $t\in K(m:\mathbf{1})^{(1)}$, $t\notin(K(m:\mathbf{1})^{(1)})_{-1}$, is a toral element. By Theorems 1 and 2 of [Wil75] we may assume that $t=\mathscr{D}_K(1+x_m^{p-1}f)$, where $f\in F[x_1,...,x_{m-1}]/(x_1^p,...,x_{m-1}^p)$. Then Jacobson's formula for pth powers ([Jac37]; cf. [Jac62; BW82, Sect. 1.1]) shows that

$$t^p=(\mathscr{D}_K(1)+\mathscr{D}_K(x_m^{p-1}f))^p=\sum_{i=1}^{p-1}s_i,$$

where is_i is the coefficient of λ^{i-1} in

$$(\mathrm{ad}(\mathscr{D}_K(1)+\lambda\mathscr{D}_K(x_m^{p-1}f)))^{p-1}\,\mathscr{D}_K(x_m^{p-1}f).$$

Using (1.3.2) we see that for $i\geqslant 2$ we have $s_i=\mathscr{D}_K(g_i)$, where $g_i\in x_mB_m$ (recall that $B_m=F[x_1,...,x_m]/(x_1^p,...,x_m^p)$), and that $s_1=\mathscr{D}_K(-f)$. Since $t^p=t$ this implies $f=-1$. Thus there is only one conjugacy class of toral elements in $K(m:\mathbf{1})^{(1)}$ outside $(K(m:\mathbf{1})^{(1)})_{-1}$. Since $2(x_m+1)D_m+\sum_{i=1}^{m-1}x_iD_i=\mathscr{D}_K(x_m+1)$ is such an element we have proved (j).

Next assume that $t \in (K(m:\mathbf{1})^{(1)})_{-1}$, $t \notin (K(m:\mathbf{1})^{(1)})_0$, is a toral element. Again by Theorems 1 and 2 of [Wil75] we may assume that $t = \mathscr{D}_K(x_1 + x_{r+1}^{p-1} f)$, where $f \in F[x_1, ..., x_r, \quad x_{r+2}, ..., x_m]/(x_1^p, ..., x_r^p, x_{r+2}^p, ..., x_m^p)$. Then, as above, Jacobson's formula for pth powers gives $t^p = \sum_{i=1}^{p-1} w_i$, where (using (1.3.2)) for some $g_1, ..., g_{p-1} \in x_{r+1} B_m$ we have $w_1 = \mathscr{D}_K(-f + g_1)$ and $w_i = \mathscr{D}_K(g_i)$ for $i \geqslant 2$. Since $t^p = t$ this implies $f = -x_1$. Thus there is only one conjugacy class of toral elements in $(K(m:\mathbf{1})^{(1)})_{-1}$ outside $(K(m:\mathbf{1})^{(1)})_0$. Since $(x_1 + 1) D_1 - x_{r+1} D_{r+1} + x_{r+1} D_m = \mathscr{D}_K((x_1 + 1) x_{r+1})$ is such an element we have proved (k).

Finally, assume that $t \in (K(m:\mathbf{1})^{(1)})_0$ is a toral element. Since $K(m:\mathbf{1})^{(1)}$ is graded we may write $t = \sum_{j \geqslant 0} t_{[j]}$, where $t_{[j]} \in (K(m:\mathbf{1})^{(1)})_{[j]}$. Clearly $t_{[0]}^p = t_{[0]}$. We claim that t is conjugate to $t_{[0]}$. To prove this observe that by Theorem 1 of [Wil75] we may assume $[t_{[0]}, t_{[j]}] = 0$ for all $j > 0$ and so by Jacobson's formula for pth powers $t^p = t_{[0]} + (\sum_{j>0} t_{[j]})^p$. This clearly implies $t_{[j]} = 0$ for $j > 0$. Then (l) follows from Theorem 2 of [Wil75]. ▮

COROLLARY 1.3.2. *Let M be a restricted subalgebra of $W(2:\mathbf{1})$ and T be a two-dimensional torus in M. Then $\mathfrak{z}_M(T) = T$.*

Proof. This is immediate from Theorem 1.3.1(c). ▮

1.4. We recall some notation from [Wil83]. Let T be a maximal torus in a restricted Lie algebra A and let $\Gamma = \Gamma(A, T)$ denote the set of roots of A with respect to T so that

$$A = \mathfrak{z}_A(T) + \sum_{\gamma \in \Gamma} A_\gamma$$

is the root space decomposition. Let $\Delta = \Delta(A, T)$ be the additive group generated by $\Gamma(A, T)$. For $X \subseteq \Delta$ let $\mathbf{Z}X$ denote the subgroup of Δ generated by X. Define

$$A^{(X)} = \sum_{\alpha \in \mathbf{Z}X} A_\alpha \tag{1.4.1}$$

and

$$A[X] = A^{(X)}/\text{solv}(A^{(X)}), \tag{1.4.2}$$

where solv B denotes the solvable radical of B. Note that $A^{(X)}$ is a restricted subalgebra of A and hence $A[X]$ is restricted. Let

$$\Psi_X : A^{(X)} \to A[X]$$

denote the canonical epimorphism (which is a homomorphism of restricted Lie algebras). We write $A^{(\alpha)}$, $A[\alpha]$, Ψ_α, $A^{(\alpha,\beta)}$, $A[\alpha,\beta]$, $\Psi_{\alpha,\beta}$ in place of $A^{(\{\alpha\})}$, $A[\{\alpha\}]$, $\Psi_{\{\alpha\}}$, $A^{(\{\alpha,\beta\})}$, $A[\{\alpha,\beta\}]$, $\Psi_{\{\alpha,\beta\}}$, respectively.

1.5.

DEFINITION 1.5.1. Let A be a restricted Lie algebra. A maximal torus $T \subseteq A$ is said to be *standard* (in A) if $\mathfrak{z}_A(T)$ contains a nil ideal I such that

$$\mathfrak{z}_A(T) = T + I.$$

LEMMA 1.5.2. *The following are equivalent:*

(a) *T is standard in A.*

(b) *The set of nilpotent elements in $\mathfrak{z}_A(T)$ forms a subalgebra.*

(c) *$[\mathfrak{z}_A(T), \mathfrak{z}_A(T)]$ is nil.*

Proof. Clearly (a) implies (c) and (b) implies (a). Thus it suffices to prove that (c) implies (b). Let $D = [\mathfrak{z}_A(T), \mathfrak{z}_A(T)]$. Assume (c) holds. By (c), $\bar{D}$ is a restricted nil ideal of $\mathfrak{z}_A(T)$. It follows that $x \in \mathfrak{z}_A(T)$ is nilpotent if and only if $(x + \bar{D}) \in \mathfrak{z}_A(T)/\bar{D}$ is nilpotent. Since $\mathfrak{z}_A(T)/\bar{D}$ is abelian it is clear that the set of nilpotent elements of $\mathfrak{z}_A(T)$ forms a subalgebra, so (b) holds. ∎

Remark 1.5.3. If A is a restricted simple Lie algebra then any maximal torus $T \subseteq A$ is standard, for $\mathfrak{z}_A(T)$ is a Cartan subalgebra of A (by [Sel67, Thm. V.7.3]) and so has the required structure by Theorem 2.1 of [Wil77].

Remark 1.5.4. There exist pairs (A, T) such that A is a restricted semisimple Lie algebra, T is a maximal torus in A, and T is not standard. We give two examples, each involving a type of algebra that will occur frequently in the sequel.

For the first example let

$$A = (\mathfrak{sl}(2) \otimes F[x]/(x^p)) + F(\partial/\partial x).$$

This is semisimple (by Theorem 9.3 of [Blo69]; cf. Theorem 1.16.1 of [BW82]). Let $\mathfrak{sl}(2)$ have the usual basis $\{e, f, h\}$. Then $T = F(h \otimes 1)$ is a maximal torus in A,

$$\mathfrak{z}_A(T) = (h \otimes (F[x]/(x^p))) + F(\partial/\partial x),$$

and $[\partial/\partial x, h \otimes x] = h \otimes 1$ is not nilpotent. Hence (see Lemma 1.5.2) T is not standard. (The adjoint representation of the algebra $\mathfrak{z}_A(T)$ on $e \otimes (F[x]/(x^p))$ is isomorphic to the example given by Seligman (p. 97 of

[Sel67]) of a representation of a nilpotent Lie algebra for which the weights are not linear.)

For the second example let $A = \overline{W(1:\mathbf{2})}$ and $T = F(x_1 D_1)$. Then T is a maximal torus in A, and $\mathfrak{z}_A(T)$ contains D_1^p and $x^{(p+1)\varepsilon_1} D_1$. Since $[D_1^p, x^{(p+1)\varepsilon_1} D_1] = x_1 D_1$ we see from Lemma 1.5.2 that T is not standard.

Remark 1.5.5. If T is a standard maximal torus in a restricted Lie algebra A, α is a root of A with respect to the Cartan subalgebra $\mathfrak{z}_A(T)$, and $\mathfrak{z}_A(T) = T + I$, where I is the nil radical of $\mathfrak{z}_A(T)$, then $\alpha \in \mathfrak{z}_A(T)^*$ and $\alpha(I) = (0)$. Therefore, identifying T^* with the subset $\{\gamma \in \mathfrak{z}_A(T)^* \mid \gamma(I) = (0)\}$ we see that $\Gamma(T)$, the set of roots with respect to $\mathfrak{z}_A(T)$, can be regarded as a subset of T^*.

1.6. We now investigate some functorial properties of standard maximal tori.

LEMMA 1.6.1. *Let β: $A \to B$ be a surjective homomorphism of restricted Lie algebras. Let T be a maximal torus in A.*

(a) *$\beta(T)$ is a maximal torus in B.*

(b) $\mathfrak{z}_B(\beta(T)) = \beta(\mathfrak{z}_A(T))$.

(c) *If T is standard in A then $\beta(T)$ is standard in B.*

Proof. Part (a) is contained in Winter's Theorem 2.16 of [Win69]. For (b) let $C = \{x \in A \mid \beta(x) \in \mathfrak{z}_B(\beta(T))\}$. Then $\beta([C, T]) = [\beta(C), \beta(T)] = (0)$ so $[C, T] \subseteq C$. Thus $C = \sum C_\gamma$. Now if $\gamma \neq 0$ then $\beta(C_\gamma) = \beta([C_\gamma, T]) = (0)$ and so $C_\gamma \subseteq \ker \beta$. Hence $C \subseteq \mathfrak{z}_A(T) + \ker \beta$. Since β is surjective this gives $\mathfrak{z}_B(\beta(T)) = \beta(C) = \beta(\mathfrak{z}_A(T))$, proving (b). Finally, $\mathfrak{z}_B(\beta(T)) = \beta(\mathfrak{z}_A(T)) = \beta(T + I) = \beta(T) + \beta(I)$. Now, as $\beta(I)$ is a nil ideal, $\beta(T)$ is standard and so (c) holds. ∎

LEMMA 1.6.2. *Let $A \supseteq B$ be restricted Lie algebras. Let T be a standard maximal torus in A and $T \subseteq B$. Then T is a standard maximal torus in B.*

Proof. The condition of Lemma 1.5.2(c) holds. ∎

1.7.1. We now investigate the relation between tori of maximal dimension in A and in a section $A[X]$.

LEMMA 1.7.1. *Let β: $A \to B$ be a surjective homomorphism of restricted Lie algebras with kernel K. Let S be a maximal torus in K and U be a maximal torus in B. Then there is a maximal torus V in A such that $\beta(V) = U$ and* $\dim V = \dim S + \dim U$.

Proof. Take V to be a maximal torus of $\beta^{-1}(U)$ which contains S.

Then $\beta(V)$ is a maximal torus in U (by Lemma 1.6.1(a)) so $\beta(V)=U$. Also $V\cap K$ is a torus in K containing the maximal torus S, so $V\cap K=S$. Now $\dim V=\dim(V\cap K)+\dim\beta(V)=\dim S+\dim U$. Since V is a maximal torus in A (by Theorem 2.16 of [Win69]) the lemma is proved. ▮

LEMMA 1.7.2. *Let β: $A\to B$ be a surjective homomorphism of restricted Lie algebras.*

(a) *If V is a torus of maximal dimension in A, then $\beta(V)$ is a torus of maximal dimension in B.*

(b) *If U is a torus of maximal dimension in B, then there is a torus V of maximal dimension in A such that $\beta(V)=U$.*

Proof. Denote $\ker\beta$ by K. Let V be as in (a) and suppose that U is a maximal torus in B with $\dim U>\dim\beta(V)$. Then by Lemma 1.7.1 there is a torus V' in A such that $\dim V'\geqslant\dim(V\cap K)+\dim U>\dim(V\cap K)+\dim\beta(V)=\dim V$, contradicting the maximality of $\dim V$. Thus (a) holds.

Now let U be as in (b). Let S be a torus of maximal dimension in K. Let V be the maximal torus of A given by Lemma 1.7.1. If V' is a torus in A with $\dim V'>\dim V$, then either $\dim(V'\cap K)>\dim(V\cap K)=\dim S$, contradicting the maximality of $\dim S$, or $\dim\beta(V')>\dim\beta(V)=\dim U$, contradicting the maximality of $\dim U$. Thus (b) holds. ▮

COROLLARY 1.7.3. *Let T be a torus of maximal dimension in a restricted Lie algebra A and $A[X]$ be a section of A (with respect to T). Let U be a torus of maximal dimension in $A[X]$. Then there is a torus V of maximal dimension in A such that $V\subseteq A^{(X)}$ and $\Psi_X(V)=U$.*

Proof. By Lemma 1.7.2(b) there is a torus V of maximal dimension in $A^{(X)}$ such that $\Psi_X(V)=U$. Since T is of maximal dimension in A and $T\subseteq A^{(X)}$ (by definition), T is of maximal dimension in $A^{(X)}$ and so $\dim T=\dim V$. Thus V is of maximal dimension in A and we are done. ▮

PROPOSITION 1.7.4. *Let T be a torus of maximal dimension in a restricted simple Lie algebra L and $L[X]$ be a section on L (with respect to T). Let U be a torus of maximal dimension in $L[X]$. Then U is standard in $L[X]$. In particular $\Psi_X(T)$ is standard and of maximal dimension in $L[X]$.*

Proof. By Corollary 1.7.3, $U=\Psi_X(V)$, where V is a torus of maximal dimension (hence a maximal torus) in L. By Remark 1.5.3, V is standard in L. By Lemmas 1.6.1(c) and 1.6.2, U is standard in $L[X]$. Since $\Psi_X(T)$ is of maximal dimension in $L[X]$ by Lemma 1.7.2, the last remark holds. ▮

1.8. The following lemmas will be useful in the sequel.

LEMMA 1.8.1. *Let T be a maximal torus in a restricted Lie algebra A. If $\alpha \in \Gamma(A, T)$ and $x \in A_\alpha$ then $x^p \in \mathfrak{z}_A(T)$ and $\alpha(x^p) = 0$.*

Proof. Since $[x, x^p] = 0$ and $(\operatorname{ad}(x^p) - \alpha(x^p))^n x = 0$ for sufficiently large n, we have the result. ∎

LEMMA 1.8.2. *Let A be a restricted Lie algebra, S an ideal in A, and J a restricted nil subalgebra of A such that $A = S + J$. Then every semisimple element of A is contained in $\bar{S}$.*

Proof. Let $x \in A$, $x = s + m$, $s \in S$, $m \in J$. By Jacobson's formula for pth powers (cf. [BW82, Sect. 1.1]), $x^{p^n} \equiv m^{p^n} \bmod \bar{S}$ for any $n \geq 0$. As J is nil this implies $x^{p^n} \in \bar{S}$ for sufficiently large n, so if x is semisimple then $x \in \bar{S}$. ∎

LEMMA 1.8.3. *Let Y be a Lie algebra with root space decomposition $\sum_{\alpha \in \Gamma \cup \{0\}} Y_\alpha$. Let M be a solvable ideal in Y. Then for any $\alpha \in \Gamma$, $\alpha([Y_\alpha, M_{-\alpha}]) = (0)$.*

Proof. Suppose $\alpha([Y_\alpha, M_{-\alpha}]) \neq (0)$. Then $Y_\alpha \subseteq M_\alpha$ so $\alpha([M_\alpha, M_{-\alpha}]) \neq (0)$. Now if M_α, $M_{-\alpha} \subseteq M^{(n)}$ then $[M_\alpha, M_{-\alpha}] \subseteq M^{(n+1)}$. Since $\alpha([M_\alpha, M_{-\alpha}]) \neq (0)$, $M_{\pm\alpha} = [[M_\alpha, M_{-\alpha}], M_{\pm\alpha}]$ so M_α, $M_{-\alpha} \subseteq M^{(n+1)}$. Thus by induction M_α, $M_{-\alpha} \subseteq M^{(n)}$ for all n. As M is solvable this implies $M_\alpha = M_{-\alpha} = (0)$, a contradiction. ∎

LEMMA 1.8.4. *Let M be a solvable restricted Lie algebra containing a maximal torus $Fr + Fz$, where $r^p = r$ is not central and z is central. Assume that $\mathfrak{z}_M(Fr + Fz) = Fr + Fz + I$, where I is a nil ideal in $\mathfrak{z}_M(Fr + Fz)$. Define α, $\beta \in (Fr + Fz)^*$ by $\alpha(r) = 1$, $\alpha(z) = 0$, $\beta(r) = 0$, $\beta(z) = 1$. Suppose $[M_{i\alpha}, M_{-i\alpha}]$ is not nil for some i. Then if V is an irreducible restricted M-module with $zV \neq (0)$, V has p weights.*

Proof. As M is solvable there is an integer m and a restricted subalgebra M_1 of M such that $\dim V = p^m$, $\dim M/M_1 = m$ and V contains a one-dimensional M_1-submodule Fv (cf. Theorem 1.13.1 of [BW82]).

Since z is central in M and V is an irreducible M-module we see that z acts as a nonzero scalar on V. Since M is solvable, Lemma 1.8.3 shows $\alpha([M_{i\alpha}, M_{-i\alpha}]) = (0)$. Therefore $[M_{i\alpha}, M_{-i\alpha}]$ (which is not nil by hypothesis) contains $z + n$ for some nilpotent $n \in M$. Thus $[M_{i\alpha}, M_{-i\alpha}]$ cannot annihilate V.

Suppose $r \in M_1$. Then v is a weight vector, say $v \in V_\gamma$. If $M = Fz + I + M_1$ then $V = V_\gamma$ and so $M_{i\alpha}$ annihilates V. Hence $[M_{i\alpha}, M_{-i\alpha}]$ annihilates V. But we have seen in the previous paragraph that this is impossible. Hence there is some root η and some root vector $y \in M_\eta$ such that $y \notin M_1$. Then $\{y^j v \mid 0 \leqslant j \leqslant p-1\}$ is linearly independent, so V has weights $\gamma + j\eta$, $0 \leqslant j \leqslant p-1$. Thus V has p weights.

Suppose $r \notin M_1$. We may write $v = v_\gamma + v_{\gamma+\alpha} + \cdots + v_{\gamma(p-1)\alpha}$, where $v_\eta \in V_\eta$. Then since $\{r^j v \mid 0 \leqslant j \leqslant p-1\}$ is linearly independent we see that $\{v_{\gamma+j\alpha} \mid 0 \leqslant j \leqslant p-1\}$ is a linearly independent set in V and hence that V has p weights. ▌

1.9. In this section we recall some properties of Winter's exponential maps E^x and e^x [Win69] and discuss the relation between root spaces for T and $e^x(T)$.

We assume throughout this section that M is a finite-dimensional restricted Lie algebra and that every torus of maximal dimension in M is standard.

Let T be a torus of maximal dimension in M, $\alpha \in T^*$, and $x \in M_\alpha$. Define

$$E^x = \sum_{i=0}^{p-1} (\operatorname{ad} x)^i / i!.$$

Then $E^x(T) = \{t - \alpha(t)x \mid t \in T\}$ is an abelian subalgebra of M. Therefore $\overline{E^x(T)}$ contains a unique maximal torus which we denote by $e^x(T)$. Note that $E^x|_T$ is injective. By Theorem 3.4(1) of [Win69], $\dim e^x(T) = \dim T$, so $e^x(T)$ is a torus of maximal dimension in M and hence is standard. Let I' denote the nil radical of $\mathfrak{z}_M(e^x(T))$, so that

$$\mathfrak{z}_M(e^x(T)) = e^x(T) \oplus I'. \tag{1.9.1}$$

LEMMA 1.9.1. (a) *$\mathfrak{z}_M(e^x(T)) = E^x(T) \oplus I'$, a direct sum of subspaces.*

(b) *Every $\lambda \in E^x(T)^*$ has a unique extension to an element of $\mathfrak{z}_M(e^x(T))^*$ (again denoted λ) which vanishes on I'.*

Proof. Since $E^x(T)$ is abelian so is $\overline{E^x(T)}$. Thus $E^x(T) \subseteq \mathfrak{z}_M(e^x(T))$. For an arbitrary element $t - \alpha(t)x \in E^x(T)$ we have

$$\begin{aligned}(t - \alpha(t)x)^{p^n} &= t^{p^n} - (\alpha(t))^{p^n}(x + x^p + \cdots + x^{p^n}) \\ &\equiv -(\alpha(t))^{p^n} x \qquad \operatorname{mod} \mathfrak{z}_M(T).\end{aligned}$$

Thus $(t - \alpha(t))x)^{p^n} = 0$ implies $\alpha(t) = 0$ and so $t - \alpha(t)x = t \in T$. Since T is a torus this implies $t = 0$. Thus $E^x(T)$ contains no nonzero nilpotent elements and so $E^x(T) \cap I' = (0)$. Then $\dim(E^x(T) + I') = \dim E^x(T) + \dim I' = \dim e^x(T) + \dim I' = \dim \mathfrak{z}_M(e^x(T))$ and so $E^x(T) + I' = \mathfrak{z}_M(e^x(T))$, proving (a). Part (b) is immediate from (a). ▌

DEFINITION 1.9.2. Pick $\xi \in \operatorname{Hom}_{\mathbf{Z}}(F, F)$ such that $\xi(u)^p - \xi(u) = u$ for all $u \in F$. (As noted in [Wil83, Sect. 4.3] such a ξ exists.) Let $\alpha \in T^*$, $x \in M_\alpha$. For $\beta \in T^*$ define $\beta_x \in \mathfrak{z}_M(e^x(T))^*$ by $\beta_x|_{I'} = (0)$ (where I' is as in (1.9.1)) and $\beta_x(E^x t) = \beta(t) - \xi(\beta(x^p))\,\alpha(t)$ for $t \in T$. (By Lemma 1.9.1 there is a unique $\beta_x \in \mathfrak{z}_M(e^x(T))^*$ with these properties.)

PROPOSITION 1.9.3. *Let M be as above. Let α be a root with respect to T, $x \in M_\alpha$. Then*:

(a) $\mathfrak{z}_M(e^x(T))$ *is a Cartan subalgebra of M.*

(b) *If $\beta \in T^*$ then* $\sum_{j \in \mathbf{Z}_p} M_{\beta + j\alpha} = \sum_{j \in \mathbf{Z}_p} M_{(\beta + j\alpha)_x}$.

(c) *If $\beta \in T^*$ then* $\dim M_{\beta_x} = \dim M_\beta$.

(d) *If $\beta \in T^*$ then* $M_{\beta_x} \subseteq \sum_{i=0}^{p-1} (\operatorname{ad} x)^i M_\beta$.

(e) $\Delta(M, e^x(T)) = \{\beta_x \mid \beta \in \Delta(M, T)\}$.

Proof. Since $e^x(T)$ is a maximal torus, $\mathfrak{z}_M(e^x(T))$ is a Cartan subalgebra, proving (a).

In proving (b) we will first show that $\sum_{j \in \mathbf{Z}_p} M_{\beta + j\alpha} \subseteq \sum_{j \in \mathbf{Z}_p} M_{(\beta + j\alpha)_x}$. For this it is enough to show

$$M_\beta \subseteq \sum_{j \in \mathbf{Z}_p} M_{(\beta + j\alpha)_x}. \tag{1.9.2}$$

To establish (1.9.2) it is sufficient to show that if

$$U(t) = \prod_{j \in \mathbf{Z}_p} (\operatorname{ad}(E^x t) - (\beta + j\alpha)_x (E^x t))$$

then $U(t)^{\dim M} M_\beta = (0)$ for all $t \in T$. Then

$$\begin{aligned} U(t) &= \prod_{j \in \mathbf{Z}_p} ((\operatorname{ad} t) - \alpha(t)(\operatorname{ad} x) - \beta(t) + \xi(\beta(x^p))\,\alpha(t) - j\alpha(t)) \\ &= ((\operatorname{ad} t) - \alpha(t)(\operatorname{ad} x) - \beta(t) + \xi(\beta(x^p))\,\alpha(t))^p \\ &\quad - \alpha(t)^{p-1}\,((\operatorname{ad} t) - \alpha(t)(\operatorname{ad} x) - \beta(t) + \xi(\beta(x^p))\,\alpha(t)) \\ &= (\operatorname{ad}(t^p) - \beta(t^p)) - \alpha(t)^{p-1}\,(\operatorname{ad} t - \beta(t)) \\ &\quad - \alpha(t)^p\,(\operatorname{ad}(x^p) - \beta(x^p)) \end{aligned}$$

(where we use Jacobson's formula for pth powers to compute $((\operatorname{ad} t) - \alpha(t)(\operatorname{ad} x))^p = \operatorname{ad}(t^p) - \alpha(t)^p\,(\operatorname{ad} x) - \alpha(t)^p\,(\operatorname{ad}(x^p))$ and also use the fact that $\xi(\beta(x^p))^p - \xi(\beta(x^p)) = \beta(x^p)$). Thus (1.9.2) holds and so $\sum_{j \in \mathbf{Z}_p} M_{\beta + j\alpha} \subseteq \sum_{j \in \mathbf{Z}_p} M_{(\beta + j\alpha)_x}$.

To prove the reverse inclusion introduce an equivalence relation $\sim$ on

T^* by $\mu \sim \nu$ if and only if $\mu \in \nu + \mathbf{Z}_p\alpha$. Write $N_\mu = \sum_{\nu\sim\mu} M_\nu = \sum_{j\in\mathbf{Z}_p} M_{\mu+j\alpha}$ and $N_{\mu_x} = \sum_{\nu\sim\mu} M_{\nu_x} = \sum_{j\in\mathbf{Z}_p} M_{(\mu+j\alpha)_x}$. We have shown $N_\beta \subseteq N_{\beta_x}$ and must show $N_{\beta_x} \subseteq N_\beta$. Let $y \in N_{\beta_x}$. Then $y = \sum_{\theta\in\Theta} y_\theta$, where Θ is a subset of $\Delta(M, T)$ such that $\beta \in \Theta$, and that if $\theta_1, \theta_2 \in \Theta$ and $\theta_1 \sim \theta_2$ then $\theta_1 = \theta_2$ and where $y_\theta \in N_\theta$. But since $N_\theta \subseteq N_{\theta_x}$ we have $y_\theta \in N_{\theta_x}$. If $\beta_x = \gamma_x$ for some $\beta, \gamma \in T^*$ then $\beta - \gamma = \xi((\beta-\gamma)(x^p))\alpha$ and so (since $\alpha(x^p) = 0$) we have $(\beta-\gamma)(x^p) = 0$ and $\beta = \gamma$. Therefore $\sum_{\gamma\in\Delta(M,T)} M_{\gamma_x}$ is direct and so $\sum_{\theta\in\Theta} N_{\theta_x}$ is direct. Therefore $y = y_\beta \in N_\beta$, as required. Thus (b) holds.

Now fix $t \in T$ satisfying $t^p = t$ and $\alpha(t) = 1$ and let $k \in \mathbf{Z}$ be large enough so that

$$(\operatorname{ad}(E^x t) - (\beta + j\alpha)_x(E^x t))^{p^k}\,(M_{(\beta+j\alpha)_x}) = (0)$$

for all $j \in \mathbf{Z}_p$. Then $M_{\beta_x} = U'(N_\beta)$, where

$$U' = \prod_{j=1}^{p-1} (\operatorname{ad}(E^x t) - (\beta + j\alpha)_x\,(E^x t))^{p^k}.$$

In particular, $M_{\beta_x} \supseteq U'(M_\beta)$ and so $\dim M_{\beta_x} \geqslant \dim U'(M_\beta)$. Now, since $(E^x t)^{p^k} = t - x - x^p - \cdots - x^{p^k}$, since $\beta(t) \in \mathbf{Z}_p$, and since $\xi(u)^{p^k} = u + u^p + \cdots + u^{p^{k-1}} + \xi(u)$ for all $u \in F$, we see that

$$\begin{aligned} U' = \prod_{j=1}^{p-1} &((\operatorname{ad} t) - (\operatorname{ad} x) - (\operatorname{ad} x^p) - \cdots - (\operatorname{ad} x^{p^k}) \\ &-\beta(t) + \beta(x^p) + \cdots + \beta(x^{p^k}) + \xi(\beta(x^p)) - j). \end{aligned} \tag{1.9.3}$$

Let τ denote the projection of N_β onto M_β. Then

$$\tau U'|_{M_\beta} = \prod_{j=1}^{p-1} \left(-\sum_{l=1}^{k} ((\operatorname{ad} x^{p^l}) - \beta(x^{p^l})) + \xi(\beta(x^p)) - j\right)\Bigg|_{M_\beta}.$$

Now $(\operatorname{ad} x^{p^l}) - \beta(x^{p^l})$ is nilpotent on M_β for all l and $\xi(\beta(x^p)) \notin \mathbf{Z}_p^*$ (for if $\xi(\beta(x^p)) \in \mathbf{Z}_p^*$ then $\beta(x^p) = \xi(\beta(x^p))^p - \xi(\beta(x^p)) = 0$ and so $\xi(\beta(x^p)) = 0$, a contradiction). Thus $\tau U'|_{M_\beta}$ is invertible. Therefore $\dim U'(M_\beta) = \dim M_\beta$ and so $\dim M_{\beta_x} \geqslant \dim M_\beta$. Thus, in view of (b), (c) holds (and (e) is immediate from (c)). Furthermore, $M_{\beta_x} = U'(M_\beta)$ and so (1.9.3) gives (d). ∎

1.10. Let L be a Lie algebra over F and K be an ideal. If V is a K-module with corresponding representation σ then the stabilizer $\operatorname{Stab}(V, L)$ of V in L is defined by

$\operatorname{Stab}(V, L) = \{x \in L \mid$ there exists $\eta \in \operatorname{Hom}_F(V, V)$ such that $\sigma[x, y] = [\eta, \sigma y]$ for all $y \in K\}$.

This is a subalgebra of L containing K. If L is restricted then so is Stab(V, L).

At characteristic 0 (over an algebraically closed field) Blattner [Bla69] has shown that if V is irreducible and if W is an irreducible module for $H = \text{Stab}(V, L)$, which as a K-module is a direct sum of copies of V, then the induced L-module $UL \otimes_{UH} W$ is irreducible; Dixmier [Dix71] has used this result to show that every irreducible L-module containing an irreducible K-submodule is isomorphic to such an induced module. As remarked in [Blo74], the Blattner–Dixmier result goes through for restricted Lie algebras at prime characteristic when the enveloping algebras UL and UH are replaced by the restricted enveloping algebras uL and uH.

If H is a restricted subalgebra of a restricted Lie algebra L, and if W is a restricted H-module, then the induced module $uL \otimes_{uH} W$ is a restricted L-module, of dimension $p^n(\dim W)$, where n is the codimension of H in L. The proof of the restricted version of the Blattner–Dixmier result is essentially unchanged from that at characteristic 0, using (in the analogue of the Blattner proof) the fact that p-truncated standard monomials give a basis of uL. The precise result we shall use is as follows.

LEMMA 1.10.1. *Let L be a restricted Lie algebra over an algebraically closed field of characteristic p, K a restricted ideal in L, M an irreducible restricted L-module, V an irreducible K-submodule of M, $H = \text{Stab}(V, L)$, and $\tilde{V}$ the sum of all K-submodules of M which are K-isomorphic to V. Then $\tilde{V}$ is an irreducible H-submodule of M, and M is isomorphic to the irreducible L-module $uL \otimes_{uH} \tilde{V}$.*

COROLLARY 1.10.2. $\dim M$ *is divisible by p^n, where n is the codimension of H in L.*

1.11. The following result is proved by a special case of an argument due to Kreknin [Kre71].

LEMMA 1.11.1. (a) $W(1 : \mathbf{1})$ *contains a unique subalgebra of codimension* 1. *This subalgebra is* $W(1 : \mathbf{1})_0$. *Furthermore, any subalgebra of codimension* 2 *in* $W(1 : \mathbf{1})$ *is contained in a subalgebra of codimension* 1.

(b) $H(2 : \mathbf{1})^{(2)}$ *contains a unique proper subalgebra of codimension* $\leqslant 2$. *This subalgebra is* $(H(2 : \mathbf{1})^{(2)})_0$ *and is of codimension* 2.

Proof. Let A denote $W(1 : \mathbf{1})$ in case (a) and $H(2 : 1)^{(2)}$ in case (b). Since A_0 clearly has the asserted properties it is sufficient to prove that if M is a subalgebra of codimension $\leqslant 2$ in A and $M \not\subseteq A_0$ then $M = A$. We may assume (replacing M by gr M) that M is graded. We write $M = \sum M_{[i]}$. Thus $M_{[-1]} \neq (0)$. Since codim $M \leqslant 2$ we have $M_{[i]} = A_{[i]}$ for some i,

$i=2, 3, 4$. Direct computation shows in either case that if $0 \neq x \in A_{[-1]}$ then $[x, A_{[i]}] = A_{[i-1]}$ for all i, $0 \leqslant i \leqslant p-3$, and that $A_{[-1]} + A_{[0]} + A_{[1]} + A_{[2]}$ generates A. Thus $M_{[-1]} \neq (0)$ and $M_{[i]} = A_{[i]}$ for $i=2, 3$, or 4 implies $M = A$, as required. ∎

1.12. The following lemma due to Schue [Sch69] will be used repeatedly to derive information on the structure of simple Lie algebras.

LEMMA 1.12.1 (Schue's Lemma). *Let L be a simple Lie algebra, H be a nilpotent subalgebra of* Der L, *and $L = \sum_{\gamma \in \Gamma} L_\gamma$ be the weight space decomposition of L with respect to H. Let $x \in \mathfrak{z}_L(H)$ and $\Gamma' = \{\gamma \in \Gamma \mid \gamma(x) \neq 0\}$. If $\Gamma' \neq \varnothing$, then $L = \sum_{\gamma \in \Gamma'} L_\gamma + \sum_{\gamma,\delta \in \Gamma'} [L_\gamma, L_\delta]$. In particular, if H is a Cartan subalgebra of L, then $H = \sum_{\gamma \in \Gamma'} [L_\gamma, L_{-\gamma}]$.*

Proof. One checks that $J = \sum_{\gamma \in \Gamma'} L_\gamma + \sum_{\gamma,\delta \in \Gamma'} [L_\gamma, L_\delta]$ is invariant under ad L_α for every $\alpha \in \Gamma$. Thus, as $\Gamma' \neq \varnothing$, J is a nonzero ideal in L, so $J = L$. ∎

2. SIMPLE LIE ALGEBRAS OF TORAL RANK ONE

We will require a number of detailed results about certain of the simple Lie algebras of toral rank one. By [Wil78] (which uses [Kap58]) a simple Lie algebra has toral rank one if and only if it is isomorphic to $\mathfrak{sl}(2)$, some $W(1:\mathbf{n})$, or some $H(2:\mathbf{n}:\Phi)^{(2)}$. For completeness we develop the theory of the algebras $H(2:\mathbf{n}:\Phi)^{(2)}$ from the beginning, even though our treatment is closely parallel to the special case $m=2$ of the treatment of $H(m:\mathbf{n})^{(2)}$ by Kostrikin and Šafarevič [KS69, Sect. 1.6.1] (and that treatment is analogous to the well-known characteristic zero case).

2.1. Let $\omega = dx_1 \wedge dx_2$. Recall (Section 1.1) that $H(2) = \{D \in W(2) \mid D\omega = 0\}$ and

$$H(2:\mathbf{n}:\Phi) = \Phi H(2) \cap W(2:\mathbf{n}). \tag{2.1.1}$$

DEFINITION 2.1.1. Let Φ be a continuous automorphism of $W(2)$ and ϕ be the unique divided automorphism of $\mathfrak{A}(2)$ such that $\Phi D = \phi D \phi^{-1}$ for all $D \in W(2)$. Define

$$J(\Phi) = D_1(\phi(x_1))\, D_2(\phi(x_2)) - D_1(\phi(x_2))\, D_2(\phi(x_1)).$$

Note that (see (1.1.1))

$$\begin{aligned}\phi\omega = \phi(dx_1 \wedge dx_2) &= d(\phi(x_1)) \wedge d(\phi(x_2)) \\ &= (D_1(\phi(x_1))\,dx_1 + D_2(\phi(x_1))\,dx_2) \\ &\quad \wedge (D_1(\phi(x_2))\,dx_1 + D_2(\phi(x_2))\,dx_2)\end{aligned}$$

and so

$$\phi\omega = J(\Phi)\omega. \tag{2.1.2}$$

Therefore $J(\Phi)$ is invertible (in fact $J(\Phi)^{-1} = \phi(J(\Phi^{-1}))$).

DEFINITION 2.1.2. Let $a \in \mathfrak{A}(2)$ be invertible. Set $a_i = a^{-1} D_i a$ for $i = 1, 2$. For $f \in \mathfrak{A}(2)$ define

$$\mathscr{D}_a(f) = ((D_2 + a_2) f) D_1 - ((D_1 + a_1) f) D_2.$$

Note that

$$(D_i + a_i) f = a^{-1} D_i(af) \qquad \text{for} \quad i = 1, 2. \tag{2.1.3}$$

LEMMA 2.1.3. *Let $\Phi \in \operatorname{Aut} W(2)$, $a = J(\Phi)$. Then $\mathscr{D}_a$ is a linear map of $\mathfrak{A}(2)$ onto $\Phi H(2)$ with kernel Fa^{-1}.*

Proof. $\Phi H(2) = \Phi\{D \in W(2) \mid D\omega = 0\} = \{\phi D \phi^{-1} \in W(2) \mid D\omega = 0\} = \{E \in W(2) \mid \phi^{-1} E \phi\omega = 0\} = \{E \in W(2) \mid E\phi\omega = 0\}$. By (2.1.2) we have

$$\Phi H(2) = \{E \in W(2) \mid E(J(\Phi)\omega) = 0\}. \tag{2.1.4}$$

Write $E = g_1 D_1 + g_2 D_2$. Then $E(a\omega) = (Ea)\omega + a(E\omega) = (g_1 D_1 a + g_2 D_2 a)\omega + a(D_1 g_1 + D_2 g_2)\omega = (D_1(ag_1) + D_2(ag_2))\omega$. Thus

$$\Phi H(2) = \{g_1 D_1 + g_2 D_2 \in W(2) \mid D_1(ag_1) + D_2(ag_2) = 0\}. \tag{2.1.5}$$

Now it is well known (cf. [Wil76, Lemma 1.2]) that if $f_1, f_2 \in \mathfrak{A}(2)$ satisfy $D_1 f_1 + D_2 f_2 = 0$ then there exists $f \in \mathfrak{A}(2)$ such that $f_1 = D_2 f$, $f_2 = -D_1 f$. Thus (as a is invertible) $E = g_1 D_1 + g_2 D_2$ belongs to $\Phi H(2)$ if and only if there exists some $g \in \mathfrak{A}(2)$ such that $ag_1 = D_2(ag)$, $ag_2 = -D_1(ag)$. In view of (2.1.3) this is equivalent to $E = \mathscr{D}_a(g)$. Thus $\mathscr{D}_a$ (which is clearly linear) maps $\mathfrak{A}(2)$ onto $\Phi H(2)$. Since $\mathscr{D}_a(g) = 0$ if and only if $D_i(ag) = 0$ for $i = 1, 2$, we see that $\mathscr{D}_a(g) = 0$ if and only if $ag \in F$, as required. ∎

LEMMA 2.1.4. *Let $D = f_1 D_1 + f_2 D_2$, $E = g_1 D_1 + g_2 D_2 \in \Phi H(2)$, and $a = J(\Phi)$. Then*

$$[D, E] = \mathscr{D}_a(g_1 f_2 - g_2 f_1).$$

Proof. We have

$$[D, E] = (f_1 D_1 g_1 + f_2 D_2 g_1 - g_1 D_1 f_1 - g_2 D_2 f_1) D_1 + (f_1 D_1 g_2 + f_2 D_2 g_2 - g_1 D_1 f_2 - g_2 D_2 f_2) D_2.$$

As $D, E \in \Phi H(2)$ we see from (2.1.3) and (2.1.5) that

$$D_1 f_1 = -D_2 f_2 - a_1 f_1 - a_2 f_2,$$
$$D_2 f_2 = -D_1 f_1 - a_1 f_1 - a_2 f_2,$$
$$D_1 g_1 = -D_2 g_2 - a_1 g_1 - a_2 g_2,$$
$$D_2 g_2 = -D_1 g_1 - a_1 g_1 - a_2 g_2.$$

Making these substitutions in the expression for $[D, E]$ gives

$$\begin{aligned}&(f_1(-D_2 g_2 - a_1 g_1 - a_2 g_2) + f_2 D_2 g_1 - g_1(-D_2 f_2 - a_1 f_1 - a_2 f_2)\\ &\quad - g_2 D_2 f_1) D_1 + (f_1 D_1 g_2 + f_2(-D_1 g_1 - a_1 g_1 - a_2 g_2)\\ &\quad - g_1 D_1 f_2 - g_2(-D_1 f_1 - a_1 f_1 - a_2 f_2)) D_2\\ &= \mathscr{D}_a(g_1 f_2 - g_2 f_1),\end{aligned}$$

as required. ∎

COROLLARY 2.1.5.

$$[\mathscr{D}_a(f), \mathscr{D}_a(g)] = \mathscr{D}_a((D_1 + a_1) g(D_2 + a_2) f - (D_1 + a_1) f(D_2 + a_2) g).$$

Let $\gamma(\mathbf{n}) = (p^{n_1} - 1)\varepsilon_1 + (p^{n_2} - 1)\varepsilon_2$. Let $\phi(\gamma(\mathbf{n}))$ be the automorphism of the divided power algebra $\mathfrak{A}(2)$ defined by $\phi(\gamma(\mathbf{n}))x_1 = x_1 + x^{\varepsilon_1 + \gamma(\mathbf{n})}$ and $\phi(\gamma(\mathbf{n}))x_2 = x_2$. Define $\Phi(\gamma(\mathbf{n}))E = \phi(\gamma(\mathbf{n}))\, E(\phi(\gamma(\mathbf{n})))^{-1}$ for $E \in W(2)$. Then $\Phi(\gamma(\mathbf{n})) \in \operatorname{Aut} W(2)$.

Recall (cf. [Wil69]) that if $y \in \mathfrak{A}(2)_1$ then $\exp y$ is defined to be $\sum_{j=0}^{\infty} y^{(j)}$. Let $\delta(\mathbf{n})$ be the automorphism of the divided power algebra $\mathfrak{A}(2)$ defined by $\delta(\mathbf{n})x_1 = x_1 + x_1(\exp(x^{p^{n_1}\varepsilon_1}) - 1)$ and $\delta(\mathbf{n})x_2 = x_2$. Define $\Delta(\mathbf{n})E = \delta(\mathbf{n})\, E(\delta(\mathbf{n}))^{-1}$ for $E \in W(2)$. Then $\Delta(\mathbf{n}) \in \operatorname{Aut} W(2)$. Note that the automorphism $\Delta(\mathbf{n})$ is the same as the automorphism $\Phi(1)$ of [Wil80] (cf. [BW82, Sect. 1.8]). Since it is never necessary to consider the automorphisms $\Phi(i)$ for $i \neq 1$, we favor the simpler notation here.

Note that $J(\Phi(\gamma(\mathbf{n}))) = 1 + x^{\gamma(\mathbf{n})}$ and $J(\Delta(\mathbf{n})) = \exp(x^{p^{n_1}\varepsilon_1})$.

When $\mathbf{n}$ is clear from context we may write $\Phi(\gamma)$ for $\Phi(\gamma(\mathbf{n}))$ and Δ for $\Delta(\mathbf{n})$. In particular, we will write $H(2 : \mathbf{n} : \Phi(\gamma))$ for $H(2 : \mathbf{n} : \Phi(\gamma(\mathbf{n})))$ and $H(2 : \mathbf{n} : \Delta)$ for $H(2 : \mathbf{n} : \Delta(\mathbf{n}))$.

Recall ([Wil80]; cf. [BW82, Sect. 1.8]) that any simple algebra

$H(2:\mathbf{m}:\Psi)^{(2)}$ is isomorphic to some $H(2:\mathbf{n}:\Phi)^{(2)}$, where $\Phi=I$, $\Phi=\Phi(\gamma(\mathbf{n}))$, or $\Phi=\Delta(\mathbf{n})$ and where if $\mathbf{m}=(m_1,m_2)$ then $\mathbf{n}=\mathbf{m}$ or $\mathbf{n}=(m_2,m_1)$.

Recall from Section 1.1 that the algebras $H(2:\mathbf{n}:\Phi)$ are filtered.

Notation 2.1.6. For any Φ we write $\mathscr{D}_\Phi$ for $\mathscr{D}_{J(\Phi)}$. In addition, we write $\mathscr{D}$ for $\mathscr{D}_1$, $\mathscr{D}_{\gamma(\mathbf{n})}$ for $\mathscr{D}_{J(\Phi(\gamma(\mathbf{n})))}$ (or, if $\mathbf{n}$ is clear from context, just $\mathscr{D}_\gamma$), and $\mathscr{D}_{\Delta(\mathbf{n})}$ for $\mathscr{D}_{J(\Delta(\mathbf{n}))}$ (or, if $\mathbf{n}$ is clear from context, just $\mathscr{D}_\Delta$).

Note that if $\Phi\in\operatorname{Aut}W(2)$ and $a=J(\Phi)$ then

$$\mathscr{D}_a(\mathfrak{A}(2)_i)\in(\Phi H(2))_{i-2}. \tag{2.1.6}$$

LEMMA 2.1.7. *$H(2:\mathbf{n}:\Phi)_0$ is a restricted subalgebra of $W(2)$.*

Proof. Let $E\in H(2:\mathbf{n}:\Phi)_0$. *Then by* (2.1.4), $E(J(\Phi)\omega)=0$ *and so* $E^p(J(\Phi)\omega)=0$. *Since* $E^p\in W(2)_0$, (2.1.4) *shows that* $E^p\in(\Phi H(2))_0$. As $E\in H(2:\mathbf{n}:\Phi)_0\subseteq W(2:\mathbf{n})$ we have $E\mathfrak{A}(2:\mathbf{n})\subseteq\mathfrak{A}(2:\mathbf{n})$ and so $(E^p)\,\mathfrak{A}(2:\mathbf{n})\subseteq\mathfrak{A}(2:\mathbf{n})$. Then $E^p\in W(2:\mathbf{n})$ so $E^p\in H(2:\mathbf{n}:\Phi)_0$. ∎

PROPOSITION 2.1.8. (a) *Let $M=H(2:\mathbf{n})$. Then*

(i) $\mathscr{D}(f)=(D_2f)D_1-(D_1f)D_2$. *In particular,* $\mathscr{D}(x_1)=-D_2$, $\mathscr{D}(x_2)=D_1$, $\mathscr{D}(x^{p^{n_1}\varepsilon_1})=-x^{(p^{n_1}-1)\varepsilon_1}D_2$, and $\mathscr{D}(x^{p^{n_2}\varepsilon_2})=x^{(p^{n_2}-1)\varepsilon_2}D_1$.

(ii) *M has basis* $\{\mathscr{D}(x^\alpha)\mid\alpha\in A(2:\mathbf{n}),\ \alpha\neq 0\}\cup\{\mathscr{D}(x^{p^{n_i}\varepsilon_i})\mid i=1,2\}$.

(iii) *$M^{(1)}$ has basis* $\{\mathscr{D}(x^\alpha)\mid\alpha\in A(2:\mathbf{n}),\ \alpha\neq 0\}$.

(iv) *$M^{(2)}$ has basis* $\{\mathscr{D}(x^\alpha)\mid\alpha\in A(2:\mathbf{n}),\ \alpha\neq 0,\ \gamma(\mathbf{n})\}$.

(v) *$M^{(2)}$ is a simple Lie algebra.*

(vi) *$\overline{M^{(2)}}$ has basis* $\{\operatorname{ad}\mathscr{D}(x^\alpha)\mid\alpha\in A(2:\mathbf{n}),\ \alpha\neq 0,\ \gamma(\mathbf{n})\}\cup\{(\operatorname{ad}D_i)^{p^j}\mid i=1,2,\ 1\leqslant j<n_i\}$.

(vii) $\operatorname{Der}(M^{(2)})$ *has basis* $\{\operatorname{ad}\mathscr{D}(x^\alpha)\mid\alpha\in A(2:\mathbf{n}),\ \alpha\neq 0\}\cup\{\operatorname{ad}\mathscr{D}(x^{p^{n_i}\varepsilon_i})\mid i=1,2\}\cup\{(\operatorname{ad}D_i)^{p^j}\mid i=1,2,\ 1\leqslant j<n_i\}\cup\{\operatorname{ad}(x_1D_1+x_2D_2)\}$.

(b) *Let $M=H(2:\mathbf{n}:\Phi(\gamma))$. Then*:

(i) $\mathscr{D}_\gamma(f)=((D_2+x^{\gamma(\mathbf{n})-\varepsilon_2})f)D_1-((D_1+x^{\gamma(\mathbf{n})-\varepsilon_1})f)D_2$. *In particular* $\mathscr{D}_\gamma(x_1)=-(1-x^{\gamma(\mathbf{n})})D_2$, $\mathscr{D}_\gamma(x_2)=(1-x^{\gamma(\mathbf{n})})D_1$, $\mathscr{D}_\gamma(x^{p^{n_1}\varepsilon_1}(1-x^{\gamma(\mathbf{n})}))=-x^{(p^{n_1}-1)\varepsilon_1}D_2$, *and* $\mathscr{D}_\gamma(x^{p^{n_2}\varepsilon_2}(1-x^{\gamma(\mathbf{n})}))=x^{(p^{n_2}-1)\varepsilon_2}D_1$.

(ii) *M has basis* $\{\mathscr{D}_\gamma(x^\alpha)\mid\alpha\in A(2:\mathbf{n}),\ \alpha\neq 0\}\cup\{\mathscr{D}_\gamma(x^{p^{n_i}\varepsilon_i}(1-x^{\gamma(\mathbf{n})}))\mid i=1,2\}$.

(iii) *$M^{(1)}$ has basis* $\{\mathscr{D}_\gamma(x^\alpha)\mid\alpha\in A(2:\mathbf{n}),\ \alpha\neq 0\}$.

(iv) *$M^{(1)}$ is a simple Lie algebra.*

(v) $\operatorname{Der}(M^{(1)})=\overline{M^{(1)}}$ *has basis* $\{\operatorname{ad}\mathscr{D}_\gamma(x^\alpha)\mid\alpha\in A(2:\mathbf{n}),\ \alpha\neq 0\}\cup\{\operatorname{ad}\mathscr{D}_\gamma(x^{p^{n_i}\varepsilon_i}(1-x^{\gamma(\mathbf{n})}))\mid i=1,2\}\cup\{(\operatorname{ad}\mathscr{D}_\gamma(x_1))^{p^j}\mid 1\leqslant j<n_2\}\cup\{(\operatorname{ad}\mathscr{D}_\gamma(x_2))^{p^j}\mid 1\leqslant j<n_1\}$.

(c) *Let* $M = H(2:\mathbf{n}:\Delta)$. *Then*:

(i) $\mathscr{D}_\Delta(f) = (D_2 f) D_1 - ((D_1 + x^{(p^{n_1}-1)\varepsilon_1}) f) D_2$. *In particular* $\mathscr{D}_\Delta(x_1) = -D_2$ *and* $\mathscr{D}_\Delta(x_2) = D_1 - x^{(p^{n_1}-1)\varepsilon_1+\varepsilon_2} D_2$.

(ii) *M has basis* $\{\mathscr{D}_\Delta(x^\alpha) \mid \alpha \in A(2:\mathbf{n})\}$.

(iii) *M is a simple Lie algebra.*

(iv) Der $M = \bar{M}$ *has basis* $\{\mathrm{ad}\ \mathscr{D}_\Delta(x^\alpha) \mid \alpha \in A(2:\mathbf{n})\} \cup \{(\mathrm{ad}\ \mathscr{D}_\Delta(x_1))^{p^j} \mid 1 \leqslant j < n_2\} \cup \{(\mathrm{ad}\ \mathscr{D}_\Delta(x_2))^{p^j} \mid 1 \leqslant j < n_1\} \cup \{\mathrm{ad}(x_1 D_1 + x_2 D_2)\}$.

Proof. Let $M = H(2:\mathbf{n}:\Phi)$, where Φ is one of I, $\Phi(\gamma)$, Δ. Let $a = J(\Phi)$, so $a = 1$, $1 + x^{\gamma(\mathbf{n})}$, $\exp(x^{p^{n_1}\varepsilon_1})$. Part (i) of (a), (b), and (c) follows immediately from Definition 2.1.2.

Our first goal is to determine $\{f \in \mathfrak{A}(2) \mid \mathscr{D}_\Phi(f) \in H(2:\mathbf{n}:\Phi)\}$. Denote this space by $U(\Phi)$. In view of Lemma 2.1.3, $U(\Phi) = \{f \in \mathfrak{A}(2) \mid \mathscr{D}_\Phi(f) \in W(2:\mathbf{n})\}$ and, by (2.1.3), $U(\Phi) = \{f \in \mathfrak{A}(2) \mid a^{-1} D_i(af) \in \mathfrak{A}(2:\mathbf{n}), i = 1, 2\}$. If $a \in \mathfrak{A}(2:\mathbf{n})$, as happens if $\Phi = I$ or $\Phi = \Phi(\gamma(\mathbf{n}))$, then $U(\Phi) = \{a^{-1}f \mid D_i f \in \mathfrak{A}(2:\mathbf{n}), i = 1, 2\}$. It is clear that $\{f \in \mathfrak{A}(2) \mid D_i f \in \mathfrak{A}(2:\mathbf{n}), i = 1, 2\}$ has basis $\{x^\alpha \mid \alpha \in A(2:\mathbf{n})\} \cup \{x^{p^{n_i}\varepsilon_i} \mid i = 1, 2\}$. This and the fact that $\ker \mathscr{D}_a = Fa^{-1}$ (by Lemma 2.1.3) prove part (ii) of (a) and (b).

Now suppose $\Phi = \Delta(\mathbf{n})$ so $a = \exp(x^{p^{n_1}\varepsilon_1})$. Since $a_1 = a^{-1} D_1 a = x^{(p^{n_1}-1)\varepsilon_1}$ and $a_2 = a^{-1} D_2 a = 0$ both belong to $\mathfrak{A}(2:\mathbf{n})$ it is clear that $\mathfrak{A}(2:\mathbf{n}) \subseteq U(\Phi)$. Also $a^{-1} = \exp(-x^{p^{n_1}\varepsilon_1}) \in U(\Phi)$. We claim that $U(\Phi) = \mathfrak{A}(2:\mathbf{n}) + Fa^{-1}$. To see this let $f = \sum f_\alpha x^\alpha \in U(\Phi)$. By adding an element of $\mathfrak{A}(2:\mathbf{n}) + Fa^{-1}$ we may assume $f_\alpha = 0$ if $\alpha \in A(2:\mathbf{n}) \cup \{p^{n_1}\varepsilon_1\}$. We will show that $f = 0$. Since $f \in U(\Phi)$ and $D_2 a = 0$, $a^{-1} D_2(af) = D_2 f \in \mathfrak{A}(2:\mathbf{n})$ and so $f_\alpha = 0$ unless $\alpha(2) = 0$ or $\alpha = r\varepsilon_1 + p^{n_2}\varepsilon_2$, $0 \leqslant r \leqslant (p^{n_1}-1)$. Thus $f = f_1 + x^{p^{n_2}\varepsilon_2} f_2$, where $f_1 = \sum_{j > p^{n_1}} g_j x^{j\varepsilon_1}$ and $f_2 = \sum_{0 \leqslant r \leqslant (p^{n_1}-1)} h_r x^{r\varepsilon_1} \in \mathfrak{A}(2:\mathbf{n})$, with the g_j and $h_r \in F$. Then $a^{-1} D_1(af) = (D_1 + a_1)(f_1 + x^{p^{n_2}\varepsilon_2} f_2) \in \mathfrak{A}(2:\mathbf{n})$. Thus $(D_1 + a_1) f_2 = 0$ and $(D_1 + a_1) f_1 \in \mathfrak{A}(2:\mathbf{n})$. Now $(D_1 + a_1) f_1 = \sum_{j \geqslant p^{n_1}} g'_j x^{j\varepsilon_1} \in \mathfrak{A}(2:\mathbf{n})$ for some $g'_j \in F$ and so $(D_1 + a_1) f_1 = 0$. But then for $i = 1, 2$, $0 = (D_1 + a_1) f_i = a^{-1} D_1(af_i)$. Since $D_2(af_i) = 0$ we have $af_i \in F$, $f_i \in Fa^{-1}$. Since f_1 is not invertible and $f_2 \in \mathfrak{A}(2:\mathbf{n})$, we conclude in each case that $f_i = 0$. Thus $f = 0$ as claimed. Hence $U(\Phi) = \mathfrak{A}(2:\mathbf{n}) + Fa^{-1}$. In view of Lemma 2.1.3, this proves part (ii) of (c).

We now prove the remaining parts of (a). Let $M = H(2:\mathbf{n})$. It follows from Lemma 2.1.4 that $M^{(1)} \subseteq \{\mathscr{D}(f) \mid f \in \mathfrak{A}(2:\mathbf{n})\}$ and from Corollary 2.1.5 that

$$[\mathscr{D}(x_1), \mathscr{D}(f)] = -\mathscr{D}(D_2 f), \qquad [\mathscr{D}(x_2), \mathscr{D}(f)] = \mathscr{D}(D_1 f), \quad (2.1.7)$$

$$[\mathscr{D}(x^{\varepsilon_1+\varepsilon_2}), \mathscr{D}(x^\alpha)] = (\alpha(1) - \alpha(2))\, \mathscr{D}(x^\alpha), \quad (2.1.8)$$

and

$$[\mathscr{D}(x^{p^{n_2}\varepsilon_2}), \mathscr{D}(x^{p^{n_1}\varepsilon_1})] = \mathscr{D}(x^{\gamma(\mathbf{n})}). \quad (2.1.9)$$

Then part (iii) of (a) follows from (2.1.7) and (2.1.9). Corollary 2.1.5 also shows that for $\alpha, \beta \in A(2:\mathbf{n})$

$$[\mathscr{D}(x^\alpha), \mathscr{D}(x^\beta)] = \left\{\binom{\alpha+\beta-\varepsilon_1-\varepsilon_2}{\alpha-\varepsilon_2} - \binom{\alpha+\beta-\varepsilon_1-\varepsilon_2}{\alpha-\varepsilon_1}\right\} \mathscr{D}(x^{\alpha+\beta-\varepsilon_1-\varepsilon_2}).$$

This, together with the fact that in F

$$\binom{p^u-1}{j} + \binom{p^u-1}{j-1} = \binom{p^u}{j} = 0 \qquad \text{for} \quad 0 < j < p^u$$

gives $[\mathscr{D}(x^\alpha), \mathscr{D}(x^\beta)] = 0$ whenever $\alpha+\beta-\varepsilon_1-\varepsilon_2 = \gamma(\mathbf{n})$. Together with (2.1.7) this proves part (iv) of (a). Then (2.1.7) and (2.1.8) show that $M^{(3)} = M^{(2)}$. Since it is known (e.g., [Wil76, Corollary 2.5]) that $M^{(\infty)}$ is simple, we have that $M^{(2)}$ is simple, proving (v) of (a).

Since Lemma 2.1.7 shows that M_0 is restricted, we have that $(\mathscr{D}(x^\alpha))^p \in M$ for all $\alpha \in A(2:\mathbf{n})$, $\alpha(1)+\alpha(2) \geqslant 2$. Now, as $\mathscr{D}(x^\alpha) \in M_{[\alpha(1)+\alpha(2)-2]}$, $(\mathscr{D}(x^\alpha))^p \in M_{p[\alpha(1)+\alpha(2)-2]}$. But, by (ii) and (iv) of (a), $M_{[j]} = M^{(2)}_{[j]}$ unless $j = p^{n_1}-2$, $p^{n_2}-2$, or $p^{n_1}+p^{n_2}-4$. Since none of these numbers is a multiple of p, $(M^{(2)})_0$ is restricted and part (vi) of (a) follows.

Since $(x_1 D_1 + x_2 D_2)\omega = 2\omega$ we have $x_1 D_1 + x_2 D_2 \in N_{W(2:n)}(H(2))$. Hence $\operatorname{ad}(x_1 D_1 + x_2 D_2) \in \operatorname{Der} H(2:\mathbf{n})^{(2)}$. Part (vii) of (a) now follows from the fact that the given set of derivations of $M^{(2)}$ is linearly independent and that by [Blo58a, Theorem 14; BW82, Lemma 1.8.3b], $\dim(\operatorname{Der} M^{(2)}/M^{(2)}) = n_1 + n_2 + 2$. (Cf. also [Cel70, p. 127], where the degree derivation $x_1 D_1 + x_2 D_2$ is omitted.)

Next we prove the remaining parts of (b). Let $M = H(2:\mathbf{n}:\Phi(\gamma))$ (so that $a = 1 + x^{\gamma(\mathbf{n})}$). It follows from Lemma 2.1.4 that $M^{(1)} \subseteq \{\mathscr{D}_\gamma(f) \mid f \in \mathfrak{A}(2:\mathbf{n})\}$. Also Corollary 2.1.5 shows that $[\mathscr{D}_\gamma(x_1), \mathscr{D}_\gamma(x_2)] = -\mathscr{D}_\gamma((1-x^{\gamma(\mathbf{n})})^2) = -\mathscr{D}_\gamma(1-2x^{\gamma(\mathbf{n})}) = -\mathscr{D}_\gamma(a^{-1}-x^{\gamma(\mathbf{n})}) = \mathscr{D}_\gamma(x^{\gamma(\mathbf{n})})$. Thus $\mathscr{D}_\gamma(x^{\gamma(\mathbf{n})}) \in M^{(1)}$. As Corollary 2.1.5 also shows that if $f \in \mathfrak{A}(2)_j$ (and thus by (2.1.6), $\mathscr{D}_\gamma(f) \in M_{j-2}$ and $\mathscr{D}_\gamma(D_i f) \in M_{j-3}$) then

$$\begin{aligned} [\mathscr{D}_\gamma(x_1), \mathscr{D}_\gamma(f)] &\equiv -\mathscr{D}_\gamma(D_2 f), \\ [\mathscr{D}_\gamma(x_2), \mathscr{D}_\gamma(f)] &\equiv \mathscr{D}_\gamma(D_1 f) \qquad \operatorname{mod} M_{j-2}, \end{aligned} \tag{2.1.10}$$

it follows that part (iii) of (b) holds and that $M^{(2)} = M^{(1)}$. Then $M^{(1)} = M^{(\infty)}$ is simple (by Corollary 2.5 of [Wil76]) and so part (iv) of (b) holds. By [BW82, Lemma 1.8.3(a); Blo58a, Theorem 14; JJac75, p. 78], we see that $\dim(\operatorname{Der} M^{(1)}/M^{(1)}) = \dim(\overline{M^{(1)}}/M^{(1)}) = n_1 + n_2$ so that $\operatorname{Der} M^{(1)} = \overline{M^{(1)}}$. Furthermore, the set

$$\{\text{ad}\,\mathscr{D}_\gamma(x^{p^{n_i}\varepsilon_i}(1-x^{\gamma(\mathbf{n})}))\mid i=1,2\}$$
$$\cup\{(\text{ad}\,\mathscr{D}_\gamma(x_1))^{p^j}\mid 1\leqslant j<n_2\}$$
$$\cup\{(\text{ad}\,\mathscr{D}_\gamma(x_2))^{p^j}\mid 1\leqslant j<n_1\}$$

of n_1+n_2 derivations is linearly independent modulo inner derivations. Thus part (v) of (b) holds.

Finally, we prove the remaining parts of (c). Let $M=H(2:\mathbf{n}:\Delta)$. By Corollary 2.1.5,

$$[\mathscr{D}_\Delta(x_2),\mathscr{D}_\Delta(x^{(p^{n_2}-1)\varepsilon_2})]=2\,\mathscr{D}_\Delta(x^{\gamma(\mathbf{n})}).$$

If $f\in\mathfrak{A}(2)_j$ then (as $\mathscr{D}_\Delta(f)\in M_{j-2}$ and $\mathscr{D}_\Delta(D_i f)\in M_{j-3}$ by (2.1.6))

$$[\mathscr{D}_\Delta(x_1),\mathscr{D}_\Delta(f)]\equiv-\mathscr{D}_\Delta(D_2 f)\qquad\text{mod }M_{j-2},$$
$$[\mathscr{D}_\Delta(x_2),\mathscr{D}_\Delta(f)]\equiv\mathscr{D}_\Delta(D_1 f)\qquad\text{mod }M_{j-2}.$$

These results show that $M=M^{(1)}$. Then, by Corollary 2.5 of [Wil76], $M=M^{(\infty)}$ is simple, proving part (iii) of (c). Since $(x_1D_1+x_2D_2)\,J(\Delta)=0$ and therefore $(x_1D_1+x_2D_2)(J(\Delta)\omega)=2J(\Delta)\omega$, we have $x_1D_1+x_2D_2\in N_{W(2:\mathbf{n})}(\Delta H(2))$. Hence $\text{ad}(x_1D_1+x_2D_2)\in\text{Der}\,H(2:\mathbf{n}:\Delta)$.

Now it is known [BW82, Proposition 1.8.5(c); Blo58b, Corollary 2] that $\dim(\text{Der}\,M/M)=\dim(\bar{M}/M)=n_1+n_2-1$. Since the set

$$\{(\text{ad}\,\mathscr{D}_\Delta(x_1))^{p^j}\mid 1\leqslant j<n_2\}\cup\{(\text{ad}\,\mathscr{D}_\Delta(x_2))^{p^j}\mid 1\leqslant j<n_1\}$$
$$\cup\{\text{ad}(x_1D_1+x_2D_2)\}$$

of n_1+n_2-1 derivations is linearly independent modulo inner derivations, we see that part (iv) of (c) holds. ∎

The special case $H(2:\mathbf{1})$ is of particular interest. The following two results, which deal with this case, are proved by direct computation.

COROLLARY 2.1.9. *$H(2:\mathbf{1})^{(2)}$ is a restricted ideal in $H(2:\mathbf{1})$ and $H(2:\mathbf{1})/H(2:\mathbf{1})^{(2)}$ is nil. Furthermore, $H(2:\mathbf{1})=(\text{Der}(H(2:\mathbf{1})^{(2)}))^{(1)}$ is a restricted ideal in $\text{Der}(H(2:\mathbf{1})^{(2)})$ and $\text{Der}(H(2:\mathbf{1})^{(2)})=(\text{Der}(H(2:\mathbf{1})^{(2)}))^{(1)}+F(x_1D_1+x_2D_2)$. The element $x_1D_1+x_2D_2$ is toral.*

COROLLARY 2.1.10. *$H(2:\mathbf{1})$ has basis*

$$\{\mathscr{D}(x_1^ix_2^j)\mid 0\leqslant i,j\leqslant p-1,(i,j)\neq(0,0)\}\cup\{x_2^{p-1}D_1,x_1^{p-1}D_2\}$$

and $H(2:\mathbf{1})^{(2)}$ has basis

$$\{\mathscr{D}(x_1^ix_2^j)\mid 0\leqslant i,j\leqslant p-1,(i,j)\neq(0,0),(p-1,p-1)\}$$

with multiplication given by

$$[\mathscr{D}(x_1^i x_2^j), \mathscr{D}(x_1^k x_2^l)] = (jk - il)\, \mathscr{D}(x_1^{i+k-1} x_2^{j+l-1}),$$

or, equivalently,

$$[\mathscr{D}((x_1+1)^i x_2^j), \mathscr{D}((x_1+1)^k x_2^l)] = (jk - il)\, \mathscr{D}((x_1+1)^{i+k-1} x_2^{j+l-1}).$$

Let $T = F\mathscr{D}((x_1+1)x_2)$, a maximal torus in $H(2:\mathbf{1})^{(2)}$. Let $\alpha \in T^*$ be defined by $\alpha(\mathscr{D}((x_1+1)x_2)) = 1$. Write

$$V_{i,l} = \operatorname{span}\{\mathscr{D}((x_1+1)^{i+j} x_2^j) \mid l \leqslant j \leqslant p-1\}$$

(so that $V_{i,p-1} \subseteq V_{i,p-2} \subseteq \cdots \subseteq V_{i,0} = H(2:\mathbf{1})_{i\alpha}$). Let I denote the nil radical of $\mathfrak{z}_{H(2:\mathbf{1})^{(2)}}(T)$.

LEMMA 2.1.11.

(a) $[\mathscr{D}((x_1+1)^2 x_2^2), \mathscr{D}((x_1+1)^{i+j} x_2^j)] = 2i\, \mathscr{D}((x_1+1)^{i+j+1} x_2^{j+1})$.

(b) *The only nonzero* ad *I-invariant subspaces of* $H(2:\mathbf{1})_{i\alpha}$ $(i \in \mathbf{Z}_p^*)$ *are the* $V_{i,l}$ $(0 \leqslant l \leqslant p-1)$.

(c) $[V_{i,l}, V_{a,b}] \subseteq V_{i+a,l+b-1}$.

Proof. Parts (a) and (c) follow from Corollary 2.1.10 and (b) follows from (a). ∎

2.2. We now prove the existence of certain tori in the algebras $\bar{S}$, where $S = W(1:\mathbf{n})$ or $H(2:\mathbf{n}:\Phi)^{(1)}$.

LEMMA 2.2.1. *Let G be an elementary abelian group of order p^n and let S be a subset of G which generates G. Let M be a vector space with basis $\{u_\alpha \mid \alpha \in S\}$. Set $u_\gamma = 0$ if $\gamma \notin S$. Suppose that there is some function $f\colon S \times S \to F$ such that the product*

$$[u_\alpha, u_\beta] = f(\alpha, \beta) u_{\alpha+\beta} \qquad \text{for all} \quad \alpha, \beta \in S \tag{2.2.1}$$

gives M the structure of a Lie algebra. Then Der M *contains an n-dimensional torus.*

Proof. We may assume that $G = (\mathbf{Z}/p\mathbf{Z})^n$. For $1 \leqslant i \leqslant n$ let $\sigma_i\colon G \to F$ denote the projection onto the ith coordinate. Define $E_i \in \operatorname{End} M$ by

$$E_i(u_\alpha) = \sigma_i(\alpha) u_\alpha$$

for all $\alpha \in S$. It is immediate from (2.2.1) that $E_i \in \operatorname{Der} M$. Clearly the E_i

span a torus in Der M, and hence it remains only to prove that the E_i are linearly independent. But as S generates G this follows immediately from the linear independence of the σ_i. ∎

COROLLARY 2.2.2. (a) *Let A be an Albert–Zassenhaus algebra of dimension p^n. Then* Der A *contains a torus of dimension n.*

(b) *Let A be a simple algebra $\mathfrak{L}(G, \delta, f)$ of Block with $|G| = p^n$ and $G = G_0$ or G_1. Then* Der A *contains a torus of dimension n.*

Proof. An Albert–Zassenhaus algebra (cf. [Sel67, p. 109]) of dimension p^n has basis $\{u_\alpha \mid \alpha \in G\}$, where G is an elementary abelian group of order p^n and multiplication is given by (2.2.1) for an appropriate f. Thus part (a) follows from the lemma.

The algebra $\mathfrak{L}(G, \delta, f)$ of Block [Blo58a] with $|G| = p^n$ and $G = G_0$ or G_1 has basis $\{v_\alpha \mid \alpha \in S\}$, where G is an elementary abelian group, $S = G - \{0\}$ or G, and multiplication is given by

$$[v_\alpha, v_\beta] = g(\alpha, \beta) v_{\alpha+\beta-\delta}$$

for some $g\colon S \times S \to F$ and some $\delta \in G$. Setting $u_\alpha = v_{\alpha+\delta}$ we see that

$$[u_\alpha, u_\beta] = g(\alpha+\delta, \beta+\delta) u_{\alpha+\beta}$$

so that (b) follows from the lemma. ∎

COROLLARY 2.2.3. (a) *If $S = W(1 : \mathbf{n})$ (where $\mathbf{n} = (n)$) then $\bar{S}$ contains a torus of dimension n.*

(b) *If $S = H(2 : \mathbf{n})^{(2)}$ (where $\mathbf{n} = (n_1, n_2)$) then $\bar{S}$ contains a torus of dimension $n_1 + n_2 - 1$.*

(c) *If $S = H(2 : \mathbf{n} : \Phi(\gamma))^{(1)}$ (where $\mathbf{n} = (n_1, n_2)$) then $\bar{S}$ contains a torus of dimension $n_1 + n_2$.*

(d) *If $S = H(2 : \mathbf{n} : \Delta)$ (where $\mathbf{n} = (n_1, n_2)$) then $\bar{S}$ contains a torus of dimension $n_1 + n_2$.*

Proof. Let $S = W(1 : \mathbf{n})$. Then S is an Albert–Zassenhaus algebra of dimension p^n (by [BIO79, Corollary 5.1]). Since Der $S = \bar{S}$ (by [Wil71a, Lemma 4]), Corollary 2.2.2(a) proves part (a).

Now let $S = H(2 : \mathbf{n})^{(2)}$. By (vi) and (vii) of Proposition 2.1.8(a) we see that (Der $S)/\bar{S}$ is spanned by the cosets of ad $\mathscr{D}(x^{\gamma(\mathbf{n})})$, ad $\mathscr{D}(x^{p^{n_1}\varepsilon_1})$, ad $\mathscr{D}(x^{p^{n_2}\varepsilon_2})$, and ad$(x_1 D_1 + x_2 D_2)$. Clearly the cosets of ad $\mathscr{D}(x^{\gamma(\mathbf{n})})$, ad $\mathscr{D}(x^{p^{n_1}\varepsilon_1})$, and ad $\mathscr{D}(x^{p^{n_2}\varepsilon_2})$ span a nil ideal in (Der $S)/\bar{S}$ and so any torus of Der S is contained in $\bar{S} + F(\mathrm{ad}(x_1 D_1 + x_2 D_2))$. Thus to prove part (b) it is sufficient to show that Der S contains a torus of dimension $n_1 + n_2$. Since by [BW82, Lemma 1.8.3(b)] S is isomorphic to a Block algebra

$\mathfrak{L}(G, \delta, f)$ with $|G| = p^{n_1+n_2}$ and $G = G_1$, part (b) follows from Corollary 2.2.2(b).

