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ON THE HERBRAND-KLEENE UNIVERSE FOR NONDETERMINISTIC COMPUTATIONS

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Abstract. For nondeterministic recursive equations over an arbitrary signature of function symbols including the nondeterministic choice operator "or" the interpretation is factorized according to the techniques developed by the present author (1982). It is shown that one can either associate an infinite tree with the equations, then interpret the function symbol "or" as a nondeterministic choice operator and so mapping the tree onto a set of infinite trees and then interpret these trees. Or one can interpret the recursive equation directly yielding a set-valued function. Both possibilities lead to the same result, i.e., one obtains a commuting diagram. However, one has to use more refined techniques than just powerdomains. This explains and solves a problem posed by Nivat (1980). Basically, the construction gives a generalization of the powerdomain approach applicable to arbitrary nonflat (nondiscrete) algebraic domains.

1. Introduction

When trying to give a denotational semantics for nondeterministic or concurrent programs, one of the most intricate questions is that of finding the appropriate fixed point definitions for recursively defined programs. Although it is not difficult at all to give the fixed point equations, in general, one has to face a number of problems when trying to find the appropriate ordering for applying the principle of least fixed points. One option is to take the orderings used in powerdomains, but these orderings for sets are just preorderings over nonflat (nondiscrete) domains. So one has to look for more sophisticated techniques. In the following we give a general construction for the definition of set-valued (nondeterministic) functions as fixed points of recursive equations.

In the algebraic semantics as described in [3] it is shown that a recursive equation can either be interpreted directly in an algebraic domain D or first transformed into an infinite tree (the Herbrand-Kleene interpretation) and then interpreted in D with identical results. As pointed out by Nivat [4] for nondeterministic equations containing the nondeterministic choice operator "or" one would like to have analogous techniques. One would like to transform a recursion equation with the nondeterministic choice operation "or" either at first into an infinite tree (without interpreting

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"or", just considering it as a binary function symbol), then interpret "or" yielding a set of infinite (deterministic) trees, that can be interpreted then. Or one could directly interpret the recursive equation over set-valued functions. Like in the deterministic case, one is interested that all these interpretations form a commuting diagram. The function mapping applications of recursively defined functions onto sets of possibly infinite trees is what we call nondeterministic Herbrand–Kleene interpretation.

As pointed out in [4], it seems difficult or even impossible to define the nondeterministic Herbrand–Kleene interpretation. In fact, it is impossible in the classical way as long as one wants to consider a domain (a 'powerdomain') of infinite trees, and just monotonic and continuous functions for it. However, applying the techniques of Broy [1], which have originally been developed to give a denotational semantics to concurrent, communicating programs, the reason for the problem can be explained and a 'nonclassical' solution can be envisaged. Combining several orderings, nondeterministic Herbrand–Kleene interpretations can be defined such that the respective constructions commute and the techniques of 'algebraic' semantics by interpretations of recursive equations as infinite trees can be carried over to nondeterministic computations.

In particular, such techniques can be applied for arbitrary nonflat (nondiscrete) domains. Thus a generalized construction for a nondeterministic Herbrand–Kleene interpretation is obtained where the problems of interpreting nondeterministic choice are separated from the problems of classical determinate interpretations.

2. Basic definitions

As in [2] a signature $\Sigma = (S, F)$ is a pair consisting of a set S of sorts and a set F of function symbols with some fixed functionality $s_1 \times \cdots \times s_n \to s_{n+1}$ for each $f \in F$ with $s_i \in S$. For our purpose it is sufficient to consider just a one-element set of sorts and an arbitrary set F of function symbols not containing $\perp$ and "or". In particular we consider

$$\Sigma = (\{s\}, F), \quad \text{where } \perp \notin F, \text{ or } \notin \text{F},$$

$$\Sigma^+ = (\{s\}, F^+), \quad \text{where } F^+ = F \cup \{\perp, \text{or}\},$$

$$\Sigma^\perp = (\{s\}, F^\perp), \quad \text{where } F^\perp = F \cup \{\perp\},$$

where the functionality of $\perp$ is $\to s$ and that of "or" is $s \times s \to s$.