Next let $S = H(2 : \mathbf{n} : \Phi(\gamma))^{(1)}$. Then $\bar{S} = \operatorname{Der} S$ by Proposition 2.1.8(b)(v) and S is isomorphic to a Block algebra $\mathfrak{L}(G, \delta, f)$ with $|G| = p^{n_1+n_2}$ and $G = G_0$ by [BW82, Lemma 1.8.3(a)]. Thus part (c) follows from Corollary 2.2.2(b).

Finally, let $S = H(2 : \mathbf{n} : \Delta)$. Then $\bar{S} = \operatorname{Der} S$ by Proposition 2.1.8(c)(iv) and S is isomorphic to an Albert–Zassenhaus algebra of dimension $p^{n_1+n_2}$ by [BIO79, Corollary 5.1]. Thus part (d) follows from Corollary 2.2.2(a). ▮

COROLLARY 2.2.4. *Let* $S \subseteq A \subseteq \operatorname{Der} S$, *where* A *is a restricted Lie algebra and* S *is a simple Lie algebra. Suppose that* A *contains no tori of dimension greater than one. Then either* $A = S = \mathfrak{sl}(2)$, $A = S = W(1 : \mathbf{1})$ *or* $S = H(2 : \mathbf{1})^{(2)}$ *and* $H(2 : \mathbf{1})^{(2)} \subseteq A \subseteq H(2 : \mathbf{1}) = (\operatorname{Der}(H(2 : \mathbf{1})^{(2)}))^{(1)}$.

Proof. Since $\bar{S} \subseteq A$ we see that S has toral rank one. Since by [Wil78], a simple Lie algebra of toral rank one is one of $\mathfrak{sl}(2)$, $W(1 : \mathbf{n})$, or $H(2 : \mathbf{n} : \Phi)^{(2)}$, the result is immediate from Corollary 2.2.3. ▮

COROLLARY 2.2.5. (a) *Let* S *be a simple Lie algebra of toral rank one such that* $\dim \bar{S}/S \geqslant 1$ *and* $\bar{S}$ *contains no tori of dimension greater than two. Then* S *is one of* $W(1 : \mathbf{2})$, $H(2 : (2, 1))^{(2)}$, $H(2 : \mathbf{1} : \Phi(\gamma))^{(1)}$, $H(2 : \mathbf{1} : \Delta)$.

(b) *Let* S *be a simple Lie algebra of toral rank one such that* $\dim \bar{S}/S \geqslant 2$ *and* $\bar{S}$ *contains no tori of dimension greater than two. Then* $S = H(2 : \mathbf{1} : \Phi(\gamma))^{(1)}$.

Proof. By Corollary 2.2.3, the only simple Lie algebras of toral rank one with no tori of dimension greater than two are $\mathfrak{sl}(2)$, $W(1 : \mathbf{1})$, $W(1 : \mathbf{2})$, $H(2 : \mathbf{1})^{(2)}$, $H(2 : (2, 1))^{(2)}$, $H(2 : \mathbf{1} : \Phi(\gamma))^{(1)}$, and $H(2 : \mathbf{1} : \Delta)$. Since $\mathfrak{sl}(2)$, $W(1 : \mathbf{1})$ and $H(2 : \mathbf{1})^{(2)}$ are restricted, part (a) holds. Since $\dim \bar{S}/S = 1$ for $S = W(1 : \mathbf{2})$ (by [Ree59, Corollary 1.2] or [Wil71a, Lemma 4]), or for $H(2 : (2, 1))^{(2)}$ or $H(2 : \mathbf{1} : \Delta)$ (by Proposition 1.8.5 of [BW82]) part (b) holds. ▮

3. DETERMINATION OF RANK ONE SEMISIMPLES

3.1. In Section 3 we study certain restricted semisimple Lie algebras A containing a one-dimensional maximal torus. Since we are only interested in such algebras when they appear as a rank one section of a simple Lie algebra L with respect to a torus of maximal dimension, we may add any hypotheses which are inherited by such algebras. Thus (see Proposition 1.7.4) we assume that A contains no tori of dimension greater than 1 and that A does contain a one-dimensional standard torus. For an analysis of the not necessarily restricted case see [BO-pre].

THEOREM 3.1.1. *Let A be a finite-dimensional restricted semisimple Lie algebra over F containing a one-dimensional maximal torus T which is standard. Assume that A contains no tori of dimension greater than one. Then one of the following occurs*:

(a) *A is isomorphic to $\mathfrak{sl}(2)$.*

(b) *A is isomorphic to $W(1:\mathbf{1})$.*

(c) *A is isomorphic to a subalgebra of $H(2:\mathbf{1}) = (\mathrm{Der}(H(2:\mathbf{1})^{(2)}))^{(1)}$ containing $H(2:\mathbf{1})^{(2)}$.*

Furthermore, in case (a), *T is conjugate to $F\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; in case* (b), *$T$ is conjugate to $F(x_1 D_1)$ or $F((x_1+1) D_1)$; and in case* (c), *T is conjugate (under an automorphism of $H(2:\mathbf{1})$ which need not leave A invariant) to $F(x_1 D_1 - x_2 D_2) = F(\mathscr{D}(x_1 x_2))$ or $F((x_1+1) D_1 - x_2 D_2) = F(\mathscr{D}((x_1+1) x_2))$.*

We first prove a preliminary lemma (which will also be used in Section 4).

LEMMA 3.1.2. *Let A be a restricted semisimple algebra containing a torus T of maximal dimension which is standard. Let I denote the nil radical of $\mathfrak{z}_A(T)$. Suppose that $S \otimes B_n \subseteq A \subseteq \mathrm{Der}(S \otimes B_n)$, where S is a simple Lie algebra and $n > 0$, and that $T \subseteq (\mathrm{Der}\, S) \otimes B_n$. Then $\mathfrak{z}_{S \otimes B_n}(T) \subseteq I$.*

Proof. Let M denote the nil ideal $S \otimes (x_1 B_n + \cdots + x_n B_n)$ in $S \otimes B_n$. Then $(S \otimes B_n)/M \cong S$ so M is a maximal ideal of $S \otimes B_n$. Thus the A-ideal generated by M (which is $\sum_{j=0}^{\infty} (\mathrm{ad}\, A)^j M$)) must equal M or $S \otimes B_n$. Since M is nil and A is semisimple it cannot equal M. Thus $\sum_{j=0}^{\infty} (\mathrm{ad}\, A)^j M = S \otimes B_n$. Now $(\mathrm{Der}\, S) \otimes B_n$ is an ideal in $\mathrm{Der}(S \otimes B_n)$ so, since $T \subseteq (\mathrm{Der}\, S) \otimes B_n$, we have $A \subseteq (\mathrm{Der}\, S) \otimes B_n + \mathfrak{z}_A(T)$. As M is invariant under $(\mathrm{Der}\, S) \otimes B_n$ we see (by the Poincaré–Birkhoff–Witt theorem or by induction) that $S \otimes B_n = \sum_{j=0}^{\infty} (\mathrm{ad}\, \mathfrak{z}_A(T))^j M$ and so $\mathfrak{z}_{S \otimes B_n}(T) = \sum_{j=0}^{\infty} (\mathrm{ad}\, \mathfrak{z}_A(T))^j \mathfrak{z}_M(T)$. Now $(\mathrm{ad}\, \mathfrak{z}_A(T))^j \mathfrak{z}_M(T) \subseteq I$ for all $j > 0$ because T is standard (see Lemma 1.5.2) and $\mathfrak{z}_M(T) \subseteq I$ since M is nil. Thus $\mathfrak{z}_{S \otimes B_n}(T) \subseteq I$. ∎

Proof of Theorem 3.1.1. We show first that if A satisfies the hypotheses of Theorem 3.3.1 then $S \subseteq A \subseteq \mathrm{Der}\, S$, where S is a simple Lie algebra of toral rank one. Let $\mathfrak{z}_A(T) = T + I$, where I is nil.

By Theorem 9.3 of [Blo69]

$$\sum_{i=1}^{r} S_i \otimes B_{n_i} \subseteq A \subseteq \mathrm{Der}\left(\sum_{i=1}^{r} S_i \otimes B_{n_i}\right)$$

for some $r \geqslant 1$, $n_1, ..., n_r \geqslant 0$, where each S_i is a simple Lie algebra. Let J_i denote the restricted ideal generated by $S_i \otimes B_{n_i}$. If $T \cap J_i = (0)$, the Engel–Jacobson theorem (cf. Section 1.10 of [BW82]) shows that J_i is nilpotent, contradicting the simplicity of S_i. Thus, as dim $T = 1$, $T \subseteq J_i$. As the sum of the J_i is direct (since J_i annihilates $S_j \otimes B_{n_j}$ for all $j \neq i$) we have $r = 1$. We simplify notation by writing S for S_1, J for J_1, and n for n_1.

Suppose $n > 0$. By Lemma 3.1.2 we have $\overline{\mathfrak{z}_{S \otimes B_n}(T)} \subseteq I$ and so $\mathfrak{z}_{S \otimes B_n}(T) \subseteq I$. Let $A = \sum_{i=0}^{p-1} A_{i\alpha}$ be the root space decomposition of A with respect to T. If $x \in A_{i\alpha}$, $i \neq 0$, then by Lemma 1.8.1, $\alpha(x^p) = 0$ so $x^p \in I$. Thus $\mathfrak{z}_J(T) \subseteq \overline{\mathfrak{z}_{S \otimes B_n}(T)} + I = I$. Since $T \subseteq J$ and so $T \subseteq \mathfrak{z}_J(T)$, this is impossible. Hence $n = 0$.

Thus $S \subseteq A \subseteq \operatorname{Der} S$, where S is simple. Then Corollary 2.2.4 shows that one of (a), (b), (c) holds.

It remains to establish our assertions about T. In case (a) this is well known. In case (b) it is Demuškin's conjugacy theorem (Theorem 1.3.1(c)) for $W(1:\mathbf{1})$. Now assume that case (c) holds. By Corollary 2.1.9, $T \subseteq H(2:\mathbf{1})^{(2)}$. By Demuškin's conjugacy theorem for $H(2:\mathbf{1})^{(2)}$ (Theorem 1.3.1(i)), T is conjugate by an automorphism of $H(2:\mathbf{1})^{(2)}$ to $F(x_1 D_1 - x_2 D_2)$ or to $F((x_1+1) D_1 - x_2 D_2)$. Since any automorphism of $H(2:\mathbf{1})^{(2)}$ extends to an automophism of $(\operatorname{Der}(H(2:\mathbf{1})^{(2)}))^{(1)}$ we have the result. ∎

3.2. The following lemma gives a way of recognizing simple Lie algebras of toral rank one.

LEMMA 3.2.1. *Let N be a restricted Lie algebra and P be a nonnilpotent subalgebra of N such that $N = \bar{P}$. Suppose T is a two-dimensional standard maximal torus of N such that $\mathfrak{z}_P(T)$ is nil. Then there is some $0 \neq \alpha \in T^*$ and some $x \in P_\alpha$ such that x is not nil. Furthermore, $x^p = u_s + u_n$ for some $0 \neq u_s \in T$ and some nilpotent u_n, $\mathfrak{z}_P(u_s)$ is a Cartan subalgebra of P, and P has toral rank one with respect to $\mathfrak{z}_P(u_s)$.*

Proof. Since P is an ideal in N, P is (ad T)-invariant. If x is nil for every $x \in P_\alpha$, $0 \neq \alpha \in T^*$, then as $\mathfrak{z}_P(T)$ is nil, the Engel–Jacobson theorem (cf. [BW82, Corollary 1.10.2]) implies that P is nilpotent. Since this contradicts our hypotheses, we have that there is some $0 \neq \alpha \in T^*$ and some $x \in P_\alpha$ such that x is not nil.

Then $u = x^p \in \mathfrak{z}_N(T)$. Let $u = u_s + u_n$, where u_s is semisimple and u_n is nilpotent, be the Jordan–Chevalley–Seligman decomposition of u into its semisimple part u_s and nilpotent part u_n. Since u_s is a p-polynomial in u, $u_s \in \mathfrak{z}_N(T)$ and so, by the maximality of T, $u_s \in T$. Since x is not nil, $u_s \neq 0$. By Lemma 1.8.1, $\alpha(x^p) = 0$ and so $\alpha(u_s) = 0$.

Since $\dim T = 2$ we have $\ker \alpha = Fu_s$ and

$$\mathfrak{z}_P(u_s) = \mathfrak{z}_P(T) + \sum_{i=1}^{p-1} P_{i\alpha}.$$

If $y \in P_{i\alpha}$ for $1 \leqslant i \leqslant p-1$ then $v = y^p \in \mathfrak{z}_N(T)$ has decomposition $v = v_s + v_n$ into semisimple and nilpotent parts. By Lemma 1.8.1, $\alpha(y^p) = 0$ so $\alpha(v_s) = 0$ and

$$(y^p)_s = v_s \in \ker \alpha = Fu_s. \tag{3.2.1}$$

Thus ad y is nilpotent on $\mathfrak{z}_P(u_s)$. Since $\mathfrak{z}_P(T)$ is nil by hypothesis, the Engel–Jacobson theorem shows that $\mathfrak{z}_P(u_s)$ is nilpotent.

Now let $y \in N_P(\mathfrak{z}_P(u_s))$. We have $(\operatorname{ad} u_s)\, y \in \mathfrak{z}_P(u_s)$ and so $(\operatorname{ad} u_s)^2\, y = 0$. But u_s is semisimple and so $y \in \mathfrak{z}_P(u_s)$. Thus $\mathfrak{z}_P(u_s)$ is a Cartan subalgebra.

Recall that T is standard in N. Let $\mathfrak{z}_N(T) = T + I$, where I is nil. Then as $\mathfrak{z}_P(T)$ is nil we have that

$$\mathfrak{z}_P(T) \subseteq I. \tag{3.2.2}$$

Also $\alpha(u_s^p) = \alpha(u_s)^p = 0$ and so $u_s^p \in \ker \alpha = Fu_s$. Thus, by (3.2.1) and (3.2.2) we see that $V = \mathfrak{z}_P(u_s) + Fu_s + I$ is a restricted Lie algebra containing $\mathfrak{z}_P(u_s)$. Since Fu_s is a restricted ideal in V and V/Fu_s is nil by (3.2.1), (3.2.2) and the Engel–Jacobson theorem, we see that P has the toral rank one with respect to $\mathfrak{z}_P(u_s)$. ∎

4. Determination of Rank Two Semisimples

In this section we study certain restricted semisimple Lie algebras A containing a two-dimensional maximal torus. Since we are only interested in such algebras when they appear as a rank two section with respect to a torus of maximal dimension of a simple Lie algebra L, we may add any hypotheses which are inherited by such algebras. Thus (see Proposition 1.7.4) we assume that A contains no tori of dimension greater than 2 and that all tori of dimension 2 are standard.

We do not obtain in the present section the classification of such algebras up to isomorphism. Instead we stop with structural characterizations of certain classes of algebras. This information is required in Section 8. For two such classes (cases (g) and (h) in Theorem 4.1.1 below) we will (using an induction on dimension which requires the results of Section 5–8) obtain more complete results in Section 9. Thus our final result on rank two semisimple algebras (Theorem 9.1.1) appears there.

4.1. The main result of Section 4 is:

THEOREM 4.1.1. *Let A be a finite-dimensional restricted semisimple Lie algebra over F. Let T be a two-dimensional torus of A. Assume that all two-dimensional tori in A are maximal and standard. Let $\mathfrak{z}_A(T) = T + I$, where I is the nil radical of $\mathfrak{z}_A(T)$. Then one of the following occurs*:

(a) $S_1 + S_2 \subseteq A \subseteq (\mathrm{Der}\, S_1)^{(1)} + (\mathrm{Der}\, S_2)^{(1)}$, *where S_1, S_2 are distinct ideals in A and each is isomorphic to one of* $\mathfrak{sl}(2)$, $W(1:\mathbf{1})$, $H(2:\mathbf{1})^{(2)}$.

(b) $S \otimes B_n \subseteq A \subseteq \mathrm{Der}(S \otimes B_n)$, $n > 0$, *and* $T \not\subseteq \overline{(S \otimes B_n)}$ *for some simple S. In this case S is one of* $\mathfrak{sl}(2)$, $W(1:\mathbf{1})$, $H(2:\mathbf{1})^{(2)}$, *and* $T \not\subseteq (\mathrm{Der}\, S) \otimes B_n$.

(c) $\bar{S} \subseteq A \subseteq \mathrm{Der}\, S$ *with* $\dim(\bar{S} \cap T) = 1$ *for some simple S. In this case* $S = H(2:\mathbf{1})^{(2)}$ *and we may assume* (*replacing A by ΦA,* $\Phi \in \mathrm{Aut}(\mathrm{Der}(H(2:\mathbf{1})^{(2)}))$ *if necessary*) *that* $H(2:\mathbf{1})^{(2)} + F(x_1 D_1 + x_2 D_2) \subseteq A \subseteq \mathrm{Der}(H(2:\mathbf{1})^{(2)})$.

(d) $S \otimes B_n \subseteq A \subseteq \mathrm{Der}(S \otimes B_n)$, $T \subseteq \overline{(S \otimes B_n)}$, *and* $n > 0$ *for some simple S. In this case* $S = H(2:\mathbf{1}:\Phi(\gamma))^{(1)}$ *and* $\mathfrak{z}_{S \otimes B_n}(T) \subseteq I$.

(e) $\bar{S} \subseteq A \subseteq \mathrm{Der}\, S$, $T \subseteq \bar{S}$, *and* $\dim(A/(S+I)) = 2$ *for some simple S. In this case* $S = H(2:\mathbf{1}:\Phi(\gamma))^{(1)}$.

(f) $\bar{S} \subseteq A \subseteq \mathrm{Der}\, S$, $T \subseteq \bar{S}$, *and* $T \cap (S+I)$ *is one-dimensional and restricted for some simple S. In this case S is one of* $W(1:\mathbf{2})$, $H(2:(2,1))^{(2)}$, $H(2:\mathbf{1}:\Delta)$.

(g) $\bar{S} \subseteq A \subseteq \mathrm{Der}\, S$, $T \subseteq \bar{S}$, *and* $T \cap (S+I)$ *is one-dimensional and non-restricted for some simple S.*

(h) $\bar{S} \subseteq A \subseteq \mathrm{Der}\, S$ *for some simple S and* $A = S + I$.

The proof of Theorem 4.1.1 occupies the next five sections.

4.2. Let A satisfy the hypotheses of Theorem 4.1.1. By Block's theorem ([Blo69]; cf. [BW82, Theorem 1.16.1])

$$\sum_{i=1}^{r} S_i \otimes B_{n_i} \subseteq A \subseteq \mathrm{Der}\left(\sum_{i=1}^{r} S_i \otimes B_{n_i}\right), \qquad (4.2.1)$$

where $r \geqslant 1$, $n_i \geqslant 0$, and the S_i are simple Lie algebras.

Let J_i denote the restricted ideal of A generated by $S_i \otimes B_{n_i}$. The Engel–Jacobson theorem shows that if $T \cap J_i = (0)$ then J_i is nil. As A is semisimple we have $T \cap J_i \neq (0)$. As $J_i \cap (\sum_{j \neq i} J_j) = (0)$ this shows that $r \leqslant 2$.

Suppose $r = 2$. Let A_i denote the restriction of A to $S_i \otimes B_{n_i}$, $i = 1, 2$. Then

$$S_i \otimes B_{n_i} \subseteq A_i \subseteq \mathrm{Der}(S_i \otimes B_{n_i})$$

and by Block's theorem A_i is semisimple. Now T maps to a torus of A_1 which is standard and of maximal dimension (by Lemmas 1.6.1 and 1.7.2) and $T \cap J_2 \neq (0)$ is in the kernel of this mapping. Thus A_1 contains a one-dimensional standard maximal torus (and no tori of larger dimension) and so Theorem 3.1.1shows that $n_1 = 0$, that S_1 is one of $\mathfrak{sl}(2)$, $W(1:\mathbf{1})$, $H(2:\mathbf{1})^{(2)}$, and that $S_1 \subseteq A_1 \subseteq (\operatorname{Der} S_1)^{(1)}$. By symmetry A_2 has the same structure. Therefore if $r > 1$, (a) is satisfied.

4.3. We continue to assume that A satisfies the hypotheses of Theorem 4.1.1. In addition we now assume that $r = 1$ in (4.2.1). We drop the subscript 1 and write

$$S \otimes B_n \subseteq A \subseteq \operatorname{Der}(S \otimes B_n), \tag{4.3.1}$$

where S is a simple Lie algebra and $n \geqslant 0$. Note ([Blo69]; cf. [BW82, Sect. 1.16]) that

$$\operatorname{Der} S \otimes B_n = (\operatorname{Der} S) \otimes B_n + F \otimes (\operatorname{Der} B_n). \tag{4.3.2}$$

Let J denote the restricted ideal of A generated by $S \otimes B_n$. We have that $J \cap T \neq (0)$.

Assume $\dim(J \cap T) = 1$. Then A/J contains the nonzero torus $(T + J)/J \cong T/(J \cap T)$. Since A contains no tori of dimension greater than two it follows from Lemma 1.7.1 that J contains no tori of dimension greater than one. Since $\bar{S} \otimes F \subseteq J$ we see that $\bar{S}$ contains no tori of dimension greater than one. Thus by Corollary 2.2.4, S is one of $\mathfrak{sl}(2)$, $W(1:\mathbf{1})$, $H(2:\mathbf{1})^{(2)}$.

If $n = 0$ then $J = \bar{S}$ and as $T \nsubseteq J$, $T \subseteq \operatorname{Der} S$ we have $\bar{S} \neq \operatorname{Der} S$. Thus $S = H(2:\mathbf{1})^{(2)}$. Then Theorem 1.18.4 of [BW82] shows that (c) holds.

Now if $n > 0$ we have $S \otimes B_n \subseteq A \subseteq \operatorname{Der}(S \otimes B_n)$, where S is one of $\mathfrak{sl}(2)$, $W(1:\mathbf{1})$, $H(2:\mathbf{1})^{(2)}$. Note that, as S is restricted, $J = \overline{S \otimes B_n} = S \otimes B_n$. Suppose $T \subseteq (\operatorname{Der} S) \otimes B_n$. Then by Lemma 3.1.2 we have $\mathfrak{z}_J(T) \subseteq I$, contradicting $J \cap T \neq (0)$. Thus $T \nsubseteq (\operatorname{Der} S) \otimes B_n$ and so (b) holds in this case.

4.4. We continue to assume that A satisfies the hypothesis of Theorem 4.1.1 and (4.3.1). In addition we assume that $n > 0$ and that $T \subseteq J$ (where J is the restricted subalgebra of A generated by $S \otimes B_n$). We will show that (d) holds in this case. Let $M = S \otimes (x_1 B_n + \cdots + x_n B_n)$.

Lemma 1.6.2 implies that T is a standard maximal torus in J. By Lemma 3.1.2, $\mathfrak{z}_{S \otimes B_n}(T)$ is nil. By Lemma 3.2.1 (applied with $N = J$ and $P = S \otimes B_n$) we see that $\mathfrak{z}_{S \otimes B_n}(u_s)$ (for some $0 \neq u_s \in T$) is a Cartan subalgebra of toral rank one in P. Then $(\mathfrak{z}_{S \otimes B_n}(u_s) + M)/M$ is a Cartan subalgebra of toral rank one in $P/M \cong S$. Since $T \subseteq J$ and $\mathfrak{z}_{S \otimes B_n}(T)$ is nil we have

$\dim J/(S \otimes B_n) \geqslant 2$ and so $\dim \bar{S}/S \geqslant 2$. Then Corollary 2.2.5(b) gives $S = H(2:\mathbf{1}:\Phi(\gamma))^{(1)}$ and so (d) holds.

4.5. We continue to assume that A satisfies the hypotheses of Theorem 4.1.1. In addition we assume that

$$\bar{S} \subseteq A \subseteq \operatorname{Der} S, \tag{4.5.1}$$

where S is a simple Lie algebra, and that

$$T \subseteq \bar{S}. \tag{4.5.2}$$

Then if $0 \neq \alpha \in T^*$ we have $A_\alpha \subseteq [T, A_\alpha] \subseteq [\bar{S}, \operatorname{Der} S] \subseteq S$ and so

$$A = S + T + I. \tag{4.5.3}$$

Thus $A/(S+I) \cong T/(T \cap (S+I))$ has dimension $\leqslant 2$. If $\dim A/(S+I) = 0$ then part (h) of Theorem 4.1.1 holds. We will consider the case $\dim A/(S+I) = 2$ in this section and the case $\dim A/(S+I) = 1$ in Section 4.6.

If $\dim A/(S+I) = 2$ then by the above remarks $(S+I) \cap T = (0)$. Thus if $u \in \mathfrak{z}_S(T)$, $u = u_s + u_n$, $u_s \in T$, $u_n \in I$, then $u_s \in (S+I) \cap T = (0)$, so u is nilpotent. Then Lemma 3.2.1 (with $N = \bar{S}$, $P = S$) shows that S has toral rank one with respect to some Cartan subalgebra. Since $T \subseteq \bar{S}$ we have that $\dim(\bar{S}/S) \geqslant 2$. Then Corollary 2.2.5(b) shows that $S = H(2:\mathbf{1}:\Phi(\gamma))^{(1)}$ and so (e) holds in this case.

4.6. We continue to assume that A satisfies the hypotheses of Theorem 4.1.1 and that (4.5.1) and (4.5.2) hold. We now assume that $\dim A/(S+I) = 1$. By the remarks following (4.5.2), $A = S + T + I$ and $\dim T \cap (S+I) = 1$.

If $T \cap (S+I)$ is nonrestricted then (g) of Theorem 4.1.1 holds.

Now suppose that $T \cap (S+I)$ is restricted. Then there exists $u \in \mathfrak{z}_S(T)$ with Jordan–Chevalley–Seligman decomposition $u = u_s + u_n$, where $0 \neq u_s = u_s^p$ such that u_s spans $T \cap (S+I)$. Fix $v \in T$ so that $v^p = v$ and $T = Fu_s + Fv$. Let $S = \mathfrak{z}_S(T) + \sum_{0 \neq \alpha \in T^*} S_\alpha$ be the root space decomposition of S with respect to T. As $u_s^p = u_s$, $v^p = v$, we have that if $\alpha \neq 0$ then $\alpha(u_s)$, $\alpha(v) \in \mathbf{Z}_p$. Define $\beta \in T^*$ by $\beta(u_s) = 0$, $\beta(v) = 1$. Then $\mathfrak{z}_S(u_s) = \mathfrak{z}_S(T) + \sum_{i=1}^{p-1} S_{i\beta}$. If $x \in S_{i\beta}$, $i \neq 0$, and $x^p = y$ has Jordan–Chevalley–Seligman decomposition $y_s + y_n$ then $y \in \mathfrak{z}_S(T)$ implies $y_s \in T$. By Lemma 1.8.1 we have $\beta(y_s) = 0$. Thus $y_s \in Fu_s$ and the restriction of $\operatorname{ad} x$ to $\mathfrak{z}_S(u_s)$ is nilpotent. Let $w \in \mathfrak{z}_S(T)$ have Jordan–Chevalley–Seligman decomposition $w = w_s + w_n$. Then $w_s \in T \cap (S+I)$ and so we have $w_s \in Fu_s = \ker \beta$. Thus the restriction of $\operatorname{ad} w$ to $\mathfrak{z}_S(u_s)$ is nilpotent.

Therefore, by the Engel–Jacobson theorem $\mathfrak{z}_S(u_s)$ is nilpotent. Since $u \in S$, $\mathfrak{z}_S(u_s)$ is equal to its own normalizer in S. Thus $\mathfrak{z}_S(u_s)$ is a Cartan subalgebra of S. We have seen that if $x \in S_{i\beta}$, $i \neq 0$, then $x^p \in Fu_s + I$ and that $Fu_s + I \subseteq \mathfrak{z}_S(u_s) + I$. Thus $\mathfrak{z}_S(u_s) + I$ is a restricted Lie algebra and $\underline{(\mathfrak{z}_S(u_s) + I)/Fu_s}$ is nil. Thus Fu_s spans a one-dimensional maximal torus $\overline{\mathfrak{z}_S(u_s)}$ and so S has toral rank one with respect to $\mathfrak{z}_S(u_s)$.

Since $T \subseteq \bar{S}$ and $T \not\subseteq S$, Corollary 2.2.5(a) applies and shows that S is one of $W(1:\mathbf{2})$, $H(2:(2,1))^{(2)}$, $H(2:\mathbf{1}:\Phi(\gamma))^{(1)}$, $H(2:\mathbf{1}:\Delta)$. It remains to show that S cannot be $H(2:\mathbf{1}:\Phi(\gamma))^{(1)}$. Suppose $S \cong H(2:\mathbf{1}:\Phi(\gamma))^{(1)}$. Corollary 1.3.2 shows that $\mathfrak{z}_S(T)$ is a torus and hence Corollary 4.74 of [BW82] shows $S \cong W(1:\mathbf{2})$ or $H(2:\mathbf{1}:\Delta)$, a contradiction. Thus in this case, S is one of the algebras listed in conclusion (f) of Theorem 4.1.1.

This completes the proof of Theorem 4.1.1. ▮

5. Proper Roots

5.1. Throughout Section 5 we will assume that A is a finite-dimensional restricted Lie algebra over F, T is a maximal torus in A, and $\mathfrak{z}_A(T) = T + I$, where I is a nil ideal of $\mathfrak{z}_A(T)$ (so that T is standard in A). Recall (Remark 1.5.5) that we identify T^* with $\{\gamma \in \mathfrak{z}_A(T)^* \mid \gamma(I) = (0)\}$. Recall also (Section 1.4) that $\Gamma(A, T) = \{\gamma \in T^* \mid \gamma \neq 0,\ A_\gamma \neq 0\}$ and that $\Delta(A, T)$ denotes the subgroup of T^* generated by $\Gamma(A, T)$. Denote the rank of $\Delta(A, T)$ by $r(A, T)$. We will usually write Δ for $\Delta(A, T)$ and Γ for $\Gamma(A, T)$.

Definition 5.1.1. For $\gamma \in \Gamma$ define

$$K_\gamma(A) = \{x \in A_\gamma \mid \gamma([x, A_{-\gamma}]) = (0)\}$$

and

$$K^{(\gamma)}(A) = \mathfrak{z}_A(T) + \sum_{i=1}^{p-1} K_{i\gamma}(A).$$

Definition 5.1.2. For $\gamma \in \Gamma$, $\delta \in T^*$ define

$$M_\gamma^\delta(A) = \{x \in A_\gamma \mid \delta([x, A_{-\gamma}]) = (0)\}$$

and

$$M^\delta(A) = \mathfrak{z}_A(T) + \sum_{\gamma \in \Gamma} M_\gamma^\delta(A).$$

Lemma 5.1.3. *For $\delta \in T^*$, $M^\delta(A)$ is a restricted subalgebra of A.*

Proof. Let $x \in M^\delta_\alpha(A)$, $y \in M^\delta_\beta(A)$, $z \in A_{-(\alpha+\beta)}$. Then

$$\begin{aligned}\delta([[xy]z]) &= \delta([[xz]y]) + \delta([x[yz]]) \in \delta([A_{-\beta}, M^\delta_\beta(A)]) \\ &\quad + \delta([M^\delta_\alpha(A), A_{-\alpha}]) = (0).\end{aligned}$$

Thus $[M^\delta_\alpha(A), M^\delta_\beta(A)] \subseteq M^\delta_{\alpha+\beta}(A)$ whenever $\alpha, \beta \in \Gamma$, $\alpha+\beta \neq 0$. Also $\delta([[\mathfrak{z}_A(T), M^\delta_\alpha(A)], A_{-\alpha}]) \subseteq \delta([A_{-\alpha}, M^\delta_\alpha(A)]) + \delta([\mathfrak{z}_A(T), \mathfrak{z}_A(T)]) \subseteq (0) + \delta(I) = (0)$. Thus $[\mathfrak{z}_A(T), M^\delta_\alpha(A)] \subseteq M^\delta_\alpha(A)$ and so $M^\delta(A)$ is a subalgebra. By Lemma 1.8.1, if $\alpha \in \Delta$, $x \in A_\alpha$ then $x^p \in \mathfrak{z}_A(T)$. Thus $M^\delta(A)$ is a restricted subalgebra of A. ∎

COROLLARY 5.1.4. *$K^{(\gamma)}(A)$ is a restricted subalgebra of A.*

Proof. As $K_{i\gamma}(A) = M^\gamma_{i\gamma}(A)$ we have that $K^{(\gamma)}(A) = M^\gamma(A^{(\gamma)})$. As $A^{(\gamma)}$ is a restricted subalgebra of A we have the result. ∎

DEFINITION 5.1.5. For $\gamma \in \Gamma$, define

$$R_\gamma(A) = \{x \in A_\gamma \mid [x, A_{-\gamma}] \subseteq I\}$$

and

$$R(A) = \mathfrak{z}_A(T) + \sum_{\gamma \in \Gamma} R_\gamma(A).$$

LEMMA 5.1.6. *$R(A)$ is a restricted subalgebra of A.*

Proof. For $\alpha, \beta \in \Gamma$ we have

$$[A_{-(\alpha+\beta)}, [R_\alpha(A), R_\beta(A)]] \subseteq [A_{-\beta}, R_\beta(A)] + [R_\alpha(A), A_{-\alpha}] \subseteq I.$$

Thus $[R_\alpha(A), R_\beta(A)] \subseteq R_{\alpha+\beta}(A)$ whenever $\alpha, \beta \in \Gamma$, $\alpha+\beta \neq 0$. Also for $\alpha \in \Gamma$ we have

$$\begin{aligned}[A_{-\alpha}, [\mathfrak{z}_A(T), R_\alpha(A)]] &\subseteq [A_{-\alpha}, R_\alpha(A)] + [\mathfrak{z}_A(T), [A_{-\alpha}, R_\alpha(A)]] \\ &\subseteq I + [\mathfrak{z}_A(T), I] \subseteq I.\end{aligned}$$

Thus $[\mathfrak{z}_A(T), R_\alpha(A)] \subseteq R_\alpha(A)$ and so $R(A)$ is a subalgebra. If $\alpha \in \Delta$ and $x \in A_\alpha$ then by Lemma 1.8.1 we have $x^p \in \mathfrak{z}_A(T)$. Thus $R(A)$ is a restricted subalgebra of A, as required. ∎

DEFINITION 5.1.7. For $\gamma \in \Gamma$ define

$$RK_\gamma(A) = \{x \in K_\gamma(A) \mid [x, K_{-\gamma}] \subseteq I\}$$

and

$$RK^{(\gamma)}(A) = \mathfrak{z}_A(T) + \sum_{i=1}^{p-1} RK_{i\gamma}(A).$$

Thus $RK_{i\gamma}(A) = R_{i\gamma}(K^{(\gamma)}(A))$ and $RK^{(\gamma)}(A) = R(K^{(\gamma)}(A))$.

COROLLARY 5.1.8. *$RK^{(\gamma)}(A)$ is a restricted subalgebra of A.*

It is clear that $A_\gamma \supseteq K_\gamma(A) \supseteq RK_\gamma(A) \supseteq R_\gamma(A)$ for all $\gamma \in \Gamma$.

5.2. It will be of interest to know the $K_\alpha(A)$ for the pairs (A, T) occurring in Theorem 3.1.1.

LEMMA 5.2.1. (a) *If $A = \mathfrak{sl}(2)$, T is any maximal torus in A (necessarily one-dimensional), and $\alpha \in \Gamma$, then $K_\alpha(A) = (0)$.*

(b) *If $A = W(1:\mathbf{1})$, $T = F(x_1 D_1)$, and $\alpha \in T^*$ is defined by $\alpha(x_1 D_1) = 1$, then $K_{i\alpha}(A) = (0)$ for $i = \pm 1$ and $K_{i\alpha}(A) = A_{i\alpha}$ for $i \in \mathbf{Z}_p^*$, $i \neq \pm 1$.*

(c) *If $A = W(1:\mathbf{1})$, $T = F((x_1+1) D_1)$, and $\alpha \in T^*$ is defined by $\alpha((x_1+1) D_1) = 1$, then $K_{i\alpha}(A) = (0)$ for all $i \in \mathbf{Z}_p^*$.*

(d) *If $H(2:\mathbf{1})^{(2)} \subseteq A \subseteq H(2:\mathbf{1})$, $T = F(\mathscr{D}(x_1 x_2))$, and $\alpha \in T^*$ is defined by $\alpha(\mathscr{D}(x_1 x_2)) = 1$, then $K_\alpha(A) = A_\alpha \cap A_2$ has basis $\{\mathscr{D}(x_1^{i+1} x_2^i) \mid 2 \leqslant i \leqslant p-2\} \cup \{\mathscr{D}(x_2^{p-1})\}$ so $\dim A_\alpha / K_\alpha(A) = 2$, $K_{-\alpha}(A) = A_{-\alpha} \cap A_2$ has basis $\{\mathscr{D}(x_1^i x_2^{i+1}) \mid 2 \leqslant i \leqslant p-2\} \cup \{\mathscr{D}(x_1^{p-1})\}$ so $\dim A_{-\alpha}/K_{-\alpha}(A) = 2$, $K_{2\alpha}(A) = A_{2\alpha} \cap A_2$ has basis $\{\mathscr{D}(x_1^{i+2} x_2^i) \mid 1 \leqslant i \leqslant p-3\} \cup \{\mathscr{D}(x_1 x_2^{p-1}),\ \mathscr{D}(x_2^{p-2})\}$ so $\dim A_{2\alpha}/K_{2\alpha}(A) = 1$, $K_{-2\alpha}(A) = A_{-2\alpha} \cap A_2$ has basis $\{\mathscr{D}(x_1^i x_2^{i+2}) \mid 1 \leqslant i \leqslant p-3\} \cup \{\mathscr{D}(x_1^{p-1} x_2),\ \mathscr{D}(x_1^{p-2})\}$ so $\dim A_{-2\alpha}/K_{-2\alpha}(A) = 1$, and $K_{i\alpha}(A) = A_{i\alpha}$ if $i \in \mathbf{Z}_p^*$, $i \neq \pm 1, \pm 2$. Finally, $T \subseteq [A, [A, I]]$ (where I is the nil radical of $\mathfrak{z}_A(T)$).*

(e) *If $H(2:\mathbf{1})^{(2)} \subseteq A \subseteq H(2:\mathbf{1})$, $T = F(\mathscr{D}((x_1+1)x_2))$, and $\alpha \in T^*$ is defined by $\alpha(\mathscr{D}((x_1+1)x_2)) = 1$, then $K_{i\alpha}(A) = A_{i\alpha} \cap A_1$ has basis $\{\mathscr{D}((x_1+1)^{i+j} x_2^j) \mid 3 \leqslant j \leqslant p-1\}$ so $\dim A_{i\alpha}/K_{i\alpha}(A) = 3$ for $i \in \mathbf{Z}_p^*$. Finally, $T \subseteq [A, [A, I]]$ (where I is the nil radical of $\mathfrak{z}_A(T)$).*

Proof. In cases (a)–(c) we have $I = (0)$. Thus Lemma 2.2.3 of [BW82] gives the result.

In cases (d) and (e) note that $T \subseteq H(2:\mathbf{1})^{(2)}$, which is an ideal in A. Then $A_{i\alpha} \subseteq H(2:\mathbf{1})^{(2)}$ for all $i \in \mathbf{Z}_p^*$ and so we may assume without loss of generality that $A = H(2:\mathbf{1})^{(2)}$.

For (d) we begin by observing (using Corollary 2.1.10) that $(H(2:\mathbf{1})^{(2)})_{i\alpha}$ contains $\operatorname{span}(\{\mathscr{D}(x_1^{i+j} x_2^j) \mid 0 \leqslant j \leqslant p-1-i\} \cup \{\mathscr{D}(x_1^j x_2^{p-i+j}) \mid 0 \leqslant j \leqslant i-1\})$ if $i \neq 0$ and $H(2:\mathbf{1})^{(2)})_0$ contains span $\{\mathscr{D}(x_1^i x_2^i) \mid 1 \leqslant i \leqslant p-2\}$.

Since by Proposition 2.1.8(a)(iv), $H(2:\mathbf{1})^{(2)}$ has basis $\{\mathscr{D}(x_1^i x_2^j) \mid 0 \leqslant i, j \leqslant p-1, (i, j) \neq (0,0), (p-1, p-1)\}$, we see that equality holds in each case and that the indicated sets of elements are bases for the root spaces.

Since $\mathscr{D}(x_1^{i+j} x_2^j) \in A_{i+2j-2}$ and $\mathscr{D}(x_1^j x_2^{p-i+j}) \in A_{p-i+2j-2}$ (see (2.1.6)) we see that $A_{i\alpha} \subseteq A_1$, for $i \neq \pm 1, \pm 2$.

Since $A_1 \cap \mathfrak{z}_A(T) \subseteq I$ we see that $A_\gamma = K_\gamma(A)$ whenever $A_1 \supseteq A_\gamma + A_{-\gamma}$. Thus $A_{i\alpha} = K_{i\alpha}(A)$ whenever $i \neq \pm 1, \pm 2$. Furthermore, as $A = A_{-1}$, we see that $[A_\gamma \cap A_2, A_{-\gamma}] \subseteq \mathfrak{z}_A(T) \cap A_1 \subseteq I$ so that $K_\gamma(A) \supseteq A_\gamma \cap A_2$ for all γ. Since

$$[\mathscr{D}(x_1), \mathscr{D}(x_1 x_2^2)] = -2\,\mathscr{D}(x_1 x_2),$$

$$[\mathscr{D}(x_1), \mathscr{D}(x_2)] = 0,$$

$$[\mathscr{D}(x_1^2 x_2), \mathscr{D}(x_1 x_2^2)] \in A_2,$$

$$[\mathscr{D}(x_1^2 x_2), \mathscr{D}(x_2)] = -2\,\mathscr{D}(x_1 x_2),$$

and

$$[\mathscr{D}(x_1^2), \mathscr{D}(x_2^2)] = -4\,\mathscr{D}(x_1 x_2),$$

we see that $K_\gamma(A) = A_\gamma \cap A_2$ and that $K_\gamma(A)$ has the indicated basis for $\gamma = \pm\alpha, \pm 2\alpha$. Clearly $\mathscr{D}(x_1^2 x_2^2) \in I$. Then $-4\,\mathscr{D}(x_1 x_2) = [\mathscr{D}(x_1), [\mathscr{D}(x_2), \mathscr{D}(x_1^2 x_2^2)]] \in [A, [A, I]]$ so $T \subseteq [A, [A, I]]$.

For (e) we begin by observing (using Corollary 2.1.10) that $(H(2:\mathbf{1})^{(2)})_{i\alpha}$ contains $\operatorname{span}\{\mathscr{D}((x_1+1)^{i+j} x_2^j) \mid 0 \leqslant j \leqslant p-1\}$ for $i \neq 0$ and that $(H(2:\mathbf{1})^{(2)})_0$ contains $\operatorname{span}\{\mathscr{D}((x_1+1)^i x_2^i) \mid 1 \leqslant i \leqslant p-2\}$. Since the given sets are clearly linearly independent, we see that they are bases for the indicated root spaces.

Since $(H(2:\mathbf{1})^{(2)})_0$ is restricted (by Lemma 2.1.7) it contains the semisimple part of each of its elements. As T is a maximal torus and $T \not\subseteq (H(2:\mathbf{1})^{(2)})_0$ we see that $\mathfrak{z}_A(T) \cap (H(2:\mathbf{1})^{(2)})_0 \subseteq I$. Thus (as $A = A_{-1}$) we see that $A_\gamma \cap A_1 \subseteq K_\gamma$ for all γ. Since the cosets of $\mathscr{D}((x_1+1)^i)$, $\mathscr{D}((x_1+1)^{i+1} x_2)$, and $\mathscr{D}((x_1+1)^{i+2} x_2^2)$ form a basis for $A_{i\alpha}/A_{i\alpha} \cap A_1$ and direct computation shows that

$$\begin{aligned}
&[\mathscr{D}((x_1+1)^{i+j} x_2^j), \mathscr{D}((x_1+1)^{p-i+l} x_2^l)] \\
&= \mathscr{D}(-i(j+l)(x_1+1)^{j+l-1} x_2^{j+l-1}) \\
&\equiv -2i\,\delta_{j+l,2}\,\mathscr{D}((x_1+1)x_2) \qquad \bmod I
\end{aligned}$$

for $0 \leqslant j, l \leqslant 2$, we see that $K_{i\alpha}$ has the indicated basis. Finally, it is clear that $\mathscr{D}((x_1+1)^2 x_2^2) \in I$ and so $-4\,\mathscr{D}((x_1+1)x_2) = [\mathscr{D}(x_1), [\mathscr{D}(x_2), \mathscr{D}((x_1+1)^2 x_2^2)]] \in [A, [A, I]]$. Thus $T \subseteq [A, [A, I]]$. ∎

5.3. We will now show that the calculations of Section 5.2 allow us to determine $\dim A_\gamma/K_\gamma$ for any finite-dimensional restricted Lie algebra A containing a torus of maximal dimension which is standard.

LEMMA 5.3.1. *Let A be a finite-dimensional restricted Lie algebra and T be a standard torus in A. Then $K^{(\gamma)}(A) \supseteq \mathrm{solv}(A^{(\gamma)})$. Furthermore,*

$$K_{i\gamma}(A[\gamma]) = (K_{i\gamma}(A) + \mathrm{solv}\, A^{(\gamma)})/(\mathrm{solv}\, A^{(\gamma)}) = \Psi_\gamma(K_{i\gamma}(A)),$$

where $K_{i\gamma}(A[\gamma])$ is taken with respect to the torus

$$\Psi_\gamma(T) = (T + \mathrm{solv}\, A^{(\gamma)})/(\mathrm{solv}\, A^{(\gamma)}) \subseteq A[\gamma].$$

Proof. Lemma 1.8.3 (with $Y = A^{(\gamma)}$, $M = \mathrm{solv}(A^{\gamma})$) gives $K^{(\gamma)}(A) \supseteq \mathrm{solv}(A^{(\gamma)})$. Now if $A^{(\gamma)} = \mathrm{solv}\, A^{(\gamma)}$ the second assertion is trivially true (as both sides are (0)). Thus we can assume $A^{(\gamma)} \neq (\mathrm{solv}\, A^{(\gamma)})$ and so γ vanishes on $T \cap \mathrm{solv}\, A^{(\gamma)}$. Thus γ induces a linear functional (again denoted γ) on $(T + \mathrm{solv}\, A^{(\gamma)})/\mathrm{solv}\, A^{(\gamma)} \cong T/(T \cap (\mathrm{solv}\, A^{(\gamma)}))$ by $\gamma(t + \mathrm{solv}\, A^{(\gamma)}) = \gamma(t)$. Then $A[\gamma]_{i\gamma} = (A_{i\gamma} + \mathrm{solv}\, A^{(\gamma)})/\mathrm{solv}\, A^{(\gamma)}$. For $x \in A_{i\gamma}$ we have $x + \mathrm{solv}\, A^{(\gamma)} \in K_{i\gamma}(A[\gamma])$ if and only if $\gamma([x + \mathrm{solv}\, A^{(\gamma)}, y + \mathrm{solv}\, A^{(\gamma)}]) = (0)$ for all $y \in A_{-i\gamma}$. But since $\gamma([x + \mathrm{solv}\, A^{(\gamma)}, y + \mathrm{solv}\, A^{(\gamma)}]) = \gamma([x, y] + \mathrm{solv}\, A^{(\gamma)}) = \gamma([x, y])$ this is equivalent to $\gamma([x, y]) = 0$ for all $y \in A_{-i\gamma}$ and hence to $x \in K_{i\gamma}$. This completes the proof. ∎

DEFINITION 5.3.2. Let $\gamma \in \Delta$, $\gamma \neq 0$. If $A[\gamma] = (0)$ we say γ is *solvable*, if $A[\gamma] \cong \mathfrak{sl}(2)$ we say γ is *classical*, if $A[\gamma] \cong W(1:\mathbf{1})$ we say γ is *Witt*, and if $H(2:\mathbf{1})^{(2)} \subseteq A[\gamma] \subseteq H(2:\mathbf{1})$ we say γ is *Hamiltonian*.

LEMMA 5.3.3. *Let A be a finite-dimensional restricted Lie algebra over F. Let T be a torus of maximal dimension in A. Assume that T is standard. Then any $\gamma \in \Delta$, $\gamma \neq 0$, is either solvable, classical, Witt, or Hamiltonian.*

Proof. As T is a torus of maximal dimension in A, Lemma 1.7.2(a) shows that $\Psi_\gamma(T)$ is a torus of maximal dimension in $A[\gamma]$. As T is standard in A, Lemma 1.6.1(c) shows this torus is standard. Since $T \cap \mathrm{solv}\, A^{(\gamma)} \supseteq \ker \gamma\,|_T$ this torus has dimension $\leqslant 1$. Hence $A[\gamma]$ has no tori of dimension >1. Thus by Theorem 3.1.1, if $A[\gamma] \neq (0)$ it must be $\mathfrak{sl}(2)$, $W(1:\mathbf{1})$ or else $H(2:\mathbf{1})^{(2)} \subseteq A[\gamma] \subseteq H(2:\mathbf{1})$, as required. ∎

LEMMA 5.3.4. *Let A be a finite-dimensional restricted Lie algebra and T be a torus of maximal dimension in A. Assume that T is standard. Let $\gamma \in \Delta$, $\gamma \neq 0$. Then one of the following occurs:*

(a) *γ is solvable and $K_{i\gamma}(A) = A_{i\gamma}$ for all $i \in \mathbf{Z}_p^*$.*

(b) *γ is classical and there is some $j \in \mathbf{Z}_p^*$ such that, for $i \in \mathbf{Z}_p^*$,* $\dim(A_{i\gamma}/K_{i\gamma}(A)) = 1$ *if $i = \pm j$,* $\dim(A_{i\gamma}/K_{i\gamma}(A)) = 0$ *if $i \neq \pm j$.*

(c) *γ is Witt, there is a surjective homomorphism ϕ of $A^{(\gamma)}$ to $W(1:\mathbf{1})$ such that $\phi(T) = F(x_1 D_1)$, and there is some $j \in \mathbf{Z}_p^*$ such that for $i \in \mathbf{Z}_p^*$*

$$\dim(A_{i\gamma}/K_{i\gamma}(A)) = \begin{cases} 1 & \text{if } i = \pm j, \\ 0 & \text{if } i \neq \pm j. \end{cases}$$

(d) *γ is Witt, there is a surjective homomorphism ϕ of $A^{(\gamma)}$ to $W(1:\mathbf{1})$ such that $\phi(T) = F((x_1+1)D_1)$, and* $\dim(A_{i\gamma}/K_{i\gamma}(A)) = 1$ *for all $i \in \mathbf{Z}_p^*$.*

(e) *γ is Hamiltonian, there is a surjective homomorphism ϕ of $A^{(\gamma)}$ to a subalgebra of $H(2:\mathbf{1})$ containing $H(2:\mathbf{1})^{(2)}$ such that $\phi(T) = F(\mathscr{D}(x_1x_2))$, and there is some $j \in \mathbf{Z}_p^*$ such that for $i \in \mathbf{Z}_p^*$*

$$\dim(A_{i\gamma}/K_{i\gamma}(A)) = \begin{cases} 2 & \textit{if } i = \pm j, \\ 1 & \textit{if } i = \pm 2j, \\ 0 & \textit{if } i \neq \pm j, \pm 2j. \end{cases}$$

(f) *γ is Hamiltonian, there is a surjective homomorphism ϕ of $A^{(\gamma)}$ to a subalgebra of $H(2:\mathbf{1})$ containing $H(2:\mathbf{1})^{(2)}$ such that $\phi(T) = F(\mathscr{D}((x_1+1)x_2))$, and* $\dim(A_{i\gamma}/K_{i\gamma}(A)) = 3$ *for all $i \in \mathbf{Z}_p^*$.*

Proof. By Lemma 5.3.3, γ is either solvable, classical, Witt, or Hamiltonian. If γ is solvable, Lemma 5.3.1 shows that $A^{(\gamma)} = K^{(\gamma)}(A)$, proving (a). If γ is classical, Lemma 5.3.1 and Lemma 5.2.1(a) give (b). If γ is Witt then there is a surjective homomorphism ψ of $A^{(\gamma)}$ to $W(1:\mathbf{1})$. Then $\psi(T)$ is a maximal torus in $W(1:\mathbf{1})$, so by Theorem 1.3.1(a, b) there is an automorphism τ of $W(1:\mathbf{1})$ such that $\tau\psi(T) = F(x_1 D_1)$ or $F((x_1+1)D_1)$. Setting $\phi = \tau\psi$ and using Lemmas 5.2.1(b, c) and 5.3.1 gives (c) and (d). If γ is Hamiltonian then there is a surjective homomorphism ψ of $A^{(\gamma)}$ onto a subalgebra B of $H(2:\mathbf{1})$ containing $H(2:\mathbf{1})^{(2)}$. Then $\psi(T)$ is a maximal torus of B. Since $H(2:\mathbf{1})/H(2:\mathbf{1})^{(2)}$ is nil $\psi(T) \subseteq H(2:\mathbf{1})^{(2)}$. By Theorem 1.3.1(g, h) there is an automorphism τ of $H(2:\mathbf{1})^{(2)}$ such that $\tau\psi(T) = F(\mathscr{D}(x_1x_2))$ or $F(\mathscr{D}((x_1+1)x_2))$. Since every automorphism of $H(2:\mathbf{1})^{(2)}$ extends to an automorphism of $H(2:\mathbf{1})$ (since $H(2:\mathbf{1}) = (\mathrm{Der}(H(2:\mathbf{1})^{(2)})^{(1)})$ we may view τ as an isomorphism of B to τB, a subalgebra of $H(2:\mathbf{1})$ containing $H(2:\mathbf{1})^{(2)}$. Then setting $\phi = \tau\psi$ and using Lemmas 5.2.1(d, e) and 5.3.1 gives (e) and (f). ∎

DEFINITION 5.3.5. A Lie algebra A is said to be *compositionally classical* if every composition factor of A is abelian or classical simple.

LEMMA 5.3.6. *Let A be a finite-dimensional restricted Lie algebra. Let T be a torus of maximal dimension in A. Assume that T is standard. Let $\alpha \in \Delta(A, T)$. Then*:

(a) *If α is Witt, $A^{(\alpha)}$ contains a unique compositionally classical subalgebra of codimension* 1 *and any compositionally classical subalgebra of codimension* 2 *is contained in the compositionally classical subalgebra of codimension* 1.

(b) *If α is Hamiltonian, $A^{(\alpha)}$ contains a unique compositionally classical subalgebra of codimension* 2 *and no such subalgebra of codimension one.*

Proof. Suppose α is Witt and that M is a compositionally classical subalgebra of codimension $\leqslant 2$ in $A^{(\alpha)}$. Then $\Psi_\alpha(M)$ is a compositionally classical (hence proper) subalgebra of codimension $\leqslant 2$ in $A[\alpha] \cong W(1:\mathbf{1})$. By Lemma 1.11.1(a), $\Psi_\alpha(M) \subseteq W(1:1)_0$ and so $M \subseteq \Psi_\alpha^{-1}(W(1:\mathbf{1})_0)$. Since $\Psi_\alpha^{-1}(W(1:\mathbf{1})_0)$ is compositionally classical and has codimension 1 in $A^{(\alpha)}$, this proves (a).

Next suppose that $H(2:\mathbf{1})^{(2)} \subseteq Q \subseteq H(2:\mathbf{1})$ and that M is a compositionally classical subalgebra of codimension $\leqslant 2$ in Q. Then $M \cap H(2:\mathbf{1})^{(2)}$ is a compositionally classical (hence proper) subalgebra of codimension $\leqslant 2$ in $H(2:\mathbf{1})^{(2)}$. Then by Lemma 1.11.1(b) we have $M \cap H(2:\mathbf{1})^{(2)} = (H(2:\mathbf{1})^{(2)})_0$. Thus $M \subseteq N_{H(2:\mathbf{1})}((H(2:\mathbf{1})^{(2)})_0) = H(2:\mathbf{1})_0$. Thus $M \subseteq Q \cap H(2:\mathbf{1})_0$ and so (since $\dim Q/(Q \cap H(2:\mathbf{1})_0) = 2$) $M = Q \cap H(2:\mathbf{1})_0$ has codimension two in Q.

Now, suppose that α is Hamiltonian and that M is a compositionally classical subalgebra of codimension $\leqslant 2$ in $A^{(\alpha)}$. Then $\Psi_\alpha(M)$ is a compositionally classical subalgebra of $A[\alpha]$ and $H(2:\mathbf{1})^{(2)} \subseteq A[\alpha] \subseteq H(2:\mathbf{1})$. By the result of the previous paragraph, $\Psi_\alpha(M) = A[\alpha] \cap H(2:\mathbf{1})_0$. Therefore $M \subseteq \Psi_\alpha^{-1}(A[\alpha] \cap H(2:\mathbf{1})_0)$ and, since this subalgebra has codimension 2 in $A^{(\alpha)}$, equality holds. Thus (b) holds. ∎

5.4. Having obtained bounds for $\dim(A_\gamma/K_\gamma(A))$ we now seek bounds for $\dim(K_\gamma(A)/RK_\gamma(A))$ and $\dim(RK_\gamma(A)/R_\gamma(A))$.

From now on we will usually write K_γ for $K_\gamma(A)$, RK_γ for $RK_\gamma(A)$, R_γ for $R_\gamma(A)$, and M_γ^δ for $M_\gamma^\delta(A)$.

LEMMA 5.4.1. $\dim(A_\gamma/K_\gamma) = \dim(A_{-\gamma}/K_{-\gamma})$.

Proof. The map of $A_\gamma \times A_{-\gamma} \to F$ given by $(x, y) \mapsto \gamma([x, y])$ is a bilinear form. Since, by definition of K_γ, $A_{-\gamma}^\perp = K_\gamma$, we have the result. ∎

LEMMA 5.4.2. *Let $0 \neq \gamma \in \Delta$. Then*

$$\dim(RK_\gamma/R_\gamma) \leqslant (\dim(A_\gamma/K_\gamma))((\dim T) - 1).$$

In particular, if $A_\gamma = K_\gamma$ then $RK_\gamma = R_\gamma$ and if γ is non-Hamiltonian then $\dim(RK_\gamma/R_\gamma) \leqslant (\dim T) - 1$.

Proof. As $[RK_\gamma, K_{-\gamma}] \subseteq I$ and $RK_\gamma \subseteq K_\gamma$ (Definition 5.1.7) we see that ad induces a map of RK_γ into $\mathrm{Hom}(A_{-\gamma}/K_{-\gamma}, T')$, where $T' = \ker \gamma|_T \cong (\ker \gamma)/I$. The kernel of this map is R_γ, giving $\dim(RK_\gamma/R_\gamma) \leqslant (\dim(A_\gamma/K_\gamma))((\dim T) - 1)$. The final assertion follows by noting that if γ is non-Hamiltonian then $\dim(A_\gamma/K_\gamma) \leqslant 1$. ∎

5.5. Obtaining a bound for $\dim(K_\gamma(A)/RK_\gamma(A))$ is much harder. Let B be a restricted Lie algebra over F and let e be a derivation of B satisfying $e^p = e$. Define $\alpha \in (Fe)^*$ by $\alpha(e) = 1$. Let $B = \sum_{i=0}^{p-1} B_{i\alpha}$ be the decomposition of B with respect to the torus Fe. Assume $B_0 = Fz + J$, where $z = z^p$ is central in B and J is a nil ideal in B_0. Then $Fe + Fz$ is a standard maximal torus in $Fe + B$. Let $n_i = \dim(B_{i\alpha}/R_{i\alpha}(Fe + B))$ (where $R_{i\alpha}(Fe + B)$ is taken with respect to the torus $Fe + Fz$).

PROPOSITION 5.5.1. *Let B and e be as above. Let $W \neq (0)$ be a restricted B-module with $zW = W$. Then*:

(a) $\dim W \geqslant p^m$, *where* $m = \max\{[(n_i + 1)/2] \mid i \in \mathbf{Z}_p^*\}$;

(b) *if W is an $(Fe + B)$-module and if $\sum_{i \in \mathbf{Z}_p^*} n_i > 2$ then* $\dim W \geqslant p^2$.

Proof. In proving (a) we may assume that W is irreducible. By the Engel–Jacobson theorem, B is nilpotent (as $(\mathrm{ad}\, B_0)$ is nil and hence every $x \in B_{i\alpha}$, $i \in \mathbf{Z}_p^*$ is ad-nilpotent), hence solvable. Then (cf. Theorem 1.13.1 of [BW82]) B contains a restricted subalgebra B_1 such that W contains a one-dimensional B_1-submodule W_1, and $\dim W = p^{[B:B_1]}$. For $i \in \mathbf{Z}_p$, let $B'_{i\alpha} = B_{i\alpha} \cap B_1$. Then $B' = \sum_{i \in \mathbf{Z}_p} B'_{i\alpha}$ is an e-invariant subalgebra of B. Now as B_1, and hence B', has a one-dimensional module $W_1 \subseteq W$ and as z acts as a nonzero scalar on W, hence on W_1, we must have $[B'_{i\alpha}, B'_{-i\alpha}] \subseteq J$ for $i \in \mathbf{Z}_p^*$. Define $f_i : B_{i\alpha} \times B_{-i\alpha} \to F$ by $[x, y] \in f_i(x, y)z + J$. Then rank $f_i = n_i$ and rank $f_i(B'_{i\alpha} \times B'_{-i\alpha}) = (0)$, so by Lemma 2.5.1 of [BW82], $\dim(B_{i\alpha}/B'_{i\alpha}) + \dim(B_{-i\alpha}/B'_{-i\alpha}) \geqslant n_i$. But $B_{i\alpha}/B'_{i\alpha} \cong (B_{i\alpha} + B_1)/B_1$ and so $\dim(B_{i\alpha}/B'_{i\alpha}) \leqslant [B : B_1]$. Thus $2[B : B_1] \geqslant n_i$ so $[B : B_1] \geqslant [(n_i + 1)/2]$, proving (a).

In proving (b) we may assume that W is irreducible as an $(Fe + B)$-module. Since $Fe + B$ is solvable, $\dim W = p^k$ for some $k \geqslant 0$ (cf. [BW82, Theorem 1.13.1]). We may assume $k \leqslant 1$. If W_2 is a B-submodule of W then $\dim W_2 \geqslant p$ (by (a), as some $n_i > 0$) and so $W_2 = W$ and $\dim W = p$. Thus, we may again assume that W is irreducible as a B-module. Let $D = \{b \in B \mid b \cdot W = (0)\}$. Then D is an ideal in $Fe + B$. If $x \in D_{i\alpha}$ and $y \in B_{-i\alpha}$ then $[x, y] \in D_0$; say $[x, y] = az + b$, $a \in F$, $b \in J$. Then $b|_W = -az|_W$ is not nil unless $a = 0$. Hence $[x, y] \in J$. Therefore

$D_{i\alpha} \subseteq R_{i\alpha}(Fe+B)$ so $\dim((B/D)_{i\alpha}/R_{i\alpha}((Fe+B)/D)) = n_i$. Thus if we have the result for B/D, which acts faithfully on W, we have it for B. Thus we may assume that $D=(0)$ and hence that $\mathfrak{z}(B)=Fz$. Now let B_1 be as in the proof of (a). Since $\dim W=p$, $[B:B_1]=1$. Now as some $n_i \neq 0$ there is some $i \in \mathbf{Z}_p^*$, $x \in B_{i\alpha}$, $x \in B'_{i\alpha}$. Then $\sum_{j=0}^{\infty}(x^j \cdot W_1)$ is a B-submodule of W, hence $=W$. Since $\dim W=p$ and $\dim W_1=1$, we have $x^{p-1}\cdot W_1 \neq (0)$. If $J\neq 0$, then since B acts faithfully on W, we have $\dim W_{l\alpha}>1$ for some l. But then, as $\dim W=p$, $W_{m\alpha}=(0)$ for some m. Since $x\cdot W_{u\alpha} \subseteq W_{(u+i)\alpha}$ we see that $x^{p-1}=0$, contradicting $x^{p-1}\cdot W \neq (0)$. Thus $J=(0)$. But then we have the hypotheses of Proposition 2.5.2(b) of [BW82] and so that result gives the desired conclusion. ∎

We can now bound the dimension of K_γ/RK_γ. In fact, we obtain a stronger result.

PROPOSITION 5.5.2. *Let A be a finite-dimensional restricted Lie algebra over F containing a torus of maximal dimension T which is two-dimensonal and standard (so $\mathfrak{z}_A(T)=T+I$, I a nil ideal in $\mathfrak{z}_A(T)$). Assume $A \subseteq \operatorname{Der} S$ for some simple Lie algebra S and $A=S+I$. Then for any $0\neq\gamma\in\Delta$ we have $\sum_{i\in\mathbf{Z}_p^*}\dim(K_{i\gamma}/RK_{i\gamma})\leqslant 2$ and $\dim(K_\gamma/RK_\gamma)\leqslant 1$. Also, $\dim(K_\gamma/RK_\gamma)=1$ if and only if $\dim(K_{-\gamma}/RK_{-\gamma})=1$.*

Proof. For $0\neq\gamma\in\Delta$ set $n_\gamma=\dim K_\gamma/RK_\gamma$. Choose $\alpha\in\Delta$ so that n_α is maximal. Since A is semisimple, $\mathfrak{z}(A)=(0)$, and so for every $t\in T$ there is some $\gamma\in\Gamma$ such that $\gamma(t)\neq 0$. Thus $\Gamma \not\subseteq \mathbf{Z}\alpha$. Then by Schue's lemma (Lemma 1.12.1) we have $\mathfrak{z}_S(T)=\sum_{\gamma\in\Gamma,\gamma\notin\mathbf{Z}\alpha}[A_\gamma, A_{-\gamma}]$. Since $\alpha(T)\neq(0)$, $\alpha(I)=(0)$, and $T\subseteq\mathfrak{z}_S(T)+I$ there exists $\beta\in\Gamma$, $\beta\notin\mathbf{Z}\alpha$ such that $\alpha([A_\beta, A_{-\beta}])\neq(0)$. Thus $A_\beta\neq M_\beta^\alpha$.

Let $W=(\sum_{i\in\mathbf{Z}_p}A_{\beta+i\alpha})/\sum_{i\in\mathbf{Z}_p}M^\alpha_{\beta+i\alpha})$. Then $W\neq(0)$ and W is a restricted $K^{(\alpha)}$-module (for $K_{i\alpha}=M^\alpha_{i\alpha}$ and M^α is a subalgebra by Lemma 5.1.3). Now $\dim W=\sum_{i\in\mathbf{Z}_p}\dim(A_{\beta+i\alpha}/M^\alpha_{\beta+i\alpha})$. But $\dim(A_{\beta+i\alpha}/M^\alpha_{\beta+i\alpha})\leqslant \dim(A_{\beta+i\alpha}/R_{\beta+i\alpha}) = \dim(A_{\beta+i\alpha}/K_{\beta+i\alpha}) + \dim(K_{\beta+i\alpha}/RK_{\beta+i\alpha}) + \dim(RK_{\beta+i\alpha}/R_{\beta+i\alpha})\leqslant 6+n_\alpha$ (as $\dim(RK_{\beta+i\alpha}/R_{\beta+i\alpha})\leqslant(\dim(A_{\beta+i\alpha}/K_{\beta+i\alpha}))$ $(\dim T-1)$ by Lemma 5.4.2 and $\dim(A_{\beta+i\alpha}/K_{\beta+i\alpha})\leqslant 3$ by Lemma 5.3.4). Thus $\dim W\leqslant p(6+n_\alpha)$.

There exists $t\in T$ such that $\alpha(t)=1$, $\beta(t)=0$. Let $B=\ker\alpha+I+\sum_{i\in\mathbf{Z}_p^*}K_{i\alpha}$, $e=\operatorname{ad} t$. If $x\in K_{i\alpha}$, $i\in\mathbf{Z}_p^*$, then $x^p\in\ker\alpha+I$. Thus B is a restricted subalgebra of A and so the hypotheses of Proposition 5.5.1 are satisfied. Furthermore, $B_{i\alpha}=K_{i\alpha}$ and $R_{i\alpha}(Fe+B)=RK_{i\alpha}$. Thus $n_i=n_{i\alpha}$ for $1\leqslant i\leqslant p-1$. Hence $\dim W\geqslant p^{[(n_\alpha+1)/2]}$ and so $p(6+n_\alpha)\geqslant p^{[(n_\alpha+1)/2]}$. Since $p>7$ this implies $n_\alpha\leqslant 2$. Then $\dim W\leqslant 8p<p^2$ (as $p>7$). So by Proposition 5.5.1(b) we have $\sum_{i\in\mathbf{Z}_p^*}n_{i\alpha}\leqslant 2$.

Now for $\gamma\in\Gamma$, $\ker\gamma$ is a one-dimensional subspace of T. Let $\ker\gamma=Fz_\gamma$. Then define $f_\gamma: K_\gamma\times K_{-\gamma}\to F$ by $[x,y]\in f_\gamma(x,y)\,z_\gamma+I$. Then with respect to f_γ we have $K_\gamma^\perp = RK_{-\gamma}$, $K_{-\gamma}^\perp = RK_\gamma$. Thus $n_\gamma = \dim K_\gamma/RK_\gamma = \dim K_\gamma/K_{-\gamma}^\perp = \dim K_{-\gamma}/K_\gamma^\perp = \dim K_{-\gamma}/RK_{-\gamma} = n_{-\gamma}$. Thus $2n_\alpha = n_\alpha + n_{-\alpha} \leqslant \sum_{i \in \mathbf{Z}_p^*} n_{i\alpha} \leqslant 2$ and so $n_\alpha \leqslant 1$ and hence $n_\gamma \leqslant 1$ for all $0 \neq \gamma \in \Delta$. Therefore $\sum_{i \in \mathbf{Z}_p^*} n_{i\gamma} \leqslant 2$ for all $0 \neq \gamma \in \Delta$ since, if some $n_{i\gamma} \neq 0$, we may take $\alpha = i\gamma$. ∎

COROLLARY 5.5.3. *Let A and T be as in Proposition* 5.5.2. *If $0 \neq \gamma \in \Gamma$ is non-Hamiltonian then* $\dim A_\gamma/R_\gamma \leqslant 3$ *and in any case* $\dim A_\gamma/R_\gamma \leqslant 7$.

Proof. This follows from Lemma 5.3.4, Lemma 5.4.2, and Proposition 5.5.2.

5.6. We now introduce two important sets of roots.

DEFINITION 5.6.1. Let

$$\Delta_P = \{0 \neq \gamma \in \Delta \mid A_{i\gamma} = K_{i\gamma} \text{ for some } i \in \mathbf{Z},\ 1 \leqslant i \leqslant p-1\}.$$

Let $\Gamma_P = \Gamma \cap \Delta_P$. Call the elements of Γ_P *proper roots.*

Let $i \in \mathbf{Z}_p^*$. Clearly $\gamma \in \Delta_P$ if and only if $i\gamma \in \Delta_P$ and so $\mathbf{Z}\gamma \cap \Delta_P = \varnothing$ or $\mathbf{Z}_p^*\gamma$. Thus Δ_P is the disjoint union of the sets $\mathbf{Z}_p^*\gamma$, $\gamma \in \Delta_P$. Hence $|\Delta_P|$ is a multiple of $p-1$.

DEFINITION 5.6.2. $n(A, T) = |\Delta_P|/(p-1)$.

LEMMA 5.6.3. *Let $0 \neq \gamma \in \Delta$. Then*:

(a) *$\gamma \in \Delta_P$ if and only if either γ is solvable, γ is classical, γ is Witt and there is a surjective homomorphism ϕ of $A^{(\gamma)}$ to $W(1:\mathbf{1})$ such that $\phi(T) = F(x_1 D_1)$, or γ is Hamiltonian and there is a surjective homomorphism of $A^{(\gamma)}$ to a subalgebra of $H(2:\mathbf{1})$ containing $H(2:\mathbf{1})^{(2)}$ such that $\phi(T) = \mathbf{D}(x_1 x_2)$.*

(b) *$\gamma \notin \Delta_P$ if and only if either γ is Witt and there is a surjective homomorphism ϕ of $A^{(\gamma)}$ to $W(1:\mathbf{1})$ such that $\phi(T) = F((x_1+1) D_1)$ or γ is Hamiltonian and there is a surjective homomorphism ϕ of $A^{(\gamma)}$ to a subalgebra of $H(2:\mathbf{1})$ containing $H(2:\mathbf{1})^{(2)}$ such that $\phi(T) = \mathscr{D}((x_1+1)x_2)$.*

Proof. This is immediate from Lemma 5.3.4.

COROLLARY 5.6.4. *Let A be a restricted Lie algebra, T be a torus of maximal dimension in A which is standard, and $\alpha \in \Delta(A, T)$. Then the following are equivalent*:

(a) *α is proper.*

(b) *T is contained in a compositionally classical subalgebra of codimension $\leqslant 2$ in $A^{(\alpha)}$.*

(c) *$\Psi_\alpha T$ is contained in a compositionally classical subalgebra of codimension $\leqslant 2$ in $A[\alpha]$.*

Proof. Since a subalgebra of $A^{(\alpha)}$ is compositionally classical of codimension $\leqslant 2$ if and only if its image in $A[\alpha]$ has the same property, (b) and (c) are equivalent. We will show that (a) and (c) are equivalent.

Suppose α is proper. If α is solvable or classical then $A[\alpha]$ is compositionally classical so (c) holds. If α is Witt or Hamiltonian then Lemma 5.6.3(a) shows that $\Psi_\alpha T \subseteq A[\alpha]_0$, so (c) holds.

Now suppose α is improper. Then by Lemma 5.6.3(b), α is Witt or Hamiltonian and $\Psi_\alpha T \nsubseteq A[A]_0$. But by Lemma 1.11.1, any proper subalgebra of codimension $\leqslant 2$ in $A[\alpha]$ is contained in $A[\alpha]_0$. Hence (c) cannot hold. ▮

DEFINITION 5.6.5. Let $\Gamma_E = \{\gamma \in \Gamma \mid K_\gamma \neq RK_\gamma\}$ $(= \{\gamma \in \Gamma \mid [K_\gamma, K_{-\gamma}] \nsubseteq I\})$. Call the elements of Γ_E *exceptional roots.*

It is clear that Proposition 5.5.2 implies:

COROLLARY 5.6.6. *Let A and T be as in Proposition* 5.5.2. *Then $\Gamma_E = -\Gamma_E$ and $|\mathbf{Z}\gamma \cap \Gamma_E| = 0$ or 2 for any $0 \neq \gamma \in \Delta$.*

DEFINITION 5.6.7. Let $\Gamma_R = \{\gamma \in \Gamma \mid A_\gamma \neq R_\gamma\}$ and $\Gamma_K = \{\gamma \in \Gamma \mid A_\gamma \neq K_\gamma\}$.

LEMMA 5.6.8. *Let A and T be as in Proposition* 5.5.2.

(a) $\Gamma_R = \Gamma_K \cup \Gamma_E$.

(b) $\Gamma_R = -\Gamma_R$.

Proof. Lemma 5.4.2 gives (a). Part (b) then follows since $\Gamma_K = -\Gamma_K$ (by Lemma 5.4.1) and $\Gamma_E = -\Gamma_E$ (by Corollary 5.6.6). ▮

LEMMA 5.6.9. *Let A and T be as in Proposition* 5.5.2. *If γ is proper then* $\dim A_\gamma / R_\gamma \leqslant 5$.

Proof. This follows from Lemma 5.3.4, Lemma 5.4.2, and Proposition 5.5.2. ▮

5.7. We will obtain some bounds on $|\Gamma_E|$. Throughout this section we will impose the following hypothesis on A:

> A is a finite-dimensional restricted Lie algebra over F containing a torus T of maximal dimension which is two-dimensional and standard ($\mathfrak{z}_A(T) = T + I$). Furthermore, there is some simple algebra S such that $S \subseteq A \subseteq \operatorname{Der} S$ and $A = S + I$. (5.7.1)

LEMMA 5.7.1. *Assume* (5.7.1) *holds. Then*:

(a) *If* $\gamma \in \Delta_P$ *then* $|\Gamma_K \cap \mathbf{Z}\gamma| = 0, 2,$ *or* 4, *and* $|\Gamma_R \cap \mathbf{Z}_\gamma| = 0, 2, 4,$ *or* 6. *Furthermore, if* $|\Gamma_K \cap \mathbf{Z}_\gamma| = 4$ *then* γ *is Hamiltonian and there is some* $j \in \mathbf{Z}_p^*$ *such that* $\Gamma_K \cap \mathbf{Z}_\gamma = \{\pm j\gamma, \pm 2j\gamma\}$.

(b) *If* $\beta \in \Gamma_E, \gamma \in \Delta - \mathbf{Z}\beta$, *and* $\beta([A_\gamma, A_{-\gamma}]) \neq (0)$ *then* $\pm\gamma + \mathbf{Z}\beta \subseteq \Gamma_R$.

(c) *If* $\beta, \gamma \in \Gamma_E$, $\beta \neq \pm\gamma$ *then* $\pm\beta + \mathbf{Z}\gamma \subseteq \Gamma_R$ and $\pm\gamma + \mathbf{Z}\beta \subseteq \Gamma_R$.

Proof. If $\gamma \in \Delta_P$ then by Lemma 5.3.4 we have $|\Gamma_K \cap \mathbf{Z}\gamma| = 0, 2,$ or 4. Furthermore, $|\Gamma_K \cap \mathbf{Z}\gamma| = 4$ implies that γ is Hamiltonian and $\Gamma_K \cap \mathbf{Z}\gamma = \{\pm j\gamma, \pm 2j\gamma\}$. Then Corollary 5.6.6 and Lemma 5.6.8(a) show that $|\Gamma_R \cap \mathbf{Z}\gamma| = 0, 2, 4,$ or 6, proving (a).

Suppose the hypotheses of (b) hold. Let $W = \sum_{i \in \mathbf{Z}_p^*} A_{\gamma+i\beta}/M^\beta_{\gamma+i\beta}$. As $\beta([A_\gamma, A_{-\gamma}]) \neq (0)$, $A_\gamma \neq M^\beta_\gamma$ and so $W \neq (0)$. As $M^\beta_{\gamma+i\beta} \supseteq R_{\gamma+i\beta}$, $\dim(A_{\gamma+i\beta}/M^\beta_{\gamma+i\beta}) \leqslant \dim(A_{\gamma+i\beta}/R_{\gamma+i\beta})$. By Corollary 5.5.3, $\dim(A_{\gamma+i\beta}/R_{\gamma+i\beta}) \leqslant 7$ and so (as $p > 7$) $\dim W < p^2$. Now W is a module for $K^{(\beta)}$ (as $K_{i\beta} = M^\beta_{i\beta}$ and M^β is a subalgebra by Lemma 5.1.3).

Since $\beta \in \Gamma_E$ there exist $0 \neq t \in T$, $x \in K_\beta$, $y \in K_{-\beta}$ such that $[x, y] \in t + I$ and $\beta(t) = 0$. Then $\gamma(t) \neq 0$ and $[x, y]$ does not annihilate any nonzero element of any irreducible $K^{(\beta)}$-constituent of W. Therefore W has no irreducible $K^{(\beta)}$-constituents of dimension one. But as $\beta(\sum_{i \in \mathbf{Z}_p^*} [K_{i\beta}, K_{-i\beta}]) = (0)$, Lemma 1.8.1 and the Engel–Jacobson theorem show that $\sum_{i \in \mathbf{Z}_p^*} K_{i\beta} + \sum_{i \in \mathbf{Z}_p^*} [K_{i\beta}, K_{-i\beta}]$ is nilpotent. Therefore $K^{(\beta)}$ is solvable. Hence every irreducible constituent of W has dimension p. Let W_1 be such a constituent. Then by Theorem 1.13.1 of [BW82], B contains a subalgebra B_1 of codimension 1 and W_1 contains a one-dimensional B_1-module W_2. Let $u \in B$, $u \notin B_1$. Then $\sum_{i \geqslant 0} u^i \cdot W_2$ is a B-submodule of W_1, hence is equal to W_1. Thus $u^{p-1} \neq 0$. Now if $K_\beta + K_{-\beta} \subseteq B_1$ then x and y act on the one-dimensional space W_2 so $[x, y]$ annihilates W_2, which we have seen is impossible. Thus we may find $u \in B_{\pm\beta}$, $u \notin B_1$ satisfying $u^{p-1} \neq 0$. But $u^{p-1} \neq 0$ implies W_1 has p nonzero weight spaces and hence that $A_{\gamma+i\beta}/M^\beta_{\gamma+i\beta} \neq 0$ for all $i \in \mathbf{Z}_p$. As $M^\beta_{\gamma+i\beta} \supseteq R_{\gamma+i\beta}$ this implies $\gamma + \mathbf{Z}\beta \subseteq \Gamma_R$. As $\Gamma_R = -\Gamma_R$ (by Lemma 5.6.8(b)), (b) follows.

Finally, (c) follows from (b) since $\beta \in \Gamma_E$ implies $\gamma([K_\beta, K_{-\beta}]) \neq (0)$ and $\gamma \in \Gamma_E$ implies $\beta([K_\gamma, K_{-\gamma}]) \neq (0)$. ∎

LEMMA 5.7.2. *Assume* (5.7.1) *holds. Suppose* $\beta, \gamma \in \Gamma_E$, $\beta \neq \pm\gamma$. *Then either* $\Delta_P \subseteq \mathbf{Z}\beta \cup \mathbf{Z}\gamma - \{0\}$ *or* $\Delta_P \supseteq \Delta - (\mathbf{Z}\beta \cup \mathbf{Z}\gamma)$.

Proof. Suppose $\eta \in \Delta - (\mathbf{Z}\beta \cup \mathbf{Z}\gamma)$, $\eta \notin \Delta_P$. Then $\mathbf{Z}_p^*\eta \subseteq \Gamma_R$. Thus for $i \in \mathbf{Z}_p^*$ we have $[A_{i\eta}, A_{-i\eta}] \nsubseteq I$. Since $\ker\beta \cap \ker\gamma = I$, Lemma 5.7.1(b) shows that for every $i \in \mathbf{Z}_p^*$ either $\pm i\eta + \mathbf{Z}\beta \subseteq \Gamma_R$ or $\pm i\eta + \mathbf{Z}\gamma \subseteq \Gamma_R$. As $p > 7$ we may assume (interchanging β and γ if necessary) that $i\eta + \mathbf{Z}\beta \subseteq \Gamma_R$

for at least six (eight if $p>13$), $i\in\mathbf{Z}_p^*$. But then if $\tau\notin\Delta-(\mathbf{Z}\beta\cup\mathbf{Z}\gamma)$ it is clear that $|\mathbf{Z}\tau\cap\Gamma_R|\geqslant 6$. (Thus the proof is complete if $p>13$.) Now if $\tau\in\Delta_P$ then $|\mathbf{Z}\tau\cap\Gamma_K|\leqslant 4$ by Lemma 5.7.1(a). Since $\Gamma_R=\Gamma_K\cup\Gamma_E$ this implies $j\tau\in\Gamma_E$ for some $j\in\mathbf{Z}_p^*$. Thus if the conclusion of the lemma fails to hold we may (replacing $j\tau$ by τ) find three roots β, γ, $\tau\in\Gamma_E$ which are pairwise linearly independent with $\tau\in\Gamma_P$. Note that, as $\eta\notin\Delta_P$, $\tau\in\Gamma_P$, we have $\eta\notin\mathbf{Z}\beta\cup\mathbf{Z}\gamma\cup\mathbf{Z}\tau$. Now as $i\in\mathbf{Z}_p^*$ implies $[A_{i\eta}, A_{-i\eta}]\nsubseteq I$ we see that no two of β, γ, τ vanish on $[A_{i\eta}, A_{-i\eta}]$ and hence that for each $i\in\mathbf{Z}_p^*$, two of the sets $\pm i\eta+\mathbf{Z}\beta$, $\pm i\eta+\mathbf{Z}\gamma$, $\pm i\eta+\mathbf{Z}\tau$ are contained in Γ_R. Thus of the 15 sets $\pm i\eta+\mathbf{Z}\beta$, $\pm i\eta+\mathbf{Z}\gamma$, $\pm i\eta+\mathbf{Z}\tau$, $1\leqslant i\leqslant 5$, at least 10 are contained in Γ_R. Therefore for $\mu=$ one of β, γ, τ there are four values of i for which $\pm i\eta+\mathbf{Z}\mu\subseteq\Gamma_R$. If $\mu=\beta$ or γ this implies that $|\mathbf{Z}\tau\cap\Gamma_R|\geqslant 8$ and hence (using Lemma 5.7.1(a)) that $\tau\notin\Gamma_P$, a contradiction. Thus we must have $\mu=\tau$ and so there are four values of i for which $\pm i\eta+\mathbf{Z}\tau\subseteq\Gamma_R$. Thus if $\nu\notin\Delta-\mathbf{Z}\tau$ we have $|\mathbf{Z}\nu\cap\Gamma_R|\geqslant 8$ and hence $\nu\notin\Delta_P$. Thus $\Delta_P=\mathbf{Z}\tau-\{0\}$. Then, as $\tau\in\Delta_P$ there are at least four values of i such that $i\tau\notin\Gamma_R$. Let a and b be two of these values. Let $\tau=l\beta+m\gamma$. Consider $\delta=al\beta+bm\gamma$. Since $a\neq b$ we have $\delta\notin\mathbf{Z}\tau$ and so $\delta\notin\Delta_P$. Hence $\delta\in\Gamma_R$. Since $[A_\delta, A_{-\delta}]\nsubseteq I$, we see by Lemma 5.7.1(b) that either $\delta+\mathbf{Z}\beta\subseteq\Gamma_R$ or $\delta+\mathbf{Z}\gamma\subseteq\Gamma_R$. But the first of these implies $bl\beta+bm\gamma=b\tau\in\Gamma_R$, a contradiction, and the second implies $al\beta+am\gamma=a\tau\in\Gamma_R$, again a contradiction. Thus $\tau\notin\Delta_P$, proving the lemma. ▮

COROLLARY 5.7.3. *Assume* (5.7.1) *holds. Suppose* β, $\gamma\in\Gamma_E$, $\beta\neq\pm\gamma$. *If* $\Delta_P\subseteq\mathbf{Z}\beta\cup\mathbf{Z}\gamma-\{0\}$, *then* $\Delta_P=\mathbf{Z}_p^*\beta$, $\Delta_P=\mathbf{Z}_p^*\gamma$, *or* $\Delta_P=\varnothing$.

Proof. If not, β, $\gamma\in\Delta_P$ (for $\mu\in\Delta_P$ implies $\mathbf{Z}_p^*\mu\subseteq\Delta_P$) and so by Lemma 5.7.1(a) (as $p>7$) we may find $0<i, j<p$ such that $i\beta, j\gamma\notin\Gamma_R$. But then Lemma 5.7.1(b) implies that $\beta([A_{i\beta+j\gamma}, A_{-i\beta-j\gamma}])=\gamma([A_{i\beta+j\gamma}, A_{-i\beta-j\gamma}])=(0)$, so that $[A_{i\beta+j\gamma}, A_{-i\beta-j\gamma}]\subseteq I$. Thus $i\beta+j\gamma\notin\Gamma_R$ and so $i\beta+j\gamma\in\Delta_P$, contradicting $\Delta_P\subseteq\mathbf{Z}\beta\cup\mathbf{Z}\gamma-\{0\}$. ▮

COROLLARY 5.7.4. *Assume* (5.7.1) *holds. If* α, β, $\gamma\in\Gamma_E$ *are pairwise linearly independent, then* $\Delta_P=\varnothing$ *or* $\Delta_P=\Delta-\{0\}$.

Proof. Suppose $\eta\in\Delta_P$. Then η belongs to at most one of $\mathbf{Z}\alpha$, $\mathbf{Z}\beta$, $\mathbf{Z}\gamma$. Hence we may assume $\eta\notin\mathbf{Z}\beta\cup\mathbf{Z}\gamma$ and so by Lemma 5.7.2, $\Delta_P\supseteq\Delta-\{\mathbf{Z}\beta\cup\mathbf{Z}\gamma\}$. This, of course, implies $\Delta_P\nsubseteq\mathbf{Z}\alpha\cup\mathbf{Z}\beta$ and $\Delta_P\nsubseteq\mathbf{Z}\alpha\cup\mathbf{Z}\gamma$ so $\Delta_P\supseteq\Delta-\{\mathbf{Z}\alpha\cup\mathbf{Z}\beta\}$ and $\Delta_P\supseteq\Delta-\{\mathbf{Z}\alpha\cup\mathbf{Z}\gamma\}$. Thus $\Delta_P=\Delta-\{0\}$. ▮

LEMMA 5.7.5. *Assume* (5.7.1) *holds. Suppose* β, $\gamma\in\Gamma_E$, $\beta\neq\pm\gamma$. *Then* $\Delta_P\neq\Delta-\{0\}$.

Proof. Suppose $\Delta_P=\Delta-\{0\}$. By Lemma 5.7.1(c), we have $\pm\beta+$

$\mathbf{Z}\gamma \subseteq \Gamma_R$ and $\pm\gamma + \mathbf{Z}\beta \subseteq \Gamma_R$. In particular, $\pm(\beta+2\gamma)$, $\pm(\beta+3\gamma)$, $\pm\frac{1}{2}(\beta+2\gamma) = \pm(\gamma+\frac{1}{2}\beta)$, $\pm\frac{1}{3}(\beta+3\gamma) = \pm(\gamma+\frac{1}{3}\beta) \in \Gamma_R$. Thus (as $p>7$) we have four multiples of $\beta+3\gamma$ contained in Γ_R with these multiples not being of the form $\pm j(\beta+3\gamma)$, $\pm 2j(\beta+3\gamma)$ for any j. Thus we have (by Lemma 5.7.1(a)) $\pm\delta \in \Gamma_E$, where $\delta = \beta+3\gamma$ or $\delta = \frac{1}{3}(\beta+3\gamma)$.

Suppose $\delta = \beta+3\gamma$. Then by Lemma 5.7.1(c), $\pm(\beta+3\gamma)+\mathbf{Z}\beta \subseteq \Gamma_R$ and so Γ_R contains $\pm\frac{3}{2}(\beta+2\gamma) = \pm((\beta+3\gamma)+\frac{1}{2}\beta)$. Thus $\pm\frac{1}{2}(\beta+2\gamma)$, $\pm(\beta+2\gamma)$, $\pm\frac{3}{2}(\beta+2\gamma) \in \Gamma_R$. Since $\pm\frac{1}{2}$, ± 1, $\pm\frac{3}{2}$ are distinct elements of $\mathbf{Z}_p^*$ and since j, $2j \in \{\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}\}$ implies $j = \pm\frac{1}{2}$, $2j = \pm 1$, we see from Lemma 5.7.1(a) that $\pm\frac{3}{2}(\beta+2\gamma) \in \Gamma_E$. Then by Lemma 5.7.1(c), $\pm\frac{3}{2}(\beta+4\gamma) = \pm(\frac{3}{2}(\beta+2\gamma)+\gamma) \in \pm\frac{3}{2}(\beta+2\gamma)+\mathbf{Z}\gamma \subseteq \Gamma_R$. Also $\pm(\beta+4\gamma) \in \pm\beta+\mathbf{Z}\gamma \subseteq \Gamma_R$, $\pm\frac{1}{4}(\beta+4\gamma) = \pm(\gamma+\frac{1}{4}\beta) \in \pm\gamma+\mathbf{Z}\beta \subseteq \Gamma_R$ and $\pm\frac{3}{4}(\beta+4\gamma) = \pm((\beta+3\gamma)-\frac{1}{4}\beta) \in \pm(\beta+3\gamma)+\mathbf{Z}\beta \subseteq \Gamma_R$. Thus Γ_R contains $j(\beta+4\gamma)$ whenever $j \in \{\pm\frac{1}{4}, \pm\frac{3}{4}, \pm 1, \pm\frac{3}{2}\}$. Since $\pm\frac{1}{4}$, $\pm\frac{3}{4}$, ± 1, $\pm\frac{3}{2}$ are distinct elements of $\mathbf{Z}_p^*$ we have $|\Gamma_R \cap \mathbf{Z}(\beta+4\gamma)| \geq 8$. Since $\beta+4\gamma \in \Delta_P$ this contradicts Lemma 5.7.1(a).

Suppose $\delta = \frac{1}{3}(\beta+3\gamma)$. Then by Lemma 5.7.1(c), $\pm\frac{1}{3}(\beta+3\gamma)+\mathbf{Z}\gamma \subseteq \Gamma_R$ and so $\pm\frac{1}{3}(\beta+2\gamma) = \pm(\frac{1}{3}(\beta+3\gamma)-\frac{1}{3}\gamma) \in \pm\frac{1}{3}(\beta+3\gamma)+\mathbf{Z}\gamma \subseteq \Gamma_R$. Thus $\pm\frac{1}{3}(\beta+2\gamma)$, $\pm\frac{1}{2}(\beta+2\gamma)$, $\pm(\beta+2\gamma) \in \Gamma_R$. Since $\pm\frac{1}{3}$, $\pm\frac{1}{2}$, ± 1 are distinct elements of $\mathbf{Z}_p^*$ and since j, $2j \in \{\pm\frac{1}{3}, \pm\frac{1}{2}, \pm 1\}$ implies $j = \pm\frac{1}{2}$, $2j = \pm 1$, we see from Lemma 5.7.1(a) that $\pm\frac{1}{3}(\beta+2\gamma) \in \Gamma_E$. Then by Lemma 5.7.1(c), $\pm\frac{1}{6}(\beta+4\gamma) = \pm(\frac{1}{3}(\beta+2\gamma)-\frac{1}{6}\beta) \in \pm\frac{1}{3}(\beta+2\gamma)+\mathbf{Z}\beta \subseteq \Gamma_R$. Also $\pm(\beta+4\gamma) \in \pm\beta+\mathbf{Z}\gamma \subseteq \Gamma_R$, $\pm\frac{1}{4}(\beta+4\gamma) = \pm(\gamma+\frac{1}{4}\beta) \in \pm\gamma+\mathbf{Z}\beta \subseteq \Gamma_R$ and $\pm\frac{1}{3}(\beta+4\gamma) = \pm(\frac{1}{3}(\beta+3\gamma)+\frac{1}{3}\gamma) \in \pm\frac{1}{3}(\beta+3\gamma)+\mathbf{Z}\gamma \subseteq \Gamma_R$. Thus Γ_R contains $j(\beta+4\gamma)$ whenever $j \in \{\pm\frac{1}{6}, \pm\frac{1}{4}, \pm\frac{1}{3}, \pm 1\}$. Since $\pm\frac{1}{6}$, $\pm\frac{1}{4}$, $\pm\frac{1}{3}$, ± 1 are distinct elements of $\mathbf{Z}_p^*$ we have $|\Gamma_R \cap \mathbf{Z}(\beta+4\gamma)| \geq 8$. Since $\beta+4\gamma \in \Delta_P$ this contradicts Lemma 5.7.1(a), completing the proof of the lemma. ∎

We now state our main result on Γ_E:

PROPOSITION 5.7.6. *Assume* (5.7.1) *holds. If* $n(A, T) \geq 2$ *then* $|\Gamma_E| = 0$ *or* 2. *If* $n(A, T) = 1$ *then* $|\Gamma_E| = 0, 2,$ *or* 4.

Proof. If $\beta, \gamma \in \Gamma_E$, $\beta \neq \pm\gamma$, then by Corollary 5.6.6, β and γ are linearly independent. Thus Corollary 5.7.4 and Lemma 5.7.5 show that $|\Gamma_E| > 4$ implies $n(A, T) = 0$. Lemma 5.7.2 and Corollary 5.7.3 show that if $|\Gamma_E| = 4$ then $n(A, T) \leq 1$ or $n(A, T) \geq p-1$. Suppose that $n(A, T) \geq p-1$ and $\Gamma_E = \{\pm\beta, \pm\gamma\}$, $\beta \neq \pm\gamma$. Pick $i \in \mathbf{Z}_p^*$, $i \neq \pm 1, \pm 2, \pm\frac{1}{2}$ (which is possible as $p>7$). Then $\pm(\beta+i\gamma) \in \Delta_P$ (by Lemma 5.7.2). Now let $j = i^{-1}$ in $\mathbf{Z}_p^*$. Then $\pm(\beta+i\gamma) \in \pm\beta+\mathbf{Z}\gamma \subseteq \Gamma_R$ and $\pm j(\beta+i\gamma) = \pm(\gamma+j\beta) \in \pm\gamma+\mathbf{Z}\beta \subseteq \Gamma_R$. But then, since $\mathbf{Z}(\beta+i\gamma) \cap \Gamma_E = \varnothing$ and $\Gamma_R \cap \mathbf{Z}(\beta+i\gamma) \supseteq \{\pm(\beta+i\gamma), \pm j(\beta+i\gamma)\}$, a set with four elements (as $i \neq \pm 1$ so $j \neq \pm 1$), we must have, by Lemma 5.7.1(a), that $j = \pm 2$ or $j = \pm\frac{1}{2}$. Since $j = i^{-1}$ this implies $i \in \{\pm\frac{1}{2}, \pm 2\}$. But this contradicts our choice of i and so completes the proof of the proposition. ∎

5.8. We will now study the Cartan decompositions of the algebras $W(2:\mathbf{1})$, $S(3:\mathbf{1})$, $H(4:\mathbf{1})$, $K(3:\mathbf{1})$. Our results will be used in Sections 7, 9, 10, and 11.

Let A be a restricted Lie algebra satisfying $B^{(1)} \subseteq A \subseteq B$, where B is one of $W(2:\mathbf{1})$, $S(3:\mathbf{1})$, $H(4:\mathbf{1})$, $K(3:\mathbf{1})$. Assume every two-dimensional torus in A is standard. Then $A = B^{(1)}$. If $B = W(2:\mathbf{1})$ or $K(3:\mathbf{1})$ this is vacuous as $B = B^{(1)}$. Suppose $B = S(3:\mathbf{1})$ and $B^{(1)} \subsetneq A \subseteq B$. Then (as $B/B^{(1)}$ is spanned by the cosets of $x_1^{p-1}x_2^{p-1}D_3$, $x_1^{p-1}x_3^{p-1}D_2$, $x_2^{p-1}x_3^{p-1}D_1$ (cf. [KS69, Section I.5.1; SF88, Section IV.3])) A contains $y = a_1(x_2+1)^{p-1}(x_3+1)^{p-1}D_1 + a_2(x_1+1)^{p-1}(x_3+1)^{p-1}D_2 + a_3(x_1+1)^{p-1}(x_2+1)^{p-1}D_3$ for some a_1, a_2, a_3 not all zero. Let T' denote the two-dimensional torus spanned by $\{t_1 = (x_1+1)\,D_1 - (x_2+1)\,D_2,\ t_2 = (x_2+1)\,D_2 - (x_3+1)\,D_3\}$. Direct computation shows that if $u = (x_1+1)^2\ (x_2+1)\,D_1 - (x_1+1)(x_2+1)^2\,D_2$ then $u \in z_A(T')$ and $[u, y] = -(a_1+a_2)(x_3+1)^{p-1}t_1$. Since $((x_3+1)^{p-1}t_1)^p = t_1^p = t_1$, this element is nil only if $a_1 + a_2 = 0$. Thus T' standard implies $a_1 + a_2 = 0$ and, similarly, $a_2 + a_3 = a_3 + a_1 = 0$. This implies $a_1 = a_2 = a_3 = 0$ contradicting the choice of y. Thus $A = B^{(1)}$ when $B = S(3:\mathbf{1})$. Similarly $B = H(4:\mathbf{1})$ implies $A = B^{(1)}$.

Let α be a root of A with respect to a maximal torus T. Then ker α is a one-dimensional restricted subalgebra of T and hence has the form Ft_α, where

$$0 \neq t_\alpha = t_\alpha^p, \qquad \alpha(t_\alpha) = 0. \tag{5.8.1}$$

Note that t_α is unique up to multiplication by elements of $\mathbf{Z}_p^*$. By Demuškin's conjugacy theorem (Theorem 1.3.1) there are, up to conjugacy, only a few possibilities for t_α. We will examine each of these possibilities.

LEMMA 5.8.1. *Let U be one of $W(2:\mathbf{1})$, $S(3:\mathbf{1})^{(1)}$, $H(4:\mathbf{1})^{(1)}$, $K(3:\mathbf{1})$. Let α be a root of U with respect to a maximal torus T. Let t_α be as in* (5.8.1). *Assume that $U^{(\alpha)} \not\subseteq U_0$. Then there exists a semisimple subalgebra $M \subseteq U^{(\alpha)}$ such that*

$$U^{(\alpha)} = M + (\mathrm{solv}(U^{(\alpha)}))$$

and

$$(U^{(\alpha)})_0 = M_0 + (\mathrm{solv}(U^{(\alpha)}))_0.$$

Furthermore, $\mathrm{solv}(U^{(\alpha)}) \subseteq U_0 + Ft_\alpha$, *$M$ is given (up to isomorphism) by Table* 5.8.1 *and* $\dim M/M_0 = 1$ *if* $M \cong W(1:\mathbf{1})$, $\dim M/M_0 = 2$ *if* $M \cong H(2:\mathbf{1})^{(1)}$.

TABLE 5.8.1

U	t_α	M
$W(2:\mathbf{1})$	any t_α	$W(1:\mathbf{1})$
$S(3:\mathbf{1})^{(1)}$	$t_\alpha \notin U_0$	$H(2:\mathbf{1})^{(1)}$
$S(3:\mathbf{1})^{(1)}$	$t_\alpha \in U_0$	$W(1:\mathbf{1})$
$H(4:\mathbf{1})^{(1)}$	any t_α	$H(2:\mathbf{1})^{(1)}$
$K(3:\mathbf{1})$	$t_\alpha \notin U_{-1}$	$H(2:\mathbf{1})^{(1)}$
$K(3:\mathbf{1})$	$t_\alpha \in U_{-1}$	$W(1:\mathbf{1})$

Proof. We write t in place of t_α.

Suppose $U = W(2:\mathbf{1})$ and $t \notin U_0$. Then by Theorem 1.3.1(a) we may assume that $t = (x_1 + 1) D_1$. Then

$$U^{(\alpha)} = \operatorname{span}\{x_2^i D_2 \mid 0 \leqslant i \leqslant p-1\} + \operatorname{span}\{(x_1+1) x_2^i D_1 \mid 0 \leqslant i \leqslant p-1\}.$$

Direct computation shows that the first summand is a subalgebra isomorphic to $W(1:\mathbf{1})$ and the second summand is an abelian ideal, hence solv$(U^{(\alpha)})$. Denoting the first summand by M, we see that $\dim M/M_0 = 1$ and the lemma holds in this case.

Now suppose that $U = W(2:\mathbf{1})$ and $t \in U_0$. Then by Theorem 1.3.1(b) we may assume that $t \in \operatorname{span}\{x_1 D_1, x_2 D_2\}$. Since $U^{(\alpha)} \nsubseteq U_0$ we see that $(\operatorname{ad} t)$ annihilates some linear combination of D_1 and D_2. Hence we may assume that $t = x_1 D_1$. Then

$$U^{(\alpha)} = \operatorname{span}\{x_2^i D_2 \mid 0 \leqslant i \leqslant p-1\} + \operatorname{span}\{x_1 x_2^i D_1 \mid 0 \leqslant i \leqslant p-1\}.$$

Direct computation shows that the first summand is a subalgebra isomorphic to $W(1:\mathbf{1})$ and the second summand is an abelian ideal, hence solv$(U^{(\alpha)})$. Denoting the first summand by M, we see that $\dim M/M_0 = 1$ and the lemma holds in this case.

Now suppose that $U = S(3:\mathbf{1})^{(1)}$ and $t \notin U_0$. Then by Theorem 1.3.1(d) we may assume that $t = (x_1+1)\, D_1 - x_2 D_2$. Recall ([Fra54]; cf. [KS69, Sect. I.5.1; [SF88, Sect. IV.3]) that $S(3:\mathbf{1})^{(1)}$ is spanned by

$$\{\mathscr{D}_{ij}(f) \mid 1 \leqslant i < j \leqslant 3, f \in \mathfrak{A}(3:\mathbf{1})\},$$

where $\mathscr{D}_{ij}(f) = D_j(f)\, D_i - D_i(f)\, D_j$. Then, since direct computation shows that

$$[t, D_{ij}((x_1+1)^a x_2^b x_3^c)] = (a - \delta_{i1} - b + \delta_{i2} + \delta_{j2})\, \mathscr{D}_{ij}((x_1+1)^a x_2^b x_3^c)$$

and that

$$\mathscr{D}_{12}(D_3 f) + \mathscr{D}_{31}(D_2 f) + \mathscr{D}_{23}(D_1 f) = 0,$$

we see that

$$U^{(\alpha)} = \operatorname{span}\{\mathscr{D}_{23}((x_1+1)^{i-1} x_2^i x_3^j) \mid 0 \leqslant i, j \leqslant p-1, (i,j) \neq (0,0)\}$$
$$+ \operatorname{span}(\{\mathscr{D}_{12}((x_1+1)^i x_2^i x_3^j) \mid i>0, j \geqslant 0\}$$
$$\cup \{\mathscr{D}_{13}(x_2^{p-1} x_3^j) \mid j>0\}).$$

Direct computation shows that the first summand is a subalgebra, which we denote by M, and that the map

$$\mathscr{D}_{23}((x_1+1)^{i-1} x_2^i x_3^j) \mapsto \mathscr{D}(x_1^i x_2^j), \qquad 0 \leqslant i, j \leqslant p-1, (i,j) \neq (0,0)$$

extends linearly to an isomorphism from M to $H(2:\mathbf{1})^{(1)}$. Note that $\dim M/M_0 = 2$. Direct computation also shows that the second summand is an abelian ideal, hence $\operatorname{solv}(U^{(\alpha)})$, and that the lemma holds in this case.

Now suppose that $U = S(3:\mathbf{1})^{(1)}$ and $t \in U_0$. Then by Theorem 1.3.1(e) we may assume that $t \in \operatorname{span}\{x_1 D_1 - x_2 D_2, \ x_2 D_2 - x_3 D_3\}$. Since $U^{(\alpha)} \not\subseteq U_0$ we see that $(\operatorname{ad} t)$ annihilates some linear combination of D_1, D_2, and D_3. Hence we may assume that $t = x_1 D_1 - x_2 D_2$. Then

$$U^{(\alpha)} = \operatorname{span}\{\mathscr{D}_{13}(x_1 x_3^j) \mid 0 \leqslant j \leqslant p-1\}$$
$$+ \operatorname{span}(\{\mathscr{D}_{12}(x_1^i x_2^i x_3^j) \mid 0 < i \leqslant p-1, 0 \leqslant j \leqslant p-1\}$$
$$\cup \{\mathscr{D}_{13}(x_1^{i+1} x_2^i x_3^j) \mid 1 \leqslant i \leqslant p-2, 0 \leqslant j \leqslant p-1\}$$
$$\cup \{\mathscr{D}_{13}(x_2^{p-1} x_3^j) \mid 0 \leqslant j \leqslant p-1\}$$
$$\cup \{\mathscr{D}_{23}(x_1^{p-1} x_3^j) \mid 0 < j \leqslant p-1\}).$$

Direct computation shows that the first summand is a subalgebra, which we denote by M, and that the map

$$\mathscr{D}_{13}(x_1 x_3^j) \mapsto -x_1^j D_1, \qquad 0 \leqslant j \leqslant p-1$$

extends linearly to an isomorphism from M to $W(1:\mathbf{1})$. Note that $\dim M/M_0 = 1$. Direct computation also shows that the second summand is a solvable ideal, hence $\operatorname{solv}(U^{(\alpha)})$, and that the lemma holds in this case.

Now suppose that $U = H(4:\mathbf{1})^{(1)}$. Recall ([AF54]; cf. [KS69, Sect. I.6.1; SF88, Sect. IV.4]) that $H(4:\mathbf{1})^{(1)}$ has basis

$$\{\mathscr{D}(x_1^a x_2^b x_3^c x_4^d) \mid 0 \leqslant a, b, c, d \leqslant p-1, (a, b, c, d)$$
$$\neq (0,0,0,0), (p-1, p-1, p-1, p-1)\},$$

where $\mathscr{D}(f) = -(D_3 f)\ D_1 - (D_4 f)\ D_2 + (D_1 f)\ D_3 + (D_2 f)\ D_4$ for $f \in \mathfrak{A}(4:\mathbf{1})$. Recall also that multiplication is given by

$$[\mathscr{D}(f), \mathscr{D}(g)] = \mathscr{D}\left(\sum_{i=1}^{2} ((D_i f)(D_{i+2} g) - (D_{i+2} f)(D_i g))\right).$$

If $t \notin U_0$ then by Theorem 1.3.1(g) we may assume that $t = \mathscr{D}((x_1+1)\, x_3)$. If $t \in U_0$ then by Theorem 1.3.1(h) we may assume that $t \in \operatorname{span}\{\mathscr{D}(x_1 x_3), \mathscr{D}(x_2 x_4)\}$. Since $U^{(\alpha)} \not\subseteq U_0$ we see that t centralizes some linear combination of D_1, D_2, D_3, and D_4. Thus we may assume $t = \mathscr{D}(x_1 x_3)$. Hence, in either case ($t \in U_0$ or not) we may assume $t = \mathscr{D}(y x_3)$, where $y = x_1$ or $x_1 + 1$. Since

$$[t, \mathscr{D}(y^a x_2^b x_3^c x_4^d)] = (c-a)\, \mathscr{D}(y^a x_2^b x_3^c x_4^d)$$

we have that

$$\begin{aligned} U^{(\alpha)} = \operatorname{span}\{&\mathscr{D}(x_2^b x_4^d) \mid 0 \leqslant b, d \leqslant p-1, (b, d) \neq (0, 0)\} \\ &+ \operatorname{span}\{\mathscr{D}(y^a x_2^b x_3^a x_4^d) \mid 0 \leqslant a, b, d \leqslant p-1, a \neq 0, (a, b, d) \\ &\neq (p-1, p-1, p-1)\}. \end{aligned}$$

The first summand (which we denote by M) is clearly a subalgebra isomorphic to $H(2:\mathbf{1})^{(1)}$. Note that $\dim M/M_0 = 2$. Direct computation shows that the second summand is a solvable ideal, hence is $\operatorname{solv}(U^{(\alpha)})$, and thus the lemma holds in this case.

Now suppose $U = K(3:\mathbf{1})$. Recall (cf. (1.3.1) and (1.3.2)) that $K(3:\mathbf{1}) = \{\mathscr{D}_K(f) \mid f \in \mathfrak{A}(3:\mathbf{1})\}$, where

$$\begin{aligned} \mathscr{D}_K(f) = (D_2 f + x_1 D_3 f)\, D_1 - (D_1 f - x_2 D_3 f)\, D_2 \\ + (2f - x_1 D_1 f - x_2 D_2 f)\, D_3, \end{aligned}$$

and that

$$\begin{aligned} &[\mathscr{D}_K(f), \mathscr{D}_K(g)] \\ &\quad = \mathscr{D}_K\{(2f - x_1 D_1 f - x_2 D_2 f)(D_3 g) - (2g - x_1 D_1 g - x_2 D_2 g)(D_3 f) \\ &\qquad - (D_1 f)(D_2 g) + (D_2 f)(D_1 g)\}. \end{aligned} \tag{5.8.2}$$

Suppose $t \notin U_{-1}$. Then, by Theorem 1.3.1(j) we may assume that $t = \mathscr{D}_K(x_3 + 1)$. Then from (5.8.2) we get

$$[t, \mathscr{D}_K(x_1^a x_2^b (x_3+1)^c)] = (a + b + 2c - 2)\, \mathscr{D}_K(x_1^a x_2^b (x_3+1)^c).$$

Thus

$$\begin{aligned} U^{(\alpha)} = \operatorname{span}\{&\mathscr{D}_K(x_1^a x_2^b (x_3+1)^{((2-a-b)/2)}) \mid \\ &0 \leqslant a, b \leqslant p-1, (a, b) \neq (0, 0)\} + Ft. \end{aligned}$$

Using (5.8.2) we see that the first summand is a subalgebra (which we denote by M) and that the map

$$\mathscr{D}_K(x_1^a x_2^b (x_3+1)^{((2-a-b)/2)}) \mapsto -\mathscr{D}(x_1^a x_2^b),$$
$$0 \leqslant a, b \leqslant p-1, \ (a, b) \neq (0, 0),$$

extends linearly to an isomorphism of M onto $H(2:\mathbf{1})^{(1)}$. Note that $\dim M/M_0 = 2$. The second summand is, of course, central and hence is $\operatorname{solv}(U^{(\alpha)})$. Thus the lemma holds in this case.

Now suppose $U = K(3:\mathbf{1})$ and $t \in U_{-1}$, $t \notin U_0$. In this case Theorem 1.3.1(k) shows that we may assume $t = \mathscr{D}_K((x_1+1)\, x_2)$. Then (5.8.2) gives

$$[t, \mathscr{D}_K((x_1+1)^a\, x_2^b (x_3 - x_1 x_2)^c)] = (a-b)\, \mathscr{D}_K((x_1+1)^a\, x_2^b (x_3 - x_1 x_2)^c).$$

Thus

$$\begin{aligned} U^{(\alpha)} = {} & \operatorname{span}\{\mathscr{D}_K((x_3 - x_1 x_2)^c) \mid 0 \leqslant c \leqslant p-1\} \\ & + \operatorname{span}\{\mathscr{D}_K((x_1+1)^a\, x_2^a (x_3 - x_1 x_2)^c) \mid \\ & \qquad 1 \leqslant a \leqslant p-1, 0 \leqslant c \leqslant p-1\}. \end{aligned}$$

As a special case of (5.8.2) we see that

$$\begin{aligned} & [\mathscr{D}_K((x_1+1)^a\, x_2^a (x_3 - x_1 x_2)^b), \mathscr{D}_K((x_1+1)^c\, x_2^c (x_3 - x_1 x_2)^d)] \\ & \quad = 2(bc - ad - b + d)\, \mathscr{D}_K((x_1+1)^{a+c}\, x_2^{a+c} (x_3 - x_1 x_2)^{b+d-1}). \end{aligned}$$

From this we see that in our expression for $U^{(\alpha)}$ the first summand (which we denote by M) is a subalgebra and that the map

$$\mathscr{D}_K((x_3 - x_1 x_2)^b) \mapsto 2 x_1^b D_1, \qquad 0 \leqslant b \leqslant p-1,$$

extends linearly to an isomorphism from M to $W(1:\mathbf{1})$. Note that $\dim M/M_0 = 1$. We also see that the second summand is a solvable ideal, hence $\operatorname{solv}(U^{(\alpha)})$, and that the lemma holds in this case.

Now suppose $U = K(3:\mathbf{1})$ and $t \in U_0$. In this case Theorem 1.3.1(l) shows that we may assume $t \in \operatorname{span}\{\mathscr{D}_K(x_1 x_2), \mathscr{D}_K(x_1 x_2 - x_3)\}$. Since $U^{(\alpha)} \not\subseteq U_0$ we see that $(\operatorname{ad} t)$ annihilates a linear combination of $\mathscr{D}_K(1)$, $\mathscr{D}_K(x_1)$, $\mathscr{D}_K(x_2)$. Hence we may assume t is one of $\mathscr{D}_K(x_1 x_2)$, $\mathscr{D}_K(x_1 x_2 - x_3)$, $\mathscr{D}_K(x_1 x_2 + x_3)$. Since the last two elements are conjugate (by the automorphism of $K(3:\mathbf{1})$ obtained by interchanging x_1 and x_2 and replacing x_3 by $-x_3$), we may assume $t = \mathscr{D}_K(x_1 x_2)$ or $\mathscr{D}_K(x_1 x_2 - x_3)$. In the first case we use (5.8.2) to see that

$$[t, \mathscr{D}_K(x_1^a x_2^b x_3^c)] = (a-b)\, \mathscr{D}_K(x_1^a x_2^b x_3^c).$$

Therefore

$$U^{(\alpha)} = \operatorname{span}\{\mathscr{D}_K(x_3^c) \mid 0 \leqslant c \leqslant p-1\}$$
$$+ \operatorname{span}\{\mathscr{D}_K(x_1^a x_2^a x_3^c) \mid 1 \leqslant a \leqslant p-1, 0 \leqslant c \leqslant p-1\}.$$

The first summand, which we denote by M, is clearly a subalgebra isomorphic to $W(1:\mathbf{1})$. Note that $\dim M/M_0 = 1$. The second summand is a solvable ideal, hence $\operatorname{solv}(U^{(\alpha)})$, and the lemma holds in this case. In the second case we use (5.8.2) to see that

$$[t, \mathscr{D}_K(x_1^a x_2^b x_3^c)] = 2(1-b-c)\, \mathscr{D}_K(x_1^a x_2^b x_3^c).$$

Hence

$$U^{(\alpha)} = \operatorname{span}\{\mathscr{D}_K(x_1^a x_2) \mid 0 \leqslant a \leqslant p-1\}$$
$$+ \operatorname{span}(\{\mathscr{D}_K(x_1^a(x_3 - x_1 x_2)) \mid 0 \leqslant a \leqslant p-1\}$$
$$\cup \{\mathscr{D}_K(x_1^a x_2^b x_3^{p+1-b}) \mid 0 \leqslant a \leqslant p-1, 2 \leqslant b \leqslant p-1\}).$$

Using (5.8.2) we see that the first summand (which we denote by M) is a subalgebra isomorphic to $W(1:\mathbf{1})$. Note that $\dim M/M_0 = 1$. The second summand is a solvable ideal, hence is $\operatorname{solv}(U^{(\alpha)})$, and the lemma holds in this case.

We have now considered all cases, so the lemma is proved. ∎

LEMMA 5.8.2. *Let $A = B^{(1)}$ where B is one of $W(2:\mathbf{1})$, $S(3:\mathbf{1})$, $H(4:\mathbf{1})$, $K(3:\mathbf{1})$. Let α be a root of A with respect to a maximal torus T.*

(a) $\operatorname{solv} A^{(\alpha)} \subseteq \operatorname{solv} B^{(\alpha)}$ *and* $B[\alpha]^{(2)} \subseteq A[\alpha] \subseteq B[\alpha]$.

(b) *The type of α (solvable, classical, Witt, or Hamiltonian) as a root of A is the same as the type of α as a root of B.*

(c) *α is proper as a root of A if and only if it is proper as a root of B.*

(d) *$A^{(\alpha)} \subseteq A_0$ if and only if α is solvable or classical.*

(e) $\operatorname{solv} A^{(\alpha)} \subseteq A_0 + T$.

(f) *If α is Witt or Hamiltonian then $\Psi_\alpha(A^{(\alpha)} \cap A_0) = A[\alpha]_0$ (i.e., $W(1:\mathbf{1})_0$ if α is Witt or $A[\alpha] \cap H(2:\mathbf{1})_0$ if α is Hamiltonian).*

(g) *α is proper if and only if $T = (T \cap A_0) + \ker \alpha$.*

(h) *$\Delta(A, T) = \Delta_P(A, T)$ if and only if $T \subseteq A_0$.*

Proof. Since $B^{(\alpha)} \supseteq A^{(\alpha)} \supseteq (B^{(\alpha)})^{(2)}$ we have, taking images under the quotient map $\Psi_\alpha \colon B^{(\alpha)} \to B[\alpha]$, that

$$B[\alpha] \supseteq (A^{(\alpha)} + \operatorname{solv} B^{(\alpha)})/(\operatorname{solv} B^{(\alpha)}) \supseteq (B[\alpha])^{(2)}.$$

Now $B[\alpha]$ is either (0), $\mathfrak{sl}(2)$, $W(1:\mathbf{1})$ or else $H(2:\mathbf{1})^{(2)} \subseteq B[\alpha] \subseteq H(2:\mathbf{1})$. In any case, all subalgebras contained between $B[\alpha]$ and $(B[\alpha])^{(2)}$ are semisimple. In particular, $(A^{(\alpha)} + \text{solv } B^{(\alpha)})/(\text{solv } B^{(\alpha)}) \cong A^{(\alpha)}/(A^{(\alpha)} \cap \text{solv } B^{(\alpha)})$ is semisimple and so solv $A^{(\alpha)} \subseteq \text{solv } B^{(\alpha)}$ and $A[\alpha] \cong (A^{(\alpha)} + \text{solv } B^{(\alpha)})/(\text{solv } B^{(\alpha)}) \subseteq B[\alpha]$. This proves (a) and (b) follows immediately. Since $B = B^{(2)} + \mathfrak{z}_B(T)$ we see that $B_{i\alpha} = A_{i\alpha}$ for all $i \in \mathbf{Z}_p^*$. Hence (c) holds.

Now B_0 is compositionally classical, so A_0 is also. Thus if $A^{(\alpha)} \subseteq A_0$ we see that α is solvable or classical. This proves one implication in (d). For the remaining implication in (d) and for (e)–(g), we may assume, in view of (a)–(c), that $A = B$. Then (d)–(f) are given by Lemma 5.8.1. To prove (g), first assume that α is solvable or classical. Then α is proper and, by (d), $T = T \cap A_0$. Thus (g) holds in this case. If α is Witt or Hamiltonian we see from Lemma 5.6.3 that α is a proper root of $A[\alpha]$ if and only if $\Psi_\alpha(T) \subseteq A[\alpha]_0$. But α is a proper root of $A[\alpha]$ if and only if α is a proper root of A. Also $\Psi_\alpha(T) \subseteq A[\alpha]_0$ if and only if $T \subseteq T \cap A_0 + \ker \alpha$ (by (f) and Lemma 5.8.1) Hence (g) holds. Since (h) is immediate from (g), the lemma is proved. ∎

Corollary 5.8.3. *Let $A = B^{(1)}$ where B is one of $W(2:\mathbf{1})$, $S(3:\mathbf{1})$, $H(4:\mathbf{1})$, $K(3:\mathbf{1})$. Let T be a two-dimensional torus in A such that all roots with respect to T are proper. Then*:

(a) (solv $A^{(\alpha)}) \subseteq A_0$ *for all* $\alpha \in \Delta(T)$.

(b) *If* $x \in A_\alpha \cap A_0$ *then* x^p *is nil.*

Proof. Part (a) follows from (e) and (h) of Lemma 5.8.2. Since A_1 is a nil ideal in A_0 and A_0/A_1 is classical (so $y^p = 0$ for every root vector y in A_0/A_1), (b) holds. ∎

Corollary 5.8.4. *Suppose $A = B^{(1)}$ where B is one of $W(2:\mathbf{1})$, $S(3:\mathbf{1})$, $H(4:\mathbf{1})$, $K(3:\mathbf{1})$. Let α be a root of A with respect to a maximal torus T (necessarily two-dimensional). Let $\alpha \in \Delta(T)$, $\alpha \notin \Delta_P(T)$. Let $x \in A_\alpha$ be such that $\alpha_x \in \Delta_P(e^x(T))$. Then $|\Delta_P(e^x(T))| > |\Delta_P(T)|$.*

Proof. Assume $|\Delta_P(e^x(T))| \leqslant |\Delta_P(T)|$. As α is improper, $T \neq T \cap A_0$ by Lemma 5.8.2(g). Since $\alpha_x \in \Delta_P(e^x(T))$ we have $|\Delta_P(T)| \neq 0$. Thus there is some $\beta \in \Delta_P(T)$ and so $T = T \cap A_0 + \ker \beta$ (by Lemma 5.8.2(g)). Hence $T \cap A_0 \neq (0)$, so dim $T \cap A_0 = 1$. As α is improper, $T \neq T \cap A_0 + \ker \alpha$, so $\ker \alpha = T \cap A_0$. Now $\ker \alpha_x = \ker \alpha$ and so $e^x(T) = e^x(T) \cap A_0 + \ker \alpha_x = e^x(T) \cap A_0 + \ker \alpha = e^x(T) \cap A_0 + T \cap A_0$. Thus $e^x(T) \subseteq A_0$ and so $\Delta_P(e^x(T)) = \Delta(e^x(T))$ by Lemma 5.8.2(h). But since $|\Delta_P(e^x(T))| \leqslant |\Delta_P(T)|$, this implies $\Delta_P(T) = \Delta(T)$, contradicting the choice of α. ∎

LEMMA 5.8.5. *Let* $A = B^{(1)}$, *where* B *is one of* $W(2:\mathbf{1})$, $S(3:\mathbf{1})$, $H(4:\mathbf{1})$, $K(3:\mathbf{1})$. *Let* $\alpha \in \Delta(A, T)$, *where* T *is a maximal torus, and assume* $\mathfrak{z}_A(T) = T + I$, *where* I *is a nil ideal in* $\mathfrak{z}_A(T)$. *Then if* $x, y \in A_\alpha \cap A_0$ *we have* $(\operatorname{ad} x)^{p-1} y \in I$.

Proof. Since $\mathfrak{z}(T) \cap A_1 \subseteq I$ it is sufficient to show that $(\operatorname{ad}(x + A_1))^{p-1} (y + A_1) = 0$ in A_0/A_1. To see this note that $\dim A_0/A_1 < p$ (as $A_0/A_1 \cong \mathfrak{gl}(2)$, $\mathfrak{sl}(3)$, or $\mathfrak{sp}(4)$). But since $\alpha(x^p) = 0$ (Lemma 1.8.1) we have $(\operatorname{ad} x)^n y = 0$ for some n and so $(\operatorname{ad}(x + A_1))^n (y + A_1) = 0$ for some n. Thus $(\operatorname{ad}(x + A_1))^{(\dim A_0/A_1)} (y + A_1) = 0$ so $(\operatorname{ad}(x + A_1))^{p-1} (y + A_1) = 0$. ∎

LEMMA 5.8.6. *Let* $H(2:\mathbf{1})^{(2)} \subseteq A \subseteq \operatorname{Der} H(2:\mathbf{1})$ *and let* T *be a two-dimensional torus in* A. *Let* $\alpha \in \Delta(T)$, $\alpha \notin \Delta_P(T)$. *Let* $x \in A_\alpha$ *be such that* $\alpha_x \in \Delta_P(e^x(T))$. *Then* $|\Delta_P(e^x(T))| > |\Delta_P(T)|$.

Proof. Corollary 2.1.9 shows that $\operatorname{Der} H(2:\mathbf{1}) = H(2:\mathbf{1}) + F(x_1 D_1 + x_2 D_2)$ so Corollary 1.3.2 shows that $\mathfrak{z}_A(T) = T$. Thus this result is a special case of Proposition 4.9 of [Wil83]. However, this lemma has a direct proof which avoids the involved arguments occurring in [Wil83]. We give that proof here.

By [BW82, Theorem 1.18.4] we see that T is conjugate to the span of $\{y_1 D_1, y_2 D_2\}$, where the pair (y_1, y_2) is one of (x_1, x_2), $(x_1 + 1, x_2)$, $(x_1 + 1, x_2 + 1)$. In the first case $T \subseteq W(2:\mathbf{1})_0$ and so Lemma 5.8.2(h) shows $\Delta_P(W(2:\mathbf{1}), T) = \Delta(W(2:\mathbf{1}), T)$ and hence $\Delta_P(A, T) = \Delta(A, T)$, contradicting the assumption that $\alpha \notin \Delta_P(A, T)$. Thus we may assume $y_1 = x_1 + 1$. Define $\alpha_1, \alpha_2 \neq T^*$ by $\alpha_i(y_j D_j) = \delta_{ij}$.

Suppose $y_2 = x_2$ and $\beta \notin \mathbf{Z}\alpha_1$. Then $x_2 D_2 \notin \ker \beta$ so $T \cap A_0 + \ker \beta = T$ and hence (by Lemma 5.8.2(g)) $\beta \in \Delta_P(W(2:\mathbf{1}), T) \subseteq \Delta_P(A, T)$. Thus $\alpha \in \mathbf{Z}\alpha_1$ and $x_2 D_2 \in e^x(T)$. Furthermore, $\beta_x(x_2 D_2) = \beta(x_2 D_2) \neq 0$ so $x_2 D_2 \notin \ker \beta_x$. Thus $e^x(T) \cap A_0 + \ker \beta_x = e^x(T)$ and so $\beta_x \in \Delta_P(W(2:\mathbf{1}), e^x(T)) \subseteq \Delta_P(A, e^x(T))$ for all $\beta \notin \mathbf{Z}\alpha_1$. Since $\alpha \in \mathbf{Z}\alpha_1$ we have $\mathbf{Z}(\alpha_1)_x \subseteq \Delta_P(A, e^x(T))$ and so $\Delta_P(A, e^x(T) = \Delta(A, e^x(T))$. Thus our result holds in this case.

Finally, suppose $y_2 = x_2 + 1$. Then direct computation (using Corollary 2.1.10) shows that $\mathscr{D}(y_1^i y_2^j)$ spans $A_{(i-1)\alpha_1 + (j-1)\alpha_2}$, that $\alpha_1 + \alpha_2$ is solvable, and that if $k \neq l$ then $k\alpha_1 + l\alpha_2$ is improper and Witt. Then α_x is a proper Witt root of A with respect to $e^x(T)$. Hence $e^x(T)$ cannot be conjugate to $\operatorname{span}\{(x_1 + 1)\ D_1,\ (x_2 + 1)\ D_2\}$ and so $|\Delta_P(A, e^x(T))| > |\Delta_P(A, T)|$ in this case. Thus the lemma is proved. ∎

6. Distinguished Maximal Subalgebras

Let A be a finite-dimensional restricted semisimple Lie algebra over F. Throughout this section we assume that every two-dimensional torus in A is maximal and standard, that $S \subseteq A \subseteq \operatorname{Der} S$ for some simple Lie algebra S, and that for any two-dimensional torus T in A we have $A = S + I$ (where I is the nil radical of $\mathfrak{z}_A(T)$) and so (see Lemma 1.8.2) $T \subseteq \bar{S}$. We wish to find a maximal subalgebra $A_0 \subseteq A$ such that if $A = A_{-k} \supseteq \cdots \supseteq A_{-1} \supseteq A_0 \supseteq A_1 \supseteq \cdots$ is a corresponding filtration we will be able to apply the Recognition Theorem (Theorem 1.2.2) to S.

To find a suitable A_0 we will impose several conditions. First we require that A_0 contain $\mathfrak{z}_A(T)$ for some two-dimensional torus $T \subseteq A$. This requirement implies that when we form a corresponding filtration and the associated graded algebra $G = \sum G_i$, we may identify T with a maximal torus of G which is contained in G_0. Thus each G_i has a decomposition into weight spaces with respect to T. In Section 6.1 we develop the necessary notation to deal with this situation.

As in [BW82] it turns out that not every two-dimensional torus is suitable for our purposes. In Section 6.2 we define a class of maximal tori which we call *optimal tori* of A and study some of their properties.

In Section 6.3 we introduce the maximal subalgebras which we will use for our A_0. We call these *distinguished maximal subalgebras*. These are certain maximal subalgebras containing $\mathfrak{z}_A(T)$ for an optimal maximal torus T.

6.1. Let T be a two-dimensional torus of A and let A_0 be a maximal subalgebra of A containing $\mathfrak{z}_A(T)$. Then each A_i is ad T-invariant and so has a weight space decomposition $A_i = \sum_{\gamma \in T^*} A_{i,\gamma}$. Since A_1 is a nil subalgebra of A (as $(\operatorname{ad} A_1)^m\ A = (\operatorname{ad} A_1)^m\ A_{-k} \subseteq A_{m-k} = (0)$ for sufficiently large m) we have $T \cap A_1 = (0)$. Thus the map $T \to (T + A_1)/A_1$, $t \mapsto t + A_1$ is an isomorphism of T onto a two-dimensional torus of G_0. Since the quotient map $A_0 \to A_0/A_1 = G_0$ is a surjective homomorphism of restricted Lie algebras we have that the image of T in G_0 (which we again denote by T) is standard and of maximal dimension (Lemmas 1.6.1(c) and 1.7.2).

Now each G_i is a G_0-module (via the adjoint representation). Hence G_i has a weight space decomposition $G_i = \sum_{\gamma \in T^*} G_{i,\gamma}$ with respect to T. Let $\Gamma_i = \{\gamma \in T^* \mid G_{i,\gamma} \neq (0)\}$. Let $\Gamma_- = \bigcup_{i<0} \Gamma_i$, $m_i(\gamma) = \dim G_{i,\gamma}$, and $m_-(\gamma) = \sum_{i<0} m_i(\gamma)$. From our identification of T with a torus of G_0, it is clear that $G_{i,\gamma} = (A_{i,\gamma} + A_{i+1})/A_{i+1}$ and that $\Gamma_i = \{\gamma \in \Gamma \cup \{0\} \mid A_{i,\gamma} \not\subseteq A_{i+1}\} = \{\gamma \in \Gamma \cup \{0\} \mid A_{i,\gamma} \neq A_{i+1,\gamma}\}$.

Lemma 6.1.1. (a) $m_-(0) = 0$.

(b) *If* $A_0 \supseteq R(A)$ *then* $\Gamma_- \subseteq \Gamma_R$.

(c) *If* $A_0 \supseteq R(A)$ *and* $\gamma \in \Delta_P$ *then* $|\Gamma_- \cap \mathbf{Z}\gamma| \leqslant 6$ *and if, in addition,* γ *is non-Hamiltonian, then* $|\Gamma_- \cap \mathbf{Z}\gamma| \leqslant 4$.

(d) *If* $0 \neq \gamma \in \Delta$, $m_-(\gamma) = m_-(-\gamma) = 0$, *and* $[G_{0,\gamma}, G_{0,-\gamma}] \subseteq (I + A_1)/A_1$ *then* $\gamma \notin \Gamma_R$ *and so* $\gamma \in \Delta_P$.

Proof. Part (a) follows from the assumption that $\mathfrak{z}_A(T) \subseteq A_0$. For (b) observe that $\gamma \in \Gamma_-$ implies $A_{i\gamma} \nsubseteq A_0$ and so $A_{i\gamma} \nsubseteq R(A)$. Thus (see Definition 5.6.7) $\gamma \in \Gamma_R$. Then Lemma 5.7.1(a) gives (c). Finally, the hypotheses of (d) show that A_γ, $A_{-\gamma} \subseteq A_0$, and $[A_\gamma, A_{-\gamma}] \subseteq I + A_1$, so $[A_\gamma, A_{-\gamma}] \subseteq I$ and hence $A_\gamma = R_\gamma(A)$, so $\gamma \notin \Gamma_R$. ∎

6.2.

Definition 6.2.1. Let M be a restricted Lie algebra over F. We say a torus $T \subseteq M$ is an *optimal torus* if T is a torus of maximal dimension in M, T is standard, and if for any standard torus $T_1 \subseteq M$ with $\dim T = \dim T_1$ we have $n(M, T) \geqslant n(M, T_1)$.

We now derive some properties of optimal tori. We use the results of Section 1.9 on Winter's exponential maps.

Lemma 6.2.2. *Let* T *be a two-dimensional torus of* A, $0 \neq \alpha \in \Delta - \Delta_P$, *and* C *be a subspace of* A_α *such that either* $A_\alpha = C + K_\alpha$ *or else* α *is Hamiltonian,* $[C, I] \subseteq C$, *and* $\dim A_\alpha/(C + K_\alpha) = 1$. *Then there is some* $x \in C$ *such that* $\alpha_x \in \Delta_P(A, e^x(T))$.

Proof. By Lemma 5.6.3(b) either α is Witt and there is a surjective homomorphism ϕ of $A^{(\alpha)}$ to $W(1:\mathbf{1})$ such that $\phi(T) = F((x_1+1)\,D_1)$ or α is Hamiltonian and there is a surjective homomorphism ϕ of $A^{(\alpha)}$ to a subalgebra of $H(2:\mathbf{1})$ containing $H(2:\mathbf{1})^{(2)}$ such that $\phi(T) = \mathscr{D}((x_1+1)\,x_2)$.

Suppose first that α is Witt. Let $t \in T$ satisfy $\phi(t) = (x_1+1)\,D_1$. As $A_\alpha = C + K_\alpha$ and $\phi(K_\alpha) = (0)$ (by Lemmas 5.3.1 and 5.2.1(c)) we have $\phi(C) = F((x_1+1)^{\alpha(t)+1}\,D_1)$. Let $x \in C$ satisfy $\phi(x) = \alpha(t)^{-1}\,(x_1+1)^{\alpha(t)+1}\,D_1$. Then $E^x t = t - \alpha(t)\,x$ and $\phi(E^x t) = (x_1+1)\ D_1 - (x_1+1)^{\alpha(t)+1}\ D_1 \in W(1:\mathbf{1})_0$. Since $\phi(E^x t)$ is not nil (being congruent to $-\alpha(t)\,x_1 D_1 \bmod W(1:\mathbf{1})_1$), its semisimple part spans $\phi(e^x(T))$. As $W(1:\mathbf{1})_0$ is restricted this implies $\phi(e^x(T)) \subseteq W(1:\mathbf{1})_0$. But by Theorem 1.3.1(c), any maximal torus of $W(1:\mathbf{1})$ contained in $W(1:\mathbf{1})_0$ has the form $\tau(Fx_1D_1)$ for some $\tau \in \operatorname{Aut} W(1:\mathbf{1})$. Then $\tau^{-1}\phi(e^x(T)) = Fx_1D_1$ so by Lemma 5.6.3(a), $\alpha_x \in \Delta_P(A, e^x(T))$.

Now suppose α is Hamiltonian. Let $t \in T$ satisfy $\phi(t) = \mathscr{D}((x_1+1)\,x_2) = (x_1+1)\ D_1 - x_2 D_2$. Then we have $\phi(K_\alpha) = \operatorname{span}\{\mathscr{D}((x_1+1)^{\alpha(t)+j}\,x_2^j) \mid 3 \leqslant j \leqslant p-1\}$ (by Lemmas 5.2.1(e) and 5.3.1) and $\phi(A_\alpha) = \operatorname{span}\{\mathscr{D}((x_1+1)^{\alpha(t)+j}$

$x_2^j) \mid 0 \leqslant j \leqslant p-1\}$. As either $A_\alpha = C + K_\alpha$ or else $[C, I] \subseteq C$ and $\dim A_\alpha/(C+K_\alpha) = 1$, we see that $\phi(C+K_\alpha)$ is an (ad $\phi(I)$)-invariant subspace of codimension $\leqslant 1$ in $\phi(A_\alpha)$. By Lemma 2.1.11, the only (ad $\phi(I)$)-invariant subspaces of $\phi(A_\alpha)$ are $\operatorname{span}\{\mathscr{D}((x_1+1)^{\alpha(t)+j} x_2^j) \mid i \leqslant j \leqslant p-1\}$ for $i = 0, ..., p-1$. Therefore $\phi(C+K_\alpha)$ contains $\operatorname{span}\{\mathscr{D}((x_1+1)^{\alpha(t)+j} x_2^j) \mid 1 \leqslant j \leqslant p-1\}$ and so there is an element $x \in C$ such that

$$\phi(x) = \alpha(t)^{-1} \left(\mathscr{D}((x_1+1)^{\alpha(t)+1} x_2) + \sum_{j=3}^{p-1} d_j \mathscr{D}((x_1+1)^{\alpha(t)+j} x_2^j) \right)$$

for some $d_3, ..., d_{p-1} \in F$. Then $E^x t = t - \alpha(t)\, x$ and

$$\begin{aligned} \phi(E^x t) &= (x_1+1)\, D_1 - x_2 D_2 - (x_1+1)^{\alpha(t)+1} \\ &\quad + (\alpha(t)+1)(x_1+1)^{\alpha(t)} x_2 D_2 \\ &\quad - \sum_{j=3}^{p-1} d_j \mathscr{D}((x_1+1)^{\alpha(t)+j} x_2^j) \\ &\in (H(2:\mathbf{1})^{(2)})_0. \end{aligned}$$

Since $\phi(E^x t)$ is not nil (being congruent to $-\alpha(t)(x_1 D_1 - x_2 D_2)$ mod $(H(2:\mathbf{1})^{(2)})_1$), its semisimple part spans $\phi(e^x(T))$. As $(H(2:\mathbf{1})^{(2)})_0$ is restricted (Lemma 2.1.7) and ϕ is a homomorphism of restricted Lie algebras, $\phi(\overline{E^x t}) = \overline{\phi(E^x t)} \subseteq (H(2:\mathbf{1})^{(2)})_0$ and so $\phi(e^x(T)) \subseteq (H(2:\mathbf{1})^{(2)})_0$. By Theorem 1.3.1(h), any maximal torus of $H(2:\mathbf{1})^{(2)}$ contained in $(H(2:\mathbf{1})^{(2)})_0$ has the form $\tau(\mathscr{D}(x_1 x_2))$ for some automorphism τ. Then $\tau^{-1}\phi(e^x(T)) = \mathscr{D}(x_1 x_2)$, so by Lemma 5.6.3(a), $\alpha_x \in \Delta_P(A, e^x(T))$. ∎

COROLLARY 6.2.3. *If T is an optimal torus of A then $n(A, T) \geqslant 1$.* ∎

6.3. We now introduce the maximal subalgebras we will use in the sequel.

DEFINITION 6.3.1. Let T be an optimal torus of A. A maximal subalgebra A_0 of A which contains $R(A)$ (with respect to T) is called a *distinguished maximal subalgebra* (*associated with* T).

LEMMA 6.3.2. *Let A_0 be a distinguished maximal subalgebra of A. Then $|\Gamma_-| \leqslant p^2 - p + 6$.*

Proof. This follows from Lemma 6.1.1(c) and Corollary 6.2.3. ∎

LEMMA 6.3.3. *Let A_0 be a distinguished maximal subalgebra of A. Then $m_-(0)=0$ and $m_-(\gamma)\leqslant 7$ for all $\gamma\in T^*$. Furthermore, if γ is non-Hamiltonian then $m_-(\gamma)\leqslant 3$ and if γ is proper then $m_-(\gamma)\leqslant 5$.*

Proof. The first claim is a restatement of Lemma 6.1.1(a), the second follows from Corollary 5.5.3 (as $A_0\supseteq R_\gamma(A)$), and the third follows from Lemma 5.6.9. ▮

6.4. Winter's map e^x allows us to prove certain properties of distinguished maximal subalgebras.

LEMMA 6.4.1. *Let A_0 be a distinguished maximal subalgebra of A associated with T, $A=A_{-k}\supseteq\cdots\supseteq A_{-1}\supseteq A_0\supseteq A_1\supseteq\cdots$ a corresponding filtration. Suppose $0\neq\alpha\in\Delta-\Delta_P$ and C is a subspace of A_α such that $[C, A_0]\subseteq A^{(\alpha)}+A_1$. Then:*

(a) $C+K_\alpha\neq A_\alpha$,

(b) *if α is Hamiltonian and $[C, I]\subseteq C$ then* $\dim A_\alpha/(C+K_\alpha)\geqslant 2$.

Proof. If $C+K_\alpha=A_\alpha$ or if α is Hamiltonian, $[C, I]\subseteq C$, and $\dim A_\alpha/(C+K_\alpha)\leqslant 1$, then Lemma 6.2.2 shows that there exists $x\in C$ such that $\alpha_x\in\Delta_P(A, e^x(T))$. We claim that if $\beta\in\Delta_P(A, T)$ then $\beta_x\in\Delta_P(A, e^x(T))$. This implies $n(A, e^x(T))>n(A, T)$ and so contradicts the optimality of T. To establish the claim note that (by Lemmas 5.6.8 and 5.7.1(a) and the fact that $p>7$) $A_{\pm i\beta}=R_{\pm i\beta}$ for some $i\in\mathbf{Z}_p^*$. By change of notation we may assume that $i=1$. Then, as $A_0\supseteq R(A)$, we have $A_{\pm\beta}\subseteq A_0$. Now by Proposition 1.9.3(d),

$$A_{\pm\beta_x}\subseteq\sum_{j=0}^{p-1}(\operatorname{ad}x)^j A_{\pm\beta}.$$

But $(\operatorname{ad}x)A_0\subseteq[C, A_0]\subseteq A^{(\alpha)}+A_1$ and

$$(\operatorname{ad}x)(A^{(\alpha)}+A_1)=[x, A^{(\alpha)}]+[x, A_1]\subseteq A^{(\alpha)}+[x, A_0]\subseteq A^{(\alpha)}+A_1.$$

Thus if $j\geqslant 1$ $(\operatorname{ad}x)^j A_{\pm\beta}\subseteq A^{(\alpha)}+A_1$ and $(\operatorname{ad}x)^j A_{\pm\beta}\subseteq A_{\pm\beta+j\alpha}$. Hence if $1\leqslant j\leqslant p-1$ we have $(\operatorname{ad}x)^j A_{\pm\beta}\subseteq(A^{(\alpha)}+A_1)\cap(A_{\pm\beta+j\alpha})\subseteq A_1$. Hence (as A_1 is ad T-invariant since $T\subseteq A_0$)

$$A_{\pm\beta_x}\subseteq\sum_{j=0}^{p-1}(\operatorname{ad}x)^j A_{\pm\beta}\subseteq A_{\pm\beta}+A_1.$$

Then

$$[A_{\beta_x}, A_{-\beta_x}] \subseteq [A_\beta + A_1, A_{-\beta} + A_1]$$
$$\subseteq [A_\beta, A_{-\beta}] + A_1 = [R_\beta, R_{-\beta}] + A_1 \subseteq I + A_1,$$

a nil subalgebra of A_0. Thus $[A_{\beta_x}, A_{-\beta_x}]$ is nil so $A_{\pm\beta_x} = R_{\pm\beta_x}$ and hence $\beta_x \in \Delta_P(A, e^x(T))$, as required. ∎

COROLLARY 6.4.2. *Let A_0 be a distinguished maximal subalgebra of A, $A = A_{-k} \supseteq \cdots \supseteq A_{-1} \supseteq A_0 \supseteq A_1 \supseteq \cdots$ a corresponding filtration. Let $\alpha \in \Delta - \Delta_P$. Then*:

(a) $A_{1,\alpha} + K_\alpha \neq A_\alpha$,

(b) *if α is Hamiltonian* $\dim A_\alpha/(A_{1,\alpha} + K_\alpha) \geqslant 2$,

(c) *if $\Gamma_0 \subseteq \mathbf{Z}\alpha \cup \mathbf{Z}\beta$ for some $\beta \in \Delta$ then $A_{0,\alpha} + K_\alpha \neq A_\alpha$*,

(d) *if $\Gamma_0 \subseteq \mathbf{Z}\alpha \cup \mathbf{Z}\beta$ for some $\beta \in \Delta$ and α is Hamiltonian then* $\dim A_\alpha/(A_{0,\alpha} + K_\alpha) \geqslant 2$.

Proof. Parts (a) and (b) follow from Lemma 6.4.1 by setting $C = A_{1,\alpha}$. If the hypothesis for (c) or (d) holds then

$$[A_{0,\alpha}, A_0] \subseteq \left(A^{(\alpha)} + A_1 + \sum_{i \in \mathbf{Z}} A_{i\beta+\alpha}\right) \cap A_0 \subseteq A^{(\alpha)} + A_1,$$

so taking $C = A_{0,\alpha}$ and applying Lemma 6.4.1 gives the result. ∎

LEMMA 6.4.3. *Let A_0 be a distinguished maximal subalgebra of A associated with T. Let $A = A_{-k} \supseteq \cdots \supseteq A_{-1} \supseteq A_0 \supseteq A_1 \supseteq \cdots$ be a corresponding filtration and $G = \sum G_i$ be the associated graded algebra. Suppose $\alpha \in \Delta(A, T)$, $x \in A_{0,\alpha}$. If $\beta \in \Delta_P(A, T)$ and $\beta_x \in \Delta_P(G_0, e^x(T))$ then $\beta_x \in \Delta_P(A, e^x(T))$.*

Proof. Suppose first that β is non-Hamiltonian. Since $\beta \in \Delta_P(A, T)$ and A_0 is a distinguished maximal subalgebra, we have $A_{i\beta} \subseteq A_0$ for all but at most four values of i (by Lemmas 6.1.1(b) and 5.7.1(a)). Thus $A_{i\beta_x} \subseteq A_0$ for all but at most four values of i. Since $\beta_x \in \Delta_P(G_0, e^x(T))$ we have $\beta_x([A_{0,i\beta_x}, A_{0,-i\beta_x}]) = (0)$ for all but at most four values of i. Hence $\beta_x([A_{i\beta_x}, A_{-i\beta_x}]) \neq (0)$ for at most eight values of i and (as $p > 7$) $\beta_x([A_{i\beta_x}, A_{-i\beta_x}]) = (0)$ for some $i \in \mathbf{Z}_p^*$. Thus $\beta_x \in \Delta_P(A, e^x(T))$. If β_x is non-Hamiltonian a similar argument gives the result.

Now suppose that β is Hamiltonian and that β_x is Hamiltonian.