A (total) *Σ-algebra* $A = \langle \{s^A\}_{s \in S}, \{f^A\}_{f \in F} \rangle$ consists of a family of carrier-sets s^A for each sort $s \in S$ and a family of functions f^A for each $f \in F$ with a functionality according to the functionality of f.

A is called an *algebraic Σ-algebra* iff

- all s^A form consistently complete countably algebraic domains (for a definition, see Section 4),
- all f^A denote continuous functions.

As usual (cf. [2]) we define the *term-algebra* W_Σ to be the Σ-algebra, the sorts of which consists of all well-formed terms (of the respective sorts) formed from the function symbols in F and the functions consisting of the corresponding 'term-construction' operations. By $I_A(t)$ we denote the interpretation of a term t in the algebra A.

W_{Σ^+} (and also $W_{\Sigma^\perp} \subseteq W_{\Sigma^+}$) can be partially ordered by (cf. [3])

$$
\begin{aligned}
t \sqsubseteq r \text{ iff } & t = r \vee t = \perp \\
& \vee [t = f(t_1, \ldots, t_n) \wedge r = f(r_1, \ldots, r_n) \wedge t_1 \sqsubseteq r_1 \wedge \cdots \wedge t_n \sqsubseteq r_n].
\end{aligned}
$$

Since "$\sqsubseteq$" defines a partial ordering with least element $\perp$, we can form the ideal completion leading to the algebra of finite and infinite terms or trees (also called 'magmas'). By $W^\infty_{\Sigma^+}$ and $W^\infty_{\Sigma^\perp}$ the ideal completion of respectively W_{Σ^+} and $W_{\Sigma^\perp}$ is denoted. As demonstrated by Nivat [3], the interpretation I_A can be continuously extended in a unique way to 'infinite' terms from $W^\infty_{\Sigma^\perp}$. This extension will be denoted by I^∞_A in the next section.

3. On nondeterministic Herbrand–Kleene interpretations

Let A be an algebraic Σ-algebra. It is our goal to define mappings (the application of I^∞_A to sets of terms is assumed to be shorthand for the element-wise application of I^∞_A):

$$
\begin{tikzcd}
 & \mathrm{P}(W^\infty_{\Sigma^\perp}) \arrow[d, "I^\infty_A"] \\
W^\infty_{\Sigma^+} \arrow[ur, bend left, "B^\infty_{\mathrm{HK}}"] \arrow[r, bend right, "B^\infty_A"'] & \mathrm{P}(A)
\end{tikzcd}
$$

where the following equations are to hold:

$$B^\infty_{\mathrm{HK}}[t] = \{t\} \quad \text{for } t \in W^\infty_{\Sigma^\perp},$$

$$B^\infty_A[t] = \{I^\infty_A(t)\} \quad \text{for } t \in W^\infty_{\Sigma^\perp},$$

$$B^\infty_{\mathrm{HK}}[\mathrm{or}(t1, t2)] = B^\infty_{\mathrm{HK}}[t1] \cup B^\infty_{\mathrm{HK}}[t2],$$

$$B^\infty_A[\mathrm{or}(t1, t2)] = B^\infty_A[t1] \cup B^\infty_A[t2],$$

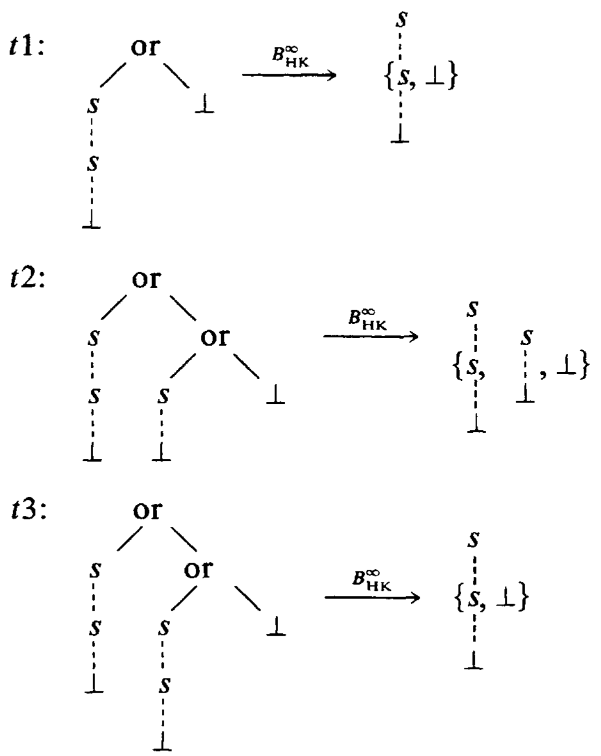
$$B^\infty_{\mathrm{HK}}[f(t_1, \ldots, t_n)] = \{f(r_1, \ldots, r_n) : \forall i, 1 \leqslant i \leqslant n : r_i \in B^\infty_{\mathrm{HK}}[t_i]\},$$

$$B^\infty_A[f(t_1, \ldots, t_n)] = \{f^A(a_1, \ldots, a_n) : \forall i, 1 \leqslant i \leqslant n : a_i \in B^\infty_A[t_i]\}.$$

Here $f \in F$ is assumed. Note that variables can be seen as nullary function symbols. The intended function B^∞_{HK} is called *nondeterministic Herbrand–Kleene interpretation*. For finite terms t the sets $B^\infty_{\mathrm{HK}}[t]$ and $B^\infty_A[t]$ are uniquely defined if we interpret the equations above as an inductive definition. However, this does not work for infinite terms t. Actually, at a first glance it is not clear whether such a function B^∞_{HK} actually axists and is uniquely determined. In spite of this we can prove that B^∞_{HK} cannot be monotonic.

Lemma 3.1. *There does not exist an ordering on* $\mathrm{P}(W_{\Sigma^\perp}^{\infty})$ *such that* B_{HK}^{∞} *is monotonic.*

Proof. Consider the following examples of trees:



We have $t1 \sqsubseteq t2 \sqsubseteq t3$; so, the monotonicity of B_{HK}^{∞} would immediately imply

$$B_{\mathrm{HK}}^{\infty}[t1] \sqsubseteq B_{\mathrm{HK}}^{\infty}[t2] \sqsubseteq B_{\mathrm{HK}}^{\infty}[t3].$$

Since $B_{\mathrm{HK}}^{\infty}[t1] = B_{\mathrm{HK}}^{\infty}[t3]$, one immediately obtains, by the antisymmetry of a partial ordering, $B_{\mathrm{HK}}^{\infty}[t1] = B_{\mathrm{HK}}^{\infty}[t2]$ which obviously gives a contradiction. □

The proof clearly shows a problem arising with nondeterministic computations. Scott's theory of computation is based on the principle of approximation: every object is determined uniquely by the set of its finite approximations (forming an ideal). For nondeterministic computations one has to consider sets of objects. A set of objects is approximated in the sense above by a set of approximatins for these objects. However, unfortunately, if such objects $x \neq y \neq z$ are in the partial order, i.e., if $x \sqsubseteq y \sqsubseteq z$ holds, then every approximation of y is also an approximation of z; every set approximating $\{x, z\}$ contains approximations of x and thus of y. So an approximation of $\{x, z\}$ is always an approximation of $\{x, y, z\}$ and vice versa. Both are not distinguishable by the classical approximation order, the Egli–Milner order. This explains one of the basic anomalies in the generalized (to nonflat domains) powerdomain approach (cf. [5, 6]).

Nondeterministic interpretations do not only describe one computation, but a set of feasible computations. So, one has two independent notions of approximations: the classical approximation 'is less defined' of computations in the sense of Scott

And the approximation 'has a wider spectrum of choices' on the level of nondeterminism.

It is not surprising that it is not possible to combine these two distinct notions of approximation into one single partial ordering without running into problems.

4. Some definitions on powersets and powerdomains

In this section we give three powerdomain constructions based on the idea of ideal completions. We choose very particular representations for the elements of these powerdomains.