Since $\beta \in \Delta_P(A, T)$ and A_0 is a distinguished maximal subalgebra, we have $A_{i\beta} \subseteq A_0$ for all but at most six values of i and these values are symmetric about 0 (by Lemmas 5.6.8(b), 5.7.1(a), and 6.1.1(b)). Then for some $i \in \mathbf{Z}_p^*$ we have $A_{i\beta}, A_{-i\beta} \subseteq A_0$ and hence $A_{i\beta_x}, A_{-i\beta_x} \subseteq A_0$. Then $\dim(A_{i\beta_x}/K_{i\beta_x}) = \dim(G_{0, i\beta_x}/K_{i\beta_x}(G_0)) < 3$ (by Lemmas 5.6.3(a) and 5.2.1(d)) so $\beta_x \in \Delta_P(A, e^x(T))$ (by Lemmas 5.6.3(b) and 5.2.1(e)). ∎

LEMMA 6.4.4. *Let A_0 be a distinguished maximal subalgebra of A associated with T. Let $A = A_{-k} \supseteq \cdots \supseteq A_0 \supseteq A_1 \supseteq \cdots$ be a corresponding filtration and $G = \sum G_i$ be the associated graded algebra. Suppose $\alpha \in \Delta(A, T)$, $\alpha \notin \Delta_P(A, T)$, and $\alpha([A_0 \cap A_\alpha, A_0 \cap A_{-\alpha}]) \neq (0)$. Then, replacing α by $-\alpha$ if necessary, there exists $x \in A_0 \cap A_\alpha$ such that $\alpha_x \in \Delta_P(A, e^x(T))$.*

Proof. We will apply Lemma 6.2.2 to the root α (with $C = A_\alpha \cap A_0$) or the root $-\alpha$ (with $C = A_{-\alpha} \cap A_0$). If α is non-Hamiltonian the hypothesis (together with Lemma 5.2.1) implies $A_\alpha = K_\alpha + (A_\alpha \cap A_0)$. If α is Hamiltonian then (noting that $I \subseteq A_0$ and so $[A_\alpha \cap A_0, I] \subseteq A_\alpha \cap A_0$) either $\dim A_\alpha/((A_\alpha \cap A_0) + K_\alpha) \leqslant 1$ or $\dim A_{-\alpha}/((A_{-\alpha} \cap A_0) + K_{-\alpha}) \leqslant 1$ (for we set $U_{\pm\alpha} = (A_{\pm\alpha} \cap A_0) + K_{\pm\alpha}$ and then by Lemmas 2.1.11(b) and 5.2.1(e) we see that if $\dim U_{\pm\alpha}/K_{+\alpha} \leqslant 1$ we have $\alpha([U_\alpha, U_{-\alpha}]) = (0)$). Hence Lemma 6.2.2 gives the result. ∎

6.5. Using e^x we can prove the following lemma on faithful restricted modules for the algebras $S(3:\mathbf{1})^{(1)}$, $H(4:\mathbf{1})^{(1)}$, $K(3:\mathbf{1})$.

LEMMA 6.5.1. *Let A be one of $S(3:\mathbf{1})^{(1)}$, $H(4:\mathbf{1})^{(1)}$, $K(3:\mathbf{1})$. Let V be a faithful restricted A-module and T be a maximal torus in A. Then V has p^2-1 nonzero weights with respect to T.*

Proof. Give V the structure of an abelian restricted Lie algebra by setting $v^p = 0$ for all $v \in V$. Let $A + V$ denote the split null extension of A by V. Then T is a maximal torus in $A + V$. Proposition 1.9.3(c) applied to A and to $A + V$ shows that if x is any root vector for T in A then the number of nonzero weights of A on V with respect to T is the same as the number of nonzero weights of A on V with respect to $e^x(T)$. Thus, in view of Lemma 6.2.2, Corollary 5.8.4, and Lemma 5.8.2(h), we may assume $T \subseteq A_0$.

Suppose T has basis $\{t_1, t_2\}$, where $t_i^p = t_i$ for $i = 1, 2$. Suppose further that there exist nonzero roots β_1, β_2 such that $\beta_i(t_j) = 0$ if $i \neq j$ and root vectors $E_{i,j}$, $1 \leqslant i, j \leqslant 2$, such that $E_{i,j} \in A_{\beta i}$ and $(\operatorname{ad} E_{i,1})^{p-1} E_{i,2}$ is congruent to a nonzero multiple of $t_{3-i} \bmod I$ for $i = 1, 2$. Then if ν is any weight of A on V with respect to T such that $\nu(t_{3-i}) \neq 0$, we see that

$(\operatorname{ad} E_{i,1})^{p-1} E_{i,2}$ is nonzero on V_ν. As $E_{i,1}, E_{i,2} \in A_{\beta_i}$ we see that $\nu + j\beta_i$ is a weight for all j, $0 \leqslant j \leqslant p-1$. Then Lemma 4.6.3 of [BW82] proves our assertion.

If $A = S(3:\mathbf{1})^{(1)}$ we may take $t_1 = x_1 D_1 - x_2 D_2$, $t_2 = x_2 D_2 - x_3 D_3$, $E_{1,1} = D_1$, $E_{2,1} = D_3$, $E_{1,2} = x_1^{p-1} x_2 D_2 - x_1^{p-1} x_3 D_3$, and $E_{2,2} = x_1 x_3^{p-1} D_1 - x_2 x_3^{p-1} D_2$. If $A = H(4:\mathbf{1})^{(1)}$ we may take $t_1 = \mathscr{D}(x_1 x_3)$, $t_2 = \mathscr{D}(x_2 x_4)$, $E_{1,1} = D_1$, $E_{2,1} = D_2$, $E_{1,2} = \mathscr{D}(x_1^{p-1} x_2 x_4)$, and $E_{2,2} = \mathscr{D}(x_1 x_2^{p-1} x_3)$. Finally, if $A = K(3:\mathbf{1})$ we may take $t_1 = \mathscr{D}_K(x_1 x_2 - x_3)$, $t_2 = \mathscr{D}_K(x_1 x_2 + x_3)$, $E_{1,1} = \mathscr{D}_K(x_1)$, $E_{2,1} = \mathscr{D}_K(x_2)$, $E_{1,2} = \mathscr{D}_K(x_2^{p-1} x_3)$, and $E_{2,2} = \mathscr{D}_K(x_1^{p-1} x_3)$. ∎

7. Determination of G_0 if $\mathfrak{z}(G_0) \neq (0)$

Throughout this section we assume that A is a finite-dimensional restricted semisimple Lie algebra over F which satisfies the conditions of Theorem 4.1.1(h). That is, we assume:

A is finite-dimensional, restricted, and semisimple over F.

All two-dimensional tori in A are maximal and standard.

T is a two-dimensional optimal torus in A and $\mathfrak{z}_A(T) = T + I$, where I is the nil radical of $\mathfrak{z}_A(T)$. (7.0.1)

$\bar{S} \subseteq A \subseteq \operatorname{Der} S$ for some simple Lie algebra S.

$A = S + I$ (and so, by Lemma 1.8.2, $T \subseteq \bar{S}$).

We let A_0 be a distinguished maximal subalgebra of A associated with T and let $A = A_{-k} \supseteq \cdots \supseteq A_{-1} \supseteq A_0 \supseteq A_1 \supseteq \cdots$ be a filtration of A constructed as in Section 1.2. Let $G = \sum_{i \geqslant -k} G_i$ be the associated graded algebra. Recall (Section 6.1) that we can and do identity T with its image, a two-dimensional standard torus, in G_0. We write $I(G_0)$ for the nil radical of $\mathfrak{z}_{G_0}(T)$ so $\mathfrak{z}_{G_0}(T) = T + I(G_0)$. We have the weight space decompositions $A_i = \sum_{\gamma \in \Gamma \cup \{0\}} A_{i,\gamma}$ and $G_i = \sum_{\gamma \in \Gamma_i} G_{i,\gamma}$ of A_i and G_i with respect to T.

We will also assume throughout this section that

$$\mathfrak{z}(G_0) \neq (0). \tag{7.0.2}$$

As the action of G_0 on G_{-1} is faithful and irreducible, $\dim(\mathfrak{z}(G_0)) = 1$ and $\mathfrak{z}(G_0)$ is a torus. Let $\{t, z\}$ be a basis for T such that $t^p = t$, $z^p = z$, and $Fz = \mathfrak{z}(G_0)$. Furthermore, if $G_0 = [G_0, G_0] + Fz$ (as a sum of restricted Lie algebras) take

$$t \in T \cap [G_0, G_0]. \tag{7.0.3}$$

Define $\gamma \in T^*$ by $\gamma(t) = 1$, $\gamma(z) = 0$, Note that $\Gamma_0 \subseteq \mathbf{Z}\gamma$. Note also that ad z acts on G_{-1} as an element of $\mathbf{Z}_p^*$. (It acts as an element of $\mathbf{Z}_p$ since $z^p = z$ and is nonzero since G_0 acts faithfully on G_{-1}.) Replacing z by a scalar multiple if necessary we may assume ad z acts as multiplication by -1 on G_{-1}. Then a simple induction argument shows that ad z acts as multiplication by i on G_i for all $i \in \mathbf{Z}$. Hence $\Gamma_i \subseteq \{\lambda \in T^* \mid \lambda(z) = i\}$. Define $\delta \in T^*$ by

$$\delta(t) = 0, \qquad \delta(z) = -1. \tag{7.0.4}$$

Then $\Gamma_i \subseteq \{-i\delta + j\gamma \mid j \in \mathbf{Z}_p\}$. Thus by Lemma 6.3.3 we have

$$\dim G_{-1} \leqslant 7p. \tag{7.0.5}$$

We use the main result of [BW82] repeatedly in this and later sections; whenever we can show $\mathfrak{z}_A(T) = T$ we are in the situation of [BW82] and so are done. We note that there is an error in Corollary 4.12.1(a) of [BW82]. The correct conclusion is: "If V is a restricted A-module with $SV \neq (0)$ then V has at least $p^2 - 2$ weights." Corollary 4.12.1(a) is used four times in [BW82]. For one of these applications [BW82, Sect. 6.2], the corrected version given above suffices. The remaining three applications are in Lemma 5.5.1 (step (5)), Lemma 5.6.1 (step (8)), and Lemma 5.7.1 (step (3)). In this Section there are generalizations of each of these lemmas (Lemmas 7.6.1, 7.7.1, and 7.8.1, respectively). We prove each of these lemmas without use of the corresponding result from [BW82], so that the proofs given here fill the gap in [BW82]. (In fact, the proof of Lemma 7.8.1 we give here is substantially simpler than the proof of the corresponding Lemma 5.7.1 of [BW82].)

7.1. Following Weisfeiler ([Wei78]; cf. [BW82, Sect. 1.5]), if $G = \sum G_i$ is any graded Lie algebra we define $M(G)$ to be the sum of all ideals of G contained in $G_- = \sum_{i<0} G_i$, hence the unique maximal ideal of G contained in G_-. We also denote by G' the subalgebra of G generated by $\sum_{i \leqslant 1} G_i$. Note that G' is a graded subalgebra of G. We define $N(G) = M(G')$.

Note (using the Poincaré–Birkhoff–Witt theorem) that if $x \in G_i$, $i < 0$, and $[x, G_1] = (0)$ then the ideal of G' generated by x is contained in G_-. Thus $N(G) = (0)$ is equivalent to the condition

$$\text{If } x \in G_i,\ i < 0, \text{ and } [x, G_1] = (0) \text{ then } x = 0. \tag{7.1.1}$$

Comparing (7.1.1) with the hypothesis (1.2.3) of the Recognition Theorem shows why we are interested in establishing $N(G) = (0)$.

Section 7 is devoted to the proof of the following result.

PROPOSITION 7.1.1. *If A satisfies* (7.0.1) *and* (7.0.2) *then either* $G_1=(0)$ *or* $G_0 \cong \mathfrak{sl}(2) \oplus Fz$ (*as restricted Lie algebras*), $N(G)=(0)$, *and* G_0 *acts faithfully on* G_1.

7.2. Since $z \in \operatorname{solv} G_0$ we have that $G_0/\operatorname{solv} G_0$ is a restricted semisimple Lie algebra in which every torus has dimension $\leqslant 1$ and all tori of dimension 1 are standard. If there are no nonzero tori then $G_0/\operatorname{solv} G_0$ is nil, hence (0). If there are nonzero tori in $G_0/\operatorname{solv} G_0$ then Theorem 3.1.1 shows that $G_0/\operatorname{solv} G_0 \cong \mathfrak{sl}(2)$, $G_0/\operatorname{solv} G_0 \cong W(1:\mathbf{1})$, or $G_0/\operatorname{solv} G_0 \cong M$, where $H(2:\mathbf{1})^{(2)} \subseteq M \subseteq H(2:\mathbf{1})$.

We will show that $I(G_0)=(0)$. We begin by considering $I(G_0) \cap \operatorname{solv} G_0$.

LEMMA 7.2.1. *Let M be a restricted Lie algebra satisfying* $H(2:\mathbf{1})^{(2)} \subseteq M \subseteq H(2:\mathbf{1})$ *and V be a nonzero restricted M-module. Let U be a maximal torus in M. Then* $V_0 \neq (0)$ (*where V_0 is the* 0-*weight space of V with respect to U*).

Proof. Since $H(2:\mathbf{1})/H(2:\mathbf{1})^{(2)}$ is nil (Corollary 2.1.9) we have $U \subseteq H(2:\mathbf{1})^{(2)}$. Then by Theorem 1.3.1(i) we may assume that U is spanned by $\mathscr{D}((x_1+a)\,x_2)$, where $a=0$ or 1. Then (by Corollary 2.1.5) $\mathscr{D}(x_1+a)$ and $\mathscr{D}((x_1+a)^{p-1}\,x_2^{p-2})$ both belong to $(H(2:\mathbf{1})^{(2)})_1$ (the 1-eigenspace for ad $\mathscr{D}((x_1+a)\;x_2)$) and $(\operatorname{ad}\mathscr{D}(x_1+a))^{p-2}\;\mathscr{D}((x_1+a)^{p-1}\;x_2^{p-2})=-\mathscr{D}((x_1+a)^{p-1})$ is a nonzero element of $H(2:\mathbf{1})^{(2)}$. Thus the ideal generated by $[M_1,[M_1,...,[M_1,M_1]\cdots]]$ ($p-1$ factors) contains U. If $UV=(0)$ then $V_0=V\neq(0)$. Hence we may assume $UV\neq(0)$ and so $[M_1, [M_1,...,[M_1,M_1]\cdots]]\,V\neq(0)$. Therefore $[M_1,[M_1,...,[M_1,M_1]\cdots]]\,V_i\neq(0)$ for some i. Thus we have $M_1^{p-1}V_i\neq 0$ and hence $V_{i+j}\neq(0)$ for $0\leqslant j\leqslant p-1$. In particular, $V_0=V_{i+(p-i)}\neq(0)$, as required. ∎

COROLLARY 7.2.2. *If* $I(G_0)\neq(0)$ *and* $\operatorname{solv} G_0 \neq \mathfrak{z}(G_0)$ *then* $I(G_0)\cap \operatorname{solv} G_0 \neq (0)$.

Proof. If $G_0=\operatorname{solv} G_0$ the result is vacuous. If $G_0/\operatorname{solv} G_0 \cong \mathfrak{sl}(2)$ or $W(1:\mathbf{1})$ then, since $I(G_0)$ maps into the nil radical of $\mathfrak{z}_{(G_0/\operatorname{solv} G_0)}(T+\operatorname{solv} G_0/\operatorname{solv} G_0)$ and since all tori in $\mathfrak{sl}(2)$ or $W(1:\mathbf{1})$ are self-centralizing, we see that $I(G_0)\subseteq \operatorname{solv} G_0$, giving the result. Finally, suppose $G_0/\operatorname{solv} G_0 \cong M$, $H(2:\mathbf{1})^{(2)}\subseteq M\subseteq H(2:\mathbf{1})$, and $\operatorname{solv} G_0\neq \mathfrak{z}(G_0)$. Then $V=\operatorname{solv} G_0/([\operatorname{solv} G_0,\ \operatorname{solv} G_0]+\mathfrak{z}(G_0))$ is a nonzero M-module. By Lemma 7.2.1 we have $(0)\neq V_0$. Thus $(\operatorname{solv} G_0)_0\neq \mathfrak{z}(G_0)$. Since $(\operatorname{solv} G_0)_0 = (I(G_0)\cap \operatorname{solv} G_0)+\mathfrak{z}(G_0)$ we have $I(G_0)\cap \operatorname{solv} G_0\neq(0)$. ∎

LEMMA 7.2.3. *Let W be an irreducible* solv G_0-*submodule of* G_{-1}. *Then either* dim $W=1$ *or else* solv G_0 *acts faithfully on W and* dim $W=p$.

Proof. We have observed (7.0.5) that $\dim G_{-1} \leqslant 7p < p^2$. Thus (cf. Theorem 1.13.1 of [BW82]) $\dim W = 1$ or p. If $\dim W = p$ then (by Lemma 1.10.1) $\mathrm{Stab}(W, G_0) = G_0$ (for $\dim G_{-1} \geqslant p^{\mathrm{codim}\,\mathrm{Stab}(W,\,G_0)}(\dim W)$) and G_{-1} is the sum of all solv G_0-submodules of G_{-1} which are isomorphic to W as solv G_0-modules. As G_{-1} is a faithful G_0-module this implies W is a faithful solv G_0-module. ▮

LEMMA 7.2.4. *Let M be a solvable subalgebra of G_0 which contains T. Then $I(G_0) \cap M$ annihilates any irreducible M-submodule of G_{-1} of dimension p.*

Proof. Let W be a p-dimensional irreducible M-submodule of G_{-1}. Suppose $(I(G_0) \cap M)\, W \neq (0)$. Then for some $i \in \mathbf{Z}_p$ we have $(I(G_0) \cap M)\, W_{\delta + i\gamma} \neq (0)$. Since $I(G_0)$ is nil, $\dim W_{\delta + i\gamma} > 1$. Thus as $\dim W = p$ we have $W_{\delta + j\gamma} = (0)$ for some $j \in \mathbf{Z}_p$. Then Lemma 1.8.4 shows that $[M_{l\gamma}, M_{-l\gamma}]$ is nil for all $l \in \mathbf{Z}_p^*$. Hence $I(G_0) \cap M + \sum_{l \in \mathbf{Z}_p^*} M_{l\gamma}$ is a nil ideal in M and so annihilates W, contradicting $(I(G_0) \cap M)\, W \neq (0)$. ▮

COROLLARY 7.2.5. *If $I(G_0) \neq (0)$ and W is an irreducible* solv G_0-*submodule of G_{-1} then* $\dim W = 1$.

Proof. If $\dim W \neq 1$ then Lemma 7.2.3 shows that solv G_0 acts faithfully on W and $\dim W = p$. We may assume W is T-invariant, for as $\dim G_{-1} < p^2$, $\mathrm{Stab}(W, G_0) = G_0$ and so G_{-1} is a direct sum of solv G_0-modules isomorphic to to W. Thus every irreducible solv G_0-module of G_{-1} is isomorphic to W. Now $T +$ solv G_0 is solvable so any irreducible $T +$ solv G_0-submodule of G_{-1} must have dimension $\leqslant p$. Since such a module must contain an irreducible solv G_0-submodule, it must be irreducible as a solv G_0-module. Also solv $G_0 \neq \mathfrak{z}(G_0)$ (as solv G_0 acts irreducibly on W and $\dim W > 1$). Thus Corollary 7.2.2 shows $I(G_0) \cap$ solv $G_0 \neq (0)$. But Lemma 7.2.4 (with $M = T +$ solv G_0) implies that $I(G_0) \cap$ solv G_0 annihilates W, a contradiction. ▮

LEMMA 7.2.6. *If $I(G_0) \neq (0)$ then* solv $G_0 = \mathfrak{z}(G_0)$.

Ptoof. By Corollary 7.2.5, G_{-1} contains a one-dimensional solv G_0-module W. Then codim $\mathrm{Stab}(W, G_0) = 0$ or 1. If codim $\mathrm{Stab}(W, G_0) = 0$ then G_{-1} is a direct sum of copies of W as a solv G_0-module. Thus solv G_0 acts on G_{-1} by scalars and since G_{-1} is a faithful G_0-module, we have solv $G_0 = \mathfrak{z}(G_0)$, as required. Therefore it remains to show that codim $\mathrm{Stab}(W, G_0) = 1$ cannot occur.

Let V be the sum of all (solv G_0)-submodules of G_{-1} which are isomorphic to W. Then, by the Blattner–Dixmier theorem (Lemma 1.10.1), V is an irreducible $\mathrm{Stab}(W, G_0)$-module and $\dim G_{-1} =$

$p^{\operatorname{codim}\operatorname{Stab}(W, G_0)}(\dim V)$. If $\operatorname{codim}\operatorname{Stab}(W, G_0) = 1$ then by (7.0.5) we have $\dim V \leqslant 7$. Also as $\operatorname{Stab}(W, G_0) \supseteq \operatorname{solv} G_0$ we have that $\operatorname{Stab}(W, G_0)/\operatorname{solv} G_0$ is a subalgebra of codimension 1 in $G_0/\operatorname{solv} G_0$. If $G_0/\operatorname{solv} G_0 = (0)$ this is impossible. If $G_0/\operatorname{solv} G_0 \cong \mathfrak{sl}(2)$ it implies that $\operatorname{Stab}(W, G_0)/\operatorname{solv} G_0$ is two-dimensional, hence solvable, so that $\operatorname{Stab}(W, G_0)$ is solvable. If $G_0/\operatorname{solv} G_0 \cong W(1{:}\mathbf{1})$ then, as $W(1:\mathbf{1})_0$ is the unique subalgebra of codimension one in $W(1:\mathbf{1})$ (by Lemma 1.11.1), we have that $\operatorname{Stab}(W, G_0)/\operatorname{solv} G_0 \cong W(1:\mathbf{1})_0$ is solvable, so that $\operatorname{Stab}(W, G_0)$ is solvable. But in either of these cases, V, being an irreducible $\operatorname{Stab}(W, G_0)$-module, has dimension p^r for some r. Since $p^2 > \dim G_{-1} = p^{\operatorname{codim}\operatorname{Stab}(W, G_0)}(\dim V) = p(\dim V)$ we have $V = W$ and $\dim G_{-1} = p$. As $I(G_0) \neq (0)$ and G_{-1} is a faithful G_0-module, we have $\dim G_{-1, \delta + i\gamma} > 1$ for some i and therefore $G_{-1, \delta + j\gamma} = (0)$ for some j. However, for any $l \in \mathbf{Z}_p^*$, $M = T + G_{0, l\gamma} + \operatorname{solv}(G_0)$ is a solvable subalgebra of G_0. Since $G_{-1, \delta + j\gamma} = (0)$, Lemma 1.8.4 shows that $[G_{0, l\gamma}, \operatorname{Solv}(G_0)_{-l\gamma}] \subseteq I(G_0)$. Thus $I(G_0) + \sum_{l \in \mathbf{Z}_p^*} \operatorname{solv}(G_0)_{l\gamma}$ is a nil ideal in G_0, therefore (0). Hence in this case $\operatorname{solv}(G_0) = Fz$. Finally, suppose $H(2:\mathbf{1})^{(2)} \subseteq G_0/\operatorname{solv}(G_0) \subseteq H(2:\mathbf{1})$. Since $\operatorname{Stab}(W, G_0)/\operatorname{solv}(G_0)$ has codimension one in $G_0/\operatorname{solv} G_0$ and since $H(2:\mathbf{1})^{(2)}$ has no proper subalgebra of codimension <2 (Lemma 1.11.1(b)), we see that $H(2:\mathbf{1})^{(2)} \subseteq \operatorname{Stab}(W, G_0)/\operatorname{solv}(G_0)$. Thus there exists some $x \in I(G_0)$ such that $x \notin \operatorname{Stab}(W, G_0)$ (for since $H(2:\mathbf{1})/H(2:\mathbf{1})^{(2)}$ is nil, $T \subseteq \operatorname{Stab}(W, G_0)$ and $I(G_0) + \operatorname{Stab}(W, G_0) = G_0$). Let $U = \{u \in \operatorname{Stab}(W, G_0) \mid uV = (0)\}$. Then U is an ideal in $\operatorname{Stab}(W, G_0)$ and

$$\begin{aligned}
&\dim(\operatorname{Stab}(W, G_0)/\operatorname{solv} G_0)/((U + \operatorname{solv} G_0)/\operatorname{solv} G_0)\\
&\quad = \dim \operatorname{Stab}(W, G_0)/(U + \operatorname{solv} G_0) \leqslant \dim \operatorname{Stab}(W, G_0)/U\\
&\quad \leqslant (\dim V)^2 \leqslant 7^2 < p^2 - 2.
\end{aligned}$$

Thus $H(2:\mathbf{1})^{(2)}$, being a simple ideal in $\operatorname{Stab}(W, G_0)/\operatorname{solv} G_0$, is contained in $(U + \operatorname{solv} G_0)/\operatorname{solv} G_0$. Furthermore, $(\operatorname{Stab}(W, G_0)/\operatorname{solv} G_0)^{(2)} \subseteq (G_0/\operatorname{solv} G_0)^{(2)} \subseteq H(2:\mathbf{1})^{(2)} \subseteq (U + \operatorname{solv} G_0)/\operatorname{solv} G_0$. Therefore $\operatorname{Stab}(W, G_0)/(U + \operatorname{solv} G_0)$ is solvable. Since $(U + \operatorname{solv} G_0)/U \cong \operatorname{solv} G_0/(U \cap \operatorname{solv} G_0)$ is solvable it follows that $\operatorname{Stab}(W, G_0)/U$ is solvable. Then as V is an irreducible module for $\operatorname{Stab}(W, G_0)/U$, we have $\dim V = p^r$ for some r. Since $7p \geqslant \dim G_{-1} = p(\dim V)$ this implies $r = 0$ and so $W = V$. Since $\dim W = 1$ and W is $\operatorname{Stab}(W, G_0)$-invariant, hence T-invariant, W is contained in a single weight space. Then $G_{-1} = W + xW + \cdots + x^{p-1}W$ is also contained in a single weight space (as $x \in I(G_0)$). Since G_0 has nonzero root spaces and G_{-1} is a faithful G_0-module, this is absurd. Thus $\operatorname{codim}\operatorname{Stab}(W, G_0) = 1$ is impossible and the lemma is proved. ∎

LEMMA 7.2.7. *Let M be a restricted Lie algebra with one-dimensional center Fz, where $z^p = z$. Suppose $H(2:\mathbf{1})^{(2)} \subseteq M/Fz \subseteq H(2:\mathbf{1})$. Let T be a two-dimensional standard torus in M. Write $\mathfrak{z}_M(T) = T + I(M)$, where $I(M)$ is nil. If V is a faithful irreducible restricted M-module then* dim $V_\lambda > 7$ *for some $\lambda \in T^*$.*

Proof. Suppose dim $V_\lambda \leqslant 7$ for all weights λ (and so dim $V \leqslant 7p < p^2$). Let $M_i = \{x \in M \mid x + Fz \in H(2:\mathbf{1})_i\}$. By Theorem 1.3.1(g), if $T \nsubseteq M_0$ we may assume that T/Fz is spanned by $\mathscr{D}((x_1+1)\,x_2)$. Then taking $\mu \in T^*$ and $x \in M_\mu$ so that $x + Fz/Fz = -\mathscr{D}(x_2)$, we see (recall Section 1.9) that $E^x(T) \subseteq M_0$ and so (since Lemma 2.1.7 shows that M_0 is restricted) $e^x(T) \subseteq M_0$. Give V the structure of an abelian restricted Lie algebra by setting $y^p = 0$ for all $y \in V$. By applying Proposition 1.9.3(c) to M and to $M + V$ (the split null extension) we see that dim $V_\lambda =$ dim V_{λ_x} for all $\lambda \in T^*$. Thus it is sufficient to prove the lemma under the additional assumption that $T \subseteq M_0$. Note that this implies $I(M) \subseteq M_2$.

Let $J = M_1 + T$, let U be an irreducible M_0-constituent of V, and let W be an irreducible J-submodule of U. Since J is solvable, dim $W = p^n$ for some n (cf. [BW82, Theorem 1.13.1], hence dim $W = 1$ or p. If dim $W = p$ then $[J, J]$ is not a nil ideal in J. It follows that either there is some $x \in J_\lambda$, $\lambda \neq 0$, with x^p not nil or that $[J_\lambda, J_\lambda]$ is not nil for some $\lambda \neq 0$. Either of these implies that W has p weights (in the second case use Lemma 1.8.4). Since dim $W = p$ this implies that dim $W_\lambda = 1$ for all λ and so $I(M)\,W = (0)$. Since dim $U < p^2$ we have Stab$(W, M_0) = M_0$ and so $I(M)\,U = (0)$ when dim $W = p$. If dim $W = 1$ then

$$((\operatorname{ad} x)^i\, I(M))\, W = (0) \tag{7.2.1}$$

for any root vector $x \in M_0$ and $0 \leqslant i \leqslant p-1$. Since dim $U < p^2$ we have $U = \tilde{W} + x\tilde{W} + \cdots + x^{p-1}\tilde{W}$, where $\tilde{W}$ is the sum of all J-submodules of U isomorphic to W and $x \in M_0$ is some root vector. In view of (7.2.1) this shows $I(M)\,U = (0)$ when dim $W = 1$. Thus, in any case $I(M)\,U = (0)$.

Now let U be an irreducible M_0-submodule of V and define $U^{(-2)} = U^{(-1)} = (0)$, $U^{(0)} = U$, and $U^{(i)} = MU^{(i-1)} + U^{(i-1)}$ for $i > 0$. Then we see by induction that $I(M)\,U^{(i)} \subseteq U^{(i-1)}$ for all $i \geqslant 0$. Thus $((\operatorname{ad} I(M))^{p-2}\, M)\, U^{(p-4)} = (0)$. Now as we may assume $T \subseteq M_0$, Theorem 1.3.1(h) shows that we may assume that T/Fz is spanned by $\mathscr{D}(x_1 x_2)$. Then $\mathscr{D}(x_1^2 x_2^2) \in (I(M) + Fz)/Fz$ and so, since $(\operatorname{ad} \mathscr{D}(x_1^2 x_2^2))^{p-2}\ \mathscr{D}(x_2) \neq 0$, we see that $(\operatorname{ad} I(M))^{p-2}\, M \neq (0)$. As V is a faithful M-module this implies $((\operatorname{ad} I(M))^{p-2}\, M)\, V \neq (0)$ and so $V \neq U^{(p-4)}$. As $p > 7$ this implies $V \neq U^{(7)}$.

Therefore, as dim $V \leqslant 7p$ some $U^{(i)}/U^{(i-1)}$ has dimension $< p$. Then (as M_1 is solvable) any irreducible J-submodule of $U^{(i)}/U^{(i-1)}$ is one-dimensional and hence is annihilated by $[J, J]$ and by $I(M)$. Since $[J, J] \supseteq$

$[T, M_1]$ and $M_1 = [T, M_1] + I(M)$ we see that M_1 annihilates $U^{(i)}/U^{(i+1)}$ and hence is nil. Thus M_1 is a nil ideal in M_0 and so M_1 annihilates U. Recall that we may assume $I(M) \subseteq M_2$. Therefore $[M, I(M)]$ annihilates U. Then, by induction on i, $I(M)\, U^{(i)} \subseteq U^{(i-2)}$ for all $i \geqslant 0$. Thus $((\operatorname{ad} I(M))^{p-2} M)\, U^{(2p-6)} = (0)$. Since $p > 7$ this implies $V \neq U^{(16)}$. But if x, y are root vectors in M satisfying $x + Fz = \mathscr{D}(x_1^2)$, $y + Fz = \mathscr{D}(x_2^2)$, then $Fx + Fy + F[x, y]$ is a subalgebra of M_0 isomorphic to $\mathfrak{sl}(2)$. Choosing a so that $[[x, y], x] = 2ax$, we see that each nonzero $U^{(i)}/U^{(i-1)}$ must contain an eigenvector for $[x, y]$ belonging to 0 or to a. This implies that some weight space of V has dimension > 8 and so contradicts our assumption that $\dim V_\lambda \leqslant 7$ for all λ. Thus the lemma is proved. ∎

COROLLARY 7.2.8. $I(G_0) = (0)$.

Proof. If $I(G_0) \neq (0)$, Corollary 7.2.6 shows $\operatorname{solv}(G_0) = Fz$. Thus if $I(G_0) \neq (0)$, $I(G_0) + \operatorname{solv}(G_0)/\operatorname{solv}(G_0) \neq (0)$ so $H(2:\mathbf{1})^{(2)} \subseteq G_0/Fz \subseteq H(2:\mathbf{1})$. Since $\dim G_{-1,\lambda} \leqslant 7$ for all λ, this contradicts Lemma 7.2.7. ∎

7.3. We now prove some preliminary results dealing with the cases $G_{-(p+1)/2} = (0)$ and $G_{-p} = (0)$.

LEMMA 7.3.1. *If* $G_{-(p+1)/2} = (0)$ *then* $[G_{-1}, G_1] = G_0$.

Proof. The proof (as well as the statement) of this lemma is identical to that of Lemma 5.3.2 of [BW82]. ∎

LEMMA 7.3.2. *If* $G_{-(p+1)/2} = (0)$ *then every root of* A *with respect to* T *is non-Hamiltonian.*

Proof. Since $G_{-(p+1)/2} = (0)$ (and so $G_{-p} = (0)$) we have $A^{(\gamma)} \subseteq A_0$ and so (as $\Gamma_0 \subseteq \mathbf{Z}_\gamma$) $A_0 = A^{(\gamma)} + A_1$. Then $G_0 = (A^{(\gamma)} + A_1)/A_1 \cong A^{(\gamma)}/(A^{(\gamma)} \cap A_1)$. By Corollary 7.2.8, $I(G_0) = (0)$ and so $I(A^{(\gamma)}/(A^{(\gamma)} \cap A_1)) = (0)$ and hence $I(A^{(\gamma)}/\operatorname{solv} A^{(\gamma)}) = (0)$. Thus $A^{(\gamma)}/\operatorname{solv} A^{(\gamma)} \cong (0)$, $\mathfrak{sl}(2)$, or $W(1:\mathbf{1})$ and so γ is not Hamiltonian.

Now let $\alpha \in \Gamma$, $\alpha \notin \mathbf{Z}\gamma$. Then $A[\alpha] = A^{(\alpha)}/\operatorname{solv} A^{(\alpha)}$ contains the maximal torus $\Psi_\alpha(T)$ (recall Ψ_α from Section 1.4). Furthermore, $\Psi_\alpha(T)$ is standard so $\mathfrak{z}_{A[\alpha]}(\Psi_\alpha(T)) = \Psi_\alpha(T) + I(A[\alpha])$, where $I(A[\alpha])$, the nil radical of $\mathfrak{z}_{A[\alpha]}(\Psi_\alpha(T))$, equals $\Psi_\alpha(I)$. Suppose $I(A[\alpha]) \neq (0)$. Then Lemma 5.2.1 shows that $\Psi_\alpha(T) \subseteq [A[\alpha], [A[\alpha], \Psi_\alpha(I)]]$. But as A_0 is a distinguished maximal subalgebra we have $I \subseteq A_0$. However, $I(G_0) = (0)$ by Lemma 7.2.8 and so $I \subseteq A_1$. Then as $I \subseteq \mathfrak{z}_A(z)$ we have $I \subseteq A_p$ and $A = A_{-(p-1)/2}$ so $[A, [A, I]] \subseteq A_1$. Therefore $\Psi_\alpha(T) \subseteq \Psi_\alpha(A_1)$. But A_1 is nil and so $\Psi_\alpha(T)$ is nil, a contradiction. Thus $I(A[\alpha]) = (0)$ and so α is non-Hamiltonian, as required. ∎

LEMMA 7.3.3. *If* $G_{-p}=(0)$ *then* $\gamma \in \Delta_P$.

Proof. Since $G_{-p}=(0)$ implies $A=A_{-p+1}$ we have $A_\gamma = A_{0,\gamma}$. Corollary 6.4.2(c) then gives the result. ∎

7.4. We now prove an important special case of Proposition 7.1.1.

LEMMA 7.4.1. *If* $G_{-3}=(0)$, $G_1 \neq (0)$, *and* $|\Gamma_{-2}| \leqslant 1$, *then* $G_0 \cong \mathfrak{sl}(2) \oplus Fz$ *(as restricted Lie algebras).*

The proof has several steps:

(1) For $i, j \in \mathbf{Z}_p$, $i \neq 0, 1, 2$, we have $A_{i\delta + j\gamma} \subseteq A_1$.

Proof. Since $\Gamma_l = \varnothing$ for $l < -2$ we have that if $\tau \in \Gamma$ and $\tau \notin (\Gamma_{-2} \cup \Gamma_{-1} \cup \Gamma_0)$ then $A_\tau \subseteq A_1$. Since $i\delta + j\gamma \in \Gamma_l$ only if $-i \equiv l \pmod p$, we have the result. ∎

(2) Let $\lambda \in \Gamma_{-1}$ and $0 \neq x \in G_{-1,\lambda}$. Then for all but at most two $\tau \in \Gamma_{-1}$ we have $G_{-1,\tau} \subseteq [x, G_0]$.

Proof. Let $c \in A_{-1,\lambda}$ satisfy $x = c + A_0$. As $c \notin A_0$ and A_0 is a distinguished maximal subalgebra, we have $c \notin R_\lambda$ and hence there exists $d \in A_{-\lambda}$ with $[c, d] \notin I$. By (1) we have $A_{-\lambda} \subseteq A_1$, so $d \in A_1$. Now for any $e \in A_{-1,\tau}$ we have $[[c, d], e] \in \tau([c, d])\, e + [I, e] \subseteq \tau([c, d])\, e + A_0$. Also $[[c, d], e] = [[c, e], d] + [c, [d, e]]$. Since $|\Gamma_{-2}| \leqslant 1$ we have $[c, e] \notin A_{-1}$ for at most one value of τ (independent of e). Since $[c, d] \notin I$ and since $\Gamma_{-1} \subseteq \delta + \mathbf{Z}\gamma$, $\tau([c, d]) = 0$ for at most one value of τ. Thus except for at most two values of τ, we have $e \in \tau([c, d])^{-1}\, [c, [d, e]] + A_0 \subseteq [c, A_0] + A_0$, and hence $G_{-1,\tau} \subseteq [x, G_0]$. ∎

(3) We may assume $\dim G_{-1} \geqslant p - 1$.

Proof. If $\dim G_{-1} < p$ then for some j (with the notation of Section 6.1), $m_{-1}(\delta) \neq m_{-1}(\delta + j\gamma)$. Then by Corollary 5.2.5 of [BW82] (which applies here as $I(G_0) = (0)$) we have $G_0 \cong \mathfrak{sl}(2) \oplus Fz$ (in which case we are done) or $W(1:\mathbf{1}) \oplus Fz$. In the latter case, G_{-1}, being a nontrivial irreducible $W(1:\mathbf{1})$-module, must have at least $p - 1$ weights (cf. Theoem 1.15.2 of [BW82]). ∎

(4) $\dim G_{-1,\tau} \leqslant 3$ for all $\tau \in \Gamma_{-1}$. Hence $\dim G_{-1} \leqslant 3p$.

Proof. By Lemma 7.3.2, τ is non-Hamiltonian. Then by Lemma 6.3.3, $m_-(\tau) \leqslant 3$ and so $\dim G_{-1,\tau} \leqslant 3$. Since $\Gamma_{-1} \subseteq \delta + \mathbf{Z}_p\gamma$ we have $|\Gamma_{-1}| \leqslant p$, giving the final result. ∎

(5) There is a solvable restricted subalgebra Q, $G_0 \supseteq Q \supseteq T$, with $\dim G_0/Q \leqslant 1$. If G_0 is solvable we take $G_0 = Q$.

Proof. This follows from Proposition 1.7 of [Wil83] applied to the restricted Lie algebra $G_0 = G_0^{(\gamma)}$ (which contains toral Cartan subalgebra T by Corollary 7.2.8). ▮

(6) We may assume that G_{-1} is an irreducible Q-module of dimension p.

Proof. Let $V \subseteq G_{-1}$ be an irreducible Q-submodule. Then (cf. Theorem 1.13.1 of [BW82]) dim V is a power of p, hence by (4) is 1 or p. Now as $T \subseteq Q$ we can find $0 \neq x \in V_\lambda$ for some $\lambda \in \Gamma_{-1}$. Then by (2) we have $G_{-1,\tau} \subseteq [x, G_0]$ for all but at most two values of τ. Since $\dim G_{-1,\tau} \leqslant 3$ by (4) this gives $\dim G_{-1} \leqslant 6 + \dim[x, G_0]$. As Q has codimension $\leqslant 1$ in G_0 we have $\dim[x, G_0] \leqslant 1 + \dim V$. Thus $\dim G_{-1} \leqslant 7 + \dim V$. By (3) this gives $\dim V \geqslant p - 8 > 1$ (as $p > 7$) so $\dim V = p$. Thus $\dim G_{-1} < 2p$. If $G_{-1} \neq V$ then $\dim G_{-1} > p$ and so for some $\tau, \eta \in \Gamma_{-1}$ we have $\dim G_{-1,\tau} \neq \dim G_{-1,\eta}$. But then Corollary 5.2.5 of [BW82] shows that either $G_0 \cong \mathfrak{sl}(2) \oplus Fz$ (in which case we are done) or $G_0 \cong W(1:\mathbf{1}) \oplus Fz$. But if $G_0 \cong W(1:\mathbf{1}) \oplus Fz$ then, since the largest irreducible restricted $W(1:\mathbf{1})$-module has dimension p (cf. Theorem 1.15.2 of [BW82]) and since G_{-1} is an irreducible G_0-module, $\dim G_{-1} > p$ is impossible. ▮

(7) $Q_1 = Fz + \sum_{i=1}^{p-1} Q_{i\gamma}$ is a nilpotent ideal of Q. For some $i \in \mathbf{Z}_p^*$ and some $0 \neq b \in Q_{i\gamma}$, $B = Fb + Fz$ is a two-dimensional restricted abelian ideal of Q and $[b, Q_1] \subseteq Fz$.

Proof. Since Q is solvable, $T \cap [Q, Q] \subseteq Fz$ (for otherwise Q contains a copy of $\mathfrak{sl}(2)$). Hence $[Q, Q] \subseteq Q_1$, so Q_1 is an ideal. By the Engel–Jacobson theorem (Theorem 1.10.1 of [BW82]), Q_1 is nilpotent. Therefore $\{x \in Q_1 \mid [x, Q_1] \subseteq Fz\} \supsetneq Fz$. This set is clearly (ad t)-invariant, hence contains an element $0 \neq b \in Q_{i\gamma}$ for some $i \in \mathbf{Z}_p^*$. Since $[b^p, b] = 0$, we have $b^p \in Fz$. Thus $Fb + Fz$ has the required properties. ▮

(8) Let $W \subseteq G_{-1}$ be an irreducible B-module, necessarily one-dimensional, say $W = Fw$. Then Stab(W, Q) has codimension 1 in Q, $t \notin$ Stab(W, Q), and W is invariant under Stab(W, Q). We may therefore assume that $b^p = z$.

Proof. Since W is one-dimensional and z is central, Stab$(W, Q) = \{x \in Q \mid [x, b] \cdot W = (0)\}$. By Lemma 1.10.1, $\dim G_{-1} = p^{\dim(Q/\mathrm{Stab}(W, Q))}(\dim \tilde{W})$. Now $G_{-1} = \tilde{W}$ is impossible since $Fb + Fz$ acts on $\tilde{W}$ as scalars but G_{-1} is a faithful G_0-module. Hence we must have that $\dim(Q/\mathrm{Stab}(W, Q)) = 1$ and $W = \tilde{W}$ so that W is invariant under Stab(W, Q). Thus if $t \in$ Stab(W, Q) we have $W \subseteq G_{-1,\lambda}$ for some $\lambda \in \Gamma_{-1}$. Now by (2) and (4) we have $\dim G_{-1} \leqslant 6 + \dim[W, G_0]$. Since $\dim(G_0/\mathrm{Stab}(W, Q)) \leqslant 2$ we see that $\dim[W, G_0] \leqslant 3$. Hence $\dim G_{-1} \leqslant 9$, contradicting (6) (as $p > 7$). Thus

$t \notin \mathrm{Stab}(W, Q)$ and so $ib \cdot W = [t, b] \cdot W \neq (0)$ so $b \cdot W \neq (0)$ and hence $b^p \cdot W \neq (0)$. Since $b^p \in Fz$ this implies that b^p is a nonzero multiple of z. Replacing b by some scalar multiple of b we can assume $b^p = z$. ▮

(9) G_0 is not solvable.

Proof. If G_0 is solvable, then $G_0 = Q$ and Q_1 is a ideal in G_0. Then $\{x \in G_{-1} \mid [x, G_1] \subseteq Q_1\}$ is a G_0-invariant subspace of G_{-1}, hence (0) or G_{-1}. By Lemma 7.3.1, it cannot be G_{-1}. Thus there exists some $e \in G_1$ such that $[w, e] \notin Q_1$. Replacing e by a scalar multiple, we may assume $[w, e] \in t + Q_1$. Now let $b \in Q_{i\gamma}$ be as in (7). Then $b \cdot w \in Fw$ and so $[b, [w, e]] = [w, e_1]$ for some $e_1 \in G_1$. But $[b, [w,e]] \in [b, t] + [b, Q_1] \subseteq -ib + Fz$. Since $-z$ acts as the identity on G_{-1} it is clear that for any $\lambda \in \Gamma_{-1}$, $0 \neq x \in G_{-1, \lambda}$, and $\mu \in F$ we have $(\mathrm{ad}(-ib + \mu z))\, G_{-1} + Fx \supseteq \mathrm{span}\{(\mathrm{ad}(-ib + \mu z))^j x \mid 0 \leqslant j \leqslant p-1\} = \mathrm{span}\{(\mathrm{ad}\, b)^j x \mid 0 \leqslant j \leqslant p-1\} = G_{-1}$ (since $b^p = z$). Thus $\dim([-ib + \mu z, G_{-1}]) \geqslant p-1$ and so $p - 1 \leqslant \dim[[w, e_1], G_{-1}] \leqslant \dim[[w, G_{-1}], e_1] + \dim[w, [e_1, G_{-1}]] \leqslant \dim[G_{-2}, e_1] + \dim[w, G_0] \leqslant \dim G_{-2} + \dim[w, Q] \leqslant 5$ (as $|\Gamma_{-2}| \leqslant 1$ by hypothesis, so if $G_{-2} \neq (0)$ then $\dim G_{-2} = \dim G_{-2, \lambda}$ for some $\lambda \in \Gamma_{-2}$ and $\dim G_{-2, \lambda} \leqslant 3$ by Lemma 6.3.3 (as λ is non-Hamiltonian by Lemma 7.3.2)). Thus $p \leqslant 6$, a contradiction. ▮

(10) $\mathfrak{z}_{G_0}(b)$ is an abelian subalgebra of codimension $\leqslant 3$ in G_0. Consequently, $G_0/\mathrm{solv}\, G_0 \not\cong W(1:\mathbf{1})$ and so we may assume $G_{-2} = (0)$ and $G_0/\mathrm{solv}\, G_0 \cong \mathfrak{sl}(2)$.

Proof. Since $[b, Q_1] \subseteq Fz$ we see that $\mathfrak{z}_{Q_1}(b)$ has codimension $\leqslant 1$ in Q_1. Since $\dim G_0/Q_1 = 2$ we have that $\mathfrak{z}_{G_0}(b)$ has codimension $\leqslant 3$ in G_0. As $b^p = z$, $-\mathrm{ad}\, z|_{G_{-1}}$ is the identity, and $\dim G_{-1} = p$ we see that $\mathrm{ad}\, b|_{G_{-1}}$ is a cyclic transformation of G_{-1}. Thus (cf. [Jac53, Corollary to Theorem 3.17]) the centralizer of $\mathrm{ad}\, b|_{G_{-1}}$ in End G_{-1} consists of polynomials in $\mathrm{ad}\, b|_{G_{-1}}$ and so is abelian. Thus $(\mathfrak{z}_{G_0}(b) + \mathrm{solv}\, G_0)/\mathrm{solv}\, G_0$ is an abelian subalgebra of codimension $\leqslant 3$ in $G_0/\mathrm{solv}\, G_0$. As b is a root vector this subalgebra is ad T-invariant. But there exists no such subalgebra if $G_0/\mathrm{solv}\, G_0 \cong W(1:\mathbf{1})$ since otherwise, as γ is proper by Lemma 7.3.3, we may assume $(T + \mathrm{solv}\, G_0)/\mathrm{solv}\, G_0 = F(xD)$ and we have $[x^i D, x^j D] = (j - i)\, x^{i+j-1} D \neq 0$ whenever $0 \leqslant i \neq j \leqslant (p+1)/2$. Since we may assume that $G_0 \not\cong \mathfrak{sl}(2) \oplus Fz$ (for in that case we are done), Corollary 5.2.5 of [BW82] shows that $|\Gamma_{-2}|$ is a multiple of p. As $|\Gamma_{-2}| \leqslant 1$ by hypothesis we have $\Gamma_{-2} = \varnothing$ and so $G_{-2} = (0)$. ▮

(11) There exist $\tau \in T^*$, $e \in G_{0, \tau}$, $f \in G_{0, -\tau}$ such that $e + \mathrm{solv}(G_0)$, $f + \mathrm{solv}(G_0)$, and $t + \mathrm{solv}(G_0)$ span $G_0/\mathrm{solv}(G_0)$, that $Q = Ft + Ff + \mathrm{solv}(G_0)$, and that there exists $c \in G_1$ such that $[w, c] \in e + Q$.

Proof. Since $G_0/\mathrm{solv}(G_0) \cong \mathfrak{sl}(2)$ and since $T \subseteq Q$, we may choose such

e and f. Since $\{x \in G_{-1} \mid [x, G_1] \subseteq Q\}$ is a Q-invariant subspace of G_{-1}, by (6) it is (0) or G_{-1}. By Lemma 7.3.1 it is not G_{-1}. This gives the result. ∎

(12) There is some $c_1 \in G_1$ such that $[[[w, c_1], f], b] \in b + Fz$.

Proof. Let c be as in (11). Then $[[w, c], f] \in [e, f] + Ff + \operatorname{solv}(G_0)$. Now $[e, f] \in at + Fz$ for some $0 \neq a \in F$. Thus $[[w, c], f] \in at + Ff + \operatorname{solv}(G_0)$. Since $Ff + \operatorname{solv}(G_0) \subseteq Q_1$ (see (7)), $[[[w, c], f], b] \in aib + Fz$. Replacing c by an appropriate scalar multiple gives the result. ∎

(13) For some $a \in F$ we have $b + az \in [[w, G_1], f]$.

Proof. This follows from (12) since $[w, b] \in Fw$, $[c_1, b] \in G_1$, and $[f, b] \in Fz$, which is central. ∎

But (13) leads to a contradiction. For $\dim[b + az, G_{-1}] \leqslant \dim[[[w, G_1], f], G_{-1}] \leqslant \dim[[w, G_1], G_{-1}] + \dim[[[w, G_1], G_{-1}], f] \leqslant 2\dim[[w, G_1], G_{-1}] \leqslant 2\dim[w, G_0]$ (as $G_{-2} = (0)$) $\leqslant 6$ (since $\operatorname{Stab}(W, Q)$ preserves Fw (by (8)) and has codimension 2 in G_0). However, as b is a root vector satisfying $b^p = z$, it is clear that $\dim[b + az, G_{-1}] \geqslant p - 1$, a contradiction. This completes the proof of the lemma. ∎

7.5. We now prove four technical results which we will need in Sections 7.6–7.8. Recall the definitions of $M(G)$ and $N(G)$ from Section 7.1.

LEMMA 7.5.1. *If $G_1 \neq (0)$, the grading of $G/M(G)$ is nondegenerate* (*in Weisfeiler's sense*; *cf. Theorem* 1.5.1 *of* [BW82]).

Proof. If not we have (cf. Theorem 1.5.1(b) of [BW82]) that $G_2 = (0)$, so $A_2 = (0)$. Now $I \subseteq A_0$ (by our choice of A_0) and since $I(G_0) = (0)$ (by Corollary 7.2.8) we have $I \subseteq A_1$. Since $I \subseteq \mathfrak{z}_A(z)$ and ad z acts as the identity on G_1 we have that $I \subseteq A_2 = (0)$. Thus G satisfies the hypothesis of Lemma 5.4.1 of [BW82] and that result gives the desired conclusion. ∎

LEMMA 7.5.2. *If $\tau \in \Gamma$, $\tau \notin \Delta_P(A, T)$, then*:

(a) $\tau \in \Delta_P(G, T)$,

(b) $\tau([G_{-1,\tau}, G_{1,-\tau}]) = (0)$ *and if* $G_{-1,\tau} = (0)$ *then* $\tau([G_{-2,\tau}, G_{2,-\tau}]) = (0)$,

(c) *if τ is non-Hamiltonian then $G_\tau = K_\tau(G)$.*

Proof. Suppose $\bar{x} \in G_\tau$, $\bar{x} \notin K_\tau(G)$. Then there exists $\bar{y} \in G_{-\tau}$ such that $\tau([\bar{x}, \bar{y}]) \neq 0$. Since G_j is nil for $j \neq 0$, we may assume $\bar{x} \in G_{i,\tau}$, $\bar{y} \in G_{-i,-\tau}$. Without loss of generality we may assume $i \geqslant 0$. Let $\bar{x} = x + A_{i+1}$, $\bar{y} = y + A_{-i+1}$, $x \in A_{i,\tau}$, $y \in A_{-i,-\tau}$. Then $\tau([x, y]) = \tau([\bar{x}, \bar{y}]) \neq 0$. This implies $x \in A_{i,\tau}$, $x \notin K_\tau(A)$. Thus $A_{0,\tau} + K_\tau(A) \neq K_\tau(A)$. If τ is non-Hamiltonian we have that $\dim A_\tau/K_\tau \leqslant 1$ (by Lemma 5.3.4) and so $A_{0,\tau} +$

$K_\tau(A) = A_\tau$. But since $\Gamma_0 \subseteq \mathbf{Z}\gamma \subseteq \mathbf{Z}\tau + \mathbf{Z}\gamma$, part (c) of Corollary 6.4.2 gives a contradiction. This proves (c) (and hence (a) and (b) in case τ is non-Hamiltonian).

Now assume that τ is Hamiltonian. Let Ψ_τ and $A[\tau]$ be as in Section 1.4. Note that $H(2:\mathbf{1})^{(2)} \subseteq A[\tau] \subseteq H(2:\mathbf{1})$. Since $\tau \notin \Delta_P(A, T)$ we can assume $\Psi(T) = \mathscr{D}((x_1+1)x_2)$ and, by Corollary 6.4.2, $\dim A_{i\tau}/(A_{0,i\tau} + K_{i\tau}) \geqslant 2$ for all $i \in \mathbf{Z}_p^*$. Since $K_{i\tau} \supseteq (\operatorname{solv} A^{(\tau)})_{i\tau}$ (by Lemma 5.3.1) we have $\dim \Psi_\tau(A_{i\tau})/\Psi_\tau(A_{0,i\tau}) \geqslant 2$. But $\Psi(A_{0,i\tau})$ is invariant under $\operatorname{ad}(I(A[\tau])) = \operatorname{ad} \Psi_\tau(I(A))$. Thus by Lemma 2.1.11 we have $\Psi_\tau(A_{0,i\tau}) \subseteq \operatorname{span}\{\mathscr{D}((x_1+1)^{i+j}x_2^j) \mid 2 \leqslant j \leqslant p-1\}$ for $i \in \mathbf{Z}_p^*$. Also $\Psi_\tau(I(A)) \subseteq \Psi_\tau(A^{(\tau)} \cap A_1)$ (as $I(G_0) = (0)$) and so, again using Lemma 2.1.11, $\Psi_\tau(A_{-1,i\tau}) \subseteq \operatorname{span}\{\mathscr{D}((x_1+1)^{i+j}x_2^j) \mid 1 \leqslant j \leqslant p-1\}$. Then $\tau([A_{-1,\tau}, A_{1,-\tau}]) = (0)$ so $\tau([G_{-1,\tau}, G_{1,-\tau}]) = (0)$. Also if $G_{-1,\tau} = (0)$ then $[\Psi_\tau(A_{-2,i\tau}), \Psi_\tau(I(A))] \subseteq \Psi_\tau(A_{0,i\tau})$ and as above this implies $\tau([G_{-2,\tau}, G_{2,-\tau}]) = (0)$. Thus (b) holds.

Finally, let τ be Hamiltonian and $\tau \notin \Delta_P(G, T)$. Then for every i we have $\tau([G_{i\tau}, G_{-i\tau}]) \neq (0)$. As in Lemma 2.1.11 set

$$V_{i,l} = \operatorname{span}\{\mathscr{D}((x_1+1)^{i+j}x_2^j) \mid l \leqslant j \leqslant p-1\}.$$

We may assume

$$\tau(\mathscr{D}((x_1+1)x_2)) = 1.$$

For each $i \in \mathbf{Z}_p^*$, we may take $j_i \in \mathbf{Z}$ such that $\tau([G_{j_i, i\tau}, G_{-j_i, -i\tau}]) \neq (0)$. We may assume that $j_{-i} = -j_i$. Then for each $i \in \mathbf{Z}_p^*$ we may take $a_i \in A_{j_i, i\tau}$ such that $\tau([a_i, a_{-i}]) \neq 0$. Then $\tau(\Psi_\tau([a_i, a_{-i}]) = \tau([a_i, a_{-i}]) \neq 0$ and so $[\Psi_\tau(a_i), \Psi_\tau(a_{-i})] \notin V_{0,2}$. Let l_i be the largest index such that $\Psi_\tau(a_i) \in V_{i,l_i}$. Then we have $l_i + l_{-i} - 1 \leqslant 1$ (by Lemma 2.1.11(c)). Pick $k \in \mathbf{Z}_p^*$ such that $j_k = \min\{j_i \mid i \in \mathbf{Z}_p^*,\ j_i \geqslant 0\}$. Also pick $m \neq k$ with $j_m \geqslant 0$. We have $a_k \in A_{j_k, k\tau} \subseteq A_{0,k\tau}$ and so $\Psi_\tau(a_k) \in V_{k,2}$ and $l_k \geqslant 2$. Therefore $l_{-k} = 0$ and $l_k = 2$. Similarly, $l_m = 2$. Then $\Psi_\tau(a_{-k}) = u\mathscr{D}((x_1+1)^{p-k}) + \sum_{j \geqslant 1} u_j \mathscr{D}((x_1+1)^{p-k+j}x_2^j)$ and $\Psi_\tau(a_m) = v\mathscr{D}((x_1+1)^{m+2}x_2^2) + \sum_{j \geqslant 3} v_j \mathscr{D}((x_1+1)^{m+j}x_2^j)$ for some scalars $u, v, u_j, v_j \in F$ with $u \neq 0$, $v \neq 0$. Then direct computation shows that $[\Psi_\tau(a_m), \Psi_\tau(a_{-k})] \notin V_{m-k,2}$. Thus $\Psi_\tau(A_{j_m - j_k, (m-k)\tau}) \nsubseteq V_{m-k,2}$, contradicting $\Psi_\tau(A_{0,(m-k)\tau}) \subseteq V_{m-k,2}$. Hence (a) holds and the lemma is proved. ∎

Lemma 7.5.3. *If* $M(G)_{-k-1} = G_{-k-1}$, $M(G)_{-k} \neq G_{-k}$, *and* $N(G) \neq M(G)$ *then* $N(G)_{-k} = G_{-k}$ *and* $N(G)_{-k+1} = G_{-k+1}$.

Proof. Let $G'' = \sum_{i \geqslant 2} G_i$. Then G'' is a subalgebra of G and $G = G' + G''$. Hence $U(G) = U(G'')\,U(G')$. Now $N(G)/M(G) \neq (0)$ so it generates a nonzero ideal $(U(G) \cdot N(G))/M(G)$ of $G/M(G)$. But $G/M(G)$ contains a unique minimal ideal which contains $\sum_{i<0} (G/M(G))_i$. Thus

$U(G)\cdot N(G)\supseteq G_i$ for all $i<0$. But $N(G)$ is an ideal in G' and so $U(G)\cdot N(G)=U(G'')\,U(G')\cdot N(G)=U(G'')\cdot N(G)$. Now as $G_{-k-1}=M(G)_{-k-1}$, $M(G)$ is an ideal in G, and $G_j=[G_{j+1}, G_{-1}]$ for all $j<-1$, we see that $G_j=M(G)_j$ for all $j\leqslant -k-1$ and

$$\sum_{j<0} G_j\subseteq U(G'')\cdot N(G)\subseteq M(G)+N(G)_{-k}+N(G)_{-k+1}+\sum_{i>-k+1} G_i.$$

Thus we have $G_{-k}=N(G)_{-k}$ and $G_{-k+1}=N(G)_{-k+1}$. ∎

LEMMA 7.5.4. *If $G_1\neq(0)$ then the grading of $G'/N(G)$ is nondegenerate.*

Proof. If not, by Weisfeiler's results (cf. [BW82, Sect. 1.5]) there is a simple Lie algebra Q such that $G'/N(G)$ contains an ideal isomorphic to $Q\otimes B_n$, $B_n=F[x_1, ..., x_n]/(x_1^p, ..., x_n^p)$, B_n is graded by setting $\deg x_i=-1$ for $1\leqslant i\leqslant u$ and $\deg x_i=0$ for $u<i\leqslant n$, and $(Q\otimes B_n)_i=Q\otimes B_{n,i}$. Furthermore, $(G'/N(G))_i=(Q\otimes B_n)_i$ for all $i<0$, $G_1\subseteq\{D\in 1\otimes \operatorname{Der} B_n \mid \deg D=1\}$, and $(G')_2=(0)$. Now as $Q\otimes B_{n,0}\subseteq G_0\subseteq \ker(\operatorname{ad} z)$ we see that T acts on $Q\otimes B_{n,0}$ as a one-dimensional torus of derivations. If $T\cap(Q\otimes B_{n,0})=(0)$ then for every $x\in(Q\otimes B_{n,0})_{i\gamma}$, $i\in \mathbf{Z}_p^*$, $(\operatorname{ad} x)^p=0$. Hence the Engel–Jacobson theorem (Theorem 1.10.1 of [BW82]) shows that $Q\otimes B_{n,0}$ is nilpotent, a contradiction. Hence $T\cap(Q\otimes B_{n,0})\neq(0)$. Now $\ker\psi$ (where $\psi: Q\otimes B_n\to Q\otimes B_n$ is defined by $\psi(s\otimes 1)=s\otimes 1$ for all $s\in Q$ and $\psi(Q\otimes x_j)=(0)$ for all j) is a nil ideal in $Q\otimes B_n$. Thus $(T\cap(Q\otimes B_{n,0}))\cap\ker\psi=(0)$. Then by Lemma 4.3.2 of [BW82], $u\otimes a\mapsto u\cdot a$ gives an injection of $(T\cap(Q\otimes B_{n,0}))\otimes B_{n,i}$ into G_i. Since $[u, u\cdot a]=0$ for all $u\in Q\otimes B_n$, $a\in B_n$, and since ad z annihilates G_0 and G_{-p}, we see that $(T\cap(Q\otimes B_{n,0}))\otimes B_{n,i}$ injects into $\mathfrak{z}_{G_i}(T)$ for $i=0, -p$. Thus $\dim B_{n,0}\leqslant 2$ and $\dim B_{n,-p}=0$. Hence $n=u=1$. Therefore $G_1\subseteq\langle\partial/\partial x_1\rangle$ and as $G_1\neq(0)$ we have $G_1=\langle\partial/\partial x_1\rangle$. Since $\dim G_1=1$, Corollary 5.2.5 of [BW82] shows that $G_0\cong \mathfrak{sl}(2)\oplus Fz$ or $W(1:\mathbf{1})\oplus Fz$. Then (recall (7.0.3), (7.0.4)) we have $G_1=G_{1,-\delta}$ and $t\cdot x_1^j\in(G'/N(G))_{-j,j\delta}$ for $1\leqslant j\leqslant p-1$.

Now for some i we have $\gamma([G_{0,i\gamma}, G_{0,-i\gamma}])\neq(0)$. If γ is not Hamiltonian then $A_{i\gamma}=A_{0,i\gamma}+K_{i\gamma}(A)$. If γ is Hamiltonian then (using Lemmas 2.1.11 and 5.2.1) we see that $\dim(A_\eta/(A_{0,\eta}+K_\eta(A)))\leqslant 1$ for $\eta=i\gamma$ or $-i\gamma$. In either case Corollary 6.4.2 shows that $\gamma\in\Delta_P$ so $|\Gamma_{-p}|\leqslant 6$ by Lemma 6.1.1(c). As $0\notin\Gamma_{-p}$ (Lemma 6.3.3) by Theorem 1.15.2 of [BW82], we see that if $G_0\cong W(1:\mathbf{1})\oplus Fz$ then $\Gamma_{-p}=\varnothing$. If $G_0\cong\mathfrak{sl}(2)\oplus Fz$ then $\Gamma_0=\{0, \pm i\gamma\}$ for some i. As $(G/M(G))_j=(Q\otimes B_1)_j$ for $j<0$ and $G_0\supseteq(Q\otimes B_1)_0$, we have $\Gamma_{-j}\supseteq\Gamma_0+j\delta=\{j\delta, j\delta\pm i\gamma\}$ for $1\leqslant j\leqslant p-1$. Furthermore, as $M(G)\cap G_{-1}=(0)$ we have $\Gamma_{-1}=\Gamma_0+\delta=\{\delta, \delta\pm i\gamma\}$. Suppose that for some k, $2\leqslant k<p$, we have $\Gamma_{-j}=\{j\delta, j\delta\pm i\gamma\}$ for all $j<k$ and $\Gamma_{-k}\neq\{k\delta, k\delta\pm i\gamma\}$. Then, as $\Gamma_{-k}\subseteq\Gamma_{-k+1}+\Gamma_{-1}$, and as Γ_{-k} is symmetric about $k\delta$

(by Theorem 1.15.1 of [BW82]), we have $k\delta \pm 2i\gamma \in \Gamma_{-k}$. But then Lemma 1.14.1 of [BW82], applied to the Heisenberg subalgebra $C = \langle \partial/\partial x_1, h, h \cdot x_1 \rangle$ (where $\{e, f, h\}$ is the usual basis for $\mathfrak{sl}(2)$) acting on the C-submodule of G generated by $G_{-k, k\delta \pm 2i\gamma}$, shows $\pm 2i\gamma \in \Gamma_{-p}$. But since $0 \notin \Gamma_{-p}$ and $|\Gamma_{-p}| \leqslant 6$, Theorem 1.15.1 of [BW82] shows that $\Gamma_{-p} \subseteq \{\pm i\gamma/2, \pm 3i\gamma/2, \pm 5i\gamma/2\}$. Thus $2 \in \{\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}\}$. As $p > 7$ this is impossible. Hence $\Gamma_{-j} = \{j\delta, j\delta \pm i\gamma\}$ for all j, $1 \leqslant j < p$. But then $\Gamma_{-p} \subseteq \Gamma_{-p+1} + \Gamma_{-1} \subseteq \{0, \pm i\gamma, \pm 2i\gamma\}$. Since $\Gamma_{-p} \subseteq \{\pm i\gamma/2, \pm 3i\gamma/2, \pm 5i\gamma/2\}$ and $p > 7$ this implies $\Gamma_{-p} = \varnothing$ so $G_{-p} = (0)$.

It now follows that $N(G) = M(G)$. For if not, Lemma 7.5.3 shows that $(G/N(G))_{-p+1} = (0)$. Then applying Lemma 1.14.1 of [BW82] to the C-submodule of G' generated by $G_{0, \pm i\gamma}$ gives a contradiction.

Now suppose $u \in G_2$. Then ad u restricts to a derivation of $\sum_{i \leqslant -1} (G/M(G))_i$ into $\sum_{i \leqslant 1} (G/M(G))_i$. Since $M(G) = N(G)$ we may identify $(G/M(G))_i$ with $Q \otimes x_1^{-i}$ for $i < 0$ and $(G/M(G))_1$ with $F(\partial/\partial x_1)$. Then $(\text{ad } u)(s \otimes x_1) = \lambda(s)\, \partial/\partial x_1$, where λ is a linear functional on Q. Assume $Q = sl(2)$. (The case $Q = W(1:\mathbf{1})$ is similar.) Then

$$\begin{aligned}
0 &= (\text{ad } u)([[e \otimes x_1, f \otimes x_1], h \otimes x_1]) \\
&= [[\lambda(e)\, \partial/\partial x_1, f \otimes x_1], h \otimes x_1] \\
&\quad + [[e \otimes x_1, \lambda(f)\, \partial/\partial x_1], h \otimes x_1] \\
&\quad + [[e \otimes x_1, f \otimes x_1], \lambda(h)\, \partial/\partial x_1] \\
&= 2\lambda(e) f \otimes x_1 + 2\lambda(f)\, e \otimes x_1 - 2\lambda(h)\, h \otimes x_1.
\end{aligned}$$

Therefore $\lambda(e) = \lambda(f) = \lambda(h) = 0$ so $[u, G_{-1}] = (0)$ and therefore $u = 0$. Thus $G_2 = (0)$ so $G = G'$ and hence $G'/N(G) = G/M(G)$ has nondegenerate grading by Lemma 7.5.1. This contradicts our assumption (that $G'/N(G)$ has degenerate grading) and so proves the lemma. ▌

7.6. We now analyze the case in which each nonzero G_i, $i < 0$, $i \not\equiv 0 \pmod p$, has p weights each of the same multiplicity. Recall that $\Gamma_i \subseteq \{-i\delta + j\gamma \mid j \in \mathbf{Z}_p\}$ and that $m_i(\tau)$ is the multiplicity of the weight τ in G_i. Note (see the remark after (7.0.5)) that our proof of the following lemma does not use Lemma 5.5.1 of [BW82] and so provides a corrected proof of that result.

LEMMA 7.6.1. *If* $A_1 \neq (0)$ *and* $m_i(-i\delta) = m_i(-i\delta + j\gamma)$ *for every* $i < 0$, $i \not\equiv 0 \pmod p$, *and every* $j \in \mathbf{Z}_p$, *then* $A = A_{-1}$.

We will assume $A \neq A_{-1}$ and derive a contradiction. Our proof has several steps.

(1) $\Gamma_P \subseteq \mathbf{Z}\gamma$ and $|\Gamma_{-p+1}| = p$.

Proof. Since $A \neq A_{-1}$ we must have $|\Gamma_{-1}| = |\Gamma_{-2}| = p$. If $|\Gamma_{-3}| = 0$ then $G_{-3} = (0)$ and so by Lemma 7.3.2 every $\tau \in \Gamma$ is non-Hamiltonian. Thus if $\tau \in \Gamma_{-1}$ we have τ, $2\tau \in \Gamma_R$ (see Definition 5.6.7) and so by Lemma 5.6.8(b), $|\Gamma_R \cap \mathbf{Z}\tau| \geqslant 4$. Since every τ is non-Hamiltonian, Lemma 5.7.1(a) implies that $|\Gamma_E \cap \mathbf{Z}\tau| = 2$ for every $\tau \in \Gamma_{-1}$. Then by Lemma 5.7.5, $\Delta_P \neq \Delta - \{0\}$ and hence we can find some $\tau \in \Gamma_{-1}$, $\tau \notin \Gamma_P$. If $|\Gamma_{-3}| \neq 0$ then $|\Gamma_{-3}| = p$ and so if $\tau \in \Gamma_{-1}$ we have τ, 2τ, $3\tau \in \Gamma_R$. Then by Lemma 5.6.8, $|\Gamma_R \cap \mathbf{Z}\tau| \geqslant 6$ and so Lemmas 5.7.1(a) and 5.7.5 again show that we can find some $\tau \in \Gamma_{-1}$, $\tau \notin \Gamma_P$. Thus in any case there is some $\tau \in \Gamma_{-1}$, $\tau \notin \Gamma_P$. Then by Lemma 6.4.2(c) we have $A_{i\tau} \neq A_{0, i\tau}$ for all i, $1 \leqslant i \leqslant p-1$. Thus $i\tau \in \Gamma_{-i} \cup \Gamma_{-i-p} \cdots$ and in any case $\Gamma_{-i} \neq \varnothing$. Therefore $|\Gamma_{-i}| = p$ for all i, $1 \leqslant i \leqslant p-1$. Thus if $\eta \in \Gamma_{-1}$ we have $\mathbf{Z}_p^* \eta \subseteq \Gamma_- \subseteq \Gamma_R$ so by Lemma 5.7.1(a), $\eta \notin \Gamma_P$. Thus $\Gamma_P \subseteq \mathbf{Z}\gamma$. ∎

(2) $m_-(i\gamma) = 0$ for all but at most six values of $i \in \mathbf{Z}_p^*$.

Proof. By Corollary 6.2.3, $n(A, T) \geqslant 1$. By (1), $\Gamma_P \subseteq \mathbf{Z}\gamma$. Hence $\gamma \in \Delta_P$. Then Lemma 6.1.1(c) gives the result. ∎

(3) $G/M(G)$ contains a simple minimal ideal Q with $(G/M(G))_i = Q_i$ for $i < 0$.

Proof. By Lemma 7.5.1 the grading of $G/M(G)$ is nondegenerate. Hence by Weisfeiler's theorem (cf. Theorem 1.5.1 of [BW82]), $G/M(G)$ contains a minimal ideal $V = Q \otimes B_n$, where Q is simple and graded, $(G/M(G))_i = V_i = Q_i \otimes B_n$ for $i < 0$, and Q_0, $Q_1 \neq (0)$. Suppose $n \geqslant 1$. Now ad t (see (7.0.3), (7.0.4)) acts on $V = Q \otimes B_n$. Thus ad $t = s + w$, where $s \in (\mathrm{Der}\, Q)_0 \otimes B_n$ and $w \in 1_Q \otimes (\mathrm{Der}\, B_n)$. As $t^p = t$ we have $w^p = w$. Suppose that $w = 0$. Then clearly $V_{-1,\tau} \cdot B_n \subseteq V_{-1,\tau}$ (for ad z acts as a scalar on V_{-1}). Let $\tau \in \Gamma_{-1}$, $v \in V_{-1,\tau}$ be such that $v \notin Q_{-1} \otimes (x_1 B_n + \cdots + x_n B_n)$. Then it is clear that $\dim V_{-1,\tau} \geqslant \dim(v \cdot B_n) = p^n$. But $\dim V_{-1,\tau} = \dim G_{-1,\tau} \leqslant 7$ by Lemma 6.3.3. Since $n \geqslant 1$ this is impossible. Hence $w \neq 0$. This implies $\dim(T \cap (\overline{Q \otimes B_n})) \leqslant 1$. Since ad t acts diagonally on $Q_0 \otimes B_n$, there exists a t-invariant (hence T-invariant) vector space complement U of $Q_0 \otimes (x_1 B_1 + \cdots + x_n B_n)$ in $Q_0 \otimes B_n$. Then $\dim U = \dim Q_0$. We can write $U = \sum U_i$ and $B_n = \sum (B_n)_i$, where U_i (respectively $(B_n)_i$) is the i-eigenspace for ad t (respectively for w). By Lemma 4.3.1 of [BW82], each $(B_n)_i$ has dimension p^{n-1}, and by Lemma 4.3.2 of [BW82], the mapping $U \otimes B_n \to Q_0 \otimes B_n$, $u \otimes b \mapsto u \cdot b$, is injective. Hence

$$\dim \sum_i U_i \cdot (B_n)_{p-i} = (\dim Q_0)\, p^{n-1}.$$

But $[t, \sum_i U_i \cdot (B_n)_{p-i}] = (0)$ and so $\sum_i U_i \cdot (B_n)_{p-i} \subseteq \mathfrak{z}_{G_0}(T) = T$,

$$(\dim Q_0)\, p^{n-1} \leqslant \dim(T \cap (Q_0 \otimes B_n)) \leqslant \dim(T \cap (\overline{Q \otimes B_n})) \leqslant 1,$$

$\dim Q_0 = 1$, $n=1$, and $T \cap (Q_0 \otimes B_n)$ is restricted and one-dimensional. Since $t \notin (\text{Der } Q)_0 \otimes B_n$ we have $\dim(T \cap ((\text{Der } Q)_0 \otimes B_n)) \leqslant 1$ and so $T \cap (Q_0 \otimes B_n) = T \cap ((\text{Der } Q)_0 \otimes B_n)$. Now if $z \notin T \cap ((\text{Der } Q)_0 \otimes B_n)$ then it follows from Lemmas 4.3.1 and 4.3.2 of [BW82] that z has p eigenvalues on G_{-1}, a contradiction (since z acts as a scalar on G_{-1}). Thus $Fz = T \cap V_0$. Since $G_{-i} = [G_{-1}, G_{-i+1}]$ for $i > 1$ we have $Q_{-i} = [Q_{-1}, Q_{-i+1}]$ for $i > 1$. Hence $Q_0 = [Q_{-1}, Q_1]$ (for otherwise $[Q_{-1}, Q_1] + \sum_{i \neq 0} Q_i$ is a proper ideal in Q). Thus $V_0 \subseteq [G_{-1}, G_1]$ and so there exists $\tau \in \Gamma_{-1}$ such that $[G_{-1,\tau}, G_{1,-\tau}] = T \otimes V_0 = Fz$. As $\tau \in \Gamma_{-1}$ this implies $\tau([G_{-1,\tau}, G_{1,-\tau}]) \neq (0)$. As $\tau \notin \Gamma_P$ this contradicts Lemma 7.5.2(b). Hence $n = 0$, as required. ∎

(4) If G_0 is solvable or $G_{-p-1} \neq (0)$ then $[G_{-1,\tau}, G_{1,-\tau}] = (0)$ for every $\tau \in \Gamma_{-1}$.