A partially ordered set DOM is called *countably algebraic domain* if:

- every directed set $S \subseteq \mathrm{DOM}$ has a least upper bound (lub),
- the set of finite approximations of an element x is directed and every element $x \in \mathrm{DOM}$ is lub of the set of its finite approximations:

$$x = \mathrm{lub}\{y \in \mathrm{DOM}\colon y \sqsubseteq x \text{ and } y \text{ finite}\},$$

- the set of finite elements is countable.

Here a set $S \subseteq \mathrm{DOM}$ is called *directed* if

$$\forall x, y \in S \;\; \exists z \in S\colon \quad x \sqsubseteq z \wedge y \sqsubseteq z.$$

An element $x \in \mathrm{DOM}$ is called *finite* if for all directed sets $S \subseteq \mathrm{DOM}$ we have

$$x \sqsubseteq \mathrm{lub}\, S \;\Rightarrow\; \exists y \in S\colon x \sqsubseteq y.$$

DOM is called *consistently complete* if every set $S \subseteq \mathrm{DOM}$ with an upper bound even has a least upper bound. Trivially every set has then a greatest lower bound (glb).

Let DOM be a consistently complete, countably algebraic domain; for $S, S1, S2 \subseteq \mathrm{DOM}$ we define:

$$\mathrm{MIN}(S) = \{x \in S : \forall y \in S : y \sqsubseteq x \Rightarrow x = y\},$$

$$\mathrm{MAX}(S) = \{x \in S : \forall y \in S : x \sqsubseteq y \Rightarrow x = y\},$$

$$\begin{aligned}\mathrm{CLOSE}(S) = \{x \in \mathrm{DOM} : \exists S0 \subseteq S\colon &\\ &((\forall a, b \in S0 \;\; \exists z \in S0 : a \sqsubseteq z \wedge b \sqsubseteq z) \wedge x = \mathrm{lub}\, S0)\\ &\vee ((\forall a, b \in S0 \;\; \exists z \in S0 : z \sqsubseteq a \wedge z \sqsubseteq b) \wedge x = \mathrm{glb}\, S0)\},\end{aligned}$$

$$\mathrm{UPC}(S) = \{x \in \mathrm{DOM} : \exists y \in S : y \sqsubseteq x\},$$

$$\mathrm{DOC}(S) = \{x \in \mathrm{DOM} : \exists y \in S : x \sqsubseteq y\},$$

$$\mathrm{CONE}(S) = \{x \in \mathrm{DOM} : \exists y, z \in S : y \sqsubseteq x \sqsubseteq z\}.$$

S is called *convex* iff $\mathrm{CONE}(S) = S$; S is called *closed* iff $\mathrm{CLOSE}(S) = S$.

We have

$$\text{CONE}(S) = \text{UPC}(S) \cap \text{DOC}(S).$$

Trivially all these functions and notions on sets can be extended to set-valued functions and functionals over those functions by applying them elementwise.

The following three preorderings are used (cf. [5, 6]):

$$S1 \sqsubseteq_E S2 \text{ iff } \forall x \in S1 \ \exists y \in S2 : x \sqsubseteq y,$$

$$S1 \sqsubseteq_M S2 \text{ iff } \forall y \in S2 \ \exists x \in S1 : x \sqsubseteq y,$$

$$S1 \sqsubseteq_{EM} S2 \text{ iff } S1 \sqsubseteq_E S2 \wedge S1 \sqsubseteq_M S2.$$

Over nonflat (nondiscrete) domains these relations just define preorderings. What sets are identified if we try to make these relations into orderings can be seen from the following lemma.