Proof. If not, then, as $\tau([G_{-1,\tau}, G_{1,-\tau}]) = (0)$ by Lemma 7.5.2(b), we see that G contains a Heisenberg subalgebra $C = Fx + Fy + F[x, y]$, where $x \in G_{-1,\tau}$, $y \in G_{1,-\tau}$. Suppose $G_{-p-1} \neq (0)$. Then $|\Gamma_{-p-1}| = p$ so $G_{-p-1,\tau+i\gamma} \neq (0)$ for $1 \leqslant i \leqslant p-1$. By Lemma 1.14.1 of [BW82] (applied to the restricted Lie algebra $T + C$ which has toral Cartan subalgebra T), since $(\tau + i\gamma)([x, y]) = i\gamma([x, y]) \neq 0$ the C-module generated by $G_{-p-1,\tau+i\gamma}$ must have nonzero intersection with $G_{-p,i\gamma} + G_{-2p,i\gamma}$ and so $m_-(i\gamma) \neq 0$. Since this must occur for all i, $1 \leqslant i \leqslant p-1$, this contradicts (2). Hence $G_{-p-1} = (0)$ and so $A = A_{-p}$. Now suppose A_0 is solvable. Since $I \subseteq A_0$ and $I(G_0) = (0)$ (by Corollary 7.2.8) we have $I \subseteq A_p$ (as $[z, I] = (0)$). Recall that $S \subseteq A \subseteq \text{Der } S$ for some simple algebra S and that $S + I = A$. Thus $A = S + A_0 = S + A_p$. Suppose that X is a proper subalgebra of S which properly contains $S \cap A_0$. Then (as $A/A_0 = (S + A_0)/A_0 \cong S/(S \cap A_0)$) $X + A_0$ is a proper subspace of A properly containing A_0. Now $A_0 = A_0 \cap (S + A_p) = S \cap A_0 + A_p$ and so $X + A_0 = X + A_p$. But $[X + A_p, X + A_p] \subseteq [X, X] + [X, A_p] + [A_p, A_p] \subseteq X + [A_{-p}, A_p] \subseteq X + A_0 = X + A_p$. Thus $X + A_0 = X + A_p$ is a proper subalgebra of A properly containing A_0. This contradicts the maximality of A_0. Hence $S \cap A_0$ is a solvable maximal subalgebra of S. It follows by results of Kuznecov [Kuz76] and Weisfeiler [Wei84] that $\dim S/(S \cap A_0) = 1$ and hence that $\dim A/A_0 = 1$. Then (as G_0 acts faithfully on G_{-1}) $\dim G_0 = 1$, which contradicts the fact that T injects into G_0. This proves our assertion, ∎

(5) G_0 is not solvable.

Proof. If G_0 is solvable then (4) shows that $[G_{-1,\tau}, G_{1,-\tau}] = (0)$ for all $\tau \in \Gamma_{-1}$. Let Q be as in (3), so that Q is a simple ideal in $G/M(G)$ with $(G/M(G))_i = Q_i$ for $i < 0$. Since G_0 is solvable, and G_0 acts irreducibly on G_{-1}, we have $\dim G_{-1} = p^n$. Since G_{-1} has p weight by hypothesis, $n \geqslant 1$. If $n > 1$ then $\dim G_{-1,\tau} \geqslant p$ for some $\tau \in \Gamma_{-1}$, contradicting Lemma 6.3.3.

Hence $\dim G_{-1}=p$. Now $\sum_{i\neq 0} Q_i+[Q_{-1}, Q_1]$ is an ideal in Q, so (as Q is simple) $Q_0=[Q_{-1}, Q_1]$. Thus $Q_{0,0}=(0)$. Now Q_0 is an ideal in G_0 and G_0 acts faithfully and irreducibly on G_{-1} so Q_0 is not nil. Thus there exists some $x\in Q_{0,i\gamma}$ for some $i\neq 0$ such that $x^p\neq 0$. Now (as $Q_{0,0}=(0)$) we have $(\operatorname{ad} x)^p Q_0=(0)$, so $x^p\in Fz$ (since $Q_0\not\subseteq T$). We may assume $x^p=-z$. Therefore (since $\dim G_{-1}=p$ and $-z$ acts as the identity on G_{-1}) $\operatorname{ad} x|_{G_{-1}}$ is a cyclic linear transformation of G_{-1}. Hence (cf. [Jac53, Corollary to Theorem 3.17]) any linear transformation of G_{-1} which commutes with $\operatorname{ad} x|_{G_{-1}}$ is a polynomial in $\operatorname{ad} x|_{G_{-1}}$. Since $x\notin Fz$ and $(\operatorname{ad} x)^{p+1}=0$ on G_0 there exists some $y\in G_{0,j\gamma}$ such that $[x,y]\neq 0$, $[x[x,y]]=0$. Now $[x,y]\in Q_{0,(i+j)\gamma}$ so (as $Q_{0,0}=(0)$) we have $i+j\not\equiv 0 \pmod p$. As $\operatorname{ad}[x,y]|_{G_{-1}}$ commutes with $\operatorname{ad} x|_{G_{-1}}$ we have $\operatorname{ad}[xy]|_{G_{-1}}\in F(\operatorname{ad} x|_{G_{-1}})^m$, where $mi\equiv i+j \bmod p$. Therefore $[[x,y],y]\neq 0$, and $[x,[[x,y],y]]=0$. Assume $(\operatorname{ad} y)^u x\neq 0$, $(\operatorname{ad} x)(\operatorname{ad} y)^u x=0$. Then $(\operatorname{ad} y)^u x\in Q_{0,(i+uj)\gamma}$ and so $i+uj\neq 0$. Thus $\operatorname{ad}((\operatorname{ad} y)^u x)|_{G_{-1}}$ is proportional to a power of $\operatorname{ad} x|_{G_{-1}}$ and so does not commute with $\operatorname{ad} y|_{G_{-1}}$. Hence $(\operatorname{ad} y)^{u+1} x\neq 0$ and $(\operatorname{ad} x)(\operatorname{ad} y)^{u+1} x=0$. Thus $(\operatorname{ad} y)^n x\neq 0$ for all n so $Q_{0,(i+nj)\gamma}\neq (0)$ for all n. If $j\neq 0$ this implies $Q_{0,0}\neq (0)$, a contradiction. Thus $[x, G_{0,j\gamma}]=(0)$ for all $j\in \mathbf{Z}_p^*$. As above this implies that if $j\in\mathbf{Z}_p^*$ and $G_{0,ij\gamma}\neq(0)$ then $(\operatorname{ad} G_{0,ij\gamma})|_{G_{-1}}\subseteq F((\operatorname{ad} x)|_{G_{-1}})^j$. Since G_0 acts faithfully on G_{-1} this implies that $\sum_{k\in\mathbf{Z}_p^*} G_{0,k\gamma}$ is an abelian subalgebra of G_0 and hence that Q_0 is abelian. Also the action of Q_0 on Q_{lp} is nil for all $l\in\mathbf{Z}$ (so Q_{lp} contains no nontrivial irreducible Q_0-submodules).

Write $Q=\sum_{i\geqslant r} Q_i$. Then the simplicity of Q implies (using [SF88, Proposition III.3.5]) that Q_r is a nontrivial irreducible Q_0-module. Since Q_0 is abelian we have $\dim Q_r=1$ and since Q_r is a nontrivial Q_0-module, $p\nmid r$. This implies that $x^p Q_r\neq(0)$ and so (as x is a root vector) $\dim Q_r\geqslant p$, a contradiction. ∎

(6) Write $J=[G_{-1}, G_1]\cap \operatorname{solv} G_0$. If $J\neq(0)$ then $\operatorname{solv} G_0$ is nilpotent and either $\mathfrak{z}_J(\operatorname{solv} G_0)=Fz$ or $\dim G_{-1}=p$.

Proof. Suppose $J\neq(0)$. Note that $T\cap(\operatorname{solv} G_0)=Fz$, since otherwise $T\subseteq \operatorname{solv} G_0$ and so $\operatorname{solv} G_0=G_0$, contradicting (5). Hence $\operatorname{solv} G_0$ is nilpotent by the Engel–Jacobson theorem. Write $X=\mathfrak{z}_J(\operatorname{solv} G_0)$. Then X is a nonzero abelian ideal in G_0. Let Y be an irreducible X-submodule, necessarily one-dimensional, of G_{-1}. Then $\operatorname{codim} \operatorname{Stab}(Y, G_0)\leqslant 1$ since $\dim G_{-1}\leqslant 7p<p^2$. If $\operatorname{codim}\operatorname{Stab}(Y, G_0)=0$ then $G_{-1}=\tilde{Y}$, a sum of copies of the one-dimensional X-module Y. As G_{-1} is a faithful G_0-module this implies $X=Fz$. Thus we may assume that $\operatorname{codim}\operatorname{Stab}(Y, G_0)=1$. As $[\operatorname{solv} G_0, X]=(0)$, we have $\operatorname{solv} G_0\subseteq \operatorname{Stab}(Y, G_0)$, and so $\operatorname{Stab}(Y, G_0)/\operatorname{solv} G_0$ is a subalgebra of codimension 1 in $G_0/\operatorname{solv} G_0\cong \mathfrak{sl}(2)$ or $W(1:1)$. Hence $\operatorname{Stab}(Y, G_0)$ is solvable. Therefore $\tilde{Y}$, an irreducible $\operatorname{Stab}(Y, G_0)$-

module, has dimension a power of p. Hence G_{-1} has dimension a power of p. Since $p \leqslant \dim G_{-1} \leqslant 7p$, we have $\dim G_{-1} = p$. ∎

(7) $[G_{-1}, G_1] \nsubseteq \operatorname{solv} G_0$, and if $J \neq (0)$ then $[G_{-1,\tau}, G_{1,-\tau}] \neq (0)$ for at least two $\tau \in \Gamma_{-1}$.

Proof. Since $[G_{-1}, G_1] \neq (0)$, the assertion is vacuous if $J = (0)$. Hence assume $J \neq (0)$. We let X and Y be as in the proof of (6). Now $G_0/\operatorname{solv} G_0 \cong \mathfrak{sl}(2)$ or $W(1:\mathbf{1})$. In either case we can find $i \in \mathbf{Z}_p^*$ and $e \in G_{0,i\gamma}$, $f \in G_{0,-i\gamma}$ so that $0 \neq \gamma([e,f])$. Then $Fe + Ff + F[e,f]$ is a subalgebra of G_0 isomorphic to $\mathfrak{sl}(2)$. Suppose $J \cap T = (0)$. Then 0 is not a weight for the action of the above copy of $\mathfrak{sl}(2)$ on X, and so $[[e,f] + \lambda e, X] = X$ for all $\lambda \in F$. Since X is an ideal in G_0 and G_{-1} is a faithful G_0-module, we have $[X, Y] \neq (0)$. Thus $F([e,f] + \lambda e) \cap \operatorname{Stab}(Y, G_0) = (0)$. But as $\operatorname{codim} \operatorname{Stab}(Y, G_0) \leqslant 1$ (since $\dim G_{-1} < p^2$) we have $(F[e,f] + Fe) \cap \operatorname{Stab}(Y, G_0) \neq (0)$ and so $e \in \operatorname{Stab}(Y, G_0)$. Similarly, $f \in \operatorname{Stab}(Y, G_0)$ and so $[e, f] = [e,f] + 0e \in \operatorname{Stab}(Y, G_0)$, a contradiction. Thus $J \cap T \neq (0)$.

For $\tau \in \Gamma_{-1}$ we have $\tau([G_{-1,\tau}, G_{1,-\tau}]) = (0)$ by Lemma 7.5.2(b). Thus if $[G_{-1,\tau}, G_{1,-\tau}] \subseteq Fz$ then $[G_{-1,\tau}, G_{1,-\tau}] = (0)$. But $(\operatorname{solv} G_0) \cap T = Fz$ and so if $[G_{-1,\tau}, G_{1,-\tau}] \subseteq \operatorname{solv} G_0$ then $[G_{-1,\tau}, G_{1,-\tau}] = (0)$. Thus, as $J \cap T \neq (0)$, we have $[G_{-1}, G_1] \nsubseteq \operatorname{solv} G_0$. Since $J \cap T \neq (0)$ we have $[G_{-1,\tau}, G_{1,-\tau}] \neq (0)$ for at least two $\tau \in \Gamma_{-1}$. ∎

(8) If $\mathfrak{z}_J(\operatorname{solv} G_0) = Fz$ and $J \nsubseteq Fz$ then $G_0/\operatorname{solv} G_0 \cong \mathfrak{sl}(2)$ and $J = Fz + Fu_{i\gamma} + Fu_{-i\gamma}$ for some $i \in \mathbf{Z}_p^*$, $u_{\pm i\gamma} \in (\operatorname{solv} G_0)_{\pm i\gamma}$.

Proof. We write $X_i = J \cap (\operatorname{solv} G_0)_i$, where $(\operatorname{solv} G_0)_i$ denotes the ith term of the upper central series of $\operatorname{solv} G_0$. Thus $X_i = \{x \in J \mid [\operatorname{solv} G_0, x] \subseteq X_{i-1}\}$ is an ideal in G_0, and $X_1 = \mathfrak{z}_J(\operatorname{solv} G_0) = Fz$. As $\operatorname{solv} G_0$ is nilpotent, by (6), and as $J \neq Fz$, we have $J = X_i$ for some $i \geqslant 2$ and $Fz \subsetneqq X_2$. Suppose X_2 is abelian and let Y_2 be an irreducible X_2-submodule of G_{-1}. Then since $\dim G_{-1} < p^2$ we have $\operatorname{codim} \operatorname{Stab}(Y_2, G_0) \leqslant 1$ and $\operatorname{Stab}(Y_2, G_0) \neq G_0$ (for if equality held G_{-1} would be $\tilde{Y}_2$ and as Y_2 is one-dimensional X_2 would not act faithfully on $\tilde{Y}_2$). Thus $\operatorname{codim} \operatorname{Stab}(Y_2, G_0) = 1$. Now $[\operatorname{solv} G_0, X_2] = Fz$ and so $\operatorname{solv} G_0 \nsubseteq \operatorname{Stab}(Y_2, G_0)$. Thus $G_0 = \operatorname{Stab}(Y_2, G_0) + \operatorname{solv} G_0$. It follows, as $\operatorname{solv} G_0$ is nilpotent, that there exists an i such that $(\operatorname{Stab}(Y_2, G_0) \cap (\operatorname{solv} G_0)^i) + (\operatorname{solv} G_0)^{i+1} \neq (\operatorname{solv} G_0)^i$. Now $G_0/\operatorname{solv} G_0 = (\operatorname{Stab}(Y_2, G_0) + \operatorname{solv} G_0)/\operatorname{solv} G_0$ acts on $(\operatorname{solv} G_0)^i/(\operatorname{solv} G_0)^{i+1}$ and $((\operatorname{Stab}(Y_2, G_0) \cap (\operatorname{solv} G_0)^i) + (\operatorname{solv} G_0)^{i+1})/(\operatorname{solv} G_0)^{i+1}$ is a submodule of codimension 1. But then 0 is a weight of the quotient module

$$(\operatorname{solv} G_0)^i/((\operatorname{Stab}(Y_2, G_0) \cap (\operatorname{solv} G_0)^i) + (\operatorname{solv} G_0)^{i+1}),$$

contradicting the fact that the 0 weight space for the action of G_0 on $(\operatorname{solv} G_0)^i$ is contained in $Fz \subseteq \operatorname{Stab}(Y_2, G_0)$. Therefore X_2 is not abelian. As

$[X_2, \text{solv}\, G_0] = Fz = \mathfrak{z}_J(\text{solv}\, G_0)$ if $i \in \mathbf{Z}_p^*$ and $0 \neq u \in (X_2)_{i\gamma}$, then $[u, (\text{solv}\, G_0)_{-i\gamma}] = Fz$. Then by Proposition 5.5.1(b) (with $B = \text{solv}\, G_0$, $W = G_{-1}$) we see that $X_2 = (X_2)_{i\gamma} + (X_2)_{-i\gamma} + Fz$ for some $i \in \mathbf{Z}_p^*$, and $\dim(X_2)_{\pm i\gamma} = 1$. Since X_2/Fz is a $G_0/(\text{solv}\, G_0)$-module this implies that $G_0/(\text{solv}\, G_0) \cong \mathfrak{sl}(2)$. Now multiplication induces a homomorphism of $\mathfrak{sl}(2)$-modules

$$((\text{solv}\, G_0)/[\text{solv}\, G_0, \text{solv}\, G_0]) \otimes (X_3/X_2) \to X_2/X_1.$$

Since $Fz = (\text{solv}\, G_0) \cap T$ is contained in X_1 and in $[\text{solv}\, G_0, \text{solv}\, G_0]$, we see that 0 is not a weight of any of the three modules $(\text{solv}\, G)/[\text{solv}\, G_0, \text{solv}\, G_0]$, X_3/X_2, X_2/X_1. Then by [Jac58, Lemma 3, Theorem 1] each of these is a direct sum of irreducible restricted $\mathfrak{sl}(2)$-modules of even dimension. Then (cf. [BW82, Proposition 1.15.3]) this homomorphsim is zero. Therefore $[\text{solv}\, G_0, X_3] \subseteq X_1$, and $X_3 = X_2 = J$. Thus J is as described in the statement. ∎

(9) There exists $\tau \in \Gamma_{-1}$ such that $[G_{-1,\tau}, G_{1,-\tau}] \neq (0)$.

Proof. If not the ideal $[G_{-1}, G_1]$ of G_0 satisfies $[G_{-1}, G_1] \cap T = (0)$. Since $G_0/\text{solv}\, G_0$ is simple (by Theorem 3.1.1 and the fact that $I(G_0) = (0)$) and $T \nsubseteq \text{solv}\, G_0$ this would imply $[G_{-1}, G_1] \subseteq \text{solv}\, G_0$, which contradicts (7). ∎

(10) If $\eta \in \Gamma_{-1}$ then $\dim G_{-1,\eta} = 1$.

Proof. Let τ be as in (9). Then, by Lemma 7.5.2(b) there exist $x \in G_{-1,\tau}$ $y \in G_{1,-\tau}$ so that $C = Fx + Fy + F[x, y]$ is a Heisenberg subalgebra of G. By equality of multiplicities, it suffices to prove that $\dim G_{-1,\tau+i\gamma} \leqslant 1$ for some $i \in \mathbf{Z}_p$. If $i \neq 0$ then Lemma 1.14.1 of [BW82] applied to the C-submodule of G generated by $G_{-1,\tau+i\gamma}$ shows that $\dim G_{-1,\tau+i\gamma} \leqslant \dim G_{-p,i\gamma} + \dim\, [G_{-1}, G_1]_{i\gamma}$. By (2), $G_{-p,i\gamma} = (0)$ for all but at most six values of i. Also, $\dim\, [G_{-1}, G_1]_{i\gamma} \leqslant \dim(G_0/\text{solv}\, G_0)_{i\gamma} + \dim J_{i\gamma}$. If $J = (0)$ or Fz then $\dim\, J_{i\gamma} = (0)$ for all (nonzero) i, while if $J \nsubseteq Fz$ and $\mathfrak{z}_J(\text{solv}\, G_0) = Fz$ then $J_{i\gamma} = (0)$ for all but at most two values of i, by (8), and we are done in these cases since $p > 7$. Finally, if $J \nsubseteq Fz$ and $\mathfrak{z}_J(\text{solv}\, G_0) \neq Fz$ then (6) gives the result. ∎

(11) If $\tau \in \Gamma_{-1}$ then τ is non-Hamiltonian.

Proof. By (4) and (9), $G_{-p-1} = (0)$. Therefore by (10) we have $\dim A_\tau/A_{0,\tau} = 1$. Then Corollary 6.4.2(d) shows τ is non-Hamiltonian. ∎

(12) $T \subseteq Q$ is impossible.

Proof. As in the proof of (3) we have $Q_0 = [Q_{-1}, Q_1]$. Assume $T \subseteq Q$, so we have $T = \sum_{\eta \in \Gamma_{-1}} [Q_{-1,\eta}, Q_{1,-\eta}]$. Thus, as $\eta([Q_{-1,\eta}, Q_{1,-\eta}]) = (0)$ by (1), (11), and Lemma 7.5.2(c) we have $[Q_{-1,\eta}, Q_{1,-\eta}] \neq (0)$ for at least two $\eta \in \Gamma_{-1}$. Since, for $\eta \notin \mathbf{Z}\gamma$, $G_\eta = K_\eta(G)$ we have $\bar{Q}_\eta = K_\eta(\bar{Q})$ and hence by Definition 5.1.7, $RK_\eta(\bar{Q}) = R_\eta(\bar{Q})$. Thus $[Q_{-1,\eta}, Q_{1,-\eta}] \neq (0)$ implies $\eta \in \Gamma_E(\bar{Q}, T)$ so we have $|\Gamma_{-1} \cap \Gamma_E(\bar{Q}, T)| \geqslant 2$. Also $\Delta_P(\bar{Q}, T) = \Delta(\bar{Q}, T) - \{0\}$. Since $T \subseteq Q$, $\bar{Q}$ satisfies (5.7.1) (with $A = \bar{Q}$, $S = Q$). But then Lemma 5.7.5 shows $|\Gamma_{-1} \cap \Gamma_E(\bar{Q}, T)| \leqslant 1$, a contradiction. ▌

(13) $T \nsubseteq Q$ is impossible.

Proof. As in (12) we have $G_\eta = K_\eta(G)$ for all $\eta \notin \mathbf{Z}\gamma$. In view of (9) this shows that if $T \nsubseteq Q$ then $T \cap Q = \ker \tau$. Then setting $M = \mathfrak{z}_Q(T \cap Q)$ we see that M is a Cartan subalgebra of Q but $[M, M]$ is not nil. This contradicts the fact that a Cartan subalgebra of a simple Lie algebra must be standard. ▌

Since (12) and (13) are contradictory, the proof of the lemma is complete. ▌

7.7. We now consider the case in which for some i, $i > 0$, $i \not\equiv 0 \pmod{p}$, there exist $\mu, \nu \in \{i\delta + j\gamma \mid j \in \mathbf{Z}_p\}$ with $m_{-i}(\mu) \neq m_{-i}(\nu)$. Note (see the remark after (7.0.5)) that our proof of the following lemma does not use Lemma 5.6.1 of [BW82] and so provides a corrected proof of that result.

LEMMA 7.7.1. *If, $A_1 \neq (0)$ and if there exist i, μ, and ν such that $i > 0$, $i \not\equiv 0 \pmod{p}$, $\mu, \nu \in \{i\delta + j\gamma \mid j \in \mathbf{Z}_p\}$, $m_{-i}(\mu) \neq m_{-i}(\nu)$, then $G_0 \cong \mathfrak{sl}(2) \oplus Fz$ (as restricted Lie algebras).*

Proof. We will assume $G_0 \ncong \mathfrak{sl}(2) \oplus Fz$ and derive a contradiction. The proof has several steps.

(1) $G_0 \cong W(1:\mathbf{1}) \oplus Fz$ by Corollary 7.2.8 above and Corollary 5.2.5 of [BW82]. ▌

(2) $\Gamma_{-p} = \varnothing$.

Proof. Recall that $\Gamma_0 \subseteq \mathbf{Z}\gamma$. By Theorem 1.15.2 of [BW82] an irreducible $W(1:\mathbf{1})$-module for which 0 is not a weight has $p-1$ distinct weights. Thus if $G_{-p} \neq (0)$ we must have $|\Gamma_{-p}| = p-1$. As $G_0 \cong W(1:\mathbf{1}) \oplus Fz$ we have $\gamma([A_{0,i\gamma}, A_{0,-i\gamma}]) \neq (0)$ for some $i \in \mathbf{Z}_p^*$. If γ is non-Hamiltonian this implies $A_{i\gamma} = A_{0,i\gamma} + K_{i\gamma}(A)$ and so Corollary 6.4.2(c) shows that $\gamma \in \Gamma_P$. If γ is Hamiltonian then, as $A_{0,\pm i\gamma}$ is ad I-invariant, we see from Lemma 2.1.11 that either $\dim A_{i\gamma}/(A_{0,i\gamma} + K_{i\gamma}(A)) \leqslant 1$ or $\dim A_{-i\gamma}/(A_{0,-i\gamma} + K_{-i\gamma}(A)) \leqslant 1$. In either case Corollary 6.4.2(d) shows that $\gamma \in \Gamma_P$. Now $|\Gamma_{-p}| \leqslant 6$ by Lemma 6.1.1(c). Thus $p-1 \leqslant 6$, a contradiction. ▌

(3) $m_-(i\delta + j\gamma) \neq 0$ for all $i, j \in \mathbf{Z}_p^*$.

Proof. Suppose $|\Gamma_{-i}|>1$, where $i>0$, $i \not\equiv 0 \pmod p$. Then by Theorem 1.15.2 of [BW82] we have $\{i\delta+j\gamma \mid j\in \mathbf{Z}_p^*\}\subseteq \Gamma_{-i}$. In particular (as G_0 acts faithfully on G_{-1}), $\{\delta+j\gamma \mid j\in \mathbf{Z}_p^*\}\subseteq \Gamma_{-1}$.

We claim that if $2\leqslant i\leqslant p-2$ and $|\Gamma_{-i}|=1$ then either $|\Gamma_{-i-1}|=0$ or $|\Gamma_{-i-1}|\geqslant p-1$. For as $|\Gamma_{-i}|=1$, every composition factor of G_{-i} is a trivial G_0-module and so every composition factor of $G_{-i-1}=[G_{-1}, G_{-i}]$ (which is a quotient of $G_{-1}\otimes G_{-i}$) is isomorphic to the (irreducible) G_0-module G_{-1}. Thus if $G_{-i-1}\neq(0)$ we have $|\Gamma_{-i-1}|\geqslant p-1$.

Now suppose $G_{-(p+1)/2}\neq(0)$. As $(p+1)/2\geqslant 6$ we see by the remarks above that $\{i\delta+j\gamma \mid j\in \mathbf{Z}_p^*\}\subseteq \Gamma_{-i}$ for $i=1$, $i=2$ or 3, and $i=4$ or 5. Thus as $\Gamma_{-}\subseteq \Gamma_R$ (as A_0 is a distinguished maximal subalgebra) and as $\Gamma_R=-\Gamma_R$ (by Lemma 5.6.8(b)), we see that if $\eta\in\{\delta+j\gamma \mid j\in \mathbf{Z}_p^*\}$ then $|\mathbf{Z}\eta\cap\Gamma_R|\geqslant 6$. Since $|\Gamma_E|\leqslant 4$ (by Proposition 5.7.6 and Corollary 6.2.3), Lemma 5.7.1 shows that for some $l\in\mathbf{Z}_p^*$, $\delta+l\gamma\notin\Gamma_P$.

Next suppose that $G_{-(p+1)/2}=(0)$ and $|\Gamma_{-2}|>1$. Then by Lemma 7.3.2, every root of A is non-Hamiltonian. As $|\Gamma_{-2}|>1$ then $\{2\delta+j\gamma \mid j\in \mathbf{Z}_p^*\}\subseteq \Gamma_{-2}$ and so if $\eta\in\{\delta+j\gamma \mid j\in \mathbf{Z}_p^*\}$ we have $|\mathbf{Z}\eta\cap\Gamma_R|\geqslant 4$. Since $|\Gamma_E|\leqslant 4$ (Proposition 5.7.6 and Corollary 6.2.3) and since every η is non-Hamiltonian, we see by Lemma 5.7.1(a) that for some $l\in\mathbf{Z}_p^*$, $\delta+l\gamma\notin\Gamma_P$.

Finally, suppose that $G_{-(p+1)/2}=(0)$ and $|\Gamma_{-2}|\leqslant 1$. If $|\Gamma_{-3}|=0$ then Lemma 7.4.1 gives the result of the lemma. Hence we may assume $|\Gamma_{-3}|\geqslant 1$ and so, by our remarks above, $\{3\delta+j\gamma \mid j\in \mathbf{Z}_p^*\}\subseteq \Gamma_{-3}$. Thus if $\eta\in\{\delta+j\gamma \mid j\in \mathbf{Z}_p^*\}$ we have $|\mathbf{Z}\eta\cap\Gamma_R|\geqslant 4$. We again have (by Lemma 7.3.2) that every root of A is non-Hamiltonian and so as $|\Gamma_E|\leqslant 4$, Proposition 5.7.1(a) shows that for some $l\in\mathbf{Z}_p^*$, $\delta+l\gamma\notin\Gamma_P$.

Thus, in any case, there is some $l\in\mathbf{Z}_p^*$such that $\delta+l\gamma\notin\Gamma_P$. Hence by Corollary 6.4.2 we have $m_{-}(i(\delta+l\gamma))\neq 0$ for all $i\in\mathbf{Z}_p^*$ so $|\Gamma_{-i}|\neq 0$. Thus $\{i\delta+j\gamma \mid j\in \mathbf{Z}_p^*\}\subseteq \Gamma_{-i}$, as required. ∎

(4) If $\tau=i\delta+j\gamma$, $i, j\in\mathbf{Z}_p^*$, then $\tau\notin\Gamma_P$.

Proof. By (3), $m_{-}(k\tau)\geqslant 1$ for all $k\in\mathbf{Z}_p^*$. Thus $|\Gamma_R\cap\mathbf{Z}\tau|=p-1$, so by Lemma 5.7.1(a), $\tau\notin\Gamma_P$. ∎

(5) If $\tau=\delta+j\gamma$, $j\in\mathbf{Z}_p^*$, then $[G_{-1,\tau}, G_{1,-\tau}]=(0)$.

Proof. By (4) and Lemma 7.5.2(b), $\tau([G_{-1,\tau}, G_{1,-\tau}])=(0)$. Suppose there exist $x\in G_{-1,\tau}$, $y\in G_{1,-\tau}$ such that $[x, y]\neq 0$. Then x, y and $[x, y]$ span a three-dimensional Heisenberg algebra. Since $G_{-p}=(0)$ (by (2)), Lemma 1.14.1 of [BW82] shows that ad y is an injection of $G_{-i,\eta}$ into $G_{-i+1,\eta-\tau}$ for all i, $1\leqslant i\leqslant p-1$, and all $\eta\in\Gamma_{-i}$, $\eta\notin\mathbf{Z}\tau$. But as $G_0\cong W(1:\mathbf{1})\oplus Fz$ we have $\dim G_{0,\lambda}=1$ for all $\lambda\in\Gamma_0$, $\lambda\neq 0$. Thus we have $\dim G_{-i,\eta}\leqslant 1$ for all i, $1\leqslant i\leqslant p-1$, $\eta\in\Delta$, $\eta\notin\mathbf{Z}\tau$. Now by (3) we also have $\dim G_{-i,\eta}\geqslant 1$ for all i, $1\leqslant i\leqslant p-1$, $\eta=i\delta+k\gamma$, $k\in\mathbf{Z}_p^*$. By Corollary 5.2.5

of [BW82], $m_{-i}(i\delta+k\gamma)=m_{-i}(i\delta+l\gamma)$ for $1\leqslant i\leqslant p-1$ and k, $l\in\mathbf{Z}_p^*$. Thus $\dim G_{-i,\eta}=1$ for all $\eta=i\delta+k\gamma$, $k\in\mathbf{Z}_p^*$. But then for each l, $1\leqslant l\leqslant p-1$, $(\text{ad } y)^l$ injects the one-dimensional space $G_{-(p-1),(p-1-l)\delta+l\tau}$ into $G_{-(p-1)+l,(p-1-l)\delta}$. Hence $\dim G_{-i,i\delta}\geqslant 1$ for $1\leqslant i\leqslant p-2$. Thus $\delta\notin\Gamma_P$ and so, by Corollary 6.4.2(c), $m_-((p-1)\delta)\neq 0$. Therefore $\dim G_{-i,i\delta}\geqslant 1$ for all i, $1\leqslant i\leqslant p-1$. But then (as $i\delta\notin\mathbf{Z}\tau$) $\dim G_{-i,i\delta}=1$ for all i, $1\leqslant i\leqslant p-1$. But this means that for all $i>0$, k, $l\in\mathbf{Z}_p$, $m_{-i}(i\delta+k\gamma)=m_{-i}(i\delta+l\gamma)$. This contradicts our hypotheses of unequal multiplicities. ∎

(6) $G/M(G)$ contains a graded simple minimal ideal Q with $(G/M(G))_i=Q_i$ for $i<0$.

Proof. By Lemma 7.5.1, the grading of $G/M(G)$ is nondegenerate. Hence by Weisfeiler's theorem (cf. Theorem 1.5.1 of [BW82]), $G/M(G)$ contains a minimal ideal $J=Q\otimes B_n$, where Q is simple and graded, $J_i=Q_i\otimes B_n$ for all i, $J_1\neq(0)$, and $(G/M(G))_i=J_i$ for all $i<0$. Since $G_0\cong W(1:\mathbf{1})\oplus Fz$ (by (1)), $\dim J_0\leqslant p+1$. As $J_0\neq(0)$ (for otherwise, as $G_{-i}=[G_{-i+1},G_{-1}]$ and so $Q_{-i}=[Q_{-i+1},Q_{-1}]$ for $i>0$, we see that $\sum_{i>0}Q_i$ is a nilpotent ideal of Q, so equals (0), and thus $Q=\sum_{i<0}Q_i$ is nilpotent, a contradiction) we have $n\leqslant 1$. Furthermore, if $n=1$ then Q_0 must be one-dimensional and so J_0 is a p-dimensional abelian ideal of $W(1:\mathbf{1})\oplus Fz$, which is impossible. Thus $n=0$, proving our assertion. ∎

(7) $Q_0\cong W(1:\mathbf{1})$ and $\dim T\cap Q=1$. Thus we may assume that the element t (see (7.0.3), (7.0.4)) has been chosen so that $T\cap Q=Ft$.

Proof. $\{x\in G_{-1}\mid[x,G_1]\subseteq W(1:\mathbf{1})\}$ is a G_0-invariant subspace of G_{-1}, hence (0) or G_{-1}. By (5) this subspace is nonzero. Thus $[G_{-1},G_1]\subseteq W(1:\mathbf{1})$. Now as Q is simple, by Schue's lemma we have $Q_0=\sum_{i>0}[Q_i,Q_{-i}]$. Since $Q_{-i}=[Q_{-i+1},Q_{-1}]$ for $i>1$ (as in the proof of (6)), we have $Q_0=[Q_{-1},Q_1]\subseteq W(1:\mathbf{1})$. But Q_0 is a nonzero ideal in G_0, so $Q_0=W(1:\mathbf{1})$ and hence $\dim T\cap Q=1$. ∎

(8) $\mathfrak{z}_Q(T\cap Q)=\sum_{i\in\mathbf{Z}}Q_{i\delta}$ is a Cartan subalgebra in Q.

Proof. Since $\mathfrak{z}_Q(T\cap Q)$ is the centralizer of a torus, it is equal to its own normalizer. Since $\delta(T\cap Q)=(0)$, $\mathfrak{z}_Q(T\cap Q)=(T\cap Q)+\sum_{j\neq 0,j\in\mathbf{Z}}Q_{j,j\delta}$ is nilpotent by the Engel–Jacobson theorem. ∎

(9) $[Q_{-1,\delta},Q_{1,-\delta}]=(0)$.

Proof. As Q is simple, $\mathfrak{z}_Q(T\cap Q)$ is standard by [Wil77]. Thus $[Q_{-1,\delta},Q_{1,-\delta}]\subseteq[\mathfrak{z}_Q(T\cap Q),\ \mathfrak{z}_Q(T\cap Q)]\subseteq Q_{0,0}$ is nil and so $[Q_{-1,\delta},Q_{1,-\delta}]=(0)$. ∎

We now have a contradiction. For by (5) we have $[G_{-1,\tau},G_{1,-\tau}]=(0)$

for $\tau \neq \delta$. Since$[Q_{-1,\delta}, Q_{1,-\delta}] = (0)$ we have $[Q_{-1,\tau}, Q_{1,-\tau}] = (0)$ for all τ. Thus $T \cap [Q_{-1}, Q_1] = (0)$. But $[Q_{-1}, Q_1]$ is an ideal in $Q_0 \cong W(1:\mathbf{1})$ so $[Q_{-1}, Q_1] = (0)$, contradicting the simplicity of Q (as in the proof of (6)). This completes the proof of the lemma. ∎

7.8. We now consider the case $G_0 \cong \mathfrak{sl}(2) \oplus Fz$ in more detail. Note (see the remark after (7.0.5)) that our proof of the following lemma does not use Lemma 5.7.1 of [BW82] and so provides a simplified and corrected proof of that result.

LEMMA 7.8.1. *If $G_0 \cong \mathfrak{sl}(2) \oplus Fz$ and $G_1 \neq (0)$, then $N(G) = (0)$ and G_0 acts faithfully on G_1.*

Proof. We first show that G_0 acts faithfully on G_1. For if not, since z acts as the identity on G_1, G_1 is a trivial $\mathfrak{sl}(2)$-module. Hence $[G_{-1}, G_1]$ is a quotient of a direct sum of copies of the $\mathfrak{sl}(2)$-module G_{-1}. Since G_0 acts faithfully and irreducibly on G_{-1}, this forces $[G_{-1}, G_1] \subseteq \mathfrak{sl}(2)$ and so $[[G_{-1}, G_1], G_1] = (0)$. Thus $[G_{-1}, G_1] + \sum_{i<0} G_i$ is an ideal in G' and so the grading of $G'/N(G)$ is degenerate. This contradicts Lemma 7.5.4.

We next show that $N(G) = (0)$. We will assume $N(G) \neq (0)$ and derive a contradiction. We let γ be as in (7.0.3). Without loss of generality (by replacing t by a suitable element of $\mathbf{Z}_p^* t$) we may assume $\Gamma_0 = \{0, \pm 2\gamma\}$.

Now Kac's recognition theorem for graded Lie algebras (Theorem 1.2.1) applies to $G'/N(G)$ (where G' denotes the subalgebra of G generated by $\sum_{i \leqslant 1} G_i$). Since G' (and hence $G'/N(G)$ is restricted (by Lemma 1.19.1 of [BW82]), we see that $G'/N(G)$ must be isomorphic to Q, where either $Q = W(2:\mathbf{1})$, $H(2:\mathbf{1})^{(2)} + F(x_1 D_1 + x_2 D_2) \subseteq Q \subseteq \operatorname{Der} H(2:\mathbf{1})$;$Q = K(3:\mathbf{1})$; or Q is classical simple. Since $N(G) \subseteq \sum_{i<0} G_i$ is nilpotent, each each quotient $N(G)^i/N(G)^{i+1}$ is a nontrivial $G'/N(G)$-module. By Lemma 6.3.4, each of these has $\leqslant p^2 - p + 6$ weights. This is impossible if $Q = W(2:\mathbf{1})$ (by [BW82, Corollary 4.11.2]) or if $Q = K(3:\mathbf{1})$ (by Lemma 6.5.1). If $H(2:\mathbf{1})^{(2)} + F(x_1 D_1 + x_2 D_2) \subseteq Q \subseteq \operatorname{Der} H(2:\mathbf{1})$ then by [BW82, Lemma 4.11.1] we see that $H(2:\mathbf{1})^{(2)}$ annihilates each composition factor of each $N(G)^i/N(G)^{i+1}$. Thus $N(G)_j = N(G)_{j,-j\delta}$ for all $j \leqslant -2$. Since $Q_{-2} = (0)$ and $G_{-1,\delta} = (0)$ we have $G_{-2} = N(G)_{-2}$ and $G_{-3} = (0)$. But then Lemma 7.3.1 gives $[G_{-1}, G_1] = G_0$, contradicting the choice of Q. Thus Q must be classical.

Since Q is classical, Kac's theorem gives that the grading of $Q = G'/N(G)$ is standard. Thus we have that $Q_{-4} = (0)$ and that the possibilities for the dimensions of the Q_{-i}, $1 \leqslant i \leqslant 3$, are given by Table 7.8.1.

First assume that Q and its grading are given by one of the last three lines of Table 7.8.1. Then we have $Q_{-2} \neq (0)$, $G_{-1,\delta} = (0)$, and (as Q is classical) $\delta([G_{-2,2\delta}, G_{2,-2\delta}]) \neq (0)$. Then Lemma 7.5.2(b) shows that δ is

TABLE 7.8.1

Q	$\dim Q_{-1}$	$\dim Q_{-2}$	$\dim Q_{-3}$
A_2	2	0	0
C_2	3	0	0
C_2	2	1	0
G_2	4	1	0
G_2	2	1	2

proper and so by Lemma 5.7.1(a), $|\{\pm 2\} \cup \{i \mid N(G)_{i\delta} \neq (0)\}| \leqslant 6$. Since $N(G)_0 = (0)$ this implies $N(G)_{i\delta} = (0)$ unless $i = \pm 1, \pm 3$. Thus $N(G)_{-4, 4\delta} = N(G)_{-5, 5\delta} = (0)$. Since $G_{-1, \delta} = (0)$, Proposition 1.15.3 of [BW82] shows that $N(G)_{-5} = (0)$. As each $N(G)^i/N(G)^{i+1}$ is a Q-module, the set of weights of $N(G)$ is invariant under the Weyl group of Q. Since Q is of type C_2 or G_2, this implies the set of weights of $N(G)$ is symmetric about 0. Since $N(G) = N(G)_{-2} + N(G)_{-3} + N(G)_{-4}$, this implies $N(G) = (0)$, contradicting our assumption.

Similarly, if $Q \cong C_2$ and $\dim Q_{-1} = 3$ then $\delta([G_{-1, \delta}, G_{1, -\delta}]) \neq (0)$, δ is proper by Lemma 7.5.2(b), and so $|\{\pm 1\} \cup \{i \mid N(G)_{i\delta} \neq (0)\} \leqslant 6$. Since $N(G)_0 = (0)$ this implies $N(G)_{i\delta} = (0)$ unless $i = \pm(p+1)/2$, $\pm(p+3)/2$. Then $N(G)_{-2, 2\delta} = (0)$. Since $N(G)_{-2}$ is a homomorphic image of $G_{-1} \otimes G_{-1}$, Lemma 1.15.3 of [BW82] shows $N(G) = (0)$.

Thus we may assume that $G'/N(G) \cong \mathfrak{sl}(3)$. Assume $\Gamma_0 = \{0, \pm 2\gamma\}$.

We claim that $\gamma \in \Gamma_P$. For we have $\gamma([A_{0, 2\gamma}, A_{0, -2\gamma}]) \neq (0)$. If γ is non-Hamiltonian this implies $A_{2\gamma} = A_{0, 2\gamma} + K_{2\gamma}(A)$ and so Corollary 6.4.2(c) shows that $\gamma \in \Gamma_P$. If γ is Hamiltonian then, as $A_{0, \pm 2\gamma}$ is ad I-invariant, we see from Lemmas 2.1.11 and 5.2.1(e) that either $\dim A_{2\gamma}/(A_{0, 2\gamma} + K_{2\gamma}(A)) \leqslant 1$ or $\dim A_{-2\gamma}/(A_{0, -2\gamma} + K_{-2\gamma}(A)) \leqslant 1$. In either case Corollary 6.4.2(d) shows that $\gamma \in \Gamma_P$.

Therefore, $\Gamma_{-lp} \subseteq \{\pm\gamma, \pm 3\gamma\}$ for $l = 1, 2, \ldots$. For since $\gamma \in \Gamma_P$ we see from Lemma 5.7.1(a) that $|\Gamma_R \cap \mathbf{Z}\gamma| \leqslant 6$. Since $\Gamma_{-lp} \subseteq \Gamma_R$ and $\pm 2\gamma \in \Gamma_R$ we have $|\Gamma_{-lp} \cup \{\pm 2\gamma\}| \leqslant 6$. Since $0 \notin \Gamma_{-lp}$ and $p > 7$ we see (cf. Theorem 1.15.1 of [BW82]) that $\Gamma_{-lp} \subseteq \{\pm\gamma, \pm 3\gamma\}$. Thus $\Gamma_{-lp} \neq \varnothing$ implies $\gamma \in \Gamma_{-lp}$. Since $m_-(\gamma) \leqslant 5$, by Lemma 6.3.3 we see that $\Gamma_{-6p} = \varnothing$.

Since $\Gamma_{-1} = \{\delta \pm \gamma\}$ we see that $\Gamma_{-2} \subseteq \{2\delta\}$ and $\Gamma_{-lp-2} \subseteq \{2\delta \pm 5\gamma, 2\delta \pm 3\gamma, 2\delta \pm \gamma\}$ for $l = 2, 3, \ldots$. Thus $\Gamma_- \cap (2\delta + \mathbf{Z}\gamma) \subseteq \{2\delta \pm 5\gamma, 2\delta \pm 3\gamma, 2\delta \pm \gamma, 2\delta\}$. Since $p > 7$ this implies $2\delta + 4\gamma \notin \Gamma_-$.

Now as $G_{-1, \delta+\gamma} + G_{1, -\delta-\gamma} + [G_{-1, \delta+\gamma}, G_{1, -\delta-\gamma}]$ and $G_{0, 2\gamma} + G_{0, -2\gamma} + [G_{0, 2\gamma}, G_{0, -2\gamma}]$ are copies of $\mathfrak{sl}(2)$ and as $G_{-2} = N(G)_{-2} = G_{-2, 2\delta}$, we see that

$$(\operatorname{ad} G_{0, -2\gamma})^i (\operatorname{ad} G_{-1, \delta+\gamma})^j G_{-2, 2\delta} \neq (0) \qquad (7.8.1)$$

for all i, j, $0 \leqslant j \leqslant p-3$, $0 \leqslant i \leqslant j$. Thus, in particular, $-2\delta-4\gamma \in \Gamma_{-p+2}$ (taking $j=p-4$, $i=0$). Now $A_{-p+2,\,-2\delta-4\gamma} \supsetneqq A_{0,\,-2\delta-4\gamma} \supseteq R(A)_{-2\delta-4\gamma}$ so $[A_{-p+2,\,-2\delta-4\gamma}, A_{2\delta+4\gamma}] \not\subseteq I$. But $2\delta+4\gamma \notin \Gamma_-$ so $[A_{-p+2,\,-2\delta-4\gamma}, A_{0,\,2\delta+4\gamma}] = [A_{-p+2,\,-2\delta-4\gamma}, A_{p-2,\,2\delta+4\gamma}] \not\subseteq I$. Hence $[G_{-p+2,\,-2\delta-4\gamma}, G_{p-2,2\delta+4\gamma}] \neq (0)$. Thus $C = G_{-p+2,\,-2\delta-4\gamma} + G_{p-2,2\delta+4\gamma} + [G_{-p+2,\,-2\delta-4\gamma}, G_{p-2,\,2\delta+4\gamma}]$ is a three-dimensional subalgebra of G which is either a Heisenberg algebra or a copy of $\mathfrak{sl}(2)$.

Suppose C is a Heisenberg algebra. Then noting that $\Gamma_{-1} = \{\delta \pm \gamma\}$ we see that $(\operatorname{ad} G_{p-2,\,2\delta+4\gamma})\; G_{-p+1,\,-\delta-7\gamma} = (0)$. Since $-\delta-7\gamma \in \Gamma_{-p+1}$ (taking $j=p-3$, $i=2$ in (7.8.1)) we see (by Lemma 1.14.1 of [BW82]) that $(\operatorname{ad} G_{-p+2,\,-2\delta-4\gamma})^{p-1} G_{-p+1,\,\delta-7\gamma} \neq (0)$, contradicting $\Gamma_{-6p} = \varnothing$.

Suppose C is a copy of $\mathfrak{sl}(2)$ with $u \in G_{p-2,\,2\delta+4\gamma}$, $v \in G_{-p+2,\,-2\delta-4\gamma}$, $(2\delta+4\gamma)[u,\ v]) = 2$. Suppose $\gamma([u,v]) \neq 0$. Then (as $|\Gamma_{-p+1}| = p-2$ by (7.8.1) and since $|\Gamma_{-1}| = 2$) we see that if $i \notin \{-5, -3, -1, 1\}$ then $(\operatorname{ad} G_{p-2,\,2\delta+4\gamma})(G_{-p+1,\,-\delta+i\gamma}) = (0)$ and $-\delta + i\gamma \in \Gamma_{-p+1}$. Since $\gamma([u,v]) \neq 0$ we can find $\nu \in \Gamma_{-p+1}$ such that $(\operatorname{ad} G_{p-2,\,2\delta+4\gamma})(G_{-p+1,\,\nu}) = (0)$ and $p-2 \leqslant \nu([u,v]) \leqslant p-1$. Thus $G_{-p^2+3p-3} \neq (0)$. This contradicts $\Gamma_{-6p} = \varnothing$.

Finally, suppose C is a copy of $\mathfrak{sl}(2)$ (with u, v as above) and $\gamma([u,v]) = 0$. Then $\delta([u,v]) = 1$ and so $\nu([u,v]) = p-1$ for any $\nu \in \Gamma_{-p+1}$. Since $\Gamma_{-p-1} \subseteq \Gamma_{-p} + \Gamma_{-1} \subseteq \{\delta, \delta \pm 2\gamma,\ \delta \pm 4\gamma\}$ and $\Gamma_{-3} \subseteq \Gamma_{-2} + \Gamma_{-1} \subseteq \{3\delta \pm \gamma\}$, we see that $(\operatorname{ad}\ G_{p-2,\,2\delta+4\gamma})\ G_{-p-1} = (0)$. Thus $G_{-p-1} \neq (0)$ implies $(\operatorname{ad} G_{-p+2,\,-2\delta-4\gamma})^{p-1} G_{-p-1} \neq (0)$, contradicting $\Gamma_{-6p} = \varnothing$. Therefore $G_{-p-1} = (0)$. Thus $\nu \in \Gamma_{-p+1}$, $[G_{p-2,\,2\delta+4\gamma}, G_{-p+1,\,\nu}] = (0)$ implies $\nu([u,v]) = 0$. Since we have noted that there are $p-4$ such ν, this contradicts $\delta([u,v]) \neq (0)$.

Thus $G'/N(G) \cong \mathfrak{sl}(3)$ is impossible, so $N(G) = (0)$ and the lemma is proved. ∎

8. Determination of G_0 if $\mathfrak{z}(G_0) = (0)$

Throughout this section we assume that A is a finite-dimensional restricted Lie algebra over F which satisfies the conditions of Theorem 4.1.1(h), that is, we assume that (7.0.1) holds.

As in Section 7, we let A_0 be a distinguished maximal subalgebra of A associated with T and let $A = A_{-k} \supseteq \cdots \supseteq A_{-1} \supseteq A_0 \supseteq A_1 \supseteq \cdots$ be a filtration of A constructed as in Section 1.2. Let $G = \sum_{i \geqslant -k} G_i$ be the associated graded algebra. We continue to use the notations Γ_i, $G_{i,\gamma}$, and $I(G_0)$ as in Section 7.

We will also assume throughout this section that

$$\mathfrak{z}(G_0) = (0). \tag{8.0.1}$$

8.1. Section 8 is devoted to the proof of the following result.

PROPOSITION 8.1.1. *If A is as above, then one of the following occurs*:

(a) $\bar{Y} \subseteq G_0 \subseteq \mathrm{Der}\, Y$ *for some simple Y and for every two-dimensional torus $T_1 \subseteq G_0$ we have $T_1 \subseteq \bar{Y}$ and $G_0 = Y + I_1$ (where I_1 is the nil radical of $\mathfrak{z}_{G_0}(T_1)$)*;

(b) *G_0 is one of* $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, $\mathfrak{sl}(2) \oplus W(1:\mathbf{1})$, *or* $W(1:\mathbf{1}) \oplus W(1:\mathbf{1})$.

Furthermore, if $G_1 \neq (0)$ and (b) *holds then $G_0 \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, $N(G) = (0)$, and G_0 acts faithfully on G_1.*

Sections 8.2–8.11 are devoted to the proof of this proposition.

8.2. We begin with an easy observation.

LEMMA 8.2.1. *G_0 is semisimple.*

Proof. Let J be a maximal abelian ideal of G_0. Since $\bar{J}$ is again an abelian ideal, J is restricted. Let $x \in J$ have Jordan–Chevalley–Seligman decomposition $x = x_s + x_n$ into its semisimple and nilpotent parts. Since J is restricted, $x_s \in J$ so $(\mathrm{ad}\, x_s)^2 = 0$ (as J is abelian). Since x_s is semisimple, $\mathrm{ad}\, x_s = 0$ so $x_s \in \mathfrak{z}(G_0) = (0)$. Thus $x = x_n$ is nil. Then by Engel's theorem J is a nil ideal of G_0. Since G_0 acts faithfully and irreducibly on G_{-1} we have $J = (0)$. ∎

It follows that G_0 satisfies the hypotheses of Theorem 4.1.1.

Since A_0 is a distinguished maximal subalgebra, $|\Gamma_{-1}| \leqslant p^2 - p + 6$ by Lemma 6.3.2. Furthermore, by Lemma 6.3.3, $\dim G_{-1,\gamma} \leqslant 7$ for every $\gamma \in \Gamma_{-1}$. We will show that most of the algebras listed in the conclusion of Theorem 4.1.1 do not have any faithful irreducible restricted representation with at most $p^2 - p + 6$ weights each of multiplicity $\leqslant 7$ and hence cannot occur as G_0. In the following sections Q is always assumed to be restricted and semisimple, T' is a two-dimensional standard maximal torus in Q, and I' is the nil radical of $\mathfrak{z}_Q(T')$.

8.3. Let $Y_1 + Y_2 \subseteq Q \subseteq (\mathrm{Der}\, Y_1)^{(1)} + (\mathrm{Der}\, Y_2)^{(1)}$, where Y_1, Y_2 are among $\mathfrak{sl}(2)$, $W(1:\mathbf{1})$, $H(2:\mathbf{1})^{(2)}$ (that is, let Q be one of the algebras listed in Theorem 4.1.1(a)). Assume at least one of the Y_i is $H(2:\mathbf{1})^{(2)}$. Then any faithful irreducible restricted Q-module has some weight space of dimension >7, so Q cannot occur as G_0.

Proof. As Y_1, Y_2 are non-nil restricted ideals of Q, we have $T' = (T' \cap Y_1) \oplus (T' \cap Y_2)$. Then we can find $0 \neq t_i \in T' \cap Y_i$, $i = 1, 2$, satisfying $t_i^p = t_i$. Define $\alpha_1, \alpha_2 \in T'^*$ by $\alpha_i(t_j) = \delta_{ij}$. Then if W is a faithful irreducible restricted Q-module we have $W=\sum_{a,b\in\mathbf{Z}} W_{a\alpha_1+b\alpha_2}$. Suppose (without loss of generality) that Y_1 is $H(2:\mathbf{1})^{(2)}$. Then for each b, $0\leqslant b\leqslant p-1$, $W(b)=\sum_{a\in\mathbf{Z}} W_{a\alpha_1+b\alpha_2}$ is a Y_1-submodule of W. Since W is a faithful Q-module, some $W(b)$ is a nontrivial Y_1-module and so (as Y_1 is simple) has a faithful irreducible constituent V. But by Lemma 7.2.7 (where we take $M=Y_1\oplus Fz$ and let z act on V as the identity) dim $V_\gamma>7$ for some γ and hence dim $W_\gamma>7$. ∎

8.4. Let $H(2:\mathbf{1})^{(2)}+Fx_1D_1\subseteq Q\subseteq \text{Der}\, H(2:\mathbf{1})^{(2)}$. (That is, assume Q is one of the algebras of Theorem 4.1.1(c).) Then by Lemma 4.11.1 of [BW82] every faithful irreducible Q-module has at least p^2-2 weights and hence Q cannot occur as G_0. ∎

8.5. Assume α, $\beta\in T'^*$, with α and β $\mathbf{Z}_p$-linearly independent, $x\in Q_\alpha$, $y\in Q_\beta$, and $\mathfrak{z}_Q(T')=Fx^p+Fy^p+I$. Then any faithful irreducible restricted Q-module has at least p^2-1 weights. Therefore Q cannot occur as G_0.

Proof. Let V be a faithful irreducible restricted Q-module. Then some $V_\gamma\neq(0)$ with $\gamma(x^p)\neq 0$. Then x^p is a bijection on V_γ and so $V_{\gamma+i\alpha}\supseteq x^iV_\gamma\neq(0)$ for $0\leqslant i\leqslant p-1$. Now $\alpha(x^p)=0$ (by Lemma 1.8.1) so $\alpha(y^p)\neq 0$. Thus there is at most one $i_0\in\mathbf{Z}_p$ such that $(\gamma+i_0\alpha)(y^p)=0$ (and thus $\gamma+i_0\alpha\in\mathbf{Z}_\beta$). Hence y^p is a bijection of $V_{\gamma+i\alpha}$ for $i\neq i_0$. Thus $V_{\gamma+i\alpha+j\beta}\supseteq y^jV_{\gamma+i\alpha}\neq(0)$ for all $j\in\mathbf{Z}_p$, $i\neq i_0$, i.e., $V_\tau\neq(0)$ if $\tau\notin\mathbf{Z}\beta$. Interchanging the roles of α and β gives the result. ∎

8.6. Suppose that for some simple algebra Y we have $Y\otimes B_n\subseteq Q\subseteq \text{Der}(Y\otimes B_n)$, where $n>0$ and $T'\subseteq(\overline{Y\otimes B_n})$. Assume Q contains no tori of dimension greater than two and that all two-dimensional tori in Q are standard. (That is, assume that Q is one of the algebras of Theorem 4.1.1(d).) Then by Lemma 3.1.2, $\mathfrak{z}_{Y\otimes B_n}(T')$ is nil and so (as $T'\subseteq\overline{Y\otimes B_n}$) there exist nonzero α, $\beta\in T'^*$, $x\in(Y\otimes B_n)_\alpha\subseteq Q_\alpha$, $y\in(Y\otimes B_n)_\beta\subseteq Q_\beta$ so that $\mathfrak{z}_Q(T')=Fx^p+Fy^p+I$. Since $\alpha(x^p)=\beta(y^p)=0$ (by Lemma 1.8.1), α and β are $\mathbf{Z}_p$-linearly independent. Then by Section 8.5, any faithful irreducible restricted Q-module has at least p^2-1 weights and so Q cannot occur as G_0.

8.7. Suppose that for some simple algebra Y we have $\bar{Y}\subseteq Q\subseteq$ Der Y, $T'\subseteq\bar{Y}$, and $\dim Q/(Y+I')=2$. (That is, assume Q is as in Theorem 4.1.1(e).) Then as $Q=Y+T'+I'$ by (4.5.3), we have $\dim T'/(T'\cap(Y+I'))=2$ and so $T'\cap(Y+I')=(0)$. Then $\mathfrak{z}_Y(T')$ is nil and so, as $T'\subseteq\bar{Y}$, we have that there exist nonzero α, $\beta\in T'^*$, $x\in Y_\alpha\subseteq Q_\alpha$, $y\in Y_\beta\subseteq Q_\beta$ so that $Fx^p+Fy^p+I'=\mathfrak{z}_Q(T')$. Since $\alpha(x^p)=\beta(y^p)=0$, α and

β are $\mathbf{Z}_p$-linearly independent. Then by Section 8.5, any faithful irreducible restricted Q-module has at least p^2-1 weights and so Q cannot occur as G_0.

8.8. Suppose that for some simple algebra Y we have $\bar{Y} \subseteq Q \subseteq \operatorname{Der} Y$, $T' \subseteq \bar{Y}$, and $\dim T' \cap (Y+I')=1$. Suppose further that $T' \cap (Y+I')$ is restricted, that Q contains no tori of dimension greater than two, and that all two-dimensional tori in Q are standard. (That is, assume that Q is as in Theorem 4.1.1(f).) If V is a faithful irreducible restricted Q-module in which each weight space has dimension $\leqslant 7$, then V has at least p^2-1 weights, and hence Q cannot occur as G_0.

Proof. We have $\dim T'/(T' \cap (Y+I'))=1$. Note that $\mathfrak{z}_Y(T')=Y \cap (T'+I') \subseteq (T' \cap (Y+I'))+I'$. Since $\overline{(T' \cap (Y+I'))}$ is one-dimensional and restricted we see that $T' \not\subseteq \overline{\mathfrak{z}_Y(T')}$. But $T' \subseteq \bar{Y}$ by hypothesis so there is some $0 \neq \alpha \in T'^*$, $x \in Q_\alpha$ such that $T' \subseteq Y+I'+Fx^p$. By Lemma 1.8.1, $\alpha(x^p)=0$ and so if $\beta \in T'^*$, $\beta \notin F\alpha$, then $\beta(x^p) \neq 0$. If $\gamma \in T'^*$, $\gamma \notin \mathbf{Z}\alpha$, and $y \in Q_\gamma$ satisfies $y^p \notin I'$ then $y^p \notin Fx^p+I'$, and our result follows by Section 8.5. Thus we may assume that $y^p \in I'$ whenever $y \in Q_\gamma$, $\gamma \notin \mathbf{Z}\alpha$. Thus, setting $N^{(\gamma)}=Y^{(\gamma)}+I'$, $N^{(\gamma)}$ is restricted.

By Schue's lemma (Lemma 1.12.1) there is some $\gamma \notin \mathbf{Z}\alpha$ such that $[Y_\gamma, Y_{-\gamma}]$ is not nil, and hence $T' \cap (Y+I') \subseteq [Y_\gamma, Y_{-\gamma}]+I'$. If $\gamma([Y_\gamma, Y_{-\gamma}])=(0)$ then $\gamma(T' \cap (Y+I'))=(0)$ and so by Section 4.6 (with γ in place of β), we have that $Y^{(\gamma)}$ is a Cartan subalgebra of Y. But then $[Y^{(\gamma)}, Y^{(\gamma)}] \supseteq [Y_\gamma, Y_{-\gamma}]$ is not nil, contradicting the fact [Wil77] that $Y^{(\gamma)}$, being a Cartan subalgebra of a simple Lie algebra, must be standard. Thus $\gamma([Y_\gamma, Y_{-\gamma}]) \neq (0)$ and so, by Lemma 1.8.3, $Y^{(\gamma)}$ (and hence $N^{(\gamma)}$) is not solvable. Now $\operatorname{solv}(N^{(\gamma)})_0$ is nil (for otherwise $\gamma(\operatorname{solv}(N^{(\gamma)})_0) \neq (0)$ and so $N^{(\gamma)}=\operatorname{solv}(N^{(\gamma)})$). Also, as noted above, we may assume that $y \in (Y^{(\gamma)})_{i\gamma} \subseteq Y_{i\gamma}$ for $i \in \mathbf{Z}_p^*$ implies y^p is nil. Thus by the Engel–Jacobson theorem, $\operatorname{solv}(N^{(\gamma)})$ is a nil ideal in the restricted algebra $N^{(\gamma)}$.

Now let V be a faithful irreducible restricted Q-module in which each weight space has dimension $\leqslant 7$. Then if $\beta \notin \mathbf{Z}\alpha$ we have (as $\beta(x^p) \neq 0$) that x^p acts bijectively on V_β and so $\dim V_{\beta+i\alpha}$ is independent of i. Let $W_i=\sum_{j=0}^{p-1} V_{\beta+i\alpha+j\gamma}$. Then W_i is an $N^{(\gamma)}$-module and as $\operatorname{solv}(N^{(\gamma)})$ is nil, each $N^{(\gamma)}$ composition factor of W_i is an $N^{(\gamma)}/\operatorname{solv}(N^{(\gamma)})$-module. The hypotheses of Theorem 3.1.1 are satisfied by $N^{(\gamma)}/(\operatorname{solv} N^{(\gamma)})$ by Lemma 1.6.1 (applied to the mapping $N^{(\gamma)}+T' \to (N^{(\gamma)}+T')/N^{(\gamma)}$) and the fact that $\operatorname{solv}(N^{(\gamma)})$ is nil. Therefore $N^{(\gamma)}/\operatorname{solv}(N^{(\gamma)}) \cong \mathfrak{sl}(2)$, $W(1:\mathbf{1})$, or $H(2:\mathbf{1})^{(2)} \subseteq N^{(\gamma)}/\operatorname{solv}(N^{(\gamma)}) \subseteq H(2:\mathbf{1})$. There exists i such that $(T' \cap (Y+I'))\, W_i \neq (0)$; for this i, some composition factor is $(N^{(\gamma)}/(\operatorname{solv} N^{(\gamma)}))^{(2)}$-nontrivial, hence faithful. By Lemma 7.2.7 (with $M=N^{(\gamma)}/(\operatorname{solv} N^{(\gamma)})+Fz$, where z acts as the identity), if $H(2:\mathbf{1})^{(2)} \subseteq N^{(\gamma)}/\operatorname{solv}(N^{(\gamma)})$ then $\dim W_{i,\tau}>7$ for some τ, so

$\dim V_\tau > 7$, a contradiction. Thus $N^{(\gamma)}/\mathrm{solv}(N^{(\gamma)}) \cong \mathfrak{sl}(2)$ or $W(1:\mathbf{1})$. In either case we have that if $T' \cap (Y+I') = Ft$, $t^p = t$ then the set $\{\sigma(t) \mid W_{i,\sigma} \neq (0)\}$ is symmetric about 0 (cf. [BW82, Theorems 1.15.1, 1.15.2]). As V is faithful there exists some $\beta \notin \mathbf{Z}\alpha$ with $V_\beta \neq (0)$. Then for every $i \in \mathbf{Z}_p$ we have $V_{\beta+i\alpha} \neq (0)$ and so $W_{i,\beta+i\alpha} \neq (0)$. Then since $\{\sigma(t) \mid W_{i,\sigma} \neq (0)\}$ is symmetric about 0 we see that if $(\beta + i\alpha + j\gamma)(t) = -(\beta + i\alpha)(t)$ then $W_{i,\beta+i\alpha+j\gamma} \neq (0)$. Note that $\alpha(t) \neq (0)$. Thus if $j \in \mathbf{Z}_p$ and $i = -(2\beta + j\gamma)(t)/2\alpha(t)$ we see that $W_{i,\beta+i\alpha+j\gamma} \neq (0)$. Since $\dim V_{\beta+i\alpha+j\gamma}$ is independent of i whenever $\beta + j\gamma \notin \mathbf{Z}\alpha$ we have that $V_\tau \neq (0)$ whenever $\tau \notin \mathbf{Z}\alpha$. But $\alpha(t) \neq 0$ and $\{\sigma(t) \mid W_{i,\sigma} \neq (0)\}$ is symmetric about 0 and has at least $p-1$ elements, the pth weight σ being in $\mathbf{Z}\alpha$, so $V_\tau \neq (0)$ whenever $\tau \neq 0$, as required. ∎

8.9. Suppose that for some simple algebra Y we have $\bar{Y} \subseteq Q \subseteq \mathrm{Der}\, Y$, $T' \subseteq \bar{Y}$ and that $T' \cap (Y+I')$ is one-dimensional and nonrestricted. Suppose further that Q contains no tori of dimension greater than two and every two-dimensional torus of Q is standard. (That is, let Q be one of the algebras of Theorem 4.1.1(g).) If V is a faithful irreducible restricted Q-module in which each weight space has dimension $\leqslant 7$, then V has at least p^2-1 weights and so Q cannot occur as G_0.

Proof. By Schue's lemma there exists two roots α, β, $\alpha \notin \mathbf{Z}\beta$, such that $[Y_\alpha, Y_{-\alpha}]$ and $[Y_\beta, Y_{-\beta}]$ are not nil. We claim that if $j \neq 0$ then $\dim V_{i\alpha+j\beta}$ is independent of i (and, similarly, if $i \neq 0$ then $\dim V_{i\alpha+j\beta}$ is independent of j). If $x \in Y_{u\alpha}$, $u \in \mathbf{Z}_p^*$, and x^p is not nil then $\alpha(x^p) = 0$ so $\beta(x^p) \neq 0$ and hence x^p acts bijectively on $V_{j\beta}$, giving the result. Hence we may assume that each such x is nil.

Since $T' \cap (Y+I')$ is one-dimensional and is not restricted, any restricted subalgebra of $\mathfrak{z}_Y(T')$ is contained in I'. In particular, for any root γ we have $\ker \gamma \subseteq I'$ and so since $[Y_\alpha, Y_{-\alpha}]$ is not nil we have $\alpha([Y_\alpha, Y_{-\alpha}]) \neq (0)$. Then by Lemma 1.8.3, $Q^{(\alpha)}$ is not solvable. Let $J = I' \cap (\mathrm{solv}\, Q^{(\alpha)}) + \sum_{i=1}^{p-1} (\mathrm{solv}\, Q^{(\alpha)})_{i\alpha}$. Lemma 1.8.3 shows that $\alpha([Y_{i\alpha}, (\mathrm{solv}\, Q^{(\alpha)})_{-i\alpha}]) = (0)$ and hence $[Y_{i\alpha}, (\mathrm{solv}\, Q^{(\alpha)})_{-i\alpha}] \subseteq I'$. Thus J is an ideal in $Q^{(\alpha)}$ and is nil by the Engel–Jacobson theorem. Since α vanishes on $\mathfrak{z}_{\mathrm{solv}\, Q^{(\alpha)}}(T')$, we have $\mathrm{solv}\, Q^{(\alpha)} = Fz + J$, where $0 \neq z = z^p$ and $\alpha(z) = 0$.

By Theorem 3.1.1, either $Q^{(\alpha)}/\mathrm{solv}\, Q^{(\alpha)} \cong \mathfrak{sl}(2)$, $Q^{(\alpha)}/\mathrm{solv}\, Q^{(\alpha)} \cong W(1:\mathbf{1})$, or $H(2:\mathbf{1})^{(2)} \subseteq Q^{(\alpha)}/\mathrm{solv}\, Q^{(\alpha)} \subseteq H(2:\mathbf{1})$. Suppose $H(2:\mathbf{1})^{(2)} \subseteq Q^{(\alpha)}/\mathrm{solv}\, Q^{(\alpha)} \subseteq H(2:\mathbf{1})$. We may apply Lemma 7.2.7 to $Q^{(\alpha)}/J$ acting on some irreducible $Q^{(\alpha)}$-constituent of $\sum_{i=0}^{p-1} V_{i\alpha+j\beta}$ and conclude that $\dim V_\gamma > 7$ for some γ, a contradiction. Thus $Q^{(\alpha)}/\mathrm{solv}\, Q^{(\alpha)} \cong \mathfrak{sl}(2)$ or $W(1:\mathbf{1})$ and so $I' \subseteq \mathrm{solv}\, Q^{(\alpha)}$ (and hence $I' \subseteq J$). Therefore Corollary 5.2.5 of [BW82] applies to $Q^{(\alpha)}/J$ acting on any irreducible $Q^{(\alpha)}$-constituent of $\sum_{i=0}^{p-1} V_{i\alpha+j\beta}$. Since $T' \cap (Y+I')$ is not restricted, $[Q^{(\alpha)}/J, Q^{(\alpha)}/J]$ cannot be restricted (for $((T'+J)/J) \cap [Q^{(\alpha)}/J, Q^{(\alpha)}/J] = ((T' \cap (Y+I')) + J)/J$). Therefore dim $V_{i\alpha+j\beta}$ is independent of i. Lemma 4.6.3 of [BW82] now gives the result. ▮

8.10. We will now investigate whether an algebra Q listed in Theorem 4.1.1(b) can occur as G_0. Thus we assume

$Y \otimes B_n \subseteq Q \subseteq \mathrm{Der}(Y \otimes B_n)$, where $Y \cong \mathfrak{sl}(2)$, $W(1:\mathbf{1})$, or $H(2:\mathbf{1})^{(2)}$.

All two-dimensional tori in Q are maximal and standard and T' is one such.

$$T' \nsubseteq \overline{Y \otimes B_n}\ (= Y \otimes B_n),\ n > 0. \tag{8.10.1}$$

Then as $Y \otimes B_n$ (containing Y) is not nil we have (by the Engel–Jacobson theorem) that $T' \cap (Y \otimes B_n) \neq (0)$. We can then find t_1, t_2 satisfying $T' = Ft_1 + Ft_2$, $t_i^p = t_i$ for $i = 1, 2$, $t_1 \in Y \otimes B_n$, $t_2 \notin Y \otimes B_n$. Define $\alpha_i \in T'^*$ for $i = 1, 2$ by $\alpha_i(t_j) = \delta_{ij}$. Then as $t_1 \in Y \otimes B_n$, ad t_1 annihilates $Q/Y \otimes B_n$ and so $Q = (Y \otimes B_n) + Q^{(\alpha_2)}$.

Recall that there is a mapping

$$(Y \otimes B_n) \otimes B_n \to Y \otimes B_n$$
$$v \otimes b \mapsto v \cdot b$$

such that

$$(y \otimes b_1) \otimes b \mapsto y \otimes (b_1 b)$$

for all $y \in Y$, b_1, $b \in B_n$. If $\phi: Q \to \mathrm{Der}\, B_n \cong (\mathrm{Der}(Y \otimes B_n))/((\mathrm{Der}\, Y) \otimes B_n)$ is the canonical homomorphism we have $[g, u \cdot b] = [g, u] \cdot b + u \cdot (\phi(g)\, b)$ for all $g \in Q$, $u \in Y \otimes B_n$, $b \otimes B_n$.

Let $\eta: Y \otimes B_n \to (Y \otimes B_n)/(Y \otimes (x_1 B_n + \cdots + x_n B_n)) \cong Y$ denote the quotient map.

LEMMA 8.10.1. *Let Q and T' satisfy* (8.10.1), $Y \cong \mathfrak{sl}(2)$, *or* $W(1:\mathbf{1})$ *and* $n = 1$. *Then*:

(a) $\mathfrak{z}_Q(T') = T'$.

(b) $(Y \otimes B_1)^{(\alpha_2)} \subseteq (Y \otimes x_1 B_1) + Ft_1$.