Lemma 4.1. *For closed sets* $S1$, $S2$ *we have*

$$S1 \sqsubseteq_E S2 \text{ iff } \text{MAX}(S1) \sqsubseteq_E \text{MAX}(S2),$$

$$S1 \sqsubseteq_M S2 \text{ iff } \text{MIN}(S1) \sqsubseteq_M \text{MIN}(S2).$$

For arbitrary sets $S1$, $S2$ *we have*

$$S1 \sqsubseteq_M S2 \text{ iff } \text{UPC}(S1) \sqsubseteq_M \text{UPC}(S2),$$

$$S1 \sqsubseteq_E S2 \text{ iff } \text{DOC}(S1) \sqsubseteq_E \text{DOC}(S2),$$

$$S1 \sqsubseteq_{EM} S2 \text{ iff } \text{CONE}(S1) \sqsubseteq_{EM} \text{CONE}(S2),$$

$$S_1 \sqsubseteq_M S2 \text{ iff } \text{UPC}(S2) \subseteq \text{UPC}(S1),$$

$$S1 \sqsubseteq_E S2 \text{ iff } \text{DOC}(S1) \subseteq \text{DOC}(S2).$$

This lemma shows one pathological property of the powerdomains based on these 'orderings': In a powerdomain particular distinct sets are considered as being equivalent, i.e., the powerdomain constructions actually consider classes of equivalent sets. But sets may not only be equivalent because they cannot be distinguished by the orderings above. Due to the principle of finite approximability and continuity two sets are considered to be equivalent in a countably algebraic powerdomain based on some ordering iff the classes of finite sets of finite elements that approximate these sets in the sense of these orderings are identical.

Let FDOM denote the set of finite elements from DOM.

We take here a very concrete set-theoretic view of powerdomains. Their elements are just represented by subsets of P(DOM), i.e., by particular elements of the powerset over DOM. These representations are chosen in a very particular way which is most convenient for our semantic descriptions.

The power domain PD(DOM) of *erratic nondeterminism* (also called Plotkin power domain or Egli–Milner power domain) is defined as follows. The set of finite elements is represented by the convex hull of finite sets of finite elements:

$$\mathrm{GD} = \{\mathrm{CONE}(\mathrm{S}) : \mathrm{S} \subseteq \mathrm{FDOM} \wedge |S| < \infty\}.$$

We immediately can prove

$$(\mathrm{GD}, \sqsubseteq_{\mathrm{EM}}) \text{ is poset.}$$

PD(DOM) is defined as the ideal-completion of $(\mathrm{GD}, \sqsubseteq_{\mathrm{EM}})$. We choose as representation for PD(DOM) a subset of P(DOM), such that every ideal $I \subseteq \mathrm{GD}$ is represented by

$$\{x \in \mathrm{DOM} : \forall S1 \in I, y \in \mathrm{FDOM} :$$
$$y \sqsubseteq x \Rightarrow \exists S2 \in I, S1 \sqsubseteq_{\mathrm{EM}} S2, z \in S2 : y \sqsubseteq z \sqsubseteq x\}.$$

By

$$\mathrm{PDOM} : \mathrm{P}(\mathrm{DOM}) \to \mathrm{PD}(\mathrm{DOM})$$

we denote the function mapping every set $S \subseteq \mathrm{DOM}$ on its power domain representation. It is defined by

$$\mathrm{PDOM}(S) = \{x \in \mathrm{DOM} : \forall S1 \in \mathrm{GD}, y \in \mathrm{FDOM} :$$
$$y \sqsubseteq x \wedge S1 \sqsubseteq_{\mathrm{EM}} S \Rightarrow$$
$$\exists S2 \in \mathrm{GD}, S1 \sqsubseteq_{\mathrm{EM}} S2 \sqsubseteq_{\mathrm{EM}} S, z \in S2 : y \sqsubseteq z \sqsubseteq x\}.$$

Note that we have chosen a $\subseteq$-maximal representation for PDOM(S), i.e., the $\subseteq$-maximal set in the class of sets that are $\sqsubseteq_{\mathrm{EM}}$-equivalent w.r.t. $\sqsubseteq_{\mathrm{EM}}$-approximations by finite sets of finite elements. A proof is given in Lemma 4.2.

The power domain of *demonic nondeterminism* (also called Smyth power domain) is defined as follows:

$$\mathrm{GM} = \{\mathrm{UPC}(S) : S \subseteq \mathrm{FDOM} \wedge |S| < \infty\}.$$

One immediately can prove

$$(\mathrm{GM}, \sqsubseteq_{\mathrm{M}}) \text{ is poset.}$$

PMO(DOM) is the ideal-completion of $(\mathrm{GM}, \sqsubseteq_{\mathrm{M}})$. We choose as representation for PMO(DOM) a subset of DOM, such that every ideal $I \subseteq \mathrm{GM}$ is represented by

$$\{x \in \mathrm{DOM} : \forall S1 \in I, y \in \mathrm{FDOM} :$$
$$y \sqsubseteq x \Rightarrow \exists S2 \in I, S1 \sqsubseteq_{\mathrm{M}} S2, z \in S2 : y \sqsubseteq z \sqsubseteq x\}.$$

By

$$\mathrm{PM\ DOMO} : \mathrm{P}(\mathrm{DOM}) \to \mathrm{PMO}(\mathrm{DOM})$$

we denote the function mapping every set $S \subseteq \mathrm{DOM}$ on its power domain representation. It is defined by

$$\mathrm{PM\ DOMO}(S) = \{x \in \mathrm{DOM} : \forall S1 \in \mathrm{GM}, y \in \mathrm{FDOM} : y \sqsubseteq x \wedge S1 \sqsubseteq_{\mathrm{M}} S \Rightarrow$$
$$\exists S2 \in \mathrm{GM}, S1 \sqsubseteq_{\mathrm{M}} S2 \sqsubseteq_{\mathrm{M}} S, z \in S2 : y \sqsubseteq z \sqsubseteq x\}.$$

All sets in PMO(DOM) are closed. Since we find it more convenient to work with a $\subseteq$-minimal representation we define

$$\mathrm{PM(DOM)} = \{\mathrm{MIN}(S) : S \in \mathrm{PMO(DOM)}\}$$

and

$$\mathrm{PM\ DOM} : \mathrm{P(DOM)} \to \mathrm{PM(DOM)}$$

with

$$\mathrm{PM\ DOM}(S) = \mathrm{MIN}(\mathrm{PM\ DOMO}(S)).$$

Note that we have chosen a $\subseteq$-minimal representation for PM DOM(S), i.e., the $\subseteq$-minimal set in the class of closed, finitely approximable sets that are $\sqsubseteq_{\mathrm{M}}$-equivalent w.r.t. $\sqsubseteq_{\mathrm{M}}$-approximations by finite sets of finite elements. A proof is given in Lemma 4.2.

The power domain of *angelic nondeterminimism* (also called Hoare power domain) is defined as follows:

$$\mathrm{GE} = \{\mathrm{DOC}(S) : S \subseteq \mathrm{FDOM} \wedge |S| < \infty\}.$$

One immediately can prove

$(\mathrm{GE}, \sqsubseteq_{\mathrm{E}})$ is poset.

PEO(DOM) is the ideal-completion of $(\mathrm{GE}, \sqsubseteq_{\mathrm{E}})$. We choose as representation for PEO(DOM) a subset of DOM such that every ideal $I \subseteq \mathrm{GE}$ is represented by

$$\{x \in \mathrm{DOM} : \forall S1 \in I, y \in \mathrm{FDOM} :$$
$$y \sqsubseteq x \Rightarrow \exists S2 \in I, S1 \sqsubseteq_{\mathrm{E}} S2, z \in S2 : y \sqsubseteq z \sqsubseteq x\}.$$

By

$$\mathrm{PE\ DOMO} : \mathrm{P(DOM)} \to \mathrm{PEO(DOM)}$$

we denote the function mapping every set $S \subseteq \mathrm{DOM}$ on its power domain representation. It is defined by

$$\mathrm{PE\ DOMO}(S) = \{x \in \mathrm{DOM} : \forall S1 \in \mathrm{GE}, y \in \mathrm{FDOM} : y \sqsubseteq x \wedge S1 \sqsubseteq_{\mathrm{E}} S \Rightarrow$$
$$\exists S2 \in \mathrm{GE}, S1 \sqsubseteq_{\mathrm{E}} S2 \sqsubseteq_{\mathrm{E}} S, z \in S2 : y \sqsubseteq z \sqsubseteq x\}.$$