(c) *If* $Y \otimes x_1 B_1$ *is invariant under* ad t_2 *then* $\eta((Y \otimes B_1)^{(\alpha_1)}) = Y$ *and* $(Y \otimes B_1)^{(\gamma)} \subseteq (Y \otimes x_1 B_1) + Ft_1$ *for all* $\gamma \notin \mathbf{Z}\alpha_1$.

(d) *If* $Y \otimes x_1 B_1$ *is not invariant under* ad t_2 *then* $\eta((Y \otimes B_1)^{(\gamma)}) = Y$ *for all* $\gamma \notin \mathbf{Z}\alpha_2$.

Proof. Clearly $Q/Y \otimes B_1 \subseteq \mathrm{Der}(Y \otimes B_1)/(Y \otimes B_1) \subseteq W(1:\mathbf{1})$. Since any maximal torus in $W(1:\mathbf{1})$ is one-dimensional and is equal to its own centralizer (cf. Corollary 1.3.2), we see that $\mathfrak{z}_Q(t_2) \subseteq Ft_2 + Y \otimes B_1$. Now $\mathfrak{z}_{Y \otimes B_1}(t_1) \subseteq Ft_1 + Y \otimes x_1 B_1$ (as $Ft_1 + Y \otimes x_1 B_1$ is a one-dimensional torus in $Y \otimes B_1 / Y \otimes x_1 B_1 \cong Y$ and any nonzero torus in Y ($\cong \mathfrak{sl}(2)$ or $W(1:\mathbf{1})$) is equal to its own centralizer). It follows that $\mathfrak{z}_{Y \otimes B_1}(t_1) = t_1 \cdot B_1$. Now $[t_2, t_1 \cdot b] = t_1 \cdot \phi(t_2)(b)$ and since (using Theorem 1.3.1(c)) $\phi(t_2)(b) = 0$ implies $b \in F$ we have $\mathfrak{z}_Q(T') = T'$, proving (a).

If $x \in (Y \otimes B_1)_{i\alpha_2}$ then $[t_1, x] = 0$ and so $\eta(x) \in \mathfrak{z}_Y(\eta(t_1)) = F\eta(t_1)$. This proves (b).

Now suppose that $Y \otimes x_1 B_1$ is invariant under ad t_2. Then ad t_2 induces a derivation of $(Y \otimes B_1)/(Y \otimes x_1 B_1) \cong Y$ and this derivation commutes with $\eta(t_1)$. Since all derivations of Y ($\cong \mathfrak{sl}(2)$ or $W(1:\mathbf{1})$) are inner and any one-dimensional torus in Y is equal to its own centralizer (cf. Corollary 1.3.2), we see that we may assume (replacing t_2 by an appropriate element of $t_2 + \mathbf{Z}t_1$) that $[t_2, Y \otimes B_1] \subseteq Y \otimes x_1 B_1$. Thus $(Y \otimes B_1)^{(\gamma)} \subseteq Y \otimes x_1 B_1 + Ft_1$ for all $\gamma \notin \mathbf{Z}\alpha_1$ and $Y \otimes B_1 = (Y \otimes B_1)^{(\alpha_1)} + Y \otimes x_1 B_1$. Since η: $Y \otimes B_1 \to Y$ is surjective, $\eta((Y \otimes B_1)^{(\alpha_1)}) = Y$. Thus (c) holds.

Finally, suppose that $Y \otimes x_1 B_1$ is not invariant under ad t_2. By Theorem 1.3.1(a) we may assume $\phi(t_2) = (x_1 + 1) D_1$. Then for any $\beta \in \Delta(Q, T')$ we have $(Y \otimes B_1)_\beta \cdot (x_1 + 1)^l = (Y \otimes B_1)_{\beta + l\alpha_2}$ and so $\eta((Y \otimes B_1)_\beta) = \eta((Y \otimes B_1)_{\beta + l\alpha_2})$. Then (as $\eta((Y \otimes B_1)^{(\alpha_2)}) = F\eta(t_1)$, by (b)) we have (for any $\gamma \notin \mathbf{Z}_p \alpha_2$) $Y = \eta(Y \otimes B_1) = \sum_{\beta \notin \mathbf{Z}_p^* \alpha_2} \eta((Y \otimes B_1)_\beta) = \eta((Y \otimes B_1)^{(\gamma)})$, so (d) holds. ∎

LEMMA 8.10.2. *Let Q and T' be as in* (8.10.1). *Let V be an irreducible restricted Q-module which is not annihilated by $Y \otimes B_n$. Suppose that* $\dim V_\gamma \leqslant 7$ *for all $\gamma \in T'^*$ and that V has $< p^2 - 1$ weights. Then*:

(a) $n = 1$.

(b) $Y \cong \mathfrak{sl}(2)$ *or* $W(1:\mathbf{1})$.

(c) $\dim V_{i\alpha_1 + j\alpha_2} = \dim V_{i\alpha_1}$ *for all* $i, j \in \mathbf{Z}$.

(d) *For some* $\alpha \in \Delta(Q, T')$, $Q^{(\alpha)} \subseteq Y \otimes B_1 + T'$ *and* $Q^{(\alpha)}$ *is not solvable.*

(e) *If* α *is as in* (d), $\alpha \in \Delta_P(Q, T')$, $\beta \in \Delta(Q, T')$, *and* $Q^{(\beta)} \subseteq Y \otimes B_1 + T'$, *then* $\beta \in \Delta_P(Q, T')$.

Proof. Let W be an irreducible $Y \otimes B_n$-submodule of V. Hence W is t_1-invariant and W is annihilated by $Y \otimes (x_1 B_n + \cdots + x_n B_n)$. Then by Lemma 1.10.1, $V \cong u(Q) \otimes_{u(\mathrm{Stab}(W, Q))} \tilde{W}$, where $\tilde{W}$ is the sum of all $Y \otimes B_n$-submodules of V isomorphic to W and $u(-)$ denotes the restricted enveloping algebra functor. Let $\tilde{W}_i = \{w \in \tilde{W} \mid t_1 w = iw\}$. Then as $Q = Y \otimes B_n + Q^{(\alpha_2)}$, it is clear that

$$\sum_{j \in \mathbf{Z}_p} \dim V_{i\alpha_1 + j\alpha_2} = (p^{\dim(Q/\mathrm{Stab}(W, Q))})(\dim \tilde{W}_i). \qquad (8.10.2)$$

Since $\dim \sum_{j \in \mathbf{Z}_p} V_{i\alpha_1 + j\alpha_2} \leqslant 7p$, by hypothesis we have $\dim(Q/\mathrm{Stab}(W, Q)) \leqslant 1$. Now $Y \otimes (x_1 \cdots x_n)^{p-1}$ does not annihilate V, for if it did $Y \otimes B_n$, the ideal in Q generated by $Y \otimes (x_1 \cdots x_n)^{p-1}$, would annihilate V, contrary to our hypotheses. Also $(\mathrm{ad}\, Q)^m (Y \otimes (x_1 \cdots x_n)^{p-1})$ does annihilate $\tilde{W}$ for $0 \leqslant m < n(p-1)$, since it is contained in $Y \otimes (x_1 B_n + \cdots + x_n B_n)$. Thus $V \nsubseteq \tilde{W} + Q\tilde{W} + \cdots + Q^{n(p-1)-1}\tilde{W}$. But $V = \tilde{W} + Q\tilde{W} + \cdots + Q^{p-1}\tilde{W}$ since $\dim(Q/\mathrm{Stab}(W, Q)) \leqslant 1$. Therefore $n(p-1) - 1 < p - 1$, and (a) holds. Also $\dim(Q/\mathrm{Stab}(W, Q)) \geqslant 1$ since $V \neq \tilde{W}$. Thus $\dim(Q/\mathrm{Stab}(W, Q)) = 1$ and so (8.10.2) gives $\dim \tilde{W}_i \leqslant 7$. Then by Lemma 7.2.7, as W is an irreducible $Y \cong (Y \otimes B_1)/(Y \otimes x_1 B_1)$-module, $Y \cong H(2:\mathbf{1})^{(2)}$ is impossible, so (b) holds. Since $\dim(Q/\mathrm{Stab}(W, Q)) = 1$, we have $Q = Fy + \mathrm{Stab}(W, Q)$ for some y. We may assume $y \in Q_{m\alpha_2}$ for some $m \in \mathbf{Z}_p$. If $t_2 \in \mathrm{Stab}(W, Q)$ then, since $Q/Y \otimes B_1$ is isomorphic to a subalgebra of $\mathrm{Der}\, B_1 \cong W(1:\mathbf{1})$ and since every one-dimensional torus in $W(1:\mathbf{1})$ is equal to its centralizer, we must have $m \in \mathbf{Z}_p^*$. Let $w \in \tilde{W}_{i\alpha_1 + s\alpha_2}$. Then $y^l \cdot w \in V_{i\alpha_1 + (s+ml)\alpha_2}$ and so $\dim V_{i\alpha_1 + j\alpha_2} = \dim \tilde{W}_i$ for all $i, j \in \mathbf{Z}$. Thus (c) holds when $t_2 \in \mathrm{Stab}(W, Q)$. If $t_2 \notin \mathrm{Stab}(W, Q)$ we may take $y = t_2$. Then if $w \in \tilde{W}_i$, the linear independence of $\{t_2^l \cdot w \mid 0 \leqslant l \leqslant p-1\}$ implies that t_2 has p distinct eigenvalues on $\mathrm{span}\{t_2^l \cdot w \mid 0 \leqslant l \leqslant p-1\}$. Thus $\dim V_{i\alpha_1 + j\alpha_2} = \dim \tilde{W}_i$ for all $i, j \in \mathbf{Z}$ and (c) holds in this case as well.

By (c) and (d) of Lemma 8.10.1 we see that there is some $\alpha \in \Delta(Q, T')$, $\alpha \notin \mathbf{Z}\alpha_2$, such that $\eta((Y \otimes B_1)^{(\alpha)}) = Y$. Since Y is simple this implies $Q^{(\alpha)}$ is not solvable. Since $t_1 \in Y \otimes B_1$ and $\alpha(t_1) \neq 0$ we have $(Y \otimes B_1)^{(\alpha)} \subseteq Y \otimes B_1 + T'$. Thus (d) holds.

Suppose α is as in (d) and $Q^{(\beta)} \subseteq Y \otimes B_1 + T'$. If $\beta \in \mathbf{Z}\alpha_2$ then by Lemma 8.10.1(b) we have $Q^{(\beta)} \subseteq Y \otimes x_1 B_1 + T'$ so $Q^{(\beta)}$ is solvable and hence $\beta \in \Delta_p(Q, T')$. If $\beta \notin \mathbf{Z}\alpha_2 \cup \mathbf{Z}\alpha$ and $Y \otimes x_1 B_1$ is invariant under $\mathrm{ad}\, t_2$, then Lemma 8.10.1(c) shows that $(Y \otimes B_1)^{(\beta)} \subseteq Y \otimes x_1 B_1 + T'$, so again $\beta \in \Delta_p(Q, T')$. Finally, if $Y \otimes x_1 B_1$ is not invariant under $\mathrm{ad}\, t_2$ then Lemma 8.10.1(d) shows that, for $\gamma \notin \mathbf{Z}\alpha_2$, $\eta((Y \otimes B_1)^{(\gamma)}) = Y$. Thus η induces an isomorphism $\eta_\gamma \colon (Y \otimes B_1)[\gamma] \to Y$ such that $\eta_\gamma(\Psi_\gamma(t_1)) = \eta(t_1)$. Thus $(Y \otimes B_1)[\alpha] \cong (Y \otimes B_1)[\beta]$ by an isomorphism mapping $\Psi_\alpha(t_1)$ onto $\Psi_\beta(t_1)$. Hence (since $Q^{(\gamma)} = (Y \otimes B_1)^{(\gamma)} + T'$ for $\gamma \notin \mathbf{Z}\alpha_2$) we see that $\alpha \in \Delta_p(Q, T')$ implies $\beta \in \Delta_p(Q, T')$. Thus (e) holds. ∎

LEMMA 8.10.3. *Let A satisfy* (7.0.1) *and* (8.0.1). *Assume that $G_0 = Q$, where Q is as in Lemma* 8.10.2, *and let α be as in* (d) *of that lemma. Then* $I(G_0) = (0)$, $\alpha \in \Delta_p(A, T)$, $G_0 \subseteq \mathfrak{sl}(2) \otimes B_1 + \mathrm{Der}\, B_1$, *and* $G_{-1} \cong V \otimes B_1$, *where V is a two-dimensional irreducible* $\mathfrak{sl}(2)$-*module. Replacing α by an element of $\mathbf{Z}_p^*\alpha$ if necessary we may assume* $\Gamma_0 = (\pm\alpha + \mathbf{Z}\alpha_2) \cup \mathbf{Z}\alpha_2$ *and* $\Gamma_{-1} = \pm(\alpha/2) + \mathbf{Z}\alpha_2$. *Furthermore,* $G_1 \neq (0)$.

Proof. By Lemma 8.10.1(a) we have $I(G_0) = (0)$. Note that $\alpha \notin \mathbf{Z}\alpha_2$, since $Q^{(\alpha)} \subseteq Y \otimes B_1 + T$ is not solvable, while $(Y \otimes B_1)^{(\alpha_2)}$ is solvable (by Lemma 8.10.1(b)). Suppose $\alpha \notin \Delta_P(A, T)$. By Lemma 6.4.4 we may find $x \in A_\alpha \cap A_0$ (replacing α by some $i\alpha$ if necessary) such that $\alpha_x \in \Delta_P(A, e^x(T))$. Now $x \in Q^{(\alpha)} \subseteq Y \otimes B_1 + T$ and $Y \otimes B_1$ is a restricted ideal in Q, so $x^p \in \mathfrak{z}_Q(T) \cap (Y \otimes B_1) = Ft_1$ (by Lemma 8.10.1(a)). Since $\alpha(x^p) = 0$ (by Lemma 1.8.1) we have $x^p = 0$ (as $\alpha \notin \mathbf{Z}\alpha_2$). Thus $\gamma_x(E^x(t_1)) = \gamma(t_1)$ for every root γ (see Definition 1.9.2) and so if $\gamma \notin \mathbf{Z}\alpha_2$ we have $\gamma_x(E^x(t_1)) \neq 0$. Since $E^x(t_1) \in Y \otimes B_1$ this implies $(Y \otimes B_1)^{(\gamma_x)} \subseteq Y \otimes B_1 + e^x(T)$. Now suppose $\beta \in \Delta_P(A, T)$ and $\beta \notin \mathbf{Z}\alpha_2$. Then Lemma 8.10.2(e) applies to α_x and β_x and shows that $\beta_x \in \Delta_P(G_0, e^x(T))$. Hence Lemma 6.4.3 shows $\beta_x \in \Delta_P(A, e^x(T))$. If $\alpha_2 \in \Delta_P(A, T)$ then $\alpha_2 \in \Delta_P(G_0, e^x(T))$. But $G_0[\alpha_2]$ is a quotient of $G_0/Y \otimes B_1$, and $x \in Y \otimes B_1$, so E^x acts as the identity on $G_0[\alpha_2]$. Thus $(\alpha_2)_x \in \Delta_P(G_0, e^x(T))$ and so, again by Lemma 6.4.3, $(\alpha_2)_x \in \Delta_P(A, e^x(T))$. Thus $|\Delta_P(A, e^x(T))| > |\Delta_P(A, T)|$, contradicting the optimality of T. Hence $\alpha \in \Delta_P(A, T)$.

Now suppose $Y \cong W(1:\mathbf{1})$. Then by Lemma 8.10.1(c), (d) we see that $(Y \otimes B_1)[\alpha] \cong W(1:\mathbf{1})$. Since G_0 acts faithfully on G_{-1} there is an irreducible $Q^{(\alpha)}$-constituent U of G_{-1} with $U_\gamma \neq (0)$ for some $\gamma \notin \mathbf{Z}\alpha_2$. As $Q^{(\alpha)} = (Y \otimes B_1)^{(\alpha)} + Ft_2$ we see that U remains irreducible under $(Y \otimes B_1)^{(\alpha)}$ and hence is annihilated by the nil ideal $\operatorname{solv}((Y \otimes B_1)^{(\alpha)})$. Thus U is a $(Y \otimes B_1)[\alpha]$-module, hence a $W(1:\mathbf{1})$-module. Therefore (cf. [BW82, Theorem 1.15.2]) t_1 has $(p-1)$ nonzero eigenvalues on U and hence by Lemma 8.10.2(c), $|\Gamma_{-1} \cap \mathbf{Z}\alpha| \geqslant p-1$. This contradicts $\alpha \in \Delta_P(A, T)$. Thus $G_0 \subseteq \mathfrak{sl}(2) \otimes B_1 + \operatorname{Der} B_1$. We may therefore assume (replacing α by some element of $\mathbf{Z}_p^* \alpha$) that $\Gamma_0 \subseteq (\alpha + \mathbf{Z}\alpha_2) \cup (-\alpha + \mathbf{Z}\alpha_2) \cup \mathbf{Z}\alpha_2$. We may also assume (replacing t_1 by some element of $\mathbf{Z}_p^* t_1$) that $\alpha(t_1) = 1$, i.e., that $\alpha \in \alpha_1 + \mathbf{Z}\alpha_2$.

Let $\mathscr{E}$ denote the set of eigenvalues of ad t_1 on G_{-1}. By Lemma 8.10.2(c) we see that $i \in \mathscr{E}$ if and only if $i\alpha \in \Gamma_{-1}$. Thus $0 \notin \mathscr{E}$ (by Lemma 6.3.3) and $|\{\pm 1\} \cup \mathscr{E}| \leqslant 6$. Now we may find $u \in G_{0,\alpha}$, $v \in G_{0,-\alpha}$ such that $[u, v] = t_1$. Thus $Fu + Fv + Ft_1$ is a subalgebra of G_0 isomorphic to $\mathfrak{sl}(2)$. Hence $\mathscr{E} \subseteq \{\pm\frac{3}{2}, \pm\frac{1}{2}\}$. If equality holds then, by Lemma 8.10.2(c), we have $\mathbf{Z}\beta \cap \Gamma_{-1} = \{\pm(\beta/2), \pm(3\beta/2)\}$ for all $\beta \in \alpha_1 + \mathbf{Z}\alpha_2$. By Proposition 5.7.6 we can find $\beta \in \alpha_1 + \mathbf{Z}\alpha_2$ with $\mathbf{Z}\beta \cap \Gamma_E = \varnothing$. Thus $\pm(\beta/2), \pm(3\beta/2) \in \mathbf{Z}\beta \cap \Gamma_K$, contradicting Lemma 5.7.1(a). Hence $\mathbf{Z}\alpha \cap \Gamma_{-1} = \{\pm(\alpha/2)\}$ and so (by Lemma 8.10.2(c)) $\Gamma_{-1} = \pm(\alpha/2) + \mathbf{Z}\alpha_2$.

Now it is clear that $V \otimes B_1$ is a faithful irreducible restricted G_0-module with set of weights equal to $\pm(\alpha_1/2) + \mathbf{Z}\alpha_2$ and with each weight space of dimension $\leqslant 7$. Since G_{-1} has these same properties, the assertion that $G_{-1} \cong V \otimes B_1$ follows if we show that G_0 has a unique faithful irreducible restricted module with set of weights equal to $\pm(\alpha_1/2) + \mathbf{Z}\alpha_2$ and with each weight space of dimension $\leqslant 7$. Let W be such a module. Let W_1 be an irreducible $\mathfrak{sl}(2) \otimes B_1$-submodule of W. Since $\mathfrak{sl}(2) \otimes x_1 B_1$ is a nil ideal of

$\mathfrak{sl}(2)\otimes B_1$, it annihilates W_1. Hence $\mathrm{Stab}(W_1, G_0)\neq G_0$. Also W_1 is an irreducible $\mathfrak{sl}(2)$-module. Since t_1 has only two weights on W, $\dim W_1=2$. Then $\dim G_0/\mathrm{Stab}(W_1, G_0)=1$ and $W\cong u(G_0)\otimes_{u(\mathrm{Stab}(W_1, G_0))}\tilde{W}_1$ (see Lemma 1.10.1). Let N denote the normalizer in G_0 of $\mathfrak{sl}(2)\otimes x_1 B_1$. Clearly $\dim G_0/N\leqslant 1$ and, since G_0 is semisimple, $\dim G_0/N=1$. We claim that $N=\mathrm{Stab}(W_1, G_0)$. To see this note that if $u\in N$, then $\operatorname{ad} u$ induces a derivation of $\mathfrak{sl}(2)\otimes B_1/\mathfrak{sl}(2)\otimes x_1 B_1\cong\mathfrak{sl}(2)$. Since all derivations of $\mathfrak{sl}(2)$ are inner we have $N=\mathfrak{sl}(2)\otimes B_1+\{v\in G_0\mid [v, \mathfrak{sl}(2)\otimes B_1]\subseteq\mathfrak{sl}(2)\otimes x_1 B_1\}\subseteq\mathrm{Stab}(W_1, G_0)+\{v\in G_0\mid [v, \mathfrak{sl}(2)\otimes B_1]\,W_1=(0)\}=\mathrm{Stab}(W_1, G_0)$. Since $\dim G_0/N=\dim G_0/\mathrm{Stab}(W_1, G_0)$ we have equality. Now $\mathfrak{sl}(2)\otimes x_1 B_1$ is a nil ideal and so $\tilde{W}_1$ is an irreducible $N/(\mathfrak{sl}(2)\otimes x_1 B_1)$-module and $N/(\mathfrak{sl}(2)\otimes x_1 B_1)\cong\mathfrak{sl}(2)\oplus N'$, where N' is a subalgebra of $W(1:\mathbf{1})_0$. Since $N'\cap W(1:\mathbf{1})_1$ is a nil ideal in N' (and hence in $\mathfrak{sl}(2)\oplus N'$) we see that $\tilde{W}_1$ is an irreducible $(\mathfrak{sl}(2)\oplus N')/(N'\cap W(1:\mathbf{1})_1)$-module. But $N'/N'\cap W(1:\mathbf{1})_1$ has dimension $\leqslant 1$. Thus $\tilde{W}_1$ is an irreducible $\mathfrak{sl}(2)$-module and so $\tilde{W}_1=W_1$. Thus the structure of $W\cong u(G_0)\otimes_{u(N)} W_1$ is uniquely determined.

Finally (as $I(G_0)=(0)$), if $G_1=(0)$, then $I=(0)$. Lemma 6.3.4 of [BW82] shows that this cannot occur. ∎

Let $\{e, f, h\}$ denote the usual basis for $\mathfrak{sl}(2)$ (so $[h, e]=2e$, $[h, f]=-2f$, $[e, f]=h$). Let V, a two-dimensional $\mathfrak{sl}(2)$-module, have basis $\{v, w\}$, where $hv=v$, $hw=-w$, $ev=fw=0$, $fv=w$, $ew=v$. In view of Lemma 8.10.3 we can, and do, identify G_{-1} with $V\otimes B_1$ so that G_{-1} has basis $\{v\otimes x_1^i\mid 0\leqslant i\leqslant p-1\}\cup\{w\otimes x_1^i\mid 0\leqslant i\leqslant p-1\}$.

LEMMA 8.10.4. *Let A and G be as in Lemma* 8.10.3. *Then $\Gamma_{-i}\subseteq\Gamma_{-1}$ if i is odd and $\Gamma_{-i}\subseteq\mathbf{Z}\alpha_2$ if i is even.*

Proof. We prove this by induction on i, the case $i=1$ being vacuous.

Assume the result holds for all $j<i$ and that i is even. Then $\Gamma_{-i}\subseteq\Gamma_{-i+1}+\Gamma_{-1}\subseteq(\alpha+\mathbf{Z}\alpha_2)\cup(-\alpha+\mathbf{Z}\alpha_2)\cup\mathbf{Z}\alpha_2$ (by the induction assumption and Lemma 8.10.3). If $\Gamma_{-i}\nsubseteq\mathbf{Z}\alpha_2$ then there is an irreducible constituent U of G_{-i} which is not annihilated by $\mathfrak{sl}(2)\otimes B_1$. Then (as $0\notin\Gamma_-$ by Lemma 6.3.3) Lemma 8.10.2(c) shows that the weights of U are contained in $(\alpha+\mathbf{Z}\alpha_2)\cup(-\alpha+\mathbf{Z}\alpha_2)$. Since $\Gamma_0\subseteq(\alpha+\mathbf{Z}\alpha_2)\cup(-\alpha+\mathbf{Z}\alpha_2)\cup\mathbf{Z}\alpha_2$, the irreducibility of U implies that its weights are contained in $\alpha+\mathbf{Z}\alpha_2$ or in $-\alpha+\mathbf{Z}\alpha_2$. This implies that $G_{0,\pm\alpha}$ annihilates U. Since $t_1\in[G_{0,\alpha}, G_{0,-\alpha}]$ this is absurd. Thus $\Gamma_{-i}\subseteq\mathbf{Z}\alpha_2$.

Now assume the result holds for $j<i$ and that i is odd. Then $\Gamma_{-i}\subseteq\Gamma_{-i+1}+\Gamma_{-1}\subseteq\mathbf{Z}\alpha_2+\Gamma_{-1}=\Gamma_{-1}$ (again by the induction assumption and Lemma 8.10.3). ∎

LEMMA 8.10.5. *Let A and G be as in Lemma* 8.10.3. *Suppose*

$G_0/(\mathfrak{sl}(2)\otimes B_1)\cong W(1:\mathbf{1})$. *Then* $\alpha_2\in\Delta_P(A,T)$, $G_{-2}=(0)$, *and* $[G_{-1},G_1]\subseteq \mathfrak{sl}(2)\otimes B_1$.

Proof. Suppose $\alpha_2\notin\Delta_P(A,T)$. By Lemma 6.4.4 we may (replacing t_2 by an element of $\mathbf{Z}_p^*t_2$ if necessary) find $x\in A_{\alpha_2}\cap A_0$ so that $(\alpha_2)_x\in\Delta_P(A,e^x(T))$. Now if $\beta\in\Delta(A,T)$, $\beta\notin\mathbf{Z}\alpha_2$, we have $G_0^{(\beta_x)}\subseteq(\mathfrak{sl}(2)\otimes B_1)+e^x(T)$, which is compositionally classical and so $\beta_x\in\Delta_P(G_0,e^x(T))$. Then Lemma 6.4.3 shows that $|\Delta_P(A,e^x(T))|>|\Delta_P(A,T)|$, contradicting the optimality of T. Thus $\alpha_2\in\Delta_P(A,T)$.

By Lemma 8.10.4, $\mathfrak{sl}(2)\otimes B_1$ acts as a nil ideal on each composition factor of the G_0-module G_{-2}. Thus since $(\mathfrak{sl}(2)\otimes B_1)^{(\alpha_2)}=\mathrm{solv}(G_0^{(\alpha_2)})$, each composition factor of G_{-2} is a $G_0[\alpha_2]\cong W(1:\mathbf{1})$-module. But as α_2 is proper, $|\Gamma_{-2}\cap\mathbf{Z}\alpha_2|\leqslant 6$. Thus (cf. [BW82, Theorem 1.1.5.2]) each composition factor of G_{-2} is a trivial $G_0[\alpha_2]$-module. Hence $G_{-2}=G_{-2,0}=(0)$.

To show $[G_{-1},G_1]\subseteq\mathfrak{sl}(2)\otimes B_1$ observe that $h\otimes 1$ is toral, hence G_1 is a sum of eigenspaces for $h\otimes 1$. Since the only eigenvalues of $h\otimes 1$ on G_{-1} are ± 1 and since all eigenspaces in G_0 for nonzero eigenvalues are contained in $\mathfrak{sl}\otimes B_1$, we see that if $g\in G_1$ and $[h\otimes 1,g]=ig$, $i\neq\pm 1$, we have $[G_{-1},g]\subseteq\mathfrak{sl}(2)\otimes B_1$.

Now let $j\in\mathbf{Z}_p$ and $g\in G_1$ satisfy $[h\otimes 1,g]=-g$, $[(x_1+1)D_1,g]=jg$. Then for $i\in\mathbf{Z}_p$ we have

$$[v\otimes(x_1+1)^i,g]=a_ih\otimes(x_1+1)^{i+j}+b_i(x_1+1)^{i+j+1}D_1$$

and

$$[w\otimes(x_1+1)^i,g]=c_if\otimes(x_1+1)^{i+j},$$

where a_i, b_i, $c_i\in F$. Then

$$[v\otimes(x_1+1)^k,[v\otimes(x_1+1)^i,g]]=(-a_i-b_ik)\,v\otimes(x_1+1)^{i+j+k},$$
$$[w\otimes(x_1+1)^k,[v\otimes(x_1+1)^i,g]]=(a_i-b_ik)\,w\otimes(x_1+1)^{i+j+k},$$

and

$$[v\otimes(x_1+1)^i,[w\otimes(x_1+1)^k,g]]=-c_kw\otimes(x_1+1)^{i+j+k}$$

for all $i,j,k\in\mathbf{Z}_p$. Since $[G_{-1},G_{-1}]=(0)$ this gives

$$a_i+b_ik=a_k+b_ki \tag{8.10.3}$$

and

$$a_i-b_ik=-c_k \tag{8.10.4}$$

for all $i, k \in \mathbf{Z}_p$. Then (8.10.4) shows that for $i, j, k \in \mathbf{Z}_p$ we have $a_i - b_i k = a_j - b_j k$ and so $a_i - a_j = k(b_i - b_j)$ for all $i, j, k \in \mathbf{Z}_p$. Since the left-hand side is independent of k, this implies that $b_i = b_j$ and $a_i = a_j$ for all $i, j \in \mathbf{Z}_p$. But then (8.10.3) implies $b_i = 0$ for all i. Hence we have $[G_{-1}, g] \subseteq \mathfrak{sl}(2) \otimes B_1$. Similarly, if $g' \in G_1$ and $[h \otimes 1, g'] = g'$ then $[G_{-1}, g'] \subseteq \mathfrak{sl}(2) \otimes B_1$. Hence $[G_{-1}, G_1] \subseteq \mathfrak{sl}(2) \otimes B_1$, as required. ▌

LEMMA 8.10.6. *Let A and G be as in Lemma* 8.10.3. *Suppose* $G_0/(\mathfrak{sl}(2) \otimes B_1) \not\cong W(1:\mathbf{1})$. *Then* $[G_{-1}, G_1] \subseteq \mathfrak{sl}(2) \otimes B_1$ *and* $G_{-2} = (0)$

Proof. Every irreducible constituent of G_{-2} as an $\mathfrak{sl}(2)$-module is annihilated by $\mathfrak{sl}(2) \otimes x_1 B_1$ and hence is an $\mathfrak{sl}(2)$-module. By Lemma 8.10.4 each such constituent is a trivial $\mathfrak{sl}(2)$-module and hence is annihilated by $h \otimes 1$. Thus the only eigenvalue of $h \otimes 1$ on G_{-2} is 0. Therefore since $[h \otimes 1, v \otimes x_1^i] = v \otimes x_1^i$ we see that $[v \otimes x_1^i, v \otimes x_1^j] = 0$ for all $i, j, 0 \leqslant i, j \leqslant p-1$. Then $[v \otimes x_1^i, [v \otimes x_1^{p-1}, G_1]] = [v \otimes x_1^{p-1}, [v \otimes x_1^i, G_1]] \subseteq V \otimes (Fx_1^{p-2} + Fx_1^{p-1})$ for all i, $1 \leqslant i \leqslant p-1$. Thus $[v \otimes x_1^{p-1}, G_1] \subseteq Fx_1^{p-2}D_1 + Fx_1^{p-1}D_1 + \mathfrak{sl}(2) \otimes B_1$. If $[v \otimes x_1^{p-1}, G_1] \not\subseteq \mathfrak{sl}(2) \otimes B_1$ this implies that $(G_0/(\mathfrak{sl}(2) \otimes B_1)) \cap (Fx_1^{p-2}D_1 + Fx_1^{p-1}D_1) \neq (0)$. Since $G_0/(\mathfrak{sl}(2) \otimes B_1) \not\subseteq \sum_{i>0} Fx_1^i D_1$ (as G_0 is semisimple) this is easily seen to imply $G_0/(\mathfrak{sl}(2) \otimes B_1) \cong W(1:\mathbf{1})$, a contradiction. Hence $[v \otimes x_1^{p-1}, G_1] \subseteq \mathfrak{sl}(2) \otimes B_1$. Since $\{u \in G_{-1} \mid [u, G_1] \subseteq \mathfrak{sl}(2) \otimes B_1\}$ is a G_0-invariant subspace of G_{-1}, this implies $[G_{-1}, G_1] \subseteq \mathfrak{sl}(2) \otimes B_1$.

Now $[G_{-1}, G_1]$ is a nonzero ideal of G_0, so it cannot be nil. Thus $t_1 \in [G_{-1}, G_1]$ and so there exist $\gamma \in \Gamma_{-1}$, $x \in G_{-1,\gamma}$, $y \in G_{1,-\gamma}$ such that $[x, y] = t_1$. Note that $\gamma \in \Gamma_{-1} = ((\alpha/2) + \mathbf{Z}\alpha_2) \cup (-(\alpha/2) + \mathbf{Z}\alpha_2)$. Suppose $G/M(G) = \sum_{i \geqslant k} (G/M(G))_i$ and k is odd, $k < -1$. Then $[G_{-1,\gamma}, (G/M(G))_{k,-\gamma+l\alpha_2}] = (0) = [G_{1,-\gamma}, (G/M(G))_{k,-\gamma+l\alpha_2}]$ (as $-2\gamma + l\alpha_2 \notin \Gamma_{k+1}$ by Lemma 8.10.4). Thus $[t_1, (G/M(G))_{k,-\gamma+l\alpha_2}] = (0)$. Since $(\gamma + l\alpha_2)(t_1) = \gamma(t_1) \neq 0$ this implies $(G/M(G))_{k,-\gamma+l\alpha_2} = (0)$. Hence (see Lemma 8.10.4) $\Gamma_k \subseteq \gamma + \mathbf{Z}\alpha_2$. Therefore $(\mathfrak{sl}(2) \otimes B_1)_{l\alpha}$ annihilates $(G/M(G))_k$ for all $l \neq 0$. Since $(\mathfrak{sl}(2) \otimes B_1)^{(\alpha)}$ is not solvable this implies t_1 annihilates $(G/M(G))_k$, so $(G/M(G))_k = (0)$, a contradiction. Now suppose that k is even. Then $\Gamma_k \subseteq \mathbf{Z}\alpha_2$ and so $[G_{-1}, G_1]$ annihilates $(G/M(G))_k$. By Weisfeiler's theorem (cf. Theorem 1.5.1 of [BW82]), $G/M(G)$ is semisimple and every nonzero ideal of $G/M(G)$ contains $(G/M(G))_k$. Let $G_- = \sum_{i<0} G_i$ and $G_+ = \sum_{i>0} G_i$. Since G_{-1} generates G_- we see that $G^\dagger = [G_{-1}, G_1] + \sum_{i \neq 0} (G/M(G))_i$ is an ideal in $G/M(G)$. Since $G/M(G)$ is semisimple we have $\mathfrak{z}(G^\dagger) \neq (0)$ and so $[G^\dagger, (G/M(G))_k] \neq (0)$. Since G_- and $[G_{-1}, G_1]$ annihilate $(G/M(G))_k$, the Poincaré–Birkhoff–Witt theorem shows that the ideal J in $G^\dagger$ generated by $[G^\dagger, (G/M(G))_k]$ is contained in $\sum_{i>k} (G/M(G))_i$. But since $(G/M(G))_k$ and $G^\dagger$ are invariant under ad G_0 we see that J is an ideal in $G/M(G)$. Since $J \neq (0)$ and $J \subseteq \sum_{i>k} (G/M(G))_i$, this contradicts the fact that $(G/M(G))_k$ is contained in every nonzero ideal of $G/M(G)$. Thus k cannot be even and so $k=-1$. Hence $G_{-2} \subseteq M(G)$. Now consider the $(Fx+Fy+Ft_1)$-module generated by $G_{-3,\gamma+l\alpha_2}$ (where $l \in \mathbf{Z}$). Since $[x, G_{-3,\gamma+l\alpha_2}] \subseteq G_{-4,2\gamma+l\alpha_2} = (0)$ and $(\operatorname{ad} y)^2 G_{-3,\gamma+l\alpha_2} \subseteq M(G)_{-1} = (0)$, we see that the set of eigenvalues of ad t_1 on this module is contained in $\{\gamma(t_1), 0\}$. Since this set must be symmetric about 0 and since $\gamma(t_1) \neq 0$, we conclude that $G_{-3,\gamma+l\alpha_2} = (0)$. Thus $\Gamma_{-3} \subseteq -\gamma + \mathbf{Z}\alpha_2$ and, as above, this implies $G_{-3} = (0)$.

Now since $\Gamma_{-1} \subseteq \pm(\alpha/2) + \mathbf{Z}\alpha_2$ and $\Gamma_0 \subseteq (\pm\alpha + \mathbf{Z}\alpha_2) \cup (\mathbf{Z}\alpha_2)$ we have $[A^{(\alpha_2)}, A_0] \subseteq A^{(\alpha_2)} + A_0 + [A^{(\alpha_2)}, A_1]$. Since $G_{-2} \subseteq M(G)$ (so that $[G_{-2}, G_1] = (0)$) we have $[A, A_1] \subseteq A_0$. Thus $[A^{(\alpha_2)}, A_0] \subseteq A^{(\alpha_2)} + A_0$ and hence $A^{(\alpha_2)} + A_0$ is a subalgebra of A. By the maximality of A_0 we have $A^{(\alpha_2)} + A_0 \subseteq A_0$. By Lemma 8.10.4 this implies $G_{-2} = (0)$. ▌

LEMMA 8.10.7. *Any element $0 \neq t \in \mathfrak{sl}(2) \otimes B_1$ satisfying $t^p = t$ is conjugate (under an automorphism of $\mathfrak{sl}(2) \otimes B_1$) to an element of $\mathbf{Z}_p^*(h \otimes 1)$.*

Proof. Since $\mathfrak{sl}(2) \otimes x_1 B_1$ is nil, $t \notin \mathfrak{sl}(2) \otimes x_1 B_1$. Thus we may assume (replacing t by an element of $\mathbf{Z}_p^* t$ and applying an automorphism of $\mathfrak{sl}(2)$) that $t = h \otimes 1 + e \otimes u_1(x_1) + h \otimes u_2(x_1) + f \otimes u_3(x_1)$, where $u_i(0) = 0$ for $i = 1, 2, 3$. Now $(\operatorname{ad}(e \otimes x_1^i))^3 = (\operatorname{ad}(f \otimes x_1^i))^3 = 0$ for $0 \leqslant i \leqslant p-1$ and so $\exp(\operatorname{ad}(c(e \otimes x_1^i)))$ and $\exp(\operatorname{ad}(c(f \otimes x_1^i)))$ are automorphisms of $\mathfrak{sl}(2) \otimes B_1$ for $0 \leqslant i \leqslant p-1$, $c \in F$. Applying such automorphisms we may assume $t = h \otimes 1 + h \otimes u_2(x_1)$. But $[h \otimes 1, h \otimes u_2(x_1)] = 0$ and $(h \otimes u_2(x_1))^p = 0$ so $t^p = t$ implies $t = h \otimes 1$. ▌

LEMMA 8.10.8. *Let A and G be as in Lemma 8.10.3. Let $\tilde{A}$ denote the ideal $\sum_{\gamma \notin \mathbf{Z}\alpha_2} A_\gamma + \sum_{\gamma,\delta \notin \mathbf{Z}\alpha_2} [A_\gamma, A_\delta]$. Then $A_0 \nsubseteq A_1 + \tilde{A}$.*

Proof. Since $\mathfrak{sl}(2) \otimes B_1 \subseteq G_0 \subseteq (\mathfrak{sl}(2) \otimes B_1) + \operatorname{Der} B_1$ and since $T \nsubseteq \mathfrak{sl}(2) \otimes B_1$, we may assume (Theorem 1.3.1(c)) that either $x_1 D_1 \in G_0$ or $(x_1+1) D_1 \in G_0$. Since G_0 is semisimple, $\mathfrak{sl}(2) \otimes x_1 B_1$ is not invariant under ad G_0. Hence if $x_1 D_1 \in G_0$ we have $D_1 \in G_0$ and so $(x_1+1) D_1 \in G_0$. Thus in any case $(x_1+1) D_1 \in G_0$. Write $y = x_1 + 1$ and note that if $i, j \in \mathbf{Z}$ and $i \equiv j \bmod p$, then $y^i = y^j$. Therefore it makes sense to write y^i for $i \in \mathbf{Z}_p$.

Let R_1 be a maximal torus in A_0 such that $R_1 + A_1/A_1 = \operatorname{span}\{h \otimes 1, yD_1\}$. By Lemma 8.10.7 we may and do assume $t_1 = h \otimes 1$. However, we do not assume $R_1 = T$. For $i \in \mathbf{Z}_p$ let V_i, $W_i \in A$ and $H_i, E_i \in A_0$ be root vectors with respect to R_1 satisfying

$$V_i + A_0 = v \otimes y^i,$$

$$W_i + A_0 = w \otimes y^i,$$

$$H_i + A_1 = h \otimes y^i,$$

$$E_i + A_1 = e \otimes y^i.$$

Also let $C_i \in A_0$ be root vectors satisfying

$$C_i + A_1 = \begin{cases} y^{i+1}D_1 & \text{if } \; y^{i+1}D_1 \in G_0, \\ 0 & \text{if } \; y^{i+1}D_1 \notin G_0. \end{cases}$$

Then for $i, j \in \mathbf{Z}_p$ the following congruences hold mod A_1 (since $G_{-2} = (0)$):

$$\begin{aligned} [V_i, W_j] &\equiv a_{i,j} H_{i+j} + b_{i,j} C_{i+j} \\ [V_i, V_j] &\equiv c_{i,j} E_{i+j} \end{aligned} \tag{8.10.5}$$

for some $a_{i,j}$, $b_{i,j}$, $c_{i,j} \in F$. Furthermore, we may assume $b_{i,j} = 0$ if $y^{i+j+1}D_1 \notin G_0$.

Now $[H_1, V_i] \equiv V_{i+1} \bmod A_1$, $[H_1, W_i] \equiv -W_{i+1} \bmod A_1$, and $([A_{-1}, A_1] + A_1)/A_1 \subseteq \mathfrak{sl}(2) \otimes B_1$. Therefore since

$$[H_1, [V_i, W_j]] = [[H_1, V_i], W_j] + [V_i, [H_1, W_j]]$$

we have $b_{i+1,j} - b_{i,j+1} = 0$ for all i, j. Hence there are elements $b_u \in F$, $u \in \mathbf{Z}_p$, such that $b_{i,j} = b_{i+j}$ for $i, j \in \mathbf{Z}_p$.

Suppose that $\dim(G_0/\mathfrak{sl}(2) \otimes B_1) > 1$. Then $C_u \neq 0$ for some $u \in \mathbf{Z}_p^*$. Now $[C_u, V_i] \equiv iV_{i+u}$ and $[C_u, W_i] \equiv iW_{i+u} \bmod A_0$ for $i \in \mathbf{Z}_p^*$. If $C_{i+j+u} \neq 0$, comparing coefficients of C_{i+j+u} in the Jacobi identity

$$[C_u, [V_i, W_j]] = [[C_u, V_i], W_j] + [V_i, [C_u, W_j]]$$

gives

$$(i+j)\, b_{i+j+u} = (i+j-u)\, b_{i+j}$$

and the convention that $b_n = 0$ whenever $C_n = 0$ shows that this identity holds for all $i, j \in \mathbf{Z}_p$. Taking $i = mu$, $j = 0$ gives $b_{(m+1)u} = ((m-1)/m)\, b_{mu}$ for $m \in \mathbf{Z}_p^*$. Taking $m = 1$ gives $b_{2u} = 0$ and then induction gives $b_k = 0$ for all $k \neq u$.

On the other hand, if $\dim(G_0/\mathfrak{sl}(2) \otimes B_1) = 1$ we have $b_k = 0$ for all $k \neq 0$.

Thus, in any case, there is some $u \in \mathbf{Z}_p$ such that $b_k = 0$ for all $k \neq u$. Now by the Jacobi identity we have

$$[V_i, [W_j, V_k]] + [W_j, [V_k, V_i]] + [V_k, [V_i, W_j]] = 0.$$

But by (8.10.5) we have

$$\begin{aligned} [V_i, [W_j, V_k]] &\equiv (a_{k,j} + ib_{k+j})\, V_{i+j+k} && \bmod A_0, \\ [W_j, [V_k, V_i]] &\equiv -c_{k,i} V_{i+j+k} && \bmod A_0, \end{aligned}$$

and

$$[V_k, [V_i, W_j]] \equiv (-a_{i,j} - kb_{i+j})\, V_{i+j+k} \qquad \bmod A_0.$$

Thus

$$-a_{i,j} - kb_{i+j} + a_{k,j} + ib_{k+j} = c_{k,i} \tag{8.10.6}$$

for all $i, j, k \in \mathbf{Z}_p$.

Now suppose $i+j=u$. Fix k so that $j+k \neq u$, $j+k-1 \neq u$, $i+k-1 \neq u$, $2k-1 \neq u$. (As $p>7$ such a k exists.) Then as $b_{k+j}=0$, (8.10.6) gives

$$kb_{i+j} = -a_{i,j} + a_{k,j} - c_{k,i},$$

and as $b_{k-1+j}=0$, replacing k by $k-1$ in (8.10.6) gives

$$(k-1)\, b_{i+j} = -a_{i,j} + a_{k-1,j} - c_{k-1,i}.$$

Thus, taking the difference,

$$b_{i+j} = (a_{k,j} - a_{k-1,j}) - (c_{k,i} - c_{k-1,i}). \tag{8.10.7}$$

Fix l so that $i+l \neq u$, $k+l \neq u$, $k+l-1 \neq u$. Since $b_{k-1+l} = b_{k+l} = 0$, replacing i by $k-1$ and j by l in (8.10.6) gives

$$\begin{aligned} c_{k,k-1} &= -a_{k-1,l} + a_{k,l} \\ &= (-a_{i,l} + a_{k,l}) - (-a_{i,l} + a_{k-1,l}). \end{aligned}$$

Setting $j=l$ in (8.10.6) and using $b_{i+l} = b_{k+l} = 0$ gives

$$-a_{i,l} + a_{k,l} = c_{k,i}.$$

Setting $j=l$ and replacing k by $k-1$ in (8.10.6) and using $b_{i+l} = b_{k-1+l} = 0$ gives

$$-a_{i,l} + a_{k-1,l} = c_{k-1,i}.$$

Thus

$$c_{k,k-1} = c_{k,i} - c_{k-1,i}. \tag{8.10.8}$$

Finally, setting $i=k-1$ in (8.10.6) and using $b_{k-1+j}=0$ gives

$$-a_{k-1,j} + a_{k,j} = c_{k,k-1}. \tag{8.10.9}$$

Then substituting (8.10.8) and (8.10.9) into (8.10.7) gives $b_{i+j}=0$, i.e., $b_u=0$. Thus $b_k=0$ for all $k \in \mathbf{Z}_p$ and so (8.10.5) gives $[V_i, W_j]+$

$A_1 \in \mathfrak{sl}(2) \otimes B_1$ for all $i, j \in \mathbf{Z}_p$. Setting $A^\dagger = \mathrm{span}(\{V_i \mid i \in \mathbf{Z}_p\} \cup \{W_i \mid i \in \mathbf{Z}_p\})$ we may express this result as

$$([A^\dagger, A^\dagger] + A_1)/A_1 \subseteq \mathfrak{sl}(2) \otimes B_1. \tag{8.10.10}$$

Now since $\sum_{\gamma \notin \mathbf{Z}\alpha_2} A_\gamma = [A, t_1]$ we have $\tilde{A} = [A, t_1] + [[A, t_1], [A, t_1]]$ and the lemma will follow if we show $(([[A, t_1], [A, t_1]] \cap A_0) + A_1)/A_1 \subseteq \mathfrak{sl}(2) \otimes B_1$. We already have $([A_{-1}, A_1] + A_1)/A_1 \subseteq \mathfrak{sl}(2) \otimes B_1$ (by Lemmas 8.10.5 and 8.10.6). Therefore, since $t_1 \in \mathfrak{sl}(2) \otimes B_1$ and $(\Gamma_{-1} + \Gamma_0) \cap \Gamma_0 = \emptyset$, we have $(([[A, t_1], [A_0, t_1]] \cap A_0) + A_1)/A_1 \subseteq \mathfrak{sl}(2) \otimes B_1$. Combining this with (8.10.10) gives the required result and so proves the lemma. ∎

COROLLARY 8.10.9. *G_0 cannot be one of the algebras listed in Theorem 4.1.1(b).*

Proof. As in the lemma let $\tilde{A} = \sum_{\gamma \notin \mathbf{Z}\alpha_2} A_\gamma + \sum_{\gamma, \delta \notin \mathbf{Z}\alpha_2} [A_\gamma, A_\delta]$. Since $A = S + I$, where S is a simple ideal in A, we have $A_\gamma = S_\gamma$ for all $0 \neq \gamma \in T^*$. Then by Schue's lemma (Lemma 1.12.1) we have $S = \tilde{A}$. Hence $A = \tilde{A} + I$. Since $I(G_0) = (0)$ this implies $A = \tilde{A} + A_1$, contradicting the lemma. ∎

8.11. We have now verified that (a) or (b) of Proposition 8.1.1 must hold. We will now prove the additional conclusions of Proposition 8.1.1 in case (b) holds and $G_1 \neq (0)$. Thus we assume $G_0 = Y_1 + Y_2$, where $Y_i \cong \mathfrak{sl}(2)$ or $W(1:\mathbf{1})$ for $i = 1, 2$. Then $T = T \cap Y_1 + T \cap Y_2$ and $T \cap Y_i = Ft_i$ for some $0 \neq t_i \in T \cap Y_i$ satisfying $t_i^p = t_i$. Define $\alpha_i \in T^*$ by $\alpha_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$. Then we have $\Gamma_0 \subseteq \mathbf{Z}\alpha_1 \cup \mathbf{Z}\alpha_2$.

LEMMA 8.11.1. *If G_0 is as above, then $A = A_{-1}$. Consequently, if $G_1 \neq (0)$ then $N(G) = (0)$.*

Proof. Since $Y_i \cong \mathfrak{sl}(2)$ or $W(1:\mathbf{1})$ we have $A_{0, \pm j\alpha_i} \neq K_{0, \pm j\alpha_i}$ for some $j \in \mathbf{Z}_p^*$. By replacing t_i by $2j^{-1}t_i$, we may assume $j = 2$. Since $\Gamma_0 \subseteq \mathbf{Z}\alpha_1 \cup \mathbf{Z}\alpha_2$, Corollary 6.4.2(c) shows that if $\alpha_i \notin \Delta_P$ then $A_{0, \pm 2\alpha_i} + K_{\pm 2\alpha_i} \neq A_{\pm 2\alpha_i}$ and hence (as $A_{0, \pm 2\alpha_i} \not\subseteq K_{\pm 2\alpha_i}$) $\dim A_{\pm 2\alpha_i}/K_{\pm 2\alpha_i} \geqslant 2$. This implies that α_i is Hamiltonian and so Corollary 6.4.2(d) shows that (still assuming $\alpha_i \notin \Delta_P$) $\dim(A_{\pm 2\alpha_i}/(A_{0, \pm 2\alpha_i} + K_{\pm 2\alpha_i})) \geqslant 2$. As $\dim A_{\pm 2\alpha_i}/K_{\pm 2\alpha_i} = 3$ (for α_i is Hamiltonian and $\alpha_i \notin \Delta_P$) this implies $\dim(A_{0, \pm 2\alpha_i}/A_{0, \pm 2\alpha_i} \cap K_{\pm 2\alpha_i})) = 1$. Since $I \subseteq A_0$, $A_{0, \pm 2\alpha_i}$ is (ad I)-invariant. By Lemmas 5.2.1(e) and 2.1.11(b) this implies $[A_{0, 2\alpha_i}, A_{0, -2\alpha_i}] \subseteq I$, a contradiction since $t_i \in [A_{0, 2\alpha_i}, A_{0, -2\alpha_i}]$. Thus $\alpha_i \in \Delta_P$ for $i = 1, 2$. Therefore $n(A, T) \geqslant 2$ and so $|\Gamma_E| \leqslant 2$. Hence $\Gamma_E \cap \mathbf{Z}\alpha_i = \emptyset$ for $i = 1$ or 2. Assume, without loss of generality, that $\Gamma_E \cap \mathbf{Z}\alpha_1 = \emptyset$. Then $|\Gamma_R \cap \mathbf{Z}\alpha_1| \leqslant 4$. Thus for $l<0$ we have $|(\Gamma_l \cap \mathbf{Z}\alpha_1) \cup \{\pm 2\alpha_1\}| \leqslant 4$. Since $0 \notin \Gamma_l$ and $\sum_{j\in\mathbf{Z}} G_{l,j\alpha_1}$ is a Y_1-module, this shows that $\Gamma_l \cap \mathbf{Z}\alpha_1 \subseteq \{\pm\alpha_1\}$.

Now let M_i be any irreducible Y_i-submodule of G_{-1}. By the above, M_1 is two-dimensional. Let N_i denote the sum of all Y_i-submodules of G_{-1} isomorphic to M_i. Clearly N_i is a G_0-submodule of G_{-1}, hence $N_i = G_{-1}$. Thus ad t_1 has only two eigenvalues, ± 1, on G_{-1}. Thus G_{-2}, being a homomorphic image of $G_{-1} \otimes G_{-1}$, is spanned by eigenvectors for ad t_1 corresponding to the eigenvalues $-2, 0, 2$. Now $\Gamma_{-2} \cap \mathbf{Z}\alpha_1 = \varnothing$. But by Proposition 1.15.3 and Corollary 1.15.4 of [BW82], $\sum_{j\in\mathbf{Z}} G_{-2,j\alpha_1}$ (the zero weight space for G_{-2} as a Y_2-module) generates G_{-2} (a homomorphic image of $G_{-1} \otimes G_{-1}$) as a Y_2-module. Since $\sum_{j\in\mathbf{Z}} G_{-2,j\alpha_1} = (0)$ this implies $G_{-2} = (0)$. Since if $G_1 \neq (0)$ we have $N(G) \subseteq \sum_{j<-1} G_j$, this gives $N(G) = (0)$. ∎

LEMMA 8.11.2. *If G_0 is as above and $G_1 \neq (0)$, then $\Gamma_{-1} = \{\pm\alpha_1 \pm \alpha_2\}$, $G_0 \cong \mathfrak{sl}(2) + \mathfrak{sl}(2)$, and G_0 acts faithfully on G_1.*

Proof. Let M_i be an irreducible $T + Y_i$-submodule of G_{-1}. Since $\mathfrak{z}(T + Y_i)$ acts as scalars on M_i, M_i is an irreducible Y_i-module. Let N_i be the sum of all irreducible Y_i-submodules of G_{-1} isomorphic to M_i. Then N_i is a G_0-submodule of G_{-1} and so $N_1 = N_2 = G_{-1}$. Let $\Gamma(M_i)$ denote the set of weights of M_i. Then (as $Y_i \cong \mathfrak{sl}(2)$ or $W(1:\mathbf{1})$) setting $m_i = \dim M_i$ we have

$$\Gamma(M_1) = \{(m_1 - 1)\,\alpha_1 + n_2\alpha_2, (m_1 - 3)\,\alpha_1 + n_2\alpha_2, ..., -(m_1 - 1)\,\alpha_1 + n_2\alpha_2\}$$

and

$$\Gamma(M_2) = \{(m_2 - 1)\,\alpha_2 + n_1\alpha_1, (m_2 - 3)\,\alpha_2 + n_1\alpha_1, ..., -(m_2 - 1)\,\alpha_2 + n_1\alpha_1\}$$

for some $0 \leqslant n_1, n_2 \leqslant p - 1$. Then since $G_{-1} = N_1 = N_2$ we see that

$$\Gamma_{-1} = \{(m_1 - 1 - 2a)\,\alpha_1 + (m_2 - 1 - 2b)\,\alpha_2 \mid 0 \leqslant a \leqslant m_1 - 1,\ 0 \leqslant b \leqslant m_2 - 1\}.$$

Since the representation of G_0 on G_{-1} is faithful, we have m_1, $m_2 \geqslant 2$. We will now show $m_1, m_2 = 2$.

Suppose $m_1 > 2$. Let $\mu \in \Gamma_1$. Write $\mu = u\alpha_1 + v\alpha_2$. Then there exists some b, $0 \leqslant b \leqslant m_2 - 1$ such that $(m_2 - 1 - 2b) \notin \{0, -v\}$. (To see this note that if $m_2 - 1 - 2b = 0$ then m_2 is odd, hence $\geqslant 3$.)

Since $\Gamma_0 \subseteq \mathbf{Z}\alpha_1 \cup \mathbf{Z}\alpha_2$ this implies

$$|((\Gamma_{-1} \cap (\mathbf{Z}\alpha_1 + (m_2 - 1 - 2b)\,\alpha_2)) + \mu) \cap \Gamma_0| \leqslant 1$$

and so as $m_1 > 2$ we may find two elements $\tau_1, \tau_2 \in \Gamma_{-1} \cap (\mathbf{Z}\alpha_1 + (m_2 - 1 - 2b)\alpha_2)$ such that $\tau_i + \mu \notin \Gamma_0$ for $i = 1, 2$. Since $m_2 - 1 - 2b \neq 0$, τ_1 and τ_2 are linearly independent. Then for $i = 1, 2$, $[G_{-1,\tau_i}, [G_{-1,-\mu}, G_{1,\mu}]] \subseteq [G_{-1,-\mu}, [G_{-1,\tau_i}, G_{1,\mu}]] = (0)$. Since τ_1, τ_2 are linearly independent this implies $[G_{-1,-\mu}, G_{1,\mu}] = (0)$. Since this holds for all $\mu \in \Gamma_1$ and since Y_1 and Y_2 are simple, we have $[G_{-1}, G_1] = (0)$, contradicting $G_1 \neq (0)$. Thus $m_1 = 2$ and similarly $m_2 = 2$. Since Y_i acts faithfully on G_{-1} this implies $Y_1 \cong Y_2 \cong \mathfrak{sl}(2)$.

Finally, suppose the action of G_0 on G_1 is not faithful. Then $[Y_i, G_1] = (0)$ for $i = 1$ or 2 and so $\Gamma_1 \subseteq \mathbf{Z}\alpha_2$ or $\Gamma_1 \subseteq \mathbf{Z}\alpha_1$, say, without loss of generality, $\Gamma_1 \subseteq \mathbf{Z}\alpha_1$. Then $\Gamma_{-1} + \Gamma_1 \subseteq \mathbf{Z}\alpha_1 \pm \alpha_2$ so $(\Gamma_{-1} + \Gamma_1) \cap \Gamma_0 = \varnothing$ and so $[G_{-1}, G_1] = (0)$. This contradicts the hypothesis that $G_1 \neq (0)$. Hence G_0 acts faithfully on G_1, so the lemma is proved. ∎

This completes the proof of Proposition 8.1.1. ∎

9. Determination of All Semisimples in Which Every Two-Dimensional Torus is Maximal and Standard

9.1. We now complete our analysis of cases (g) and (h) in the conclusion of Theorem 4.1.1. From this analysis we obtain:

THEOREM 9.1.1. *Let F be an algebraically closed field of characteristic $p > 7$. Let A be a finite-dimensional restricted semisimple Lie algebra over F. Let T be a two-dimensional torus of A. Assume that all two-dimensional tori in A are maximal and standard. Let $\mathfrak{z}_A(T) = T + I$, where I is the nil radical of $\mathfrak{z}_A(T)$. Then one of the following occurs*:

(a) $S_1 + S_2 \subseteq A \subseteq (\mathrm{Der}\, S_1)^{(1)} + (\mathrm{Der}\, S_2)^{(1)}$, *where S_1, S_2 are distinct ideals in A and each is isomorphic to one of* $\mathfrak{sl}(2)$, $W(1:\mathbf{1})$, $H(2:\mathbf{1})^{(2)}$.

(b) $S \otimes B_n \subseteq A \subseteq \mathrm{Der}(S \otimes B_n)$, $n > 0$, *and* $T \nsubseteq (\mathrm{Der}\, S) \otimes B_n$ *for some simple S. In this case S is one of* $\mathfrak{sl}(2)$, $W(1:\mathbf{1})$, $H(2:\mathbf{1})^{(2)}$.

(c) $\bar{S} \subseteq A \subseteq \mathrm{Der}\, S$ *with* $\dim(\bar{S} \cap T) = 1$ *for some simple S. In this case* $S = H(2:\mathbf{1})^{(2)}$ *and we may assume (replacing A by ΦA,* $\Phi \in \mathrm{Aut}(\mathrm{Der}(H(2:\mathbf{1})^{(2)}))$ *if necessary)* $H(2:\mathbf{1})^{(2)} + Fx_1D_1 \subseteq A \subseteq \mathrm{Der}(H(2:\mathbf{1})^{(2)})$.

(d) $S \otimes B_n \subseteq A \subseteq \mathrm{Der}(S \otimes B_n)$, $T \subseteq \overline{(S \otimes B_n)}$, *and* $n > 0$ *for some simple S. In this case* $S = H(2:\mathbf{1}:\Phi(\gamma))^{(1)}$ *and* $\mathfrak{z}_{S \otimes B_n}(T) \subseteq I$.

(e) $\bar{S} \subseteq A \subseteq \mathrm{Der}\, S$, $T \subseteq \bar{S}$, *and* $\dim(A/(S + I)) = 2$ *for some simple S. In this case* $S = H(2:\mathbf{1}:\Phi(\gamma))^{(1)}$.

(f) $\bar{S} \subseteq A \subseteq \mathrm{Der}\, S$, $T \subseteq \bar{S}$, *and $T \cap (S + I)$ is one-dimensional and restricted for some simple S. In this case S is one of* $W(1:\mathbf{2})$, $H(2, (2, 1))^{(2)}$, $H(2:\mathbf{1}:\Delta)$.

(g) $\bar{S} \subseteq A \subseteq \operatorname{Der} S$, $T \subseteq \bar{S}$, *and* $T \cap (S+I)$ *is one-dimensional and non-restricted for some simple* S. *In this case* S *is one of* $W(1:\mathbf{2})$, $H(2:(2,1))^{(2)}$, $H(2:\mathbf{1}:\Delta)$.

(h) $\bar{S} \subseteq A \subseteq \operatorname{Der} S$ *and* $A = S + I$ *for some simple* S. *Consequently* $A = S$ *is one of the following simple algebras*: A_2, C_2, G_2, $W(2:\mathbf{1})$, $S(3:\mathbf{1})^{(1)}$, $H(4:\mathbf{1})^{(1)}$, $K(3:\mathbf{1})$.

9.2. Before we can prove Theorem 9.1.1 we need to accumulate some information about the algebras $S(3:\mathbf{1}:\Phi)$ and $H(4:\mathbf{1}:\Phi)$.

LEMMA 9.2.1. *Let* $U = S(3:\mathbf{1}:\Phi)$. *Assume that* $\overline{U^{(2)}}$ *contains no tori of dimension greater than two. Then* $U \cong S(3:\mathbf{1})$.

Proof. We may assume, by [Wil80], that Φ is either the identity, $\Phi(1)$, or $\Phi(\gamma(\mathbf{1}))$.

If $\Phi = \Phi(1)$ then $U^{(2)}$ contains $E = D_1 + x_1^{p-1} x_2 D_2$. Since $E^p = -x_2 D_2$ and since $x_1 D_1 - x_2 D_2$, $x_1 D_1 - x_3 D_3 \in U^{(2)}$ we have that $\overline{U^{(2)}}$ contains the three-dimensional torus spanned by $\{x_i D_i \mid 1 \leqslant i \leqslant 3\}$, contradicting our hypothesis. Note that this calculation also shows that $S(3:\mathbf{1}:\Phi(1))$ is not restricted.

Now consider

$$Q = \{E \in W(3:\mathbf{1}) \mid E((x_1+1)^{p-1} \times (x_2+1)^{p-1}(x_3+1)^{p-1}\, dx_1 \wedge dx_2 \wedge dx_3) = 0\}.$$

Then $Q = S(3:\mathbf{1}:\Psi)$ for some $\Psi \in \operatorname{Aut} W(3)$ and hence is isomorphic to $S(3:\mathbf{1})$, $S(3:\mathbf{1}:\Phi(1))$, or $S(3:\mathbf{1}:\Phi(\gamma(\mathbf{1})))$. But Q contains a three-dimensional torus spanned by $\{(x_i+1) D_i \mid i = 1, 2, 3\}$ and so $Q \not\cong S(3:\mathbf{1})$, since (as $S(3:\mathbf{1})/S(3:\mathbf{1})^{(1)}$ is nil) any torus in $S(3:\mathbf{1})$ is contained in $S(3:\mathbf{1})^{(1)}$ and Theorem 1.3.1(f) shows that all maximal tori in $S(3:\mathbf{1})^{(1)}$ are two-dimensional. Since Q is clearly restricted we must have $Q \cong S(3:\mathbf{1}:\Phi(\gamma(\mathbf{1})))$, so $S(3:\mathbf{1}:\Phi(\gamma(\mathbf{1})))$ contains a three-dimensional torus and the lemma is proved. ∎

LEMMA 9.2.2. *Let* $U = H(4:\mathbf{1}:\Phi)$. *Assume that* $\overline{U^{(2)}}$ *contains no tori of dimension greater than two. Then* $\overline{U^{(2)}} \subseteq U$.

Proof. Since $H(4:\mathbf{1})^{(2)} \subseteq \operatorname{gr} U^{(2)} \subseteq \operatorname{gr} \overline{U^{(2)}} \subseteq \operatorname{gr} \operatorname{Der} H(4:\mathbf{1}:\Phi) \subseteq \operatorname{Der} H(4:\mathbf{1})$ and $\operatorname{Der} H(4:\mathbf{1}) = H(4:\mathbf{1})^{(2)} + (\operatorname{Der} H(4:\mathbf{1}))_0$ (see [Cel70] for the computation of $\operatorname{Der} H(4:\mathbf{1})$), we have $\overline{U^{(2)}} \subseteq U^{(2)} + (\overline{U^{(2)}})_0$. Now $(\overline{U^{(2)}})_0$ contains the restricted subalgebra $(\overline{U^{(2)}})_0 \cap U_0$ which contains a two-dimensional torus (as $U_0 = (U^{(2)})_0 + U_1$ and $U_0/U_1 \cong \mathfrak{sp}(4)$). Thus, as $(\overline{U^{(2)}})_0$ contains no tori of dimension greater than two, $(\overline{U^{(2)}})_0/$

$((\overline{U^{(2)}})_0 \cap U_0) \cong ((\overline{U^{(2)}})_0 + U_0)/U_0$ is nil. But $(\text{Der } H(4:\mathbf{1}))_0 = F(\sum_{i=1}^4 x_i D_i) + H(4:\mathbf{1})_0$ so $(\text{Der } H(4:\mathbf{1}))_0/H(4:\mathbf{1})_0$ is a torus and thus $(\text{Der } U)_0/U_0$ is a torus. Hence $(\overline{U^{(2)}})_0 \subseteq U_0$ and so $\overline{U^{(2)}} \subseteq U^{(2)} + (\overline{U^{(2)}})_0 \subseteq U$, as required. ▌

LEMMA 9.2.3. *Let $U = H(4:\mathbf{1}:\Phi)$. Assume that $\overline{U^{(2)}}$ contains no tori of dimension greater than two. Then $U \cong H(4:\mathbf{1})$.*

Proof. By Lemma 9.2.2 we have $\overline{U^{(2)}} \subseteq U$. Then $H(4:\mathbf{1})^{(2)} \subseteq \text{gr } \overline{U^{(2)}} \subseteq H(4:\mathbf{1})$ and so, by Lemma 8.3 of [Kac74] applied to the restricted Lie algebra $\overline{U^{(2)}}$ (and taking $\phi \in \text{Aut } W(4)$ such that $\Phi D = \phi D \phi^{-1}$ for all $D \in W(4)$), we may assume that $\phi\omega = (1 + c_1(x_1x_3)^{p-1})\, dx_1 \wedge dx_3 + (1 + c_2(x_2x_4)^{p-1})\, dx_2 \wedge dx_4$. Let S_1 denote $U^{(2)} \cap \text{span}\{x_1^{\alpha(1)}x_3^{\alpha(3)}D_i \mid i = 1, 3,\ 0 \leqslant \alpha(1), \alpha(3) \leqslant p-1\}$ and S_2 denote $U^{(2)} \cap \text{span}\{x_2^{\alpha(2)}x_4^{\alpha(4)}D_i \mid i = 2, 4,\ 0 \leqslant \alpha(2), \alpha(4) \leqslant p-1\}$. It is clear that if $c_i \neq 0$ then $\bar{S}_i$ contains a subalgebra isomorphic to $\overline{H(2:\mathbf{1}:\phi(\gamma))^{(2)}}$ and so (by Corollary 2.2.3(c)) contains a two-dimensional torus, while if $c_i = 0$ then $\bar{S}_i$ contains a subalgebra isomorphic to $H(2:\mathbf{1})^{(2)}$ and so contains a one-dimensional torus. Since $\bar{S}_1 \cap \bar{S}_2 = (0)$, $[\bar{S}_1, \bar{S}_2] = (0)$, and $\overline{U^{(2)}}$ contains no tori of dimension greater than two, we see that we must have $c_1 = c_2 = 0$, so $U \cong H(4:\mathbf{1})$. ▌

9.3. We are now ready to begin the proof of Theorem 9.1.1. In view of Theorem 4.1.1, it is only necessary to prove that if A satisfies (g) or (h) of Theorem 4.1.1 then it satisfies the same part of Theorem 9.1.1. In this section we will do this for algebras satisfying (h) of Theorem 4.1.1.

Let A satisfy Theorem 4.1.1(h). Let A_0 be a distinguished maximal subalgebra of A containing the two-dimensional standard maximal torus T. Give A a corresponding filtration and let $G = \sum G_i$ be the associated graded algebra. Assume that if U is any algebra satisfying Theorem 4.1.1(h) and $\dim U < \dim A$, then $\bar{S} \subseteq U \subseteq \text{Der } S$, where S is one of A_2, C_2, G_2, $W(2:\mathbf{1})$, $S(3:\mathbf{1})^{(1)}$, $H(4:\mathbf{1})^{(1)}$, $K(3:\mathbf{1})$.

If $\mathfrak{z}(G_0) \neq (0)$ then by Proposition 7.1.1 either

$$G_1 = (0) \tag{9.3.1}$$

or

$$G_0 \cong \mathfrak{sl}(2) + Fz,\ N(G) = 0, \text{ and } G_0 \text{ acts faithfully on } G_1. \tag{9.3.2}$$

Also $I(G_0) = (0)$ by Corollary 7.2.8 and so if (9.3.1) holds, we have $I = (0)$. But then A satisfies the hypotheses of Theorem 7.2.1 of [BW82] and so A is one of the algebras listed in Theorem 9.1.1. Thus if $\mathfrak{z}(G_0) \neq (0)$ we may assume that (9.3.2) holds. But then G satisfies the hypotheses of Kac's theorem on graded algebras (Theorem 1.2.1) so G is either classical simple or of Cartan type (and hence isomorphic to $W(2:\mathbf{1})$ or $K(3:\mathbf{1})$ since $G_0 \cong \mathfrak{sl}(2)+Fz$). Thus G must be simple and so $A=S$ is simple. Then Kac's recognition theorem for restricted simple Lie algebras of Cartan type (cf. Proposition 1.1.1 and Theorem 1.2.2) shows that A is classical, $W(2:\mathbf{1})$, or $K(3:\mathbf{1})$.

We may therefore assume that $\mathfrak{z}(G_0)=(0)$. Then by Proposition 8.1.1 one of the following occurs:

$$\bar{Y} \subseteq G_0 \subseteq \operatorname{Der} Y \text{ for some simple } Y \text{ and for every two-dimensional torus } T_1 \subseteq G_0 \text{ we have } T_1 \subseteq \bar{Y} \text{ and } G_0 = Y + I_1; \tag{9.3.3}$$

$$G_0 \cong \mathfrak{sl}(2)\oplus\mathfrak{sl}(2),\quad \mathfrak{sl}(2)\oplus W(1:\mathbf{1}),\quad \text{or}\quad W(1:\mathbf{1})\oplus W(1:\mathbf{1}) \text{ and } G_1=(0); \tag{9.3.4}$$

$$G_0 \cong \mathfrak{sl}(2)\oplus\mathfrak{sl}(2),\ N(G)=(0), \text{ and } G_0 \text{ acts faithfully on } G_1. \tag{9.3.5}$$

If (9.3.4) holds then since $I(G_0)=(0)$ and $G_1=(0)$ we have $I=(0)$. Hence Theorem 7.2.1 of [BW82] applies and so A is one of the algebras listed in Theorem 9.1.1. If (9.3.5) holds then G satisfies the hypotheses of Kac's theorem on graded algebras [Kac70] so G is either classical or of Cartan type. But there is no G of classical or Cartan type satisfying $G_0 \cong \mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$. Thus (9.3.5) cannot occur.