The sets in PEO(DOM) are closed. We may represent them also by their maximal elements. This leads to a $\subseteq$-minimal representation for the powerdomain of angelic nondeterminism. We define

$$\mathrm{PE(DOM)} = \{\mathrm{MAX}(S) : S \in \mathrm{PEO(DOM)}\}$$

and

$$\mathrm{PE\,DOM} : \mathrm{P(DOM)} \to \mathrm{PE(DOM)}$$

with

$$\mathrm{PE\,DOM} = \mathrm{MAX}(\mathrm{PE\,DOMO}(S)).$$

Note that we have chosen a $\subseteq$-minimal representation for PE DOM(S), i.e., the $\subseteq$-minimal set in the class of closed, finitely approximable sets that are $\sqsubseteq_{\mathrm{E}}$-equivalent w.r.t. $\sqsubseteq_{\mathrm{E}}$-approximations by finite sets of finite elements. (For a proof, see Lemma 4.2.)

Basically these powerdomains contain just those sets for which the respective relations form orderings and which can be approximated by finite sets of finite elements. They are isomorphic to (continuous) ideal completions of the representation class of finite sets of finite elements.

For sets $S1$, $S2$ we define equivalence-relations as follows:

$$S1 \sim_{\mathrm{EM}} S2 \text{ iff } \mathrm{PDOM}(S1) = \mathrm{PDOM}(S2),$$

$$S1 \sim_{\mathrm{M}} S2 \text{ iff } \mathrm{PM\,DOM}(S1) = \mathrm{PM\,DOM}(S2),$$

$$S1 \sim_{\mathrm{E}} S2 \text{ iff } \mathrm{PE\,DOM}(S1) = \mathrm{PE\,DOM}(S2).$$

These equivalence relations can also be described in another way according to the following lemma.

Lemma 4.2. *For* $S1, S2 \subseteq \mathrm{DOM}$ *we have*:

(1) $S1 \sim_{\mathrm{EM}} S2$ *iff* $\forall S \subseteq \mathrm{FDOM}, |S| < \infty : (S \sqsubseteq_{\mathrm{EM}} S1 \Leftrightarrow S \sqsubseteq_{\mathrm{EM}} S2)$,

(2) $S1 \sim_{\mathrm{M}} S2$ *iff* $\forall S \subseteq \mathrm{FDOM}, |S| < \infty : (S \sqsubseteq_{\mathrm{M}} S1 \Leftrightarrow S \sqsubseteq_{\mathrm{M}} S2)$,

(3) $S1 \sim_{\mathrm{E}} S2$ *iff* $\forall S \subseteq \mathrm{FDOM}, |S| < \infty : (S \sqsubseteq_{\mathrm{E}} S1 \Leftrightarrow S \sqsubseteq_{\mathrm{E}} S2)$,

(4) $\mathrm{PDOM}(S) = \bigcup\{S0 \subseteq \mathrm{DOM} : S \sim_{\mathrm{EM}} S0\}$,

(5) $\mathrm{PM\,DOM}(S) = \mathrm{MIN}(\bigcup\{S0 \subseteq \mathrm{DOM} : S \sim_{\mathrm{M}} S0\})$,

(6) $\mathrm{PE\,DOM}(S) = \mathrm{MAX}(\bigcup\{S0 \subseteq \mathrm{DOM} : S \sim_{\mathrm{E}} S0\})$.

Proof. Parts (1)+(2)+(3) are trivial in one direction, since in the definitions of PDOM, PM DOM, PE DOM it can simply be seen that PDOM(S) (and PM DOM(S) and PE DOM(S) respectively) only depends on the set of finite sets $S0$ of finite elements with $S0 \sqsubseteq_{\mathrm{EM}} S$. Now assume that $S1 \sim_{\mathrm{EM}} S2$ and there is some $S \subseteq \mathrm{FDOM}$, $|S| < \infty$ with $S \sqsubseteq_{\mathrm{EM}} S1$ but $\neg(S \sqsubseteq_{\mathrm{EM}} S2)$. Then either there exists some $x \in S$ such that there does not exist a $y2 \in S2$ with $x \sqsubseteq y2$ and there exists some

$y1 \in S1$ with $x \sqsubseteq y1$; then $y1 \in \mathrm{PDOM}(S1)$ but $\neg(y1 \in \mathrm{PDOM}(S2))$; or there exists some $y2 \in S2$ such that there does not exist an $x \in S$ with $x \sqsubseteq y2$ and thus $y2 \in \mathrm{PDOM}(S2)$ but $\neg(y2 \in \mathrm{PDOM}(S1))$. The proofs of (2) and (3) follow analogously.