Suppose G_0 satisfies (9.3.3). Since $\dim G_0 < \dim A$ the induction assumption implies that Y is one of the algebras listed in Theorem 9.1.1(h), i.e., Y is classical or isomorphic to some $W(2:\mathbf{1})$, $S(3:\mathbf{1})^{(1)}$, $H(4:\mathbf{1})^{(1)}$, $K(3:\mathbf{1})$. But $\bar{Y}$ acts faithfully on G_{-1} and by Lemma 6.3.2, $|\Gamma_{-1}| \leqslant p^2-p+6$. Since $W(2:\mathbf{1})$ has no faithful restricted representations with fewer than p^2-2 weights (by Corollary 4.11.2 of [BW82]) and $\bar{Y}$ has no faithful restricted representations with fewer than p^2-1 weights (by Lemma 6.5.1), if $Y=S(3:\mathbf{1})^{(1)}$, $H(4:\mathbf{1})^{(1)}$, or $K(3:\mathbf{1})$ we see that G_0 must be classical. Also if $G_1=(0)$ we have $I=(0)$ so A satisfies the hypotheses of Theorem 7.1.1 of [BW82] and hence A is one of the algebras listed in Theorem 9.1.1. Therefore we are done unless

$$G_0 \text{ is classical simple and } G_1 \neq (0). \tag{9.3.6}$$

We claim that G_0 acts faithfully on G_1. For if not (as G_0 is simple) we have $G_1=G_{1,0}$. But $[G_{-1},G_1]$ is a nonzero ideal in G_0 so $G_0=[G_{-1},G_1]$. Hence $T\subseteq G_{0,0}=[G_{-1},G_1]_0=[G_{-1},G_{1,0}]_0=[G_{-1,0},G_{1,0}]=0$, a contradiction. Then Kac's theorem on graded algebras Theorem 1.2.1 applies to $G'/N(G)$, where G' is the subalgebra of G generated by $\sum_{i\leqslant 1} G_i$. It follows that $G'/N(G)$ is of classical or Cartan type. Since G_0 is classical simple (of rank two), $G'/N(G)$ cannot be classical and so must satisfy $S(3:\mathbf{n})^{(1)} \subseteq G'/N(G) \subseteq S(3:\mathbf{n})$ or $H(4:\mathbf{n})^{(1)} \subseteq G'/N(G) \subseteq H(4:\mathbf{n})$. Since A

is restricted so are G and G' (by Lemma 1.19.1 of [BW82]). Thus $\mathbf{n} = \mathbf{1}$, so $S(3:\mathbf{1})^{(1)} \subseteq G'/N(G) \subseteq S(3:\mathbf{1})$ or $H(4:\mathbf{1})^{(1)} \subseteq G'/N(G) \subseteq H(4:\mathbf{1})$. In either case $G'/N(G)$ acts on $N(G)/[N(G), N(G)]$ and thus if $N(G) \neq (0)$ we have that $\bar{X}$ (where $X = S(3:\mathbf{1})^{(1)}$ or $H(4:\mathbf{1})^{(1)}$) acts faithfully on $N(G)/[N(G), N(G)]$. But then by Lemma 6.5.1 we see that $N(G)/[N(G), N(G)]$ has $\geqslant p^2 - 1$ weights. This contradicts the fact that the set of weights of $N(G)/[N(G), N(G)]$ is contained in Γ_- and $|\Gamma_-| \leqslant p^2 - p + 6$ by Lemma 6.3.2. Thus $N(G) = (0)$ and so the hypotheses of Kac's theorem on graded algebras apply to G. Thus either $S(3:\mathbf{1})^{(1)} \subseteq G \subseteq S(3:\mathbf{1})$ or $H(4:\mathbf{1})^{(1)} \subseteq G \subseteq H(4:\mathbf{1})$. In either case we see that $A = A_{-1}$ and $\Gamma_{-1} \cap \Gamma_0 = \varnothing$. Since $I \subseteq A_1$ we have $[I, A] \subseteq A_0$ and since $\Gamma_{-1} \cap \Gamma_0 = \varnothing$ we have $[I, A] \subseteq A_1$. Thus $I \subseteq A_2$ and so $A = S + I = S + A_2$. Setting $S_i = S \cap A_i$ we see by Lemma 2.2 of [Wil76] that S_0 is a maximal subalgebra of S and $S = S_{-1} \supseteq S_0 \supseteq \cdots$ is a corresponding filtration. Moreover, $S_{-1}/S_0 \cong G_{-1}$, $S_0/S_1 \cong G_0$, $S_1/S_2 \cong G_1$ so that S satisfies the hypotheses of the Recognition Theorem for algebras of Cartan type. Thus $S = S(3:\mathbf{n}:\Phi)^{(1)}$ or $H(4:\mathbf{n}:\Phi)^{(1)}$. As $\dim S \leqslant \dim G$ we must have $\mathbf{n} = \mathbf{1}$. Thus $S = S(3:\mathbf{1}:\Phi)^{(1)}$ or $H(4:\mathbf{1}:\Phi)^{(1)}$. By Lemmas 9.2.1 and 9.2.3 we may assume $\Phi =$ identity. Thus S is among the algebras listed in part (h) of Theorem 9.1.1. By the remark at the beginning of Section 5.8 we have $S = A$. ∎

9.4. We now complete our proof of Theorem 9.1.1 by showing that if A satisfies (g) of Theorem 4.1.1 then S is one of $W(1:\mathbf{2})$, $H(2:(2,1))^{(2)}$, $H(2:\mathbf{1}:\Delta)$. We prove this by showing that if A satisfies (g) of Theorem 4.1.1 then A contains a two-dimensional torus T' such that $T' \cap (S + I')$ is restricted.

LEMMA 9.4.1. *Let U be a nonsolvable restricted Lie algebra in which every two-dimensional torus is maximal and standard. Assume that $\mathfrak{z}(U) = Fz + N$, where $z^p = z$ and N is nil. Then there exists a two-dimensional torus $R' \subseteq U$ with $R' \cap ([U, U] + I')$ nonzero and restricted.*

Proof. We may assume that $U/(\operatorname{solv} U) \cong \mathfrak{sl}(2)$. For by Theorem 3.1.1, $U/(\operatorname{solv} U)$ is isomorphic to $\mathfrak{sl}(2)$, $W(1:\mathbf{1})$, or a subalgebra of $H(2:\mathbf{1})$ containing $H(2:\mathbf{1})^{(2)}$. Any of these algebras contains a restricted subalgebra isomorphic to $\mathfrak{sl}(2)$. Letting V denote the inverse image of this subalgebra in U, we see that V satisfies the hypotheses of the lemma and that $V/(\operatorname{solv} V) \cong \mathfrak{sl}(2)$. If the lemma is proved for V with $V/(\operatorname{solv} V) \cong \mathfrak{sl}(2)$ then there exists $R' \subseteq V$ with $R' \cap ([V, V] + I'_V)$ nonzero and restricted (where I'_V is the nil radical of $\mathfrak{z}_V(R')$). But then, as $R' \cap ([U, U] + I') \supseteq R' \cap ([V, V] + I'_V)$, either $R' \cap ([U, U] + I') = R'$, which is restricted, or else $R' \cap ([U, U] + I') = R' \cap ([V, V] + I'_V)$, which is restricted.

From now on we assume $U/(\operatorname{solv} U) \cong \mathfrak{sl}(2)$.

Suppose M is a nonzero restricted nil ideal in U. Then U/M again satisfies the hypotheses of the lemma and the result follows by induction. Hence we may assume that U contains no nonzero nil ideals.

Let R be any two-dimensional torus in U. Clearly $z \in R$. Suppose that $z \in [U, U] + I$. Since R maps onto a maximal torus of $U/(\text{solv } U) \cong \mathfrak{sl}(2)$ and since $\mathfrak{sl}(2)$ is equal to its derived algebra, we have that $R \cap ([U, U] + (\text{solv } U)) \nsubseteq \text{solv } U$. Thus $R \cap ([U, U] + Fz + I) \nsubseteq \text{solv } U$. As we are assuming $z \in [U, U] + I$, we have $R \cap ([U, U] + I) \nsubseteq \text{solv } U$. Thus $R = R \cap ([U, U] + I)$ is restricted and we are done. Hence we may assume $z \notin [U, U] + I$.

Now since $U/(\text{solv } U) \cong \mathfrak{sl}(2)$ we have $I \subseteq \text{solv } U$. Let $J = I + \sum_{\alpha \neq 0} (\text{solv } U)_\alpha$. We claim that J is an ideal in U. Clearly it is enough to check that $[U_\alpha, (\text{solv } U)_{-\alpha}] \subseteq I$ for all roots α. If not, we have $z \in [U_\alpha, (\text{solv } U)_{-\alpha}] + I \subseteq [U, U] + I$, contradicting our assumption. If $x \in (\text{solv } U)_\alpha$ then $x^p \in \mathfrak{z}_{\text{solv } U}(R) = Fz + I$. Thus $(\text{ad } x)^p = \text{ad}(x^p)$ is nilpotent. By the Engel–Jacobson theorem J is nilpotent.

Suppose $J \neq (0)$. Then $\mathfrak{z}(J) \neq (0)$ and (as $\text{solv } U = J + Fz$) $\mathfrak{z}(J)$ is a $U/(\text{solv } U)$-module. Let M be an irreducible $U/(\text{solv } U)$-submodule of $\mathfrak{z}(J)$. Then M is a nonzero ideal of U so, by our assumption, M cannot be nil. Thus there exists $x \in M_\alpha$ for some α such that $x^p = z$. Let $R = Fr + Fz$, where $r^p = r$. Assume (replacing r by an element of $\mathbf{Z}r$ if necessary) that $[r, x] = x$. Let H denote the image of r in $U/(\text{solv } U)$ and A, B denote root vectors in $U/(\text{solv } U)$ with respect to FH satisfying $[A, B] = H$. Let $a, b \in U$ be root vectors such that $A = a + \text{solv } U$, $B = b + \text{solv } U$. Let $h = [a, b]$. Then $h = r + \mu z + n$, where $\mu \in F$ and $n \in I$. Let $\lambda \in F$ satisfy $\mu^p - \lambda^p - \mu = 0$. Then $[a - \lambda[a, x], \quad b - \lambda[b, x]] = [a, b] - \lambda[[a, b], x] = h - \lambda[h, x]$. Since $I \subseteq J$ and $[J, x] = 0$ we have $[h, x] = rx = x$. Thus $[U, U]$ contains $h - \lambda x$.

Write $h - \lambda x = h' = h'_s + h'_n$, where h'_s is semisimple and h'_n is nilpotent. Then $R' = Fh'_s + Fz$ is a two-dimensional torus and $R' \cap ([U, U] + I')$ contains h'_s. It is therefore enough to show that $(h'_s)^p = h'_s$ (for then $R' \cap ([U, U] + I')$ is restricted), hence enough to show that $(h')^p - h'$ is nil. Now $(h')^p = (h - \lambda x)^p = h^p - \lambda^p x^p + \sum s_i$, where is_i is the coefficient of v^{i-1} in $(\text{ad}(vh - \lambda x))^{p-1} h = (\text{ad}(vh - \lambda x))^{p-2} (\lambda x) = v^{p-2} \lambda x$. Thus $(h')^p = h^p - \lambda^p z - \lambda x$ and so $(h')^p - h' = (r^p + \mu^p z + n^p - \lambda^p z - \lambda x) - (r + \mu z + n - \lambda x) = r^p - r + (\mu^p - \lambda^p - \mu) z + n^p + n = n^p + n \in I$. Thus $(h')^p - h'$ is nil, as required. Thus if $J \neq (0)$ the lemma holds, so we may assume that $J = (0)$.

If $J = (0)$ we have $\mathfrak{z}(U) = Fz$ and $U/\mathfrak{z}(U) \cong \mathfrak{sl}(2)$. Thus U has basis $\{a, b, r, z\}$, where $Fz = \mathfrak{z}(U)$, $z^p = z$, $r^p = r$, $[r, a] = 2a$, $[r, b] = -2b$, and $[a, b] = r + \mu z$, $\mu \in F$. If $a^p \neq 0$ we may assume (replacing a by a scalar multiple if necessary) that $a^p = z$. Let $\lambda \in F$ satisfy $\mu^p - \lambda^p - \mu = 0$. Then $[a, b + (\lambda/2) r] = r + \mu z - \lambda a$ and $(r + \mu z - \lambda a)^p = r^p + \mu^p z^p - \lambda^p z - \lambda a = r + \mu z - \lambda a$. Thus $R' = F(r + \mu z - \lambda a) + Fz$ is a two-dimensional torus in U

and $r + \mu z - \lambda a \in R' \cap [U, U]$ so $R' \cap ([U, U] + I)$ is nonzero and restricted. Hence we may assume $a^p = 0$ and similarly we may assume that $b^p = 0$. Then consider $a + b = [r, a - b]/2 \in [U, U]$. Also, $(a+b)^p \equiv a^p + b^p = 0 \bmod [U, U]$. On the other hand, the linear map $\phi: U \to \mathfrak{sl}(2)$ defined by $\phi(a) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\phi(b) = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $\phi(r) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\phi(z) = 0$ is a homomorphism of restricted Lie algebras with kernel Fz. Thus $\phi((a+b)^p - (a+b)) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^p - \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = 0$ so $(a+b)^p \equiv a + b \bmod Fz$. Thus $(a+b)^p = a + b + h$, where $h \in Fz \cap [U, U] = (0)$. Thus $(a+b)^p = a + b$. Setting $R' = F(a+b) + Fz$ we have $R' \cap ([U, U] + I')$ is restricted. This completes the proof of the lemma. ∎

Now suppose that A satisfies (g) of Theorem 4.1.1, that is, $\bar{S} \subseteq A \subseteq \operatorname{Der} S$, where S is simple, $T \subseteq \bar{S}$, and $T \cap (S + I)$ is one-dimensional and nonrestricted. Since S is simple, so $S = [S, S]$, we have that $[S_\alpha, S_{-\alpha}] \not\subseteq I$ for some α. If $\alpha([S_\alpha, S_{-\alpha}]) = (0)$ then $T \cap (S + I) = \ker \alpha$, which is restricted, contradicting our hypotheses. Hence $S^{(\alpha)}$ is not solvable. We may therefore apply Lemma 9.4.1 to $U = T + \overline{S^{(\alpha)}} \subseteq \bar{S}$ to obtain a two-dimensional torus $T' \subseteq U$ such that $T' \cap ([U, U] + I')$ is nonzero and restricted. But $[U, U] \subseteq S$ so $T' \cap (S + I')$ contains a nonzero restricted subalgebra. Since T' is two-dimensional this implies that $T' \cap (S + I')$ is restricted. Note that $T' \subseteq U \subseteq \bar{S}$. If $T' \subseteq S + I'$ then $A = S + I'$, so A satisfies Theorem 4.1.1(h) and hence (by Section 9.3) satisfies Theorem 9.1.1(h). Thus S is restricted. But then as $T \subseteq \bar{S}$ we have $T \subseteq (S + I)$, contradicting our assumption that (g) of Theorem 4.1.1 holds. Thus we must have $\dim T' \cap (S + I') = 1$. By (f) of Theorem 4.1.1, S is one of $W(1 : \mathbf{2})$, $H(2 : (2, 1))^{(2)}$, $H(2 : \mathbf{1} : \Delta)$. Thus (g) of Theorem 9.1.1 holds and the proof of Theorem 9.1.1 is complete.

10. All Roots Can Be Made Proper

In this section we prove that if L is a restricted simple Lie algebra then L contains a torus T of maximal dimension such that all roots with respect to T are proper. We begin (Section 10.1) with some technical results about Cartan decompositions of the algebra $H(2 : (2, 1))^{(2)}$ analogous to the results of Section 5.8. (These results will also be used in Section 11.) In Section 10.2 we show that certain of the algebras of Theorem 9.1.1 cannot occur as sections of L. In Sections 10.3 and 10.4 we show that if T is an optimal torus (recall Definition 6.2.1) in L then $\Gamma(L, T) = \Gamma_p(L, T)$.

10.1. We begin by investigating the Cartan decompositions of $H(2 : (2, 1))^{(2)}$.

LEMMA 10.1.1. *Let* $S = H(2:(2,1))^{(2)}$, $\bar{S} \subseteq A \subseteq \operatorname{Der} S$, *and* T *be a two-dimensional torus in* A. *Assume that* A *contains no tori of dimension* >2. *Let* $\alpha \in \Delta(A, T)$ *and let* t_α *be as in* (5.8.1). *Then*:

(a) *If* $t_\alpha \notin S$ *then* $A[\alpha] \supseteq H(2:\mathbf{1})^{(2)}$.

(b) *If* $t_\alpha \in S$ *then* $A[\alpha] = (0)$.

(c) *If* $t_\alpha \in S_0$ *then* $x^p \in S_0$ *for all* $x \in A_\alpha$.

(d) $\Delta(A, T) = \Delta_P(A, T)$ *if and only if* $T \cap S_0 \neq (0)$.

(e) *If* $\alpha \in \Delta(A, T)$, $\alpha \notin \Delta_P(A, T)$, $x \in A_\alpha$, *and* $\alpha_x \in \Delta_P(A, e^x(T))$ *then* $\Delta(A, e^x(T)) = \Delta_P(A, e^x(T))$ (*and so* $|\Delta_P(A, e^x(T))| > |\Delta_P(A, T)|$).

(f) *If* $\Delta(A, T) = \Delta_P(A, T)$ *then* $T \nsubseteq S$ *and there exist* $\beta \in \Delta(T)$ *and* $x \in A_\beta$ *such that* $A[\beta] = (0)$ *and* x^p *is not nil.*

Proof. Write t for t_α, so that $A^{(\alpha)} = \mathfrak{z}_A(t)$.

Since $\bar{S}$ does contain a two-dimensional torus (Corollary 2.2.3(b)), the hypothesis that A contains no tori of dimension greater than two implies that $A/\bar{S}$ is nil. Hence $T \subseteq \bar{S}$ and so we may assume, without loss of generality, that $A = \bar{S}$.

Now suppose $t \equiv a(\operatorname{ad} D_1^p) \bmod S$, where $0 \neq a \in F$. Let $S_{(l)} = \{E \in S \mid [t, E] = lE\}$. Then it is clear that $\operatorname{gr} S_{(l)} \subseteq \mathfrak{z}_S((\operatorname{ad} D_1)^p) = \operatorname{span}(\{\mathscr{D}(x^{i\varepsilon_1 + j\varepsilon_2}) \mid 0 \leqslant i, \quad j \leqslant p-1, \quad (i,j) \neq (0,0)\} \cup \{\mathscr{D}(x^{p\varepsilon_1})\})$. Thus $\mathfrak{z}_S((\operatorname{ad} D_1)^p)$ is a p^2-dimensional subspace of $H(2:\mathbf{1})$ containing $H(2:\mathbf{1})^{(2)}$ and so $\dim S_{(l)} \leqslant p^2$ for all l. Since $S = \sum_{l=0}^{p-1} S_{(l)}$ has dimension $p^3 - 2$ we see that $\dim S_{(l)} = p^2$ for some l and so $\operatorname{gr} S_{(l)} = \mathfrak{z}_S((\operatorname{ad} D_1)^p)$ for this l. But then, since $[S_{(u)}, S_{(v)}] \subseteq S_{(u+v)}$ we see that $\operatorname{gr} S_{(kl)} \supseteq H(2:\mathbf{1})^{(2)}$ for all k. In particular, $\operatorname{gr} S_{(0)} \supseteq H(2:\mathbf{1})^{(2)}$. Thus by [Kac74, Wil76], $S_{(0)} \supseteq H(2:\mathbf{1}:\Phi)^{(2)}$ for some Φ and so we have $A[\alpha] \supseteq H(2:\mathbf{1}:\Phi)^{(2)}$. Then by Theorem 3.1.1 we must have $A[\alpha] \supseteq H(2:\mathbf{1})^{(2)}$, proving (a).

Now suppose $t \in S$, $t \notin S_0$. Since $t^p = t$ this implies that (after applying a suitable automorphism) $t \equiv D_2 \pmod{S_0}$. Then $\operatorname{gr}(\mathfrak{z}_S(t)) \subseteq \mathfrak{z}_S(D_2) = \operatorname{span}(\{\mathscr{D}(x^{i\varepsilon_1}) \mid 1 \leqslant i \leqslant p^2 - 1\} \cup \{\mathscr{D}(x^{\varepsilon_2})\})$. It is then clear that $(\mathfrak{z}_S(t))_0 + Ft$ is a solvable ideal of codimension $\leqslant 1$ in $\mathfrak{z}_S(t)$. Hence $\mathfrak{z}_S(t)$ is solvable, so (b) holds in this case.

Suppose $t \in S_0$. Then if $\bar{t} = (t + S_1) \in S_0/S_1 \cong \mathfrak{sl}(2)$ we see that $\operatorname{ad} \bar{t}$ is a linear transformation of trace zero on the two-dimensional space S_{-1}/S_0. Since $t \notin S_1$ (for S_1 is nil), $\operatorname{ad} \bar{t}$ acts nontrivially on S_{-1}/S_0 and hence $\mathfrak{z}_S(t) \subseteq S_0$ and so $\mathfrak{z}_S(t) \subseteq Ft + S_1$ is solvable. This proves (b) and (c) follows since S_0 is restricted.

Now $S_0 \cap A^{(\alpha)}$ is a subalgebra of codimension $\leqslant 3$ in $A^{(\alpha)}$. Thus if α is Hamiltonian (so $t_\alpha \notin S$), $S_0 \cap A^{(\alpha)} + Ft_\alpha$ is a compositionally classical subalgebra of codimension $\leqslant 2$ in $A^{(\alpha)}$. By Lemma 5.3.6 it is the unique subalgebra subalgebra with these properties. Thus, by Corollary 5.6.4, a

Hamiltonian root α is proper if and only if $T \subseteq S_0 \cap A^{(\alpha)} + Ft_\alpha$ or, equivalently, $T \cap S_0 \neq (0)$. Thus if $T \cap S_0 \neq (0)$, every Hamiltonian root is proper. Since (by (a), (b)) there are no Witt roots we see that $\Delta(A, T) = \Delta_P(A, T)$. Conversely, if $\Delta(A, T) = \Delta_P(A, T)$ and there is some Hamiltonian root, then $T \cap S_0 \neq (0)$, while if there is no Hamiltonian root then every $t_\alpha \in S$, so $T \subseteq S$. But as T is restricted this implies $T \subseteq FD_2 + S_0$ so $T \cap S_0 \neq (0)$. Thus (d) holds.

If $\alpha \in \Delta(A, T)$, $\alpha \notin \Delta_P(A, T)$, then (by (a)–(c)) α must be Hamiltonian, so α_x is also Hamiltonian. Since α_x is proper, $S_0 \cap e^x(T) \neq (0)$, so $\Delta(A, e^x(T)) = \Delta_P(A, e^x(T))$. Thus (e) holds.

Now assume that $\Delta(A, T) = \Delta_P(A, T)$ so $S_0 \cap T \neq (0)$. Since $S_0 \cap T$ is restricted there is some root β vanishing on $S_0 \cap T$. Write $S_0 \cap T = Fu$. Then $\mathrm{ad}(u + S_1)$ is a nonzero linear transformation of trace zero on S/S_0, so $\mathfrak{z}_S(u) \subseteq S_0$. Since $S_0/S_1 \cong \mathfrak{sl}(2)$ and S_1 is nil this implies $T \nsubseteq S$. Hence there is some $v \in T$ with $v^p = v$ and $v \equiv a(\mathrm{ad}\, D_1)^p \bmod S$, $a \neq 0$. We have seen above that if $S_{(l)} = \{E \in S \mid [v, E] = lE\}$ then $\mathrm{gr}\, S_{(l)} \supseteq H(2:\mathbf{1})^{(2)}$ for any l, in particular for $l = \beta(v)$. Then $u + S_1 \in \mathrm{gr}\, S_\beta$. Since u is not nil this shows that A_β contains a nonnil element, proving (f). ∎

10.2. We now show that certain of the toral rank two semisimple algebras of Theorem 9.1.1 cannot occur as sections of a simple Lie algebra.

If A is a semisimple restricted Lie algebra with maximal torus T and if $\alpha \in \Delta(A, T)$, we define $A\{\alpha\}$ to be the subalgebra of A (in fact of $A^{(\alpha)}$) generated by $\sum_{i=1}^{p-1} A_{i\alpha}$.

LEMMA 10.2.1. *Let A be a restricted semisimple Lie algebra containing a torus T of maximal dimension. Assume that T is standard (so $\mathfrak{z}_A(T) = T + I$, where I is a nil subalgebra of $\mathfrak{z}_A(T)$) and that* $\dim T \leqslant 2$. *Assume further that there exist $\alpha, \beta \in \Delta(A, T)$ and $x \in \overline{A\{\alpha\}} \cap \mathfrak{z}_A(T)$, such that $\alpha(x) = 0$, $\beta(x) \neq 0$, and $\alpha([A_\beta, A_{-\beta}]) \neq (0)$. Then*:

(a) $\dim T = 2$.

(b) *If M is a nonzero restricted ideal of A then $T \subseteq M$.*

(c) *A is one of the algebras listed in* (e)–(h) *of Theorem* 9.1.1.

Furthermore, if $x \in A\{\alpha\} \cap \mathfrak{z}_A(T)$ then:

(d) *If M is a nonzero ideal of A then $T \subseteq M + I$.*

(e) *A is one of the algebras listed in Theorem* 9.1.1(h).

Proof. Since α and β are linearly independent (as $\alpha \neq 0$ and $\alpha(x) = 0$, $\beta(x) \neq 0$) we have $\dim T \geqslant 2$, so $\dim T = 2$. Thus (a) holds.

Now let M be a restricted ideal in A. If $\alpha(\mathfrak{z}_M(T)) = (0)$ then $M_\beta \neq A_\beta$ (since $\alpha([A_\beta, A_{-\beta}]) \neq (0)$). Therefore $\beta(\mathfrak{z}_M(T)) = (0)$ and so (since α, β

are linearly independent) $\mathfrak{z}_M(T) \subseteq I$. Then the Engel–Jacobson theorem implies that M is nil. Since A is semisimple this implies $M = (0)$. Thus if $M \neq (0)$ we have $\alpha(\mathfrak{z}_M(T)) \neq (0)$ so $A\{\alpha\} \subseteq M$ and hence $x \in M$. This implies $\beta(\mathfrak{z}_M(T)) \neq (0)$, so $[A_\beta, A_{-\beta}] \subseteq M$. But (as $\dim T = 2$) $\mathfrak{z}_A(T) = Fx + [A_\beta, A_{-\beta}] + I$ so $A = M + I$. Since M is restricted, Lemma 1.8.2 shows that $T \subseteq M$. Thus (b) is proved.

Observe that the algebras of (a)–(c) in Theorem 9.1.1 each contain a restricted ideal not containing T (S_1 in case (a), $A \cap ((\operatorname{Der} S) \otimes B_n)$ in case (b), $\bar{S}$ in case (c)). If A is as in (d) then $A_\beta, A_{-\beta} \subseteq S \otimes B_n$ so $[A_\beta, A_{-\beta}] \subseteq \mathfrak{z}_{S \otimes B_n}(T) \subseteq I$, contradicting $\alpha([A_\beta, A_{-\beta}]) \neq (0)$. Thus A cannot be one of the algebras of (a)–(d) in Theorem 9.1.1. This proves (c).

Now suppose $x \in A\{\alpha\} \cap \mathfrak{z}(T)$. If M is a nonzero ideal in A then $\bar{M}$ is a nonzero restricted ideal, so $A = \bar{M} + I$. Then $A\{\alpha\}$, $[A_\beta, A_{-\beta}] \subseteq M$ so $Fx + [A_\beta, A_{-\beta}] \subseteq M$ and hence $A = M + I$. This proves (d). If A is one of the algebras of (e)–(g) in Theorem 9.1.1 then S is a nonzero ideal with $S + I \neq A$. Thus A cannot be one of these algebras, so (e) holds. ▌

Corollary 10.2.2. *Let L be a restricted simple Lie algebra and T be a torus of maximal dimension in L. Let $\alpha \in \Delta(L, T)$ and $u \in \overline{L\{\alpha\}} \cap \mathfrak{z}_L(T)$. Assume that $\alpha(u) = 0$ and $u \notin I$. Then there exists $\beta \in \Delta(L, T)$ such that $\beta(u) \neq 0$, $\alpha([L_\beta, L_{-\beta}]) \neq (0)$ and $L[\alpha, \beta]$ is one of the algebras listed in* (e)–(h) *of Theorem* 9.1.1. *Furthermore, if $u \in L\{\alpha\} \cap \mathfrak{z}_L(T)$ then $L[\alpha, \beta]$ is listed in* (h) *of Theorem* 9.1.1.

Proof. Schue's lemma (Lemma 1.12.1) shows that $\mathfrak{z}_L(T) = \sum_{\gamma(u) \neq 0} [L_\gamma, L_{-\gamma}]$. Since $\alpha(\mathfrak{z}_L(T)) \neq (0)$, the required β exists. Then applying Lemma 10.2.1 to $A = L[\alpha, \beta]$ (noting that the hypotheses on $\Psi_{\alpha,\beta} T$ are satisfied in view of Proposition 1.7.4 and replacing x by $\Psi_{\alpha,\beta} u$) gives the result. ▌

Lemma 10.2.3. *Let L be a restricted simple Lie algebra and T be a torus of maximal dimension in L. Let $\alpha \in \Delta_P(L, T)$, $x, y \in L_\alpha$. Suppose that $x, y \in M$, where M is a compositionally classical subalgebra of codimension $\leqslant 2$ in $L^{(\alpha)}$, and that $\alpha((\operatorname{ad} x)^{p-1} y) = 0$. Then $(\operatorname{ad} x)^{p-1} y \in I$.*

Proof. Suppose $(\operatorname{ad} x)^{p-1} y \notin I$. Then applying Corollary 10.2.2 (with $u = (\operatorname{ad} x)^{p-1} y$) we see that for some β we have that $L[\alpha, \beta] = S$, where S is classical, $W(2:\mathbf{1})$, $S(3:\mathbf{1})^{(1)}$, $H(4:\mathbf{1})^{(1)}$, or $K(3:\mathbf{1})$. Furthermore, $\beta((\operatorname{ad}(\Psi_{\alpha,\beta} x))^{p-1} (\Psi_{\alpha,\beta} y)) \neq 0$. This is clearly impossible if S is classical (for then $\mathbf{Z}\alpha \cap \Delta(S, T) = \{\pm\alpha\}$). We claim that $\Psi_{\alpha,\beta} x$, $\Psi_{\alpha,\beta} y \in L[\alpha, \beta]_0$. If α is solvable or classical this follows from Lemma 5.8.2(d). If α is Witt or Hamiltonian we note that x, $y \in M$, a compositionally classical subalgebra of codimension $\leqslant 2$ in $L^{(\alpha)}$. Thus $\Psi_{\alpha,\beta} x$, $\Psi_{\alpha,\beta} y \in \Psi_{\alpha,\beta}(M)$, a compositionally classical subalgebra of codimension $\leqslant 2$ in $L[\alpha, \beta]^{(\alpha)}$. By

Lemma 5.8.2(f), $(L[\alpha, \beta]^{(\alpha)})_0 + \Psi_{\alpha,\beta}\,\text{solv}\, L^{(\alpha)}$ is a compositionally classical subalgebra of codimension 1 in $L[\alpha, \beta]^{(\alpha)}$ if α is Witt and of codimension 2 in $L[\alpha, \beta]^{(\alpha)}$ if α is Hamiltonian. Then, by Lemma 5.3.6, $\Psi_{\alpha,\beta}x$, $\Psi_{\alpha,\beta}y \in (L[\alpha, \beta]^{(\alpha)})_0 + \Psi_{\alpha,\beta}\,\text{solv}\, L^{(\alpha)}$. By Lemma 5.8.2(e), $(L[\alpha, \beta]^{(\alpha)})_0 + \Psi_{\alpha,\beta}\,\text{solv}\, L^{(\alpha)} \subseteq (L[\alpha, \beta]^{(\alpha)})_0 + \Psi_{\alpha,\beta}T$ and since α is proper, Corollary 5.8.2(g) shows that $[\Psi_{\alpha,\beta}T, (L[\alpha, \beta]^{(\alpha)})_0] \subseteq (L[\alpha, \beta]^{(\alpha)})_0$. Thus $\Psi_{\alpha,\beta}x$, $\Psi_{\alpha,\beta}y \in (L[\alpha, \beta]^{(\alpha)})_0$. This contradicts Lemma 5.8.5. ▮

COROLLARY 10.2.4. *Let A be a restricted semisimple Lie algebra. Assume all two-dimensional tori in A are maximal and standard. Let T be a two-dimensional torus in A, $\alpha \in \Delta(A, T)$, $x, y \in A_\alpha$, $\alpha((\text{ad}\, x)^{p-1} y) = 0$, and $(\text{ad}\, x)^{p-1} y \notin I$. Let $M \subseteq A^{(\alpha)}$ be a compositionally classical subalgebra of codimension $\leqslant 2$. Assume x, $y \in M$. Then A cannot be a rank two section (with respect to a torus of maximal dimension) of a restricted simple Lie algebra.*

Proof. If A were a rank two section of a restricted simple Lie algebra L, say $A = \Psi_{\gamma,\delta}L$, then (Lemma 1.7.2(b)) there would be a torus R of maximal dimension in A such that $T = \Psi_{\gamma,\delta}R$. Then we may extend α, β to elements of R^* vanishing on $R \cap \ker \Psi_{\gamma,\delta}$ so that $A = \Psi_{\alpha,\beta}L$. Then Lemma 10.2.3 gives the result. ▮

COROLLARY 10.2.5. *Let A be a restricted semisimple Lie algebra satisfying $S \otimes B_n \subseteq A \subseteq \text{Der}(S \otimes B_n)$, $n > 0$, $A/(A \cap ((\text{Der}\, S) \otimes B_n))$ not nil, $S \cong \mathfrak{sl}(2)$, $W(1:\mathbf{1})$, or $H(2:\mathbf{1})^{(2)}$. (That is, let A be one of the algebras of Theorem 9.1.1(b).) Assume A is a rank two section (with respect to a torus of maximal dimension) of a restricted simple Lie algebra. Then $n \leqslant 2$.*

Proof. Since $S \otimes B_n$ is a restricted ideal which is not nil, it contains an element $h \neq 0$ satisfying $h^p = h$. Since $(A + ((\text{Der}\, S) \otimes B_n))/((\text{Der}\, S) \otimes B_n)$ is not nil and $[h, A] \subseteq S \otimes B_n$, we have that $\mathfrak{z}_A(h)/Fh$ is not nil. Thus $\mathfrak{z}_A(h)$ contains a torus of dimension $\geqslant 2$. Since A is a rank two section of a restricted simple algebra, $\mathfrak{z}_A(h)$ contains no tori of dimension greater than 2. Hence (by Theorem 3.1.1) $\mathfrak{z}_A(h)/(\text{solv}\, \mathfrak{z}_A(h))$ is (0), $\mathfrak{sl}(2)$, $W(1:\mathbf{1})$ or contained between $H(2:\mathbf{1})^{(2)}$ and $H(2:\mathbf{1})$. Then (Lemma 5.3.6) $\mathfrak{z}_A(h)$ contains a unique compositionally classical subalgebra M of minimal codimension, $\dim(\mathfrak{z}_A(h)/M) \leqslant 2$, and M contains a two-dimensional torus containing h. Furthermore, M is restricted.

Let $J = S \otimes (x_1 B_n + \cdots + x_n B_n)$. We claim that $\dim A/N_A(J) = n$. Since $\dim(\text{Der}(S \otimes B_n)/N_{\text{Der}(S \otimes B_n)}(J)) = n$ it is clear that $\dim A/N_A(J) \leqslant n$. Now $N_A(J)$ stabilizes $J^{n(p-1)} = S \otimes x_1^{p-1} \cdots x_n^{p-1}$. Since $N_A(J)$ is a restricted subalgebra of A we see that if $A = \sum_{i=1}^{l} Fu_i + N_A(J)$ then the ideal of A generated by $J^{n(p-1)}$ is $u(A)\, J^{n(p-1)} \subseteq J^{n(p-1)-l(p-1)}$. If $l < n$ this ideal is contained in J and hence is nil, contradicting the semisimplicity of A. Thus we must have $l \geqslant n$ so $\dim A/N_A(J) = n$.

Since $A=\mathfrak{z}_A(h)+S\otimes B_n$ this implies that

$$n\leqslant \dim(M/N_M(J))+\dim(\mathfrak{z}_A(h)/M). \tag{10.2.1}$$

If $u\in A$ write u' for the image of u in $\mathrm{Der}(S\otimes B_n)/((\mathrm{Der}\, S)\otimes B_n)\cong W(n:\mathbf{1})$.

Now $N_M(J)$ is a restricted Lie algebra containing the central torus Fh.

Suppose that $n>2$ and that Fh is not a maximal torus in $N_M(J)$, so $N_M(J)$ contains a two-dimensional torus $R=Fr+Fh$, where $r=r^p$. Then (10.2.1) shows that $M\neq N_M(J)$ and so we may find $E\in M$, $E\notin N_M(J)$ such that E is either a root vector (with respect to R) or an element of $\mathfrak{z}_A(R)$. Then as r stabilizes J we may assume (Theorem 1.3.1(b)) that $x_1, ..., x_n$ are eigenvectors for r', say $r'x_j=c_jx_j$, where $c_1, ..., c_n\in \mathbf{Z}_p$. We may also write $E\equiv\sum_{i=1}^n a_iD_i \bmod N_A(J)$ and may assume, without loss of generality, that $a_1=1$. Now $[r,E]=bE$ $(b\in\mathbf{Z}_p)$ so $[r',E']=bE'$. Thus $[r',E']\,x_1=r'E'x_1-E'r'x_1\equiv -c_1\equiv b\ (\mathrm{mod}(x_1B_n+\cdots+x_nB_n))$. Define $\alpha\in R^*$ ($\alpha=0$ is a possibility) by $\alpha(h)=0$, $\alpha(r)=b$. Then $E\in A_\alpha$ and $x_1^{p-1}\cdot h\in A_\alpha$. Clearly $(\mathrm{ad}\, E)^{p-1}\ (x_1^{p-1}\cdot h)\equiv -h \bmod J$ so $(\mathrm{ad}\, E)^{p-1}\ (x_1^{p-1}\cdot h)$ is not nil. If $b=0$ this contradicts the fact that R must be standard. If $b\neq 0$ this contradicts Corollary 10.2.4. Thus this case cannot occur.

Now suppose $n>2$ and Fh is a maximal torus in $N_M(J)$. Then M contains a two-dimensional torus $R=Fr+Fh$ with $r=r^p$ and $r\notin N_M(J)$. Suppose $M\neq Fr+N_M(J)$. Then there exists an element $E\in M$, $E\notin Fr+N_M(J)$ such that E is either a root vector (with respect to R) or an element of $\mathfrak{z}_A(R)$. By Theorem 1.3.1(a) we may assume $r'=(x_1+1)\,D_1$. We may also write $E\equiv\sum_{i=1}^n a_iD_i \bmod N_A(J)$ and may assume (since $E\notin Fr+N_M(J)$) that $a_2=1$. Now $[r,E]=bE$ for some $b\in\mathbf{Z}_p$. Define $\alpha\in R^*$ ($\alpha=0$ is a possibility) by $\alpha(h)=0$, $\alpha(r)=b$. Then $E\in A_\alpha$ and $(x_1+1)^b\,x_2^{p-1}\cdot h\in A_\alpha$. Clearly $(\mathrm{ad}\, E)^{p-1}\,((x_1+1)^b\,x_2^{p-1}\cdot h)\equiv -h \bmod J$ so $(\mathrm{ad}\, E)^{p-1}\ ((x_1+1)^b\,x_2^{p-1}\cdot h)$ is not nil. If $b=0$ this contradicts the fact that R must be standard. If $b\neq 0$ this contradicts Corollary 10.2.4. Thus this case cannot occur.

Finally, suppose that $n>2$, Fh is a maximal torus in $N_M(J)$, $R=Fr+Fh$ is a two-dimensional torus in M with $r=r^p$, and that $M=Fr+N_M(J)$. Thus $\dim M/N_M(J)=1$ and so (recalling $\dim(\mathfrak{z}_A(h)/M\leqslant 2)$ (10.2.1) shows that $\dim(\mathfrak{z}_A(h)/M)=2$. By Lemma 1.11.1 this implies $H(2:\mathbf{1})^{(2)}\subseteq\mathfrak{z}_A(h)/\mathrm{solv}(\mathfrak{z}_A(h))\subseteq H(2:\mathbf{1})$ and $M/\mathrm{solv}(\mathfrak{z}_A(h))=(\mathfrak{z}_A(h)/\mathrm{solv}(\mathfrak{z}_A(h)))_0$. Therefore $M/\mathrm{solv}\, M\cong\mathfrak{sl}(2)$. Since Fh is a maximal torus in the restricted subalgebra $N_M(J)$ and $h\in\mathfrak{z}(M)$, the Engel–Jacobson theorem shows that $N_M(J)$ is nilpotent. Hence $N_M(J)+\mathrm{solv}\, M$ is solvable. Since M is not solvable and $\dim M/N_M(J)=1$ we see that $N_M(J)\supseteq\mathrm{solv}\, M$. Then $N_M(J)/\mathrm{solv}\, M$ is a two-dimensional nilpotent subalgebra of $\mathfrak{sl}(2)$. Since $\mathfrak{sl}(2)$ has no such subalgebra, this case cannot occur and the proof is complete. ∎

10.3. We now assume that A is one of the algebras of Theorem

9.1.1 excluding case (b) or is one of the algebras of Corollary 10.2.5. We show that if A has an improper root relative to T, it is possible to switch to a torus with respect to which A has more proper roots.

LEMMA 10.3.1. *Let A be a finite-dimensional restricted semisimple Lie algebra over F. Assume that all tori of maximal dimension in A are two-dimensional and standard. Let T be a two-dimensional torus in A. If $S\otimes B_n \subseteq A \subseteq \mathrm{Der}(S\otimes B_n)$, $n>0$, and $A/(A\cap((\mathrm{Der}\, S)\otimes B_n))$ is not nil (i.e., if A is one of the algebras of Theorem 9.1.1(b)), assume that $n\leqslant 2$. Let $\alpha\in\Delta(T)-\Delta_P(T)$ and $x\in A_\alpha$. Assume $\alpha_x\in\Delta_P(e^x(T))$. Then $|\Delta_P(e^x(T))|>|\Delta_P(T)|$.*

Proof. Since A satisfies the hypotheses of Theorem 9.1.1 it is sufficient to prove the result for each of the algebras listed in cases (a)–(h) of that theorem.

Suppose A is one of the algebras listed in (a) of Theorem 9.1.1. Thus $S_1+S_2\subseteq A\subseteq \mathrm{Der}(S_1+S_2)$, where S_1, S_2 are among $\mathfrak{sl}(2)$, $W(1:\mathbf{1})$, $H(2:\mathbf{1})^{(2)}$. Thus S_1 and S_2 are nonnil restricted ideals and so $T\cap S_i\neq(0)$ for $i=1,2$. Hence $T=Ft_1+Ft_2$, where $t_i\in S_i$ and $t_i^p=t_i$. Define α_i by $\alpha_i(t_j)=\delta_{ij}$ for $i,j=1,2$. Then $\Delta(T)\subseteq \mathbf{Z}\alpha_1\cup\mathbf{Z}\alpha_2$. We may assume, without loss of generality, that $\alpha=\alpha_1$. Then $x\in A_\alpha\subseteq[t_1,A]\subseteq S_1$, so $E^x(t_2)=t_2$ and $[x,A_{j\alpha_2}]\subseteq S_1\cap S_2=(0)$ for $1\leqslant j\leqslant p-1$. Then by Proposition 1.9.3 we have $A_{(j\alpha_2)_x}=A_{j\alpha_2}$ for $1\leqslant j\leqslant p-1$. Since $(\alpha_2)_x(t_2)=(\alpha_2)_x(E^xt_2)=\alpha_2(t_2)-\xi(\alpha_2(x^p))\,\alpha_1(t_2)=\alpha_2(t_2)$ we have $(\alpha_2)_x([A_{(j\alpha_2)_x}, A_{-(j\alpha_2)_x}])=\alpha_2([A_{j\alpha_2},A_{-j\alpha_2}])$ for all j, $1\leqslant j\leqslant p-1$. Thus $(\alpha_2)_x$ is proper if and only if α_2 is proper, and the lemma holds in this case.

Next suppose that A is one of the algebras listed in (b) of Theorem 9.1.1. Thus $S\otimes B_n\subseteq A\subseteq\mathrm{Der}(S\otimes B_n)$, $n>0$, and $A/(A\cap((\mathrm{Der}\,S)\otimes B_n))$ is not nil. Then $T=Ft_1+Ft_2$, where $t_1^p=t_1$, $t_2^p=t_2$, $t_1\notin(\mathrm{Der}\,S)\otimes B_n$, and $t_2\in(S\otimes B_n)$. Define α_i for $i=1,2$ by $\alpha_i(t_j)=\delta_{ij}$. Set $J=S\otimes(x_1B_n+\cdots+x_nB_n)$. Assume that $n\leqslant 2$.

Suppose first that $T\nsubseteq N_A(J)$. Then by Theorem 1.3.1(a) we may assume that $t_1\equiv(x_1+1)D_1 \bmod((\mathrm{Der}\,S)\otimes B_n)$. Let β be a root, $\beta\notin\mathbf{Z}\alpha_1$. Then $\beta(t_2)\neq 0$ and so $A_\beta\subseteq S\otimes B_n$. Then $(x_1+1)^i\cdot A_\beta\subseteq A_{\beta+i\alpha_1}$ and (since $(x_1+1)^p=1$) the map $\theta\colon A_\beta\to A_{\beta+\alpha_1}$ defined by $\theta(u)=(x_1+1)\cdot u$ is a bijection. Then the linear map $\Theta\colon A^{(\beta)}\to A^{(\beta+\alpha_1)}$ defined by $\Theta\,|_{A_{j\beta}}=\theta^j$ is an isomorphism of Lie algebras fixing T. Thus β is proper if and only if $\beta+\alpha_1$ is proper, and hence, for any $i\in\mathbf{Z}$, β is proper if and only if $\beta+i\alpha_1$ is proper.

Now suppose $\alpha\notin\mathbf{Z}\alpha_1$. By hypothesis α is improper, so by the above remark β is improper for every $\beta\notin\mathbf{Z}\alpha_1$. Since $\alpha_1(T\cap(S\otimes B_n))=(0)$ we see by the Engel–Jacobson theorem that $(S\otimes B_n)^{(\alpha_1)}$ is solvable. Hence $A[\alpha_1]$ is a quotient of $A/(S\otimes B_n)$. Since $x\in S\otimes B_n$, E^x induces the identity map on $A/(S\otimes B_n)$. Thus $A[\alpha_1]\cong A[(\alpha_1)_x]$ by an isomorphism mapping $\Psi_{\alpha_1}T$ to $\Psi_{(\alpha_1)_x}T$ and so α_1 is proper if and only if $(\alpha_1)_x$ is proper. Thus the lemma holds when $\alpha\notin \mathbf{Z}\alpha_1$.

If $\alpha\in\mathbf{Z}\alpha_1$ and β is improper for some $\beta\notin\mathbf{Z}\alpha_1$, then every root is improper so the lemma holds vacuously in this case.

Thus we may assume that $\alpha\in\mathbf{Z}\alpha_1$ and that β is proper for every β $\mathbf{Z}\alpha_1$. Let η_i denote the homomorphism of restricted Lie algebras of $(S\otimes B_n)^{(\alpha_2+i\alpha_1)}$ into S obtained by composing the inclusion $(S\otimes B_n)^{(\alpha_2+i\alpha_1)}\subseteq S\otimes B_n$ and the isomorphism $S\cong(S\otimes B_n)/J$. Since $(x_1+1)^{kj}\cdot A_{k(\alpha_2+i\alpha_1)}=A_{k(\alpha_2+(i+j)\alpha_1)}$ we see that the images of $\eta_0, ..., \eta_{p-1}$ are all the same. Since $S\otimes B_n=\sum_i A^{(\alpha_2+i\alpha_1)}$ we see that S is the sum of the images of the η_i. Hence each η_i is surjective. Since every $\beta\in\mathbf{Z}\alpha_1$ is proper it follows that the roots of S with respect to Fh, where $h=t_2+J$, are all proper. Now let η_i' denote the homomorphism of restricted Lie algebras of $(S\otimes B_n)^{((\alpha_2)_x+i(\alpha_1)_x)}$ into S obtained by composing the inclusion $(S\otimes B_n)^{((\alpha_2)_x+i(\alpha_1)_x)}\subseteq S\otimes B_n$ and the isomorphism $S\cong(S\otimes B_n)/J$. Since $x\in A_{\alpha_1}$ and $\alpha_1(t_2)=0$ we have $t_2\in e^x(T)$ and so $e^x(T)$ maps onto Fh. Since the roots of S with respect to Fh are proper and since the kernel of η_i' is a nil ideal, the roots of $(S\otimes B_n)^{((\alpha_2)_x+i(\alpha_1)_x)}$ are proper for every i. Thus β_x is proper for every $\beta\notin\mathbf{Z}\alpha_1$, hence for every root β. This gives our result in this case.

Now suppose that $T\subseteq N_A(J)$. Since $n\leqslant 2$ and $(S\otimes B_n)^{(\alpha_1)}$ is solvable we see that $N_A(J)\cap A^{(\alpha_1)}$ is a compositionally classical subalgebra of codimension $\leqslant 2$ in $A^{(\alpha_1)}$. Then Corollary 5.6.4 shows that α_1 is proper. Thus $\alpha\notin\mathbf{Z}\alpha_1$ and so $A_\alpha\subseteq S\otimes B_n$. Hence $x\in S\otimes B_n$ and so $e^x(T)\subseteq N_A(J)$. Since $\beta\notin\mathbf{Z}\alpha_1$ implies $A_\beta\subseteq S\otimes B_n$ we see that $\Delta_P(A,T)=\Delta_P(T+S\otimes B_n, T)$ and similarly $\Delta_P(A, e^x(T))=\Delta_P(e^x(T)+S\otimes B_n, e^x(T))$. Let $\phi\colon N_A(J)\to \operatorname{Der} S$ be the homomorphism of restricted Lie algebras defined by

$$\phi(u)(s)=[u,s]+J\in S\otimes B_n/J\cong S$$

for $u\in N_A(J)$, $s\in S$. Let $U=\ker\phi\,|_T$. Suppose $\beta(U)\neq(0)$. Then $(S\otimes B_n)_\beta=[u,(S\otimes B_n)_\beta]\subseteq J$ and so $\beta\in\Delta_P(T+S\otimes B_n, T)$. This implies $\alpha(U)=(0)$ and so $\beta_x(E^xU)=\beta(U)$. Thus $\beta_x\in\Delta_P(e^x(T)+S\otimes B_n, e^x(T))$ whenever $\beta(U)\neq(0)$. Then if $U\neq(0)$ we see that $\alpha_x\in\Delta_P(A, e^x(T))$ implies $\Delta_P(A, e^x(T))=\Delta(A, e^x(T))$ and so our result holds in this case. Thus we may assume $U=(0)$ and so $\operatorname{Der} S$ contains a two-dimensional torus $\phi(T)$. Thus $H(2:\mathbf{1})^{(2)}\subseteq\Phi(T)+S\subseteq H(2:\mathbf{1})$, $\Delta_P(A,T)=\Delta_P(\phi(T)+S,\phi(T))$, and $\Delta_P(A, e^x(T))=\Delta_P(e^{\phi(x)}\phi(T)+S, e^{\phi(x)}\phi(T))$. The result (in this case) now follows from Lemma 5.8.6.

Now suppose that $\bar{S}\subseteq A\subseteq\operatorname{Der} S$, where $S=H(2:\mathbf{1})^{(2)}$, $W(1:\mathbf{2})$, $H(2:\mathbf{1}:\Phi(\gamma))^{(1)}$, $H(2:\mathbf{1}:\Delta)$, or $W(2:\mathbf{1})$. Then $A\subseteq W(2:\mathbf{1})$, so any two-dimensional torus in A is equal to its centralizer. Thus Proposition 4.9 of

[Wil83] applies and gives the result. This shows that the result holds in cases (c) and (e) of Theorem 9.1.1 and in some situations in the remaining cases.

Suppose that (d) of Theorem 9.1.1 holds. Thus $S \otimes B_n \subseteq A \subseteq \operatorname{Der}(S \otimes B_n)$, $A/(A \cap ((\operatorname{Der} S) \otimes B_n))$ is nil, and $\mathfrak{z}_{S \otimes B_n}(T) \subseteq I$. Then any root with respect to T is proper and so the conclusion of the lemma is (vacuously) true in this case.

Since the result clearly holds if A is classical (for then all roots with respect to any torus are proper), we are left only with the cases $\bar{S} \subseteq A \subseteq \operatorname{Der} S$, $S = H(2:(2,1))^{(2)}$, and $A = S(3:\mathbf{1})^{(1)}$, $H(4:\mathbf{1})^{(1)}$, $K(3:\mathbf{1})$. The first of these cases is covered by Lemma 10.1.1(e) and the rest by Corollary 5.8.4. ∎

10.4. We now prove the main result of Secton 10. Recall the definition (Definition 6.2.1) of an optimal torus.

PROPOSITION 10.4.1. *Let L be a finite-dimensional restricted simple Lie algebra over F. Let T be an optimal torus in L. Then all roots with respect to T are proper.*

Proof. If not then there exist $\alpha \in \Delta(T) - \Delta_P(T)$ and $x \in L_\alpha$ such that $\alpha_x \in \Delta_P(e^x(T))$. Let X denote $\mathbf{Z}\Delta$, the subgroup of T^* generated by Δ. Let $\sim$ denote the equivalence relation on X defined by $\beta \sim \gamma$ if and only if $\mathbf{Z}\alpha + \mathbf{Z}\beta = \mathbf{Z}\alpha + \mathbf{Z}\gamma$. Let Y be a complete set of representatives of equivalence classes for $\sim$ containing 0. Then it is clear that

$$\begin{aligned}\Delta_P(L, T) &= \Delta_P(L^{(\alpha)}, T)\\ &\quad \cup \bigcup_{\beta \in Y - \{0\}} (\Delta_P(L^{(\alpha,\beta)}, T) - \Delta_P(L^{(\alpha)}, T))\\ &= \Delta_P(L[\alpha], \Psi_\alpha T)\\ &\quad \cup \bigcup_{\beta \in Y - \{0\}} (\Delta_P(L[\alpha,\beta], \Psi_{\alpha,\beta} T) - \Delta_P(L^{(\alpha)}, T)).\end{aligned}$$

We also have the corresponding expression for $\Delta_P(L, e^x(T))$. Now

$$|\Delta_P(L[\alpha], \Psi_\alpha T)| < |\Delta_P(L[\alpha_x], \Psi_{\alpha_x} e^x(T))|$$

by hypothesis. Furthermore,

$$\begin{aligned}&|\Delta_P(L[\alpha,\beta], \Psi_{\alpha,\beta} T) - \Delta_P(L[\alpha], \Psi_\alpha T)|\\ &\quad \leqslant |\Delta_P(L[\alpha_x, \beta_x], \Psi_{\alpha_x,\beta_x} e^x(T)) - \Delta_P(L[\alpha_x], \Psi_{\alpha_x} e^x(T))|\end{aligned}$$

by Lemma 10.2.5 and Corollary 10.3.1. Therefore $|\Delta_P(T)| < |\Delta_P(e^x(T))|$. This contradicts the optimality of T. Hence the proposition is proved. ∎

11. Sections of a Simple Algebra

In this section we will show that certain of the semisimple algebras listed in Theorem 9.1.1 cannot occur as a section $L[\alpha, \beta]$, where L is a restricted simple Lie algebra over F.

Throughout this section we will let L denote a finite-dimensional restricted simple Lie algebra over F and T denote an optimal torus in L. By Proposition 10.4.1 every root of L with respect to T is proper.

11.1. We first need to accumulate some fairly detailed information about the Cartan decompositions of the algebras which occur in (e)–(g) of Theorem 9.1.1. These results are analogous to the results of Sections 5.8 and 10.1.

Lemma 11.1.1. *Let $S = W(1:\mathbf{2})$, $A = \bar{S}$, and let T be a two-dimensional torus in A. Let $\alpha \in \Delta(A, T)$ and let t_α be as in* (5.8.1). *Then*:

(a) *If $t_\alpha \notin S$ then $A[\alpha] \cong W(1:\mathbf{1})$.*

(b) *If $t_\alpha \in S$ then $A[\alpha] = (0)$ and $x^p \in S_0$ for all $x \in A_\alpha$.*

(c) *$T \not\subseteq S$.*

(d) *If $\Delta(A, T) = \Delta_P(A, T)$ then there exists $\beta \in \Delta(T)$ and $x \in A_\beta$ such that $A[\beta] = (0)$, and x^p is not nil.*

Proof. Write t for t_α.

Suppose $t \notin S$. Then we must have $t \equiv a(\operatorname{ad} D_1)^p + b(\operatorname{ad} D_1) \bmod S_0$ for some nonzero $a, b \in F$. Let $Q = \operatorname{span}\{x^{j\varepsilon_1} D_1 \mid j \geq p\}$. Clearly Q contains no eigenvectors for t. Since $(\operatorname{ad} t)^p - (\operatorname{ad} t) = 0$ we have $S = \sum_{l=0}^{p-1} S_{(l)}$, where $S_{(l)} = \{E \in W(1:\mathbf{2}) \mid [t, E] = lE\}$. Since $S_{(l)} \cap Q = (0)$ we have $\dim S_{(l)} \leq p$ for all l. Since $\dim S = p^2$ we have $\dim S_{(l)} = p$ and $S = S_{(l)} + Q$ for all l. Since $Q = S_{p-1}$ this implies $\operatorname{gr} S_{(0)} \cong W(1:\mathbf{1})$. Then $S_{(0)}$ contains a subalgebra of codimension one and $S_{(0)}$ is simple (for if $(0) \neq J$ is a proper ideal in $S_{(0)}$ then $\operatorname{gr} J \subseteq \operatorname{gr} S_{(0)}$ is a proper ideal in $\operatorname{gr} S_{(0)}$). Thus (by [Kac74] or [Wil76, Theorem 1]) $S_{(0)} \cong W(1:\mathbf{1})$. Thus (a) holds.

Now suppose $t \in S$. Then (after multiplication by a suitable integer) $t \equiv x_1 D_1 \bmod S_1$ and so $\mathfrak{z}_S(t) \subseteq S_0$. Since S_0 is solvable, $\mathfrak{z}_S(t)$ is solvable. Since $[\bar{S}, \bar{S}] \subseteq S$ this implies $\mathfrak{z}_{\bar{S}}(t)$ is solvable. Hence $A[\alpha] = (0)$. Furthermore, $A_\alpha \subseteq S \cap \mathfrak{z}_{\bar{S}}(t) = \mathfrak{z}_S(t) \subseteq S_0$. Since S_0 is a restricted subalgebra of $\bar{S}$ we see that if $x \in A_\alpha$ then $x^p \in S_0$, proving (b).

If $T \subseteq S$ then $T \cap S_0 \neq (0)$ (as $\dim S/S_0 = 1$). As $\dim S_0/S_1 = 1$ and S_1 is nil we have $S_0 = (T \cap S_0) + S_1$. Consequently $S = T + S_0$ and $[S, S_0] \subseteq [T + S_0, (T \cap S_0) + S_1] \subseteq S_0$, contradicting the simplicity of S. Thus (c) holds.

Since $T \not\subseteq S$ and T is spanned by $\{t_\gamma \mid \gamma \in \Delta(T)\}$ we see that there is some $\gamma \in \Delta(T)$ such that $t_\gamma \notin S$. By (a) we have that γ is a Witt root. Therefore Lemma 5.3.6 shows that $A^{(\gamma)}$ contains a compositionally classical subalgebra of codimension 1 which contains every compositionally classical subalgebra of codimension $\leqslant 2$. Therefore this subalgebra contains $S_0 \cap A^{(\gamma)}$ and since γ is proper, Corollary 5.6.4 shows that it also contains T. Thus $T \cap S_0 \neq (0)$ so $T \cap S_0 = Fr$, where $r^p = r$. Let $\beta \in \Delta(A, T)$ satisfy $\beta(r) = 0$ and let $x \in A_\beta$. Since $[r, x] = 0$ we have $x \in S_0$. If x^p is nil then $x \in S_1$ (for S_0/S_1 is a torus). But $x \in S_1$ and $[r, x] = 0$ implies $x \in S_p$ and (as $T \not\subseteq S$) this implies $x = 0$. Thus for any $0 \neq x \in A_\beta$ we have that x^p is not nil. Since $A[\beta] = (0)$ by (b), (d) is proved. ▮

LEMMA 11.1.2. *Let* $S = H(2 : \mathbf{1} : \Phi(\gamma))^{(1)}$, $A = \bar{S}$, *and* T *be a two-dimensional torus in* A. *Then*:

(a) $\mathfrak{z}_A(T) = T$, $T \cap S = (0)$, *and* $A[\alpha] = (0)$ *for every* $\alpha \in \Delta(T)$.

(b) *There exist some* $\alpha \in \Delta(T)$ *and* $x \in A_\alpha$ *such that* x *is not nil.*

Proof. By Corollary 1.3.2 we have $\mathfrak{z}_A(T) = T$. We must have $T \not\subseteq S$, since otherwise Theorem 9.1.1(h) would contradict $S = H(2 : \mathbf{1} : \Phi(\gamma))^{(1)}$. Then (since S is not restricted) by Lemmas 4.6.4, 4.7.2, and 4.8.1 of [BW82] we see that A has $p^2 - 1$ roots, each of multiplicity one. But $\dim S = p^2 - 1$ so $T \cap S = (0)$ and $S^{(\alpha)} = \sum_{i=1}^{p-1} A_{i\alpha}$. If $y \in S_{i\alpha}$, $\operatorname{ad} y$ is nilpotent on $S^{(\alpha)}$, so by the Engel–Jacobson theorem $S^{(\alpha)}$ is nilpotent, hence solvable. Thus $A[\alpha] = (0)$, proving (a).

Finally, $S = \sum_{\alpha \neq 0} S_\alpha$. Since S is not nilpotent, then Engel–Jacobson theorem implies that there is some $\alpha \in \Delta(T)$ and some $x \in S_\alpha$ such that x is not nil. ▮

LEMMA 11.1.3. *Let* $S = H(2 : \mathbf{1} : \Delta)$ *and* $A = \bar{S}$. *Let* T *be a two-dimensional torus in* A. *Assume that all roots with respect to* T *are proper. Then*:

(a) $T \subseteq A_0$.

(b) *If* $A^{(\alpha)} \not\subseteq A_0$ *then* $A[\alpha] \cong W(1 : \mathbf{1})$. *Hence if* $A[\alpha] \cong \mathfrak{sl}(2)$ *or* $A[\alpha] = (0)$ *and* $x \in A_\alpha$ *then* $x^p = 0$.

Proof. As in the proof of the previous lemma, we have that $\mathfrak{z}_A(T) = T$, that $T \not\subseteq S$, and that A has $p^2 - 1$ roots, each of multiplicity one. Now (by Theorem 1.3.1(c)) we may assume (replacing A by ΦA for some $\Phi \in \operatorname{Aut} W(2 : \mathbf{1})$) that T is one of $\operatorname{span}\{(x_1 + 1) D_1,\ (x_2 + 1) D_2\}$, $\operatorname{span}\{(x_1 + 1) D_1, x_2 D_2\}$, $\operatorname{span}\{x_1 D_1, x_2 D_2\}$.

Suppose $T = \operatorname{span}\{(x_1 + 1) D_1, (x_2 + 1) D_2\}$. Then as every root α is proper, Corollary 3.8 of [Wil83] shows that for any root α either $A[\alpha] \cong (0)$ or $A[\alpha] \cong \mathfrak{sl}(2)$ and therefore $A_{i\alpha} \subseteq K_{i\alpha}(W(2 : \mathbf{1}))$ for at least $p - 3$ values of i, $1 \leqslant i \leqslant p - 1$. Thus $(0) \neq A_{i\alpha} = K_{i\alpha}(W(2 : \mathbf{1}))$ for some i.

Then by Lemma 3.7(f) of [Wil83], $A \supseteq K(W(2:\mathbf{1})) = \sum_{\alpha \neq 0} K_\alpha(W(2:\mathbf{1}))$, a (p^2-1)-dimensional T-invariant subalgebra of $W(2:\mathbf{1})$. Thus $A = T + K(W(2:\mathbf{1}))$ so $S = A^{(1)} \subseteq K(W(2:\mathbf{1}))$. This contradicts the fact that $\dim S = p^2$. Hence $T = \operatorname{span}\{(x_1+1)D_1, (x_2+1)D_2\}$ is impossible and so $T \cap A_0 \neq (0)$. Therefore $T \cap A_0$ contains an element t_1 satisfying $t_1^p = t_1 \neq (0)$. Suppose $T \not\subseteq A_0$. Then $\operatorname{ad}(t_1 + A_1)$ annihilates an element of A/A_0, hence is a rank one transformation. We may therefore assume (replacing t_1 by Φt_1 and A by ΦA for some $\Phi \in \operatorname{Aut}(H(2:\mathbf{1}))$ that $t_1 \equiv x_1 D_1$ or $x_2 D_2 \bmod W(2:\mathbf{1})_1$. Then $\operatorname{gr} \mathfrak{z}_S(t_1) = \operatorname{span}\{\mathscr{D}(x_1 x_2^i) \mid 0 \leqslant i \leqslant p-1\}$ if $t_1 \equiv x_1 D_1$ or $\operatorname{span}\{\mathscr{D}(x_1^i x_2) \mid 0 \leqslant i \leqslant p-1\}$ if $t = x_2 D_2$. In either case $\operatorname{gr} \mathfrak{z}_S(t_1) \cong W(1:\mathbf{1})$ and so $\mathfrak{z}_S(t_1) \cong W(1:\mathbf{1})$. Then since all roots are proper we must have (Proposition 1.7 of [Wil83]) that $T \cap S \subseteq A_0$ and so $T \subseteq A_0$, contradicting our assumption. Therefore $T \subseteq A_0$ and (a) is proved. Our computation of $A[\alpha]$ also shows if $A^{(\alpha)} \not\subseteq A_0$ then $A[\alpha] \cong W(1:\mathbf{1})$. Thus $A[\alpha] \cong (0)$ or $\mathfrak{sl}(2)$ implies $A^{(\alpha)} \subseteq A_0$. But A_1 is a nil ideal in A_0 and $A_0/A_1 \cong \mathfrak{sl}(2)$, so if $x \in A_\alpha$ we have $x^p = 0$. This completes the proof of the lemma. ∎

11.2. We now show that only certain of the algebras listed in Theorem 9.1.1 can occur as a rank two section of a restricted simple Lie algebra.

LEMMA 11.2.1. *Let M be one of the algebras listed in Theorem 9.1.1(e)–(h). Let T be an optimal torus in M (so $\Delta(M, T) = \Delta_P(M, T)$) and $\Delta(M, T) \subseteq \mathbf{Z}\alpha + \mathbf{Z}\beta$. Suppose $M[\alpha] = (0)$ and that, for some $x \in M_\alpha$, x^p is not nil. Then $\gamma \in \Delta(M, T)$ and $\gamma(x^p) \neq 0$ imply that $\alpha([M_\gamma, M_{-\gamma}]) = (0)$.*

Proof. If M is classical then $M[\alpha] = (0)$ implies $M_\alpha = (0)$. Thus M cannot be classical. Corollary 5.8.3 shows that M cannot be one of the nonclassical algebras of Theorem 9.1.1(h). Thus we may assume $\bar{S} \subseteq M \subseteq \operatorname{Der} S$, where S is one of $W(1:\mathbf{2})$, $H(2:(2,1))^{(2)}$, $H(2:\mathbf{1}:\Phi(\gamma))^{(1)}$, $H(2:\mathbf{1}:\Delta)$.

If $S = H(2:\mathbf{1}:\Delta)$ then by Proposition 2.1.8(c) we have $M = \bar{S}$. But then Lemma 11.1.3 shows $S = H(2:\mathbf{1}:\Delta)$ is impossible.

If $S = H(2:(2,1))^{(2)}$ then by (d) and (f) of Lemma 10.1.1, $T \cap S = T \cap S_0 \neq (0)$. If $M[\alpha] = (0)$ then $t_\alpha \in S$ (where t_α is as in (5.8.1)) by Lemma 10.1.1(a). Thus $\alpha(T \cap S) = (0)$ and so, as S_0 is restricted (Lemma 2.1.7), $\alpha(\mathfrak{z}_S(T)) = (0)$. Thus $\alpha([M_\gamma, M_{-\gamma}]) = (0)$ for all γ and our result holds.

If $S = W(1:\mathbf{2})$ then, as $\bar{S} = \operatorname{Der} S$ (cf. [Wil71a, Lemma 4]), we have $M = \bar{S}$. Hence Lemma 11.1.1 shows that since $M[\alpha] = (0)$ we have $x^p \in S_0$. Since x^p is not nil, $x^p \notin S_1$ and so (as $\dim S_0/S_1 = 1$) $S_0 = Fx^p + S_1$. But

$\alpha(x^p) = 0$ (by Lemma 1.8.1) and S_1 is nil, so $\alpha(\mathfrak{z}_{S_0}(T)) = (0)$. But we have $\mathfrak{z}_S(T) = \mathfrak{z}_{S_0}(T)$, for otherwise $[S, S_0] = [\mathfrak{z}_S(T) + S_0, Fx^p + S_1] \subseteq S_0$, contradicting the simplicity of S. Thus $\alpha([M_\gamma, M_{-\gamma}]) = (0)$ for all γ and our result holds.

Finally, if $S = H(2:\mathbf{1}:\Phi(\gamma))^{(2)}$ then $\bar{S} = \operatorname{Der} S$ and so $M = \bar{S}$. Then Lemma 11.1.2 shows that $\mathfrak{z}_M(T) \cap S = (0)$ for all γ. Hence our result holds in this case as well. ∎

COROLLARY 11.2.2. *Let L be a restricted simple Lie algebra and T be an optimal torus in L. Let $\alpha \in \Delta(L, T)$. If $L[\alpha] = (0)$ and $x \in L_\alpha$ then x^p is nil.*

Proof. If not then (as $\alpha(x^p) = 0$ by Lemma 1.8.1) Corollary 10.2.2 (with $u = x^p$) implies there exists $\beta \in \Delta(L, T)$ such that $\beta(u) \neq 0$, $\alpha([L_\beta, L_{-\beta}]) \neq (0)$, and $L[\alpha, \beta]$ is one of the algebras listed in Theorem 9.1.1(e)–(h). Lemma 11.2.1 shows that this is impossible. ∎

COROLLARY 11.2.3. *Let L be a restricted simple Lie algebra. Let T be an optimal torus in L and $\alpha, \beta \in \Delta(L, T)$. Let $A = L[\alpha, \beta]$ and $\gamma \in \Delta(A, \Psi_{\alpha,\beta}(T))$. Suppose $A[\gamma] = (0)$ and $x \in A_\gamma$. Then x^p is nil.*

Proof. We may assume (replacing α, β by different generators for $\mathbf{Z}\alpha + \mathbf{Z}\beta$ if necessary) that γ is the element of $\Psi_{\alpha,\beta}(T)^*$ induced by α. Then $A[\gamma] = (0)$ implies $L[\alpha] = (0)$, so by Corollary 11.2.2, y^p is nil for all $y \in L_\alpha$. Since $\Psi_{\alpha,\beta}\colon L^{(\alpha,\beta)} \to A$ is a surjective homomorphism of restricted Lie algebras, our result follows. ∎

PROPOSITION 11.2.4. *Let L be a restricted simple Lie algebra over F. Let T be an optimal torus in L. Let $\alpha, \beta \in \Delta(L, T)$. Let $A = L[\alpha, \beta]$. Then one of the following occurs:*

(a) $S_1 + S_2 \subseteq A \subseteq (\operatorname{Der} S_1)^{(1)} + (\operatorname{Der} S_2)^{(1)}$, *where S_1, S_2 are distinct ideals in A and each is isomorphic to one of* $\mathfrak{sl}(2)$, $W(1:\mathbf{1})$, $H(2:\mathbf{1})^{(2)}$.

(b) $S \otimes B_n \subseteq A \subseteq \operatorname{Der}(S \otimes B_n)$, $n = 1$ *or* 2, $\Psi_{\alpha,\beta}(T) \not\subseteq (\operatorname{Der} S) \otimes B_n$, S *is one of* $\mathfrak{sl}(2)$, $W(1:\mathbf{1})$, $H(2:\mathbf{1})^{(2)}$, *and* $S \otimes (x_1 B_n + \cdots + x_n B_n)$ *is invariant under* ad $\Psi_{\alpha,\beta}(T)$.

(c) $H(2:\mathbf{1})^{(2)} + Fx_1 D_1 \subseteq A \subseteq \operatorname{Der}(H(2:\mathbf{1})^{(2)})$.

(d) $A = \bar{S}$, *where* $S = H(2:\mathbf{1}:\Delta)$.

(e) $A = S$, *where S is one of the following simple algebras*: A_2, C_2, G_2, $W(2:\mathbf{1})$, $S(3:\mathbf{1})^{(1)}$, $H(4:\mathbf{1})^{(1)}$, $K(3:\mathbf{1})$.

Proof. We know that A is listed in Theorem 9.1.1. If A is listed in (a), (c), or (h) of that theorem, then it is also listed here (in (a), (c), or (e)). Suppose A is listed in (d) of Theorem 9.1.1. Then, since $\mathfrak{z}_{S\otimes B_n}(\Psi_{\alpha,\beta}(T))$ is nil and $S \otimes B_n$ is not nilpotent, the Engel–Jacobson theorem shows that for some $\gamma \in \Delta(A, \Psi_{\alpha,\beta}(T))$ and some $x \in A_\gamma$, x^p is not nil. Furthermore, since $[A_\gamma, A_{-\gamma}] \subseteq \mathfrak{z}_{S \otimes B_n}(\Psi_{\alpha,\beta}(T))$ we have $A[\gamma] = (0)$. This contradicts Corollary 11.2.3.

Next suppose $\bar{S} \subseteq A \subseteq \text{Der } S$. If $S = W(1 : \mathbf{2})$ then (as $\bar{S} = \text{Der } S$) we have $A = \bar{S}$ and Lemma 11.1.1(d) and Corollary 11.2.3 give a contradiction.

If $S = H(2 : (2, 1))^{(2)}$ then Lemma 10.1.1(f) and Corollary 11.2.3 give a contradiction.

If $S = H(2 : \mathbf{1} : \Phi(\gamma))^{(2)}$ then Lemma 11.1.2(b) and Corollary 11.2.3 give a contradiction.

It remains to show that if A is one of the algebras of Theorem 9.1.1(b) then A satisfies (b) of the proposition. Since $n \leqslant 2$ by Corollary 10.2.5, we need only show that $S \otimes (x_1 B_n + \cdots + x_n B_n)$ is invariant under ad $\Psi_{\alpha,\beta}(T)$. If not we may find $t_1, t_2 \in \Psi_{\alpha,\beta}(T)$ satisfying $t_1^p = t_1$, $t_2^p = t_2$, $t_1 \notin (\text{Der } S) \otimes B_n$, and $t_2 \in S \otimes B_n$. By Theorem 1.3.1(a) we may assume that $t_1 \equiv (x_1 + 1) D_1 \mod (\text{Der } S) \otimes B_n$. We may assume, replacing $\{\alpha, \beta\}$ by another base for $\mathbf{Z}\alpha + \mathbf{Z}\beta$ if necessary, that $\alpha(t_1) \neq 0$ and $\alpha(t_2) = 0$. Then $((x_1 + 1) \cdot t_2)^p = t_2$. Furthermore, $(x_1 + 1) \cdot t_2 \in \text{solv}(A^{(\alpha)})$. Choose $v \in L_\alpha \cap (\text{solv } L^{(\alpha)})$ such that $\Psi_{\alpha,\beta} v = (x_1 + 1) \cdot t_2$. Then v^p is not nil. Then by Corollary 10.2.2 we may find $\gamma \in \Delta(T)$ so that $\gamma(v^p) \neq 0$, $\alpha([L_\gamma, L_{-\gamma}]) \neq (0)$, and $L[\alpha, \gamma]$ is one of the algebras in parts (e)–(h) of Theorem 9.1.1. Now as $\gamma(v^p) \neq 0$ and $v \in \text{solv}(L^{(\alpha)})$, we have $\text{solv}(L[\alpha, \gamma]^{(\alpha)}) \neq (0)$, which implies that $L[\alpha, \gamma]$ cannot be classical. But Corollary 5.8.3 shows $L[\alpha, \gamma]$ cannot be any of the nonclassical algebras of Theorem 9.1.1(h). Furthermore, if $L[\alpha, \gamma] \cong \overline{H(2 : \mathbf{1} : \Delta)}$ then (as every root multiplicity is one in $\overline{H(2 : \mathbf{1} : \Delta)}$) $\text{solv}(L[\alpha, \gamma]^{(\alpha)}) \neq 0$ implies $L[\alpha] \cong (0)$ or $\mathfrak{sl}(2)$. Hence Lemma 11.1.3 shows $L[\alpha, \gamma] \ncong \overline{H(2 : \mathbf{1} : \Delta)}$. But we have already shown that none of the other algebras of Theorem 9.1.1(e)–(g) can occur as $L[\alpha, \gamma]$. Thus $S \otimes (x_1 B_n + \cdots + x_n B_n)$ must be invariant under ad $\Psi_{\alpha,\beta}(T)$. This completes the proof of the proposition. ∎

Lemma 11.2.5. *Let L be a restricted simple Lie algebra over F. Let T be an optimal torus in L. Assume that $S \otimes B_n \subseteq L[\alpha, \beta] \subseteq \text{Der}(S \otimes B_n)$, where $n > 0$, $\Psi_{\alpha,\beta}(T) \nsubseteq S \otimes B_n$, $S = \mathfrak{sl}(2)$, $W(1 : \mathbf{1})$, or $H(2 : \mathbf{1})^{(2)}$ and $S \otimes (x_1 B_n + \cdots + x_n B_n)$ is invariant under* ad $\Psi_{\alpha,\beta}(T)$. *Let $t_1, t_2 \in T$ satisfy $t_1^p = t_1$, $t_2^p = t_2$, $\Psi_{\alpha,\beta}(T) = \text{span}\{\Psi_{\alpha,\beta}(t_i) \mid i = 1, 2\}$, $\Psi_{\alpha,\beta}(t_1) \in S \otimes B_n$, $\Psi_{\alpha,\beta}(t_2) \notin S \otimes B_n$. Rename the roots so that $\alpha(t_1) = 0$. Then $L[\alpha]$ is nonclassical* (*hence $L[\alpha] \cong W(1 : \mathbf{1})$ or $L[\alpha] \subseteq H(2 : \mathbf{1})$*). *Furthermore, if $L[\alpha]_0$ denotes the usual subalgebra of derivations of degree $\geqslant 0$ in $L[\alpha]$ and $(L^{(\alpha)})_0$ denotes the inverse image of $L[\alpha]_0$ in $L^{(\alpha)}$, then $S \otimes (x_1 B_n + \cdots + x_n B_n)$ is invariant under* ad $\Psi_{\alpha,\beta}((L^{(\alpha)})_0)$.

Proof. Since $L[\alpha, \beta]$ is semisimple, $S \otimes (x_1 B_n + \cdots + x_n B_n)$ is not invariant under ad $\Psi_{\alpha,\beta}(L^{(\alpha)})$. Thus we may assume (renaming the roots if necessary) that there is some $x \in L_\alpha \cup \mathfrak{z}_L(T)$ such that ad $\Psi_{\alpha,\beta}(x)$ does not stabilize $S \otimes (x_1 B_n + \cdots + x_n B_n)$. Without loss of generality we may assume that $\Psi_{\alpha,\beta}(x) \equiv D_1$ modulo the stabilizer of $S \otimes (x_1 B_n + \cdots + x_n B_n)$. If $x \in \mathfrak{z}_L(T)$ then we have $x_1 \cdot t_1 \in \mathfrak{z}_{L[\alpha,\beta]}(\Psi_{\alpha,\beta}(T))$ and $[\Psi_{\alpha,\beta}(x), x_1 \cdot t_1] \equiv [D_1, x_1 \cdot t_1] = r_1 \bmod S \otimes (x_1 B_n + \cdots + x_n B_n)$. Thus $[\Psi_{\alpha,\beta}(x), x_1 \cdot t_1]$ is not nil, contradicting the fact that T (being a maximal torus in a simple Lie algebra) is standard. Thus we must have $x \in L_\alpha$. This implies $x_1^{p-1} \cdot t_1 \in L[\alpha, \gamma]_\alpha \cap \operatorname{solv}(L[\alpha, \gamma]^{(\alpha)})$. Let $y \in L_\alpha \cap \operatorname{solv}(L^{(\alpha)})$ satisfy $\Psi_{\alpha,\beta}(y) = x_1^{p-1} \cdot t_1$. Then $\Psi_{\alpha,\beta}((\operatorname{ad} x)^{p-1} y) \equiv -\Psi_{\alpha,\beta}(t_1) \bmod S \otimes (x_1 B_n + \cdots + x_n B_n)$ so $(\operatorname{ad} x)^{p-1} y$ is not nil. It follows from Corollary 10.2.2 that there is some γ such that $\gamma((\operatorname{ad} x)^{p-1} y) \neq 0$ and $L[\alpha, \gamma]$ is one of the algebras of Theorem 9.1.1(h), necessarily one of the nonclassical algebras (since $(\operatorname{ad} \Psi_{\alpha,\beta}(x))^{p-1} \neq 0$). By Lemma 5.8.5 this implies that $L[\alpha, \gamma]_\alpha \not\subseteq L[\alpha, \gamma]_0$ and hence $L[\alpha]$ is nonclassical by Lemma 5.8.2(d) and $\Psi_\alpha(x)$ does not belong to the subalgebra $A[\alpha]_0$ by Lemma 5.8.2(f). ∎

11.3. We now study the situation in which every $L[\alpha] \cong (0)$ or $\mathfrak{sl}(2)$.

LEMMA 11.3.1. *Let L be a finite-dimensional restricted simple Lie algebra over F. Let T be an optimal torus in L. Assume that $L[\alpha] \cong (0)$ or $\mathfrak{sl}(2)$ for all α. Then every $L[\alpha, \beta]$ is classical semisimple.*

Proof. If $L[\alpha, \beta]$ has toral rank <2 then it is also $L[\gamma]$ for some $\gamma \in \mathbf{Z}\alpha + \mathbf{Z}\beta$ and hence is classical semisimple by hypothesis. Thus we may assume that $L[\alpha, \beta]$ is one of the algebras listed in Proposition 11.2.4. If $L[\alpha, \beta]$ is listed in Proposition 11.2.4(a) then $L[\alpha, \beta] \supseteq S_1 + S_2$, where S_1, S_2 are restricted simple. Renaming the roots if necessary, we have $L[\alpha] \supseteq S_1$ and $L[\beta] \supseteq S_2$. Hence $S_1 \cong S_2 \cong \mathfrak{sl}(2)$ so $L[\alpha, \beta] = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ is classical semisimple.

By Lemma 11.2.5, $L[\alpha, \beta]$ is not one of the algebras of Proposition 11.2.4(b).

Suppose $L[\alpha, \beta]$ is one of the algebras of Proposition 11.2.4(c). Then by Theorem 1.18.4 of [BW82] we may assume $\Psi_{\alpha,\beta}(T)$ contains $u = x_1 D_1$ or $v = (x_1 + 1) D_1$. As $\mathfrak{z}_{H(2:\mathbf{1})}(u) = \operatorname{span}\{\mathscr{D}(x_1 x_2^i) \mid 0 \leqslant i \leqslant p-1\} \cong W(1:\mathbf{1})$ and $\mathfrak{z}_{H(2:\mathbf{1})}(v) = \operatorname{span}\{\mathscr{D}((x_1+1) x_2^i) \mid 0 \leqslant i \leqslant p-1\} \cong W(1:\mathbf{1})$ we see that $L[\alpha, \beta]$ cannot be one of the algebras of Proposition 11.2.4(c).

By Lemma 11.1.3(b), $L[\alpha, \beta]$ cannot be the algebra of Proposition 11.2.4(d).

If A is one of the nonclassical algebras of Proposition 11.2.4(e) and $A^{(\alpha)} \not\subseteq A_0$ then $A[\alpha]$ is nonclassical by Lemma 5.8.2(d). Thus the lemma is proved. ∎

COROLLARY 11.3.2. *Let L be a finite-dimensional restricted simple Lie algebra over F. Let T be an optimal torus in L. Assume $L[\alpha] \cong (0)$ or $\mathfrak{sl}(2)$ for all α. Then L is classical.*

Proof. This follows from Lemma 11.3.1 and Proposition 7.1 of [Wil83]. ∎

12. CONCLUSION

In this section we let L be a finite-dimensional restricted simple Lie algebra over F. If L is nonclassical we will construct a maximal subalgebra L_0 such that the hypotheses of the Recognition Theorem (Theorem 1.2.2) are satisfied by the pair (L, L_0). Hence we have that L is of Cartan type and the classification of the finite-dimensional restricted simple Lie algebras over F (Theorem 12.5.1) is complete.

The key idea in our proof is the definition (Section 12.1) for a restricted Lie algebra A containing a torus T of maximal dimension such that all roots with respect to T are proper of a certain subspace $Q(A) = Q(A, T)$. In case $A = L$ is simple we will use the results of Section 11 to show (Section 12.2) that $Q(L)$ is a subalgebra, that $Q(L) = L$ implies L is classical, and that $Q(L)$ is a very "large" (see Lemma 12.2.3 for the precise statement) subalgebra of L. Assuming L is nonclassical we then (Section 12.3) let $L_0 \supseteq Q(L)$ be a maximal subalgebra of L. We construct a corresponding filtration and the associated graded algebra G as usual. It follows (Section 12.3) from Lemma 12.2.3 that $G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$ is classical semisimple and that the graded algebra G satisfies (1.2.3). Section 12.4 is devoted to showing that $G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$ cannot have more than one summand. This involves some detailed computations in $\sum_{i=-2}^{3} G_i$. When this is done we see (Section 12.5) that L satisfies the hypotheses of the Recognition Theorem and so prove the main theorem (Theorem 12.5.1).

12.1. We begin by establishing a property of proper roots.

LEMMA 12.1.1. *Let A be a restricted Lie algebra and T be a torus of maximal dimension in A. Assume that T is standard in A. Let $\alpha \in \Delta_p(A, T)$. Then $A^{(\alpha)}$ contains a unique compositionally classical subalgebra of maximal dimension and this subalgebra contains T.*

Proof. If α is solvable or classical, then $A^{(\alpha)}$ is compositionally classical and the required subalgebra is A itself. If α is Witt or Hamiltonian, Lemma 5.3.6 and Corollary 5.6.4 give the result. ∎

DEFINITION 12.1.2. Let A be a restricted Lie algebra and T be a torus of maximal dimension in A. Assume that T is standard in A. Let $\alpha \in \Delta_P(A, T)$. Define $U^{(\alpha)} = U^{(\alpha)}(A, T)$ to be the unique compositionally classical subalgebra of $A^{(\alpha)}$ of maximal dimension given by Lemma 12.1.1.

LEMMA 12.1.3. *Let A be a restricted Lie algebra and T be a torus of maximal dimension in A. Assume that T is standard in A. Let $\alpha \in \Delta_P(A, T)$. Then*:

(a) $U^{(\alpha)} \supseteq \mathfrak{z}_A(T)$.

(b) $\dim A^{(\alpha)}/U^{(\alpha)} = \begin{cases} 0 & \text{if } \alpha \text{ is solvable or classical;} \\ 1 & \text{if } \alpha \text{ is Witt;} \\ 2 & \text{if } \alpha \text{ is Hamiltonian.} \end{cases}$

(c) *If α is Witt then there is some $i \in \mathbf{Z}_p^*$ such that* $\dim A_{j\alpha}/(U^{(\alpha)})_{j\alpha} = \delta_{i,j}$.

(d) *If α is Hamiltonian then there is some $i \in \mathbf{Z}_p^*$ such that* $\dim A_{j\alpha}/(U^{(\alpha)})_{j\alpha} = \delta_{i,j} + \delta_{-i,j}$.

Proof. Since $\operatorname{solv}(A^{(\alpha)}) \subseteq U^{(\alpha)}$ (for $U^{(\alpha)} + \operatorname{solv}(A^{(\alpha)})$ is compositionally classical) we may assume that $\operatorname{solv}(A^{(\alpha)}) = (0)$. If α is solvable or classical then $A^{(\alpha)} = U^{(\alpha)}$ and there is nothing to prove. If α is Witt (respectively, Hamiltonian) it follows from the proof of Lemma 5.3.6 that $U^{(\alpha)} = W(1:\mathbf{1})_0$ (respectively, $A^{(\alpha)} \cap H(2:\mathbf{1})_0$). Since $T \subseteq U^{(\alpha)}$, Theorem 1.3.1 shows that we may assume T is spanned by $x_1 D_1$ (respectively, $x_1 D_1 - x_2 D_2$). Direct computation shows that $\operatorname{ad} x_1 D_1$ has the single eigenvalue -1 on $A^{(\alpha)}/U^{(\alpha)}$ (respectively, $\operatorname{ad}(x_1 D_1 - x_2 D_2)$ has the two eigenvalues ± 1 on $A^{(\alpha)}/U^{(\alpha)}$), proving the lemma. ∎

DEFINITION 12.1.4. Let A be a restricted Lie algebra and T be a torus of maximal dimension in A. Assume that T is standard and that every root with respect to A is proper. Define $Q(A) = Q(A, T)$ to be $\sum_{\alpha \in \Gamma(A,T)} U^{(\alpha)}(A, T)$.

LEMMA 12.1.5. *Let A be a restricted Lie algebra and T be a torus of maximal dimension in A. Assume that T is standard and that every root with respect to T is proper. Let $\alpha, \beta \in \Delta(A, T)$. Then*:

(a) $Q(A, T) \cap A^{(\alpha)} = Q(A^{(\alpha,\beta)}, T) \cap A^{(\alpha)}$;

(b) *If $x \in A_\alpha$ then $x \in Q(A, T)_\alpha$ if and only if* $\Psi_{\alpha,\beta}(x) \in Q(A[\alpha, \beta], \Psi_{\alpha,\beta}(T))$.

Proof. Definition 12.1.4 implies that $Q(A, T) \cap A^{(\alpha)} = U^{(\alpha)}(A, T) = U^{(\alpha)}(A^{(\alpha)}, T)$. Since $(A^{(\alpha,\beta)})^{(\alpha)} = A^{(\alpha)}$, part (a) holds. By Lemma 12.1.3, $\dim A^{(\alpha)}/U^{(\alpha)}(A, T) = \dim A[\alpha, \beta]^{(\alpha)}/U^{(\alpha)}(A[\alpha, \beta], \Psi_{\alpha,\beta}(T))$. Let $V = \Psi_{\alpha,\beta}^{-1}(U^{(\alpha)}(A[\alpha, \beta], \Psi_{\alpha,\beta}(T)))$. Then $\dim A^{(\alpha)}/U^{(\alpha)}(A, T) = \dim A^{(\alpha)}/V$ and, since ker $\Psi_{\alpha,\beta} = \mathrm{solv}(A^{(\alpha,\beta)})$ is solvable, V is compositionally classical. Thus $\dim V = \dim U^{(\alpha)}(A, T)$ and so, by Lemma 12.1.1, $V = U^{(\alpha)}(A, T)$. This proves (b). ∎

12.2. From now on we will assume that L is a finite-dimensional restricted simple Lie algebra over F. Recall (Definition 6.2.1 and Proposition 10.4.1) that if T is an optimal torus in L then T is of maximal dimension in L, T is standard, and all roots with respect to T are proper.

PROPOSITION 12.2.1. *Let L be a finite-dimensional restricted simple Lie algebra over F and T be an optimal torus in L. If $Q(L, T) = L$ then L is a classical simple Lie algebra.*

Proof. By Definition 12.1.4, if $Q(L, T) = L$ then $L^{(\alpha)}$ is compositionally classical and so $L[\alpha] = (0)$ or $\mathfrak{sl}(2)$ for every α. Then Corollary 11.3.2 gives the result. ∎

DEFINITION 12.2.2. Let A be a restricted Lie algebra and T be a torus of maximal dimension in A. Assume that T is standard and that every root with respect to T is proper. Let $\Omega = \Omega(A, T) = \{\alpha \in \Delta(A, T) \mid A_\alpha \neq Q(A, T)_\alpha\}$. We say that the pair (A, T) is *amenable* if the following conditions are satisfied:

$$Q(A, T) \text{ is a subalgebra of } A. \tag{12.2.1}$$

$$\text{If } \alpha \in \Omega(A, T) \text{ and } \beta \in \Delta(A, T) \text{ then } |\Omega(A, T) \cap (\alpha + \mathbf{Z}\beta)| \leqslant 3 \text{ and } (\mathrm{ad}\, A_\beta)^2 A \subseteq Q(A, T). \tag{12.2.2}$$

$$\text{If } \alpha, \beta, \alpha+\beta \in \Omega(A, T) \text{ and if } A_{\alpha+\beta} = [A_\alpha, A_\beta] + Q(A, T)_{\alpha+\beta} \text{ then } A[\alpha, \beta] \cong K(3:\mathbf{1}),\ [A_\beta, Q(A, T)_{\alpha-\beta}] + Q(A, T)_\alpha = A_\alpha, \text{ and } [A_{\alpha+\beta}, Q(A, T)_{-\beta}] + Q(A, T)_\alpha = A_\alpha. \tag{12.2.3}$$

LEMMA 12.2.3. *Let L be a finite-dimensional restricted simple Lie algebra over F and T' be an optimal torus in L. Then (L, T') is amenable.*

Proof. In view of Lemma 12.1.5(b) it is sufficient to prove that $(L[\mu, \nu], \Psi_{\mu,\nu}(T'))$ is amenable for every pair of roots $\mu, \nu \in \Delta(L, T')$. Thus, writing $A = L[\mu, \nu]$, $T = \Psi_{\mu,\nu}(T')$ we must show that (A, T) is amenable for every A which can occur as an $L[\mu, \nu]$.

If $\dim T \leqslant 1$ then $A = A^{(\gamma)}$ for some γ. Then $Q(A, T) = Q(A, T) \cap A^{(\gamma)} = U^{(\gamma)}$ is a subalgebra of A (Definition 12.1.2) so (12.2.1) holds. Also $\dim A/Q(A, T) = \dim A/U^{(\gamma)} \leqslant 2$. This clearly implies (12.2.2) and (12.2.3).

Thus we may assume that $\dim T = 2$ and hence that A is one of the algebras listed in the conclusion of Proposition 11.2.4. We will consider each of the cases in the conclusion of that theorem.

Suppose A is one of the algebras of Proposition 11.2.4(a). Thus

$S_1 + S_2 \subseteq A \subseteq \mathrm{Der}(S_1 + S_2)$, where S_1, S_2 are distinct ideals and are among $\mathfrak{sl}(2)$, $W(1:\mathbf{1})$, $H(2:\mathbf{1})^{(2)}$. Let J_i be the restricted ideal generated by S_i. Then by the Engel–Jacobson theorem, $T \cap J_i \neq (0)$ for $i = 1, 2$. Let $0 \neq t_i \in T \cap J_i$ satisfy $t_i^p = t_i$. Then $T = Ft_1 + Ft_2$. Define α_1, α_2 by $\alpha_i(t_j) = \delta_{ij}$. Since $[t_1, S_2] + [t_2, S_1] \subseteq S_1 \cap S_2 = (0)$ we have $S_i + \mathfrak{z}_A(T) = A^{(\alpha_i)}$ for $i = 1, 2$. Thus $A_\gamma = (0)$ unless $\gamma \in \mathbf{Z}\alpha_1 \cup \mathbf{Z}\alpha_2$. This clearly implies (12.2.2) and (12.2.3). For (12.2.1) we must show that $Q(A, T) = Q(A, T)^{(\alpha_1)} + Q(A, T)^{(\alpha_2)}$ is a subalgebra. Since $Q(A, T)^{(\alpha_i)}$ is a subalgebra for $i = 1, 2$ it is sufficient to show that $[Q(A, T)_{u\alpha_1}, Q(A, R)_{v\alpha_2}] \subseteq Q(A, T)$ whenever u, $v \in \mathbf{Z}$. If $u\alpha_1 = 0$ or $v\alpha_2 = 0$ then as $\mathfrak{z}_A(T) \subseteq Q(A, T)^{(\alpha_i)}$ for $i = 1, 2$ the result holds. If $u\alpha_1, v\alpha_2 \neq 0$ then $[Q(A, T)_{u\alpha_1}, Q(A, T)_{v\alpha_2}] \subseteq [S_1, S_2] = (0)$. Thus (12.2.1) holds and (A, T) is amenable.

Next suppose A is one of the algebras of Proposition 11.2.4(b). Thus $S \otimes B_n \subseteq A \subseteq \mathrm{Der}(S \otimes B_n)$, where $n = 1$ or 2, $T \not\subseteq S \otimes B_n$, $S = \mathfrak{sl}(2)$, $W(1:\mathbf{1})$, or $H(2:\mathbf{1})^{(2)}$ and $S \otimes (x_1 B_n + \cdots + x_n B_n)$ is invariant under T. Furthermore, by Lemma 11.2.5, if $t_1, t_2 \in T$ satisfy $t_i^p = t_i$, $T = Ft_1 + Ft_2$, $t_1 \in S \otimes B_n$, $t_2 \notin S \otimes B_n$ and if the roots α_1, α_2 are defined by $\alpha_i(t_j) = \delta_{ij}$, then $S \otimes (x_1 B_n + \cdots + x_n B_n)$ is invariant under $\mathrm{ad}(Q(A, T) \cap A^{(\alpha_2)})$ (and, since $A^{(\alpha_1)} \subseteq S \otimes B_n + \mathfrak{z}_A(T)$, $S \otimes (x_1 B_n + \cdots + x_n B_n)$ is invariant under $\mathrm{ad}(Q(A, T) \cap A^{(\alpha_1)})$).

Suppose $S \cong \mathfrak{sl}(2)$. Let $M = (Q(A, T) \cap A^{(\alpha_2)}) + S \otimes B_n$. Then M is compositionally classical and $M = Q(A, T)$. Thus (12.2.1) holds and (in view of Lemma 12.1.3) $|\{\gamma \mid A_\gamma \neq Q(A, T)_\gamma\}| \leqslant 2$, which implies (12.2.2) and (12.2.3) hold, so (A, T) is amenable.

Suppose $S \cong W(1:\mathbf{1})$. Then $\mathrm{ad}\, t_2$ induces a derivation of $(S \otimes B_n)/(S \otimes (x_1 B_n + \cdots + x_n B_n)) \cong S \cong W(1:\mathbf{1})$. Since $[t_1, t_2] = 0$, since all derivations of $W(1:\mathbf{1})$ are inner, and since any maximal torus in $W(1:\mathbf{1})$ is equal to its centralizer, we may (replacing t_2 by some element of $t_2 + \mathbf{Z}t_1$) assume that $[t_2, S \otimes B_n] \subseteq S \otimes (x_1 B_n + \cdots + x_n B_n)$. This implies $A^{(\gamma)} \subseteq S \otimes (x_1 B_n + \cdots + x_n B_n)$ whenever $\gamma \notin \mathbf{Z}\alpha_1 \cup \mathbf{Z}\alpha_2$. Thus $\{\gamma \mid A_\alpha \neq Q(A, T)_\gamma\} \subseteq \mathbf{Z}\alpha_1 \cup \mathbf{Z}\alpha_2$, which implies that (12.2.2) and (12.2.3) hold. Furthermore, $Q(A, T) = (Q(A, T) \cap A^{(\alpha_1)}) + (Q(A, T) \cap A^{(\alpha_2)}) + S \otimes (x_1 B_n + \cdots + x_n B_n)$, which is a subalgebra (recall that $S \otimes (x_1 B_n + \cdots + x_n B_n)$ is invariant under $\mathrm{ad}(Q(A, T) \cap A^{(\alpha_i)})$ for $i = 1, 2$). Thus (12.2.1) holds and so (A, T) is amenable.

Now suppose $S \cong H(2:\mathbf{1})^{(2)}$. Then t_1 and t_2 induce a torus of dimension $\leqslant 2$ acting on $(S \otimes B_n)/(S \otimes (x_1 B_n + \cdots + x_n B_n)) \cong S \cong H(2:\mathbf{1})^{(2)}$. We may therefore assume that $\mathrm{ad}\, t_2$ induces the derivation $l(\mathrm{ad}(x_1 D_1 + x_2 D_2))$ for some $l \in \mathbf{Z}$ and that $\mathrm{ad}\, t_1$ induces the derivation $\mathrm{ad}(x_1 D_1 - x_2 D_2)$. It is immediate that $Q(A, T) \supseteq A_\tau$ unless $\tau \in \mathbf{Z}\alpha_2$, $\tau = \alpha_1 - l\alpha_2$, or $\tau = -\alpha_1 - l\alpha_2$. Thus $Q(A, T) = (Q(A, T) \cap A^{(\alpha_2)}) + M$, where $M = (H(2:\mathbf{1})^{(2)})_0 \otimes 1 + S \otimes (x_1 B_n + \cdots + x_n B_n)$. Now $(H(2:\mathbf{1})^{(2)})_0$ is a subalgebra of $H(2:\mathbf{1})^{(2)}$ (so M is a subalgebra of A) and is invariant under every derivation of $H(2:\mathbf{1})^{(2)}$

which commutes with $\mathrm{ad}(x_1D_1 - x_2D_2)$ (by Proposition 2.1.8(vii)). Now if $x \in Q(A, T) \cap A^{(\alpha_2)}$ we have that $\mathrm{ad}\, x$ stabilizes $S \otimes (x_1B_n + \cdots + x_nB_n)$ and so induces a derivation of $H(2:\mathbf{1})^{(2)} \cong (S \otimes B_n)/(S \otimes (x_1B_n + \cdots + x_nB_n))$. Since $[x, t_1] = 0$ this derivation commutes with $\mathrm{ad}(x_1D_1 - x_2D_2)$ and so stabilizes $(H(2:\mathbf{1})^{(2)})_0$. Thus $\mathrm{ad}\, x$ stabilizes M. Hence (12.2.1) holds. Also $\Omega \subseteq \{\pm m\alpha_2, \pm\alpha_1 - l\alpha_2\}$ so (12.2.2) holds. Finally, $\alpha, \beta, \alpha + \beta \in \Omega$ forces $\{\alpha, \beta\} = \{\pm\alpha_1 - l\alpha_2\}$. Thus $[A_\alpha, A_\beta] \subseteq (S \otimes B_n)^{(\alpha_2)} \subseteq Q(A, T)$ so $[A_\alpha, A_\beta] + Q(A, T)_{\alpha+\beta} \subseteq Q(A, T)_{\alpha+\beta}$. Since $\alpha + \beta \in \Omega$ implies $A_{\alpha+\beta} \neq Q(A, T)_{\alpha+\beta}$ we see that $[A_\alpha, A_\beta] + Q(A, T)_{\alpha+\beta} = A_{\alpha+\beta}$ is impossible. Thus (12.2.3) holds and (A, T) is amenable.

Next suppose A is one of the algebras listed in Proposition 11.2.4(c). Thus $H(2:\mathbf{1})^{(2)} + Fx_1D_1 \subseteq A \subseteq \mathrm{Der}(H(2:\mathbf{1})^{(2)})$. By Theorem 1.18.4 of [BW82] we may assume T is one of $Fx_1D_1 + Fx_2D_2$, $F(x_1+1)D_1 + Fx_2D_2$, $F(x_1+1)D_1 + F(x_2+1)D_2$. Since all roots with respect to T are proper, $T = Fx_1D_1 + Fx_2D_2$. Then if $A_0 = A \cap (\mathrm{Der}(H(2:\mathbf{1})^{(2)}))_0$ we have A_0 is compositionally classical and $\dim A/A_0 \leqslant 2$. But D_1 and D_2 are root vectors for independent roots and $D_1, D_2 \notin Q(A, T)$ (since $\{\mathscr{D}(x_1^i x_2) \mid 0 \leqslant i \leqslant p-1\}$ and $\{\mathscr{D}(x_1 x_2^i) \mid 0 \leqslant i \leqslant p-1\}$ span subalgebras of $H(2:\mathbf{1})^{(2)}$ isomorphic to $W(1:\mathbf{1})$). Thus $\dim A/Q(A, T) \geqslant 2$. Hence $\dim A/Q(A, T) = 2$ and $Q(A, T) = A_0$. Thus (12.2.1)–(12.2.3) hold and (A, T) is amenable.

Next suppose A is the algebra of Proposition 11.2.4(d), i.e., suppose that $A \cong \overline{H(2:\mathbf{1}:\Delta)}$. Then Lemma 11.1.3(a) shows $T \subseteq \overline{H(2:\mathbf{1}:\Delta)}_0$. Since $\overline{H(2:\mathbf{1}:\Delta)}_0$ is compositionally classical, Lemma 11.1.3(b) shows that $Q(\overline{H(2:\mathbf{1}:\Delta)}, T) = \overline{H(2:\mathbf{1}:\Delta)}_0$. Since $\overline{H(2:\mathbf{1}:\Delta)}_0$ is a subalgebra of codimension two in $\overline{H(2:\mathbf{1}:\Delta)}$, (12.2.1)–(12.2.3) hold and hence (A, T) is amenable.

Finally, suppose that A is one of the algebras listed in Proposition 11.2.4(e). Then A is classical or isomorphic to one of $W(2:\mathbf{1})$, $S(3:\mathbf{1})^{(1)}$, $H(4:\mathbf{1})^{(1)}$, $K(3:\mathbf{1})$. If A is classical then $A = Q(A, T)$, while if A is non-classical then $T \subseteq A_0$ by Corollary 5.8.2(h). It is then immediate that $Q(A, T) = A_0$ ($= (X(m:\mathbf{1})^{(2)})_0$, where $A = X(m:\mathbf{1})^{(2)}$). Thus (12.2.1) holds and explicit computation of the Cartan decomposition for each type shows that (12.2.2)–(12.2.3) hold. ∎

12.3. We continue to assume that L is a finite-dimensional restricted simple Lie algebra over F and that T is an optimal torus in L.

Suppose L is nonclassical. Then by Proposition 12.2.1 and Lemma 12.2.3, $Q(L, T)$ is a proper subalgebra of L. Let L_0 be a maximal subalgebra of L containing $Q(L, T)$. Note that L_0 is restricted by Corollary 1.1.2 of [BW82]. Let $L \supseteq \cdots \supseteq L_{-1} \supseteq L_0 \supseteq L_1 \supseteq \cdots$ be a corresponding filtration and $G = \sum G_i$ be the associated graded algebra. We will now derive some results on the structure of G. Recall (Section 6.1) that $G_i = \sum_{\gamma \in \Gamma_i} G_{i,\gamma}$.

LEMMA 12.3.1. *$G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$ is a direct sum of classical simple algebras.*

Proof. Since G_0 acts faithfully on G_{-1} and since $\dim G_{-1,\lambda} = 1$ for all $\lambda \in \Gamma_{-1}$ (by Lemma 12.1.3) we have $I(G_0) = (0)$. If $x \in G_{0,\gamma}$, $y \in G_{0,-\gamma}$, $[x, y] \neq 0$ and $\gamma([x, y]) = 0$ then by Lemma 1.14.1 of [BW82] we see that $(\operatorname{ad} x)^2 G_{-1} + (\operatorname{ad} y)^2 G_{-1} \neq (0)$, contradicting (12.2.2) (which holds by Lemma 12.2.3). Hence, setting $J_\beta = \{x \in G_{0,\beta} \mid [x, G_{0,-\beta}] = (0)\}$, we see that if $G_{0,\beta} \neq J_\beta$ then there exist $x_{\pm\beta} \in G_{0,\pm\beta}$ such that $\beta([x_\beta, x_{-\beta}]) = 2$. Set $h_\beta = [x_\beta, x_{-\beta}]$. By (12.2.2) the eigenvalues of $\operatorname{ad} h_\beta$ on G_{-1} are contained in $\{0, \pm 1\}$ and so the eigenvalues of $\operatorname{ad} h_\beta$ on G_0 are contained in $\{0, \pm 1, \pm 2\}$. Then (applying [Jac58] to the $Fx_\beta + Fx_{-\beta} + Fh_\beta$-submodule $\sum_i G_{0,i\beta}$ of G_0) we see that $\dim G_{0,\pm\beta} = 1$. Suppose $[G_{0,\beta}, J_\gamma] \nsubseteq J_{\beta+\gamma}$. Then the Jacobi identity shows $G_{0,\beta} \neq J_\beta$ and $h_{\beta+\gamma} \in Fh_\beta$. Since the eigenvalues of $\operatorname{ad} h_\beta$ and of $\operatorname{ad} h_{\beta+\gamma}$ on G_{-1} are contained in $\{0, \pm 1\}$ we see that $h_{\beta+\gamma} = \pm h_\beta$. Now $h_{\beta+\gamma} = -h_\beta$ implies $\gamma(h_\beta) = (\beta+\gamma)(h_\beta) - \beta(h_\beta) = -4$ which is impossible. Thus $h_{\beta+\gamma} = h_\beta$ so $\gamma(h_\beta) = (\beta+\gamma)(h_\beta) - \beta(h_\beta) = 0$. Then if $\lambda \in \Gamma_{-1}$, $\lambda(h_\beta) = 1$ we have $(\operatorname{ad} x_{\beta+\gamma})(\operatorname{ad} x_{-\beta})\, G_{-1,\lambda} \neq (0)$. This implies $\lambda + \gamma \in \Gamma_{-1}$ and $(\lambda+\gamma)(h_\beta) = 1$. Iterating gives $\lambda + 3\gamma \subseteq \Gamma_{-1}$, contradicting (12.2.2). Thus $[G_{0,\beta}, J_\gamma] \subseteq J_{\beta+\gamma}$, so $\sum J_\gamma$ is a nil ideal in G_0. As G_0 acts irreducibly on G_{-1} this implies $J_\gamma = (0)$ for all γ.

It is now clear that $G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$ satisfies the hypotheses of Block's classification theorem [Blo66], giving the result. ∎

COROLLARY 12.3.2. $L_0 = Q(L, T)$.

Proof. Since L_1 is a nilpotent ideal in L_0, the lemma shows that L_0 is compositionally classical. Thus for every $\alpha \in \Delta(L, T)$ we have that $L_0 \cap L^{(\alpha)}$ is compositionally classical. Since $L_0 \cap L^{(\alpha)} \supseteq Q(L, T) \cap L^{(\alpha)} = U^{(\alpha)}(A, T)$ (by Definition 12.1.4) the maximality of $U^{(\alpha)}(A, T)$ implies $L_0 \cap L^{(\alpha)} = Q(L, T) \cap L^{(\alpha)}$ for all α so $L_0 = Q(L, T)$. ∎

LEMMA 12.3.3. *If $\lambda \in \Gamma_{-1}$ then every exist $v_\lambda \in G_{-1,\lambda}$, $w_{-\lambda} \in G_{1,-\lambda}$ such that $\lambda([v_\lambda, w_{-\lambda}]) = 2$. Thus $Fv_\lambda + Fw_{-\lambda} + F[v_\lambda, w_{-\lambda}] \cong \mathfrak{sl}(2)$.*

Proof. As $\lambda \in \Gamma_{-1}$, λ is Witt or Hamiltonian. Suppose λ is Witt. Then $L[\lambda] \cong W(1:\mathbf{1})$, $L^{(\lambda)} \cap L_0 \supseteq \operatorname{solv} L^{(\lambda)}$, and $L^{(\lambda)} \cap L_0/\operatorname{solv} L^{(\lambda)} \cong W(1:\mathbf{1})_0$. Thus there exist $x_\lambda \in L_\lambda$, $x_\lambda \notin L_0$, and $y_{-\lambda} \in L_{-\lambda} \cap L_0$ such that $\lambda([x_\lambda, y_{-\lambda}]) \neq 0$. Now either $y_{-\lambda} \in L_1$, in which case we are done (taking $v_\lambda = x_\lambda + L_0$, $w_{-\lambda} = y_{-\lambda} + L_2$), or else $G_{0,-\lambda} \neq (0)$. But by Lemma 12.3.1, if $G_{0,\lambda} \neq (0)$ then $G_{0,\lambda} + G_{0,-\lambda} + [G_{0,\lambda}, G_{0,-\lambda}] \cong \mathfrak{sl}(2)$. Then $L^{(\lambda)} \cap L_0$ is not solvable, contradicting the fact that $L^{(\lambda)} \cap L_0/\operatorname{solv} L^{(\lambda)}$ is isomorphic to the solvable algebra $W(1:\mathbf{1})_0$. Thus the lemma holds if λ is Witt. Now suppose that λ is Hamiltonian. Then $H(2:\mathbf{1})^{(2)} \subseteq L[\lambda] \subseteq H(2:\mathbf{1})$, $L^{(\lambda)} \cap L_0 \supseteq$ solv $L^{(\lambda)}$, and $L^{(\lambda)} \cap L_0/\text{solv } L^{(\lambda)} \supseteq (H(2:\mathbf{1})^{(2)})_0$. Thus there exist $x_\lambda \in L_\lambda$, $x_\lambda \notin L_0$, and $y_{-\lambda} \in L_{-\lambda} \cap L_0$ such that $\lambda([x_\lambda, y_{-\lambda}]) \neq 0$. Again it is sufficient to show that $y^{-\lambda} \in L_1$. If not $G_{0,-\lambda} \neq (0)$. But as $L^{(\lambda)} \cap L_0/\text{solv } L^{(\lambda)} \supseteq (H(2:\mathbf{1})^{(2)})_0$ we have $G_{0,\pm 2\lambda} \neq (0)$. As $G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$ is a direct sum of classical algebras, this is impossible. Hence the lemma holds if λ is Hamiltonian. ▌

LEMMA 12.3.4. $G_{-3} = (0)$.

Proof. If not, since $G_{-3} = [G_{-2}, G_{-1}]$, we can find $u \in G_{-2,\alpha}$, $v \in G_{-1,\beta}$ such that $0 \neq [u, v] \in G_{-3,\alpha+\beta}$. Write $u = x + L_{-1}$, $v = y + L_0$, where $x \in L_\alpha$, $y \in L_\beta$. Then $x \notin L_{-1}$, $y \notin L_0$, and $[x, y] \notin L_{-2}$. Thus $L_\gamma \neq Q(L, T)_\gamma$ for $\gamma = \alpha, \beta, \alpha+\beta$, and so by (12.2.3) (which applies to the pair (L, T) since (L, T) is amenable by Lemma 12.2.3) we have $[L_\beta, Q(L, T)_{\alpha-\beta}] + Q(L, T)_\alpha = L_\alpha$. Thus $L_\alpha \subseteq [L_\beta, L_0] + L_0$. But as $\dim(L_\beta/L_\beta \cap L_0) \leqslant 1$ (by Lemma 12.1.3) and $y \in L_\beta \cap L_{-1}$, $y \notin L_0$, we have $L_\beta \subseteq L_{-1}$. Thus $L_\alpha \subseteq [L_\beta, L_0] + L_0 \subseteq [L_{-1}, L_0] + L_0 \subseteq L_{-1}$, a contradiction. ▌

LEMMA 12.3.5. $N(G) = (0)$.

Proof. If $N(G) \neq (0)$ then as $N(G) \subseteq G_{-2} = [G_{-1}, G_{-1}]$ (using Lemma 12.3.4) and as $\dim G_{-2,\lambda} = 1$ for all $\lambda \in \Gamma_{-2}$ (by Lemma 12.1.3), there exist some $\alpha, \beta \in \Gamma_{-1}$ with $(0) \neq [G_{-1,\alpha}, G_{-1,\beta}]$ and $[[G_{-1,\alpha}, G_{-1,\beta}], G_1] = (0)$. Let $u \in L_{\alpha+\beta}$ satisfy $L_{\alpha+\beta} = Fu + L_{\alpha+\beta} \cap L_{-1}$. Then we must have $[u, L_1] \subseteq L_0$. By (12.2.3) (which holds for the pair (L, T) by Lemma 12.2.3) we have $[L_{\alpha+\beta}, L_{-\beta} \cap L_0] + L_\alpha \cap L_0 = L_\alpha$. Now $\dim L_{\alpha+\beta}/(L_{\alpha+\beta} \cap L_0) \leqslant 1$ (by Lemma 12.1.3) and $\dim L_{\alpha+\beta}/(L_{\alpha+\beta} \cap L_{-1}) \geqslant 1$ so $L_{\alpha+\beta} \cap L_{-1} \subseteq L_0$. Thus $[L_{\alpha+\beta}, L_{-\beta} \cap L_0] + L_\alpha \cap L_0 = [u, L_{-\beta} \cap L_0] + L_\alpha \cap L_0 = L_\alpha$. Since $L_\alpha \neq L_\alpha \cap L_0$ (as $\alpha \in \Gamma_{-1}$) and $[u, L_1] \subseteq L_0$ we must have $L_{-\beta} \cap L_0 \nsubseteq L_1$. Now (12.2.3) shows that $L[\alpha, \beta] \cong K(3:\mathbf{1})$ and hence $L_\gamma = Q(L, T)_\gamma$ for all $\gamma \in \mathbf{Z}\alpha + \mathbf{Z}\beta$, $\gamma \neq \alpha, \beta, \alpha+\beta$. Note that this implies that β is a Witt root. Since $L_{-\beta} \cap L_0 \nsubseteq L_1$ we have $G_{0,-\beta} \neq (0)$. Then by Lemma 12.3.1, $L^{(\beta)} \cap L_0$ is not solvable. But β is a Witt root and so $L^{(\beta)} \cap L_0$ is solvable, a contradiction. Hence $N(G) = (0)$. ▌

LEMMA 12.3.6. *G_0 acts faithfully on G_1.*

Proof. By Lemma 12.3.3, $\Gamma_1 \supseteq -\Gamma_{-1}$. Since G_0 acts faithfully on G_{-1}, this implies that T acts faithfully on G_1. Since any nonzero ideal of G_0 has nonzero intersection with T, the result follows. ▌

12.4. We will now show that it is impossible for $G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$ to have more than one summand.

LEMMA 12.4.1. *$G_0 = G_0^{(1)} + T$ and $\mathfrak{z}(G_0^{(1)}) \subseteq \mathfrak{z}(G_0)$.*

Proof. As each $G_{-1,\lambda}$ is one-dimensional we have that $I(G_0)$, the nil radical of $\mathfrak{z}_{G_0}(T)$, annihilates G_{-1}. But G_0 acts faithfully on G_{-1} so $I(G_0) = (0)$ and hence $\mathfrak{z}_{G_0}(T) = T$. Thus $G_0 = G_0^{(1)} + T$. Now if $G_{-1,\lambda} \neq (0)$ then $U(G_0^{(1)})\, G_{-1,\lambda}$ is a nonzero G_0-submodule of G_{-1} hence is equal to G_{-1}. Thus $\mathfrak{z}(G_0^{(1)})$ acts as scalars on G_{-1}, so $\mathfrak{z}(G_0^{(1)}) \subseteq \mathfrak{z}(G_0)$. ∎

LEMMA 12.4.2. *Suppose $G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$ has more than one summand. Then $G_0^{(1)} = J_1 + J_2$, where J_1, J_2 are ideals of G_0, $J_1 \cap J_2 \subseteq \mathfrak{z}(G_0)$ and $[J_1, J_2] = (0)$.*

Proof. If $G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$ has more than one summand we may write $G_0^{(1)}/\mathfrak{z}(G_0^{(1)}) = I_1 \oplus I_2$, where each of I_1 and I_2 is a nonzero sum of classical simple algebras. Since $I_i^{(1)} = I_i$ for $i = 1, 2$ we see that I_1 and I_2 are invariant under any derivation of $G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$. Since G_0 acts on $G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$ by derivations, we see that if J_i is the inverse image of I_i in $G_0^{(1)}$ then J_i is an ideal in G_0. Clearly $G_0^{(1)} = J_1 + J_2$ and $J_1 \cap J_2 \subseteq \mathfrak{z}(G_0)$. If $[J_1, J_2] \neq (0)$ then some $I_{1,\beta} \neq (0)$, $I_{2,-\beta} \neq (0)$. But I_1 is a direct sum of classical simple algebras so $\beta([I_{1,\beta}, I_{1,-\beta}]) \neq 0$. This implies $I_{2,-\beta} \subseteq I_1$, a contradiction. ∎

LEMMA 12.4.3. *If $G_{-2} \neq (0)$ then $G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$ has no more than one summand.*

Proof. Suppose $G_{-2} \neq (0)$. Then by (12.2.3) (which holds for (L, T) by Lemma 12.2.3) there exist $\lambda \in \Gamma_{-1}$ and $\alpha \in \Gamma_0$ such that $[G_{-1,\lambda}, G_{-1,\alpha+\lambda}] \neq (0)$ (so that $\alpha + 2\lambda \in \Gamma_{-2}$). Now suppose $\beta \in \Gamma_0$. Since $G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$ is a direct sum of classical simple algebras we can find $g_{\pm\beta} \in G_{0,\pm\beta}$ such that $\beta([g_\beta, g_{-\beta}]) = 2$. Write $h_\beta = [g_\beta, g_{-\beta}]$. Then since $(\operatorname{ad} g_\beta)^2\, G_{-1} = (\operatorname{ad} g_\beta)^2\, G_{-2} = (0)$ (by (12.2.2) and Corollary 12.3.2) we see that $\tau(h_\beta) \in \{-1, 0, 1\}$ for all $\tau \in \Gamma_-$. In particular, $\lambda(h_\beta) \in \{-1, 0, 1\}$ and $(\alpha + 2\lambda)(h_\beta) \in \{-1, 0, 1\}$. Suppose $\alpha(h_\beta) = 0$. Then $(\alpha + 2\lambda)(h_\beta) = 2\lambda(h_\beta) \in \{-1, 0, 1\} \cap \{-2, 0, 2\}$. Thus $\alpha(h_\beta) = 0$ implies $\lambda(h_\beta) = 0$. Now either $J_{1,\alpha} = (0)$ or $J_{2,\alpha} = (0)$ (for otherwise $I_{1,\alpha} \neq (0)$, $I_{2,\alpha} \neq 0$, and, as $\alpha([I_{1,\alpha}, I_{1,-\alpha}]) \neq (0)$ since I_1 is a sum of classical simple algebras, we have $I_{2,\alpha} \subseteq I_1$, a contradiction). Hence we may assume that $J_{1,\alpha} = (0)$. Now suppose that $J_{2,\beta} \neq (0)$. Then $h_\beta \in J_2$ so $\alpha(h_\beta) = 0$ and hence $\lambda(h_\beta) = 0$. Thus $J_2^{(1)}$ annihilates $G_{-1,\lambda}$. Since $J_2^{(1)}$ is an ideal in G_0 and the action of G_0 on G_{-1} is faithful and irreducible, this is impossible. Hence $G_{-2} = (0)$. ∎

LEMMA 12.4.4. *If $G_{-2} = (0)$ then $G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$ has no more than one summand.*

Proof. Suppose $G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$ has more than one summand. Let J_1 and J_2 be as in Lemma 12.4.2.

Let $\lambda \in \Gamma_{-1}$. Suppose $G_3 = (0)$. Then if v_λ, $w_{-\lambda}$ are as in Lemma 12.3.3 we see that if $\mu \in \Delta(L, T)$ then $\mu([v_\lambda, w_{-\lambda}]) \in \{-3, -2, -1, 0, 1, 2, 3\}$. But $\lambda \in \Gamma_{-1}$ implies λ is a Witt or Hamiltonian root and so $n\lambda \in \Delta(L, T)$ for all n, $1 \leqslant n \leqslant p-1$. As $p > 7$ this implies $G_3 \neq (0)$. Thus $[G_{-1}, [G_{-1}, [G_{-1}, G_3]]]$ is a nonzero ideal in G_0 which therefore contains a nonzero element of T. Hence we may find λ, μ, $v \in \Gamma_{-1}$, $v_\lambda \in G_{-1,\lambda}$, $v_\mu \in G_{-1,\mu}$, $v_v \in G_{-1,v}$, and $u_{-\lambda-\mu-v} \in G_{3,-\lambda-\mu-v}$ such that $[v_\lambda, [v_\mu, [v_v, u_{-\lambda-\mu-v}]]] \neq 0$. Then there exists some $v_\tau \in G_{-1,\tau}$ such that $0 \neq [v_\tau, [v_\lambda, [v_\mu, [v_v, u_{-\lambda-\mu-v}]]]] = [v_\lambda, [v_\tau, [v_\mu, [v_v, u_{-\lambda-\mu-v}]]]] = [v_\mu, [v_\tau, [v_\lambda, [v_v, u_{-\lambda-\mu-v}]]]] = [v_v, [v_\tau, [v_\lambda, [v_\mu, u_{-\lambda-\mu-v}]]]]$ (as $G_{-2} = (0)$). Then we have that $G_{0,\tau-\lambda} \neq (0)$, $G_{0,\tau-\mu} \neq (0)$, $G_{0,\tau-v} \neq (0)$. Thus if $\Gamma_{J_i} = \{\alpha \neq 0 \mid J_{i,\alpha} \neq (0)\}$ we have $\tau-\lambda$, $\tau-\mu$, $\tau-v \in \Gamma_{J_1} \cup \Gamma_{J_2} \cup \{0\}$. Hence (interchanging λ, μ, v if necessary) we may assume $\tau-\lambda$, $\tau-\mu \in \Gamma_{J_i} \cup \{0\}$ for $i = 1$ or 2. Now write $u_{-\lambda-\mu} = [v_v, u_{-\lambda-\mu-v}]$ so that $0 \neq [v_\lambda, [v_\mu, u_{-\lambda-\mu}]]$. Then we may find some $\alpha \neq \lambda$, $\alpha \in \Gamma_{-1}$ such that $[v_\alpha, [v_\lambda, [v_\mu, u_{-\lambda-\mu}]]] \neq 0$. For if not, $E = \mathrm{ad}([v_\lambda, [v_\mu, u_{-\lambda-\mu}]])$ is a rank one transformation of G_{-1} with $E^2 \neq 0$. Then Lemma 2 of [Wil71b] gives $G_0 \cong \mathfrak{gl}(G_{-1})$, contradicting our assumption that $G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$ has more than one summand. Since $0 \neq [v_\alpha, [v_\lambda, [v_\mu, u_{-\lambda-\mu}]]] = [v_\lambda, [v_\alpha, [v_\mu, u_{-\lambda-\mu}]]]$ (as $G_{-2} = (0)$), $G_{0,\alpha-\lambda} \neq (0)$. Similarly, $G_{0,\alpha-\mu} \neq (0)$. Assume (interchanging J_1 and J_2 if necessary) that $J_{1,\alpha-\lambda} \neq (0)$.

Note that if $\lambda \in \Gamma_{-1}$, $\beta \in \Gamma_0$, $v_\lambda \in G_{-1,\lambda}$, $g_\beta \in G_{0,\beta}$, and $[g_\beta, v_\lambda] \neq 0$, then there exists $g_{-\beta} \in G_{0,-\beta}$ such that $[g_{-\beta}, [g_\beta, v_\lambda]] \neq 0$. To see this take $g_{-\beta}$ so that $\beta([g_\beta, g_{-\beta}]) = 2$ (which is possible as $G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$ is a direct sum of classical simple algebras). Then (as $(\mathrm{ad}\, g_\beta)^2\, G_{-1} = (0)$) $[g_\beta, v_\lambda] \neq 0$ implies $\lambda([g_\beta, g_{-\beta}]) = -1$ and $[g_{-\beta}, v_\lambda] = 0$. Thus $v_\lambda = [[g_\beta, g_{-\beta}], v_\lambda] = -[g_{-\beta}, [g_\beta, v_\lambda]]$, giving the result.

Now we have $[g_{\alpha-\lambda}, v_\lambda] \neq 0$. Since $J_2^{(1)}$ is a nonzero ideal of G_0 we have $[g_\beta, v_\lambda] \neq 0$ for some $g_\beta \in J_{2,\beta}$. Thus $0 \neq [g_\beta, [g_{-\alpha+\lambda}, [g_{\alpha-\lambda}, v_\lambda]]]$ and so (as $[J_1, J_2] = (0)$) $[g_{\alpha-\lambda}, v_{\lambda+\beta}] \neq 0$ for $v_{\lambda+\beta} \in G_{-1,\lambda+\beta}$. Thus $0 \neq [v_{\lambda+\beta}, [v_\alpha, [v_\mu, u_{-\lambda-\mu}]]] = [v_\alpha, [v_{\lambda+\beta}, [v_\mu, u_{-\lambda-\mu}]]]$ so $m_\beta = [v_{\lambda+\beta}, [v_\mu, u_{-\lambda-\mu}]] \in J_{2,\beta}$ is a nonzero root vector. Since $J_1^{(1)}$ is a nonzero ideal of G_0 we have $[g_\gamma, v_\alpha] \neq 0$ for some $g_\gamma \in J_{1,\gamma}$. Thus $0 \neq [g_\gamma, [m_{-\beta}, [m_\beta, v_\alpha]]]$ and so (as $[J_1, J_2] = (0)$) $[m_\beta, v_{\alpha+\gamma}] \neq 0$ for $v_{\alpha+\gamma} \in G_{-1,\alpha+\gamma}$. Thus $0 \neq [v_{\alpha+\gamma}, [v_{\lambda+\beta}, [v_\mu, u_{-\lambda-\mu}]]]$ so $0 \neq [v_{\alpha+\gamma}, [v_{\lambda+\beta}, u_{-\lambda-\mu}]]$. We now have that $G_{0,\alpha-\mu} \neq (0)$ (for $0 \neq [v_\lambda, [v_\alpha, [v_\mu, u_{-\lambda-\mu}]]]$), $J_{2,\beta} \neq (0)$, $G_{0,\alpha-\mu+\beta} \neq (0)$ (for $0 \neq [v_\alpha, [v_{\lambda+\beta}, [v_\mu, u_{-\lambda-\mu}]]] = [v_\mu, [v_{\lambda+\beta}, [v_\alpha, u_{-\lambda-\mu}]]]$), $J_{1,\gamma} \neq (0)$, and $G_{0,\alpha-\mu+\beta+\gamma} \neq (0)$ (for $0 \neq [v_{\alpha+\gamma}, [v_{\lambda+\beta}, u_{-\lambda-\mu}]]$). Now if $G_{0,\theta}, G_{0,\phi}, G_{0,\theta+\phi} \neq (0)$ (where $\theta, \phi, \theta+\phi \neq 0$) then $[G_{0,\theta}, G_{0,\phi}] = G_{0,\theta+\phi}$ by Lemma II.4.1 of [Sel67] (which applies since $G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$ is classical semisimple). Therefore if $G_{0,\theta}, J_{i,\phi}, G_{0,\theta+\phi} \neq (0)$ we have $J_{i,\theta}, J_{i,\theta+\phi} \neq (0)$. We can therefore conclude that $J_{2,\alpha-\mu} \neq (0)$, $J_{2,\alpha-\mu+\beta} \neq (0)$, $J_{1,\alpha-\mu+\beta} \neq (0)$, and

$J_{1,\alpha-\mu+\beta+\gamma} \neq (0)$. Thus $\alpha-\mu+\beta=0$ and so we have $J_{1,\alpha-\lambda} \neq (0)$, $J_{2,\alpha-\mu}=J_{2,-\beta} \neq (0)$. But we also have that $\tau-\lambda$, $\tau-\mu \in \Gamma_{J_i} \cup \{0\}$ for $i=1$ or 2 and thus $\lambda-\mu$ vanishes on $T \cap J_1$ or on $T \cap J_2$. Since $\alpha-\lambda$ vanishes on $T \cap J_2$ and $\alpha-\mu$ vanishes on $T \cap J_1$, this implies that $\alpha-\lambda=0$ or $\alpha-\mu=0$, contradicting λ, $\mu \in \Gamma_{-1}$. ▮

LEMMA 12.4.5. *$G_0^{(1)}$ is a restricted ideal in G_0.*

Proof. If $x \in G_{0,\alpha}$, $\alpha \neq 0$, then $(\operatorname{ad} x)^2 G_{-1}=(0)$ (by (12.2.2) and Lemma 12.2.3). Thus $(\operatorname{ad} x)^p=\operatorname{ad}(x^p)$ annihilates G_{-1} and so $x^p=0$. Also $T \cap G_0^{(1)}$ is spanned by elements of the form $[x, y]$, where $x \in G_{0,\alpha}$, $y \in G_{0,-\alpha}$, $\alpha([x, y])=2$ (since $G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$ is a sum of classical simple algebras). For such x and y, since $(\operatorname{ad} x)^2 G_{-1}=(\operatorname{ad} y)^2 G_{-1}=(0)$ we have that the eigenvalues of $\operatorname{ad}[x, y]$ on G_{-1} are contained in $\{-1, 0, 1\}$. Thus on G_{-1}, $\operatorname{ad}[x, y]=(\operatorname{ad}[x, y])^p=\operatorname{ad}[x, y]^p$ and so $[x, y]^p=[x, y]$. Thus $T \cap G_0^{(1)}$ is restricted and hence $G_0^{(1)}$ is. ▮

LEMMA 12.4.6. *Let $k \in \mathbf{N}$. Up to isomorphism there is only one structure of restricted Lie algebra on $\mathfrak{gl}(pk)$ which extends the natural restricted Lie algebra structure of $\mathfrak{sl}(pk)$ and there is only one structure of restricted Lie algebra on $\mathfrak{pgl}(pk)+Fz$ which extends the restricted Lie algebra structure of $\mathfrak{psl}(pk)+Fz$ (a direct sum of restricted ideals where $\mathfrak{psl}(pk)$ has the natural restricted Lie algebra structure and $z^p=z$).*

Proof. Let E_{11} denote the usual matrix unit in $\mathfrak{gl}(pk)$. Then $(\operatorname{ad} E_{11})^p-(\operatorname{ad} E_{11})$ annihilates $\mathfrak{gl}(pk)$ and so if $x \mapsto x^{[p]}$ is any pth power map on $\mathfrak{gl}(pk)$ we have $E_{11}^{[p]}-E_{11}=aI$, where $a \in F$ and I denotes the identity matrix. Let b satisfy $b^p-b-a=0$. Then $(E_{11}-bI)^{[p]}-(E_{11}-bI)=0$. Thus the linear map ϕ: $\mathfrak{gl}(pk) \to \mathfrak{gl}(pk)$ defined by $\phi|_{\mathfrak{sl}(pk)}=$ identity and $\phi(E_{11})=E_{11}-bI$ is an isomorphism of restricted Lie algebras from $\mathfrak{gl}(pk)$ with the natural restricted Lie algebra structure to $\mathfrak{gl}(pk)$ with the pth power map $x \mapsto x^{[p]}$.

Similarly, let $\bar{E}_{11}$ denote the image of E_{11} in $\mathfrak{pgl}(pk)$. Then $(\operatorname{ad} \bar{E}_{11})^p-(\operatorname{ad} \bar{E}_{11})$ annihilates $\mathfrak{pgl}(pk)+Fz$ and so if $x \mapsto x^{[p]}$ is any pth power map on $\mathfrak{pgl}(pk)+Fz$ then $\bar{E}_{11}^{[p]}-\bar{E}_{11}=az$, where $a \in F$. Let b satisfy $b^p-b-a=0$. Then $(\bar{E}_{11}-bz)^{[p]}-(\bar{E}_{11}-bz)=0$. Thus the linear map ϕ: $\mathfrak{pgl}(pk)+Fz \to \mathfrak{pgl}(pk)+Fz$ defined by $\phi|_{\mathfrak{psl}(pk)+Fz}=$ identity and $\phi(\bar{E}_{11})=\bar{E}_{11}-bz$ is an isomorphism of restricted Lie algebras from $\mathfrak{pgl}(pk)+Fz$ (direct sum of restricted ideals with natural pth power structure on $\mathfrak{pgl}(pk)$ and $z^p=z$) to $\mathfrak{pgl}(pk)+Fz$ with the pth power map $x \mapsto x^{[p]}$. ▮

COROLLARY 12.4.7. *The restricted Lie algebra G_0 has one of the following structures:*

(i) $G_0 = Fz$,

(ii) G_0 *is isomorphic to a classical simple Lie algebra,*

(iii) $G_0 \cong \mathfrak{sl}(pk)$, $\mathfrak{pgl}(pk)$, *or* $\mathfrak{gl}(pk)$ *for some* $k \in \mathbf{N}$,

(iv) $G_0 \cong J \oplus Fz$ *(a direct sum of restricted Lie algebras), where* z *is central and* J *is classical simple or* $\mathfrak{pgl}(pk)$ *for some* $k \in \mathbf{N}$.

Proof. By Lemmas 12.4.3 and 12.4.4, $G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$ has at most one summand. If $G_0^{(1)}/\mathfrak{z}(G_0^{(1)}) = (0)$ then, as $G_0 = G_0^{(1)} + T$ and $\mathfrak{z}(G_0^{(1)}) \subseteq \mathfrak{z}(G_0) \subseteq T$, we have $G_0 \subseteq T$ so $G_0 = \mathfrak{z}(G_0)$. Thus as G_0 acts faithfully and irreducibly on G_{-1} we see that (i) holds. Thus we may assume that $G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$ has one summand, i.e., that it is a classical simple Lie algebra.

Suppose $\mathfrak{z}(G_0) = (0)$. Then $\mathfrak{z}(G_0^{(1)}) = (0)$ and $G_0^{(1)}$ is classical simple. Then $G_0 \subseteq \operatorname{Der} G_0^{(1)}$. By [Blo62, Lemma 7.1], $\operatorname{Der} G_0^{(1)} = G_0^{(1)}$ unless $G_0^{(1)} \cong \mathfrak{psl}(pk)$, $k \in \mathbf{N}$, and $\operatorname{Der}(\mathfrak{psl}(pk)) = \mathfrak{pgl}(pk)$. Thus (ii) or (iii) holds.

Next suppose that $\mathfrak{z}(G_0^{(1)}) \neq (0)$. Then $G_0^{(1)}$ is a nonsplit central extension of the classical simple Lie algebra $G_0^{(1)}/\mathfrak{z}(G_0^{(1)})$. By [Blo68, Theorem 3.1] this implies $G_0^{(1)}/\mathfrak{z}(G_0^{(1)}) \cong \mathfrak{psl}(pk)$ and $G_0^{(1)} \cong \mathfrak{sl}(pk)$. Thus $G_0 \subseteq \operatorname{Der}(\mathfrak{sl}(pk))$ and so $G_0 \cong \mathfrak{gl}(pk)$ or $\mathfrak{sl}(pk)$. By Lemma 12.4.6 this is an isomorphism of restricted Lie algebras. Thus (iii) holds.

Finally, suppose that $\mathfrak{z}(G_0) \neq (0)$ and $\mathfrak{z}(G_0^{(1)}) = (0)$. Thus $G_0^{(1)}$ is classical simple and $G_0 \supseteq G_0^{(1)} \oplus Fz$ a direct sum of restricted ideals by Lemma 12.4.5. Thus $G_0 \subseteq \operatorname{Der} G_0^{(1)} + Fz$. If $G_0^{(1)}$ is not of the form $\mathfrak{psl}(pk)$ then by [Blo62, Lemma 7.1], $G_0^{(1)} = \operatorname{Der} G_0^{(1)}$ so G_0 is listed in (iv). If $G_0^{(1)} \cong \mathfrak{psl}(pk)$ then $G_0 = \mathfrak{psl}(pk) + Fz$ or $\mathfrak{pgl}(pk) + Fz$. In the latter case, Lemma 12.4.6 shows that this is a direct sum of restricted ideals. Thus (iv) again holds and the lemma is proved. ∎

12.5. We now state and prove the main theorem of the paper.

THEOREM 12.5.1. *Let* L *be a finite-dimensional restricted simple Lie algebra over* F, *an algebraically closed field of characteristic* $p > 7$. *Then* L *is either classical simple or an algebra of Cartan type.*

Proof. Let T be an optimal torus in L. By Proposition 10.4.1, all roots with respect to T are proper. By Proposition 12.2.1 we may assume $L \neq Q(L, T)$. Let L_0 be a maximal subalgebra containing $Q(L, T)$ (indeed, $L_0 = Q(L, T)$ by Corollary 12.3.2), $L \supseteq \cdots \supseteq L_{-1} \supseteq L_0 \supseteq L_1 \supseteq \cdots$ be a corresponding filtration, and G be the associated graded algebra. By Lemmas 12.3.5 and 12.3.6, $N(G) = (0)$ and the action of G_0 on G_1 is faithful. Thus the graded Lie algebra G is transitive. By Corollary 12.4.7 we see that G_0 is a direct sum of restricted ideals each of which is abelian, classical simple, or isomorphic to $\mathfrak{sl}(pk)$, $\mathfrak{pgl}(pk)$, or $\mathfrak{gl}(pk)$. Thus the hypotheses of the Recognition Theorem (Theorem 1.2.2) are satisfied and that theorem gives our result. ∎

ACKNOWLEDGMENTS

Both authors gratefully acknowledge the hospitality of the Mathematical Sciences Research Institute and partial support from the National Science Foundation under Grants MCS-8120790 (at MSRI), MCS-8201828 and DMS-8502799 (for R.E.B.), and DMS-8301664 and DMS-8603151 (for R.L.W.) during the preparation of this work.

REFERENCES

[AF54] A. A. ALBERT AND M. S. FRANK, Simple Lie algebras of characteristic p, *Rend. Sem. Mat. Univ Politec. Torino* **14** (1954–1955), 117–139.

[BIO79] G. M. BENKART, I. M. ISAACS, AND J. M. OSBORN, Albert–Zassenhaus Lie algebras and isomorphisms, *J. Algebra* **57** (1979), 310–338.

[BO-pre] G. M. BENKART AND J. M. OSBORN, Toral rank one Lie algebras, preprint.

[Bla69] R. BLATTNER, Induced and produced representations of Lie algebras, *Trans. Amer. Math. Soc.* **144** (1969), 457–474.

[Blo58a] R. E. BLOCK, New simple Lie algebras of prime characteristic, *Trans. Amer. Math. Soc.* **89** (1958), 421–449.

[Blo58b] R. E. BLOCK, On torsion-free abelian groups and Lie algebras, *Proc. Amer. Math. Soc.* **11** (1958), 613–620.

[Blo62] R. E. BLOCK, Trace forms on Lie algebras, *Canad. J. Math.* **14** (1962), 553–564.

[Blo66] R. E. BLOCK, On the Mills–Seligman axioms for Lie algebras of classical type, *Trans. Amer. Math. Soc.* **121** (1266), 378–392.

[Blo68] R. E. BLOCK, On the extensions of Lie algebras, *Canad. J. Math.* **20** (1968), 1439–1450.

[Blo69] R. E. BLOCK, Determination of the differentiably simple rings with a minimal ideal, *Ann. of Math.* **90** (1969), 433–459.

[Blo74] R. E. BLOCK, Irreducible representations of Lie algebra extensions, *Bull. Amer. Math. Soc.* **80** (1974), 868–872.

[Blo82] R. E. BLOCK, The classification problem for simple Lie algebras of characteristic p, *in* "Lie Algebras and Related Topics, Proceedings, 1981 Rutgers Lie Algebras Conference" (D. Winter, Ed.), Lecture Notes in Mathematics, Vol. 933, pp. 38–56, Springer-Verlag, New York/Berlin, 1982.

[Blo85] R. E. BLOCK, Commutative Hopf algebras, Lie coalgebras and divided powers, *J. Algebra* **96** (1985), 275–306.

[BW82] R. E. BLOCK AND R. L. WILSON, The simple Lie p-algebras of rank two, *Ann. of Math.* **115** (1982), 93–186.

[BW84] R. E. BLOCK AND R. L. WILSON, The restricted simple Lie algebras are of classical or Cartan type, *Proc. Nat. Acad. Sci. USA* **81** (1984), 5271–5274.

[BW86] R. E. BLOCK AND R. L. WILSON, Restricted simple Lie algebras, *in* "Lie Algebras and Related Topics—Proceedings of a Summer Seminar, June 26–July 6, 1984" (D. J. Britten, F. W. Lemire, and R. V. Moody, Eds.), *Canad. Math. Soc. Conf. Proc.* **5** (1986), 3–17.

[Car09] E. CARTAN, Les groupes de transformations continus, infinis, simples, *Ann. Sci. École Norm. Sup.* **26** (1909), 93–161; "Oeuvres completes," Vol. 2, Partie II, pp. 857–925, Gauthier–Villars, Paris, 1953.

[Cel70] M. JU. CELOUSOV, Derivations of Lie algebras of Cartan type, *Izv. Vyssk. Uchebn. Zaved. Mat* **98** (1970), 126–134. [Russian]

[Cha41] H. J. CHANG, Über Wittische Lie-Ringe, *Abh. Math. Sem. Univ. Hamburg* **14** (1941), 151–184.

[Dem70] S. P. DEMUŠKIN, Cartan subalgebras of the simple Lie p-algebras W_n and S_n, *Sibirsk. Mat. Zk.* **11** (1970), 310–325 [Russian]; *Siberian Math. J.* **11** (1970), 233–245. [English transl.]

[Dem72] S. P. DEMUŠKIN, Cartan subalgebras of simple nonclassical Lie p-algebras, *Izv. Akad. Nauk SSSR Ser. Mat.* **36** (1972), 915–932 [Russian]; *Math. USSR-Izv.* **6** (1972), 905-924. [English transl.]

[Dix71] J. DIXMIER, Sur les representations induites des algebres de Lie, *J. Math. Pures Appl.* **50** (1971), 1–24.

[Fra54] M. S. FRANK, A new class of simple Lie algebras, *Proc. Nat. Acad. Sci. USA* **40** (1954), 713–719.

[Fra64] M. S. FRANK, Two new classes of simple Lie algebras, *Trans. Amer. Math. Soc.* **112** (1964), 456–482.

[JJac75] J. B. JACOBS, Necessary conditions for isomorphism of Lie algebras of Block, *Trans. Amer. Math. Soc.* **208** (1975), 73–79.

[Jac37] N. JACOBSON, Abstract derivation and Lie algebras, *Trans. Amer. Math. Soc.* **42** (1937), 206–224.

[Jac43] N. JACOBSON, Classes of restricted Lie algebras of characteristic p, II, *Duke Math. J.* **10** (1943), 107–121.

[Jac53] N. JACOBSON, "Lectures in Abstract Algebra, II. Linear Algebra," Van Nostrand, New York, 1953.

[Jac58] N. JACOBSON, A note on three-dimensional simple Lie algebras, *J. Math. Mech.* **7** (1958), 828–831.

[Jac62] N. JACOBSON, "Lie Algebras," Interscience, New York, 1962.

[Kac70] V. G. KAC, The classification of the simple Lie algebras over a field with nonzero characteristic, *Izv. Akad. Nauk SSSR Ser. Mat.* **34** (1970), 385–408 [Russian]; *Math. USSR-Izv.* **4** (1970), 391–413. [English transl.]

[Kac74] V. G. KAC, Description of filtered Lie algebras with which graded Lie algebras of Cartan type are associated, *Izv. Akad. Nauk SSSR Ser. Mat.* **38** (1974), 800–834; Errata, **40** (1976), 1415 [Russian]; *Math. USSR-Izv.* **8** (1974), 801–835; Errata, **10** (1976), 1339. [English transl.]

[Kap58] I. KAPLANSKY, Lie algebras of characteristic p, *Trans. Amer. Math. Soc.* **89** (1958), 149–183.

[Kap71] I. KAPLANSKY, "Lie Algebras and Locally Compact Groups," Univ. of Chicago Press, Chicago, 1971.

[KS66] A. I. KOSTRIKIN AND I. R. ŠAFAREVIČ, Cartan pseudogroups and Lie p-algebras, *Dokl. Akad. Nauk SSSR* **168** (1966), 740–742 [Russian]; *Soviet Math. Dokl.* **7** (1966), 715–718. [English transl.]

[KS69] A. I. KOSTRIKIN AND I. R. ŠAFAREVIČ, Graded Lie algebras of finite characteristic, *Izv. Akad. Nauk SSSR Ser. Mat.* **33** (1969), 251–322 [Russian]; *Math. USSR-Izv.* **3** (1969), 237–304. [English transl.]

[Kre71] V. A. KREKNIN, The existence of a maximal invariant subalgebra in simple Lie algebras of Cartan type, *Mat. Zametki* **9** (1971), 211–222 [Russian]; *Math. Notes* **9** (1971), 124–130. [English transl.]

[Kuz76] M. KUZNECOV, Simple modular Lie algebras with a solvable maximal subalgebra, *Mat. Sb.* **101** (1976), 77–86 [Russian]; *Math. USSR-Sb.* **30** (1976), 68–76. [English transl.]

[MS57] W. H. MILLS AND G. B. SELIGMAN, Lie algebras of classical type, *J. Math. Mech.* **6** (1957), 519–548.

[Ree59] R. REE, Note on generalized Witt algebras, *Canad. J. Math.* **11** (1959), 345–352.

[Sch69] J. R. SCHUE, Cartan decompositions for Lie algebras of prime characteristic, *J. Algebra* **11** (1969), 25–52; Errata, **13** (1969), 558.

[Sel67] G. B. SELIGMAN, Modular Lie Algebras, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 40, Springer-Verlag, New York, 1967.

[SF88] H. STRADE AND R. FARNSTEINER, "Modular Lie Algebras and Their Representations," Dekker, New York, 1988.

[Wei68] B. JU. WEISFEILER, Infinite dimensional filtered Lie algebras and their connection with graded Lie algebras, *Funktsional. Anal. i Prilozhen.* **2** (1968), 94–95. [Russian]

[Wei78] B. JU. WEISFEILER, On the structure of the minimal ideal of some graded Lie algebras of characteristic $p>0$, *J. Algebra* **53** (1978), 344–361.

[Wei84] B. JU. WEISFEILER, On subalgebras of simple Lie algebras of characteristic $p>0$, *Trans. Amer. Math. Soc.* **286** (1984), 471–503.

[Wil69] R. L. WILSON, Nonclassical simple Lie algebras, *Bull. Amer. Math. Soc.* **75** (1969), 987–991.

[Wil71a] R. L. WILSON, Classification of generalized Witt algebras over algebraically closed fields, *Trans. Amer. Math. Soc.* **153** (1971), 191–210.

[Wil71b] R. L. WILSON, Irreducible Lie algebras of infinite type, *Proc. Amer. Math. Soc.* **29** (1971), 243–249.

[Wil75] R. L. WILSON, Automorphisms of graded Lie algebras of Cartan type, *Comm. Algebra* **3** (1975), 591–613.

[Wil76] R. L. WILSON, A structural characterization of the simple Lie algebras of generalized Cartan type over fields of prime characteristic, *J. Algebra* **40** (1976), 418–465.

[Wil77] R. L. WILSON, Cartan subalgebras of simple Lie algebras, *Trans. Amer. Math. Soc.* **234** (1977), 435–446; Correction, to appear.

[Wil78] R. L. WILSON, Simple Lie algebras of toral rank one, *Trans. Amer. Math. Soc.* **236** (1978), 287–295.

[Wil80] R. L. WILSON, Simple Lie algebras of type *S*, *J. Algebra* **62** (1980), 292–298.

[Wil83] R. L. WILSON, Classification of the restricted simple Lie algebras with toral Cartan subalgebras, *J. Algebra* **83** (1983), 531–570.

[Wil-pre] R. L. WILSON, Simple Lie algebras over fields of prime characteristic, *in* "Proceedings, 1986 International Congress of Mathematicians," to appear.

[Win69] D. J. WINTER, On the toral structure of Lie *p*-algebras, *Acta Math.* **123** (1969), 69–81.

[Zas39] H. ZASSENHAUS, Über Liesche Ringe mit Primzahlcharakteristik, *Abh. Math. Sem. Univ. Hamburg* **13** (1939), 1–100.

Printed in Belgium