(4) If $x \in S0$ for some $S0 \sim_{\mathrm{EM}} S$, then for every pair of approximations $y \sqsubseteq x$, $S1 \sqsubseteq_{\mathrm{EM}} S$ (and thus $S1 \sqsubseteq_{\mathrm{EM}} S0$) with $S1 \in \mathrm{GD}$, $y \in \mathrm{FDOM}$ there exist approximations $S2 \in \mathrm{GD}$ with $S1 \sqsubseteq_{\mathrm{EM}} S2 \sqsubseteq_{\mathrm{EM}} S$ (and thus $S2 \sqsubseteq_{\mathrm{EM}} S0$) with $z \in S2$ such that $y \sqsubseteq z \sqsubseteq x$; hence $x \in \mathrm{PDOM}(S)$.

Now if $x \in \mathrm{PDOM}(S)$, then $S \sim_{\mathrm{EM}} S \cup \{x\}$ and thus $x \in \bigcup\{S1 \subseteq \mathrm{DOM} : S \sim_{\mathrm{EM}} S1\}$.

(5) Follows in analogy to (4).

(6) Follows in analogy to (5). □

The concept of finite observability over algebraic domains simply means that two objects are equal iff their classes of finite approximations are identical.

Lemma 4.3. *The following diagram commutes*:

$$\begin{tikzcd}
 & & \mathrm{PD}(\mathrm{DOM}) \arrow[dl, "\mathrm{MAX}"'] \arrow[dr, "\mathrm{MIN}"] & \\
\mathrm{P}(\mathrm{DOM}) \arrow[urr, bend left, "\mathrm{PDOM}"] \arrow[r, "\mathrm{PE\,DOM}"] \arrow[rrr, bend right, "\mathrm{PM\,DOM}"'] & \mathrm{PE}(\mathrm{DOM}) & & \mathrm{PM}(\mathrm{DOM})
\end{tikzcd}$$

In particular, MIN *and* MAX *are continuous functions.*

Proof. The lemma is a corollary of Lemma 4.2. □

On function domains we use the classical ordering. Given a domain $D1$ ordered by $\sqsubseteq$, then the set of functions

$$\{f : D2 \to D1\}$$

with some given set (or domain) $D2$ can simply be ordered by

$$f1 \sqsubseteq^* f2 \ \text{ iff } \ \forall x \in D2 : f1(x) \sqsubseteq f2(x).$$

Analogously we write lub* for the lub on the function domain ordered by $\sqsubseteq^*$.

Unfortunately, the simple powerset without the empty set ordered by inclusion ordering does not form a domain. For very obvious reasons we do not accept the empty set as element, since the set of possible computations of a nondeterministic program can never be empty. However, $(\mathrm{P(DOM)}\backslash\{\emptyset\}, \subseteq)$ forms a predomain, i.e., it has all properties of a domain besides the existence of a least element. We restrict ourselves to closed sets, i.e., to sets S where with every directed set in S its least upper bound is also in S. This is motivated by the concept of finite observability. Every object should be determined by its finite approximations.

Accordingly, the *power predomain of closed sets* is defined as follows:

$$\mathrm{PC(DOM)} = \{S \subseteq \mathrm{DOM} : S = \mathrm{CLOSE}(S)\}.$$

A function

$$f : \mathrm{P(DOM)} \to \mathrm{P(DOM)}$$

is called closely union continuous iff

$$f(\mathrm{CLOSE}(\bigcup x_i)) = \mathrm{CLOSE}(\bigcup f(x_i))$$

for every $\subseteq$-chain $\{x_i\}_{i\in\mathbb{N}}$, $x_i \in \mathrm{P(DOM)}$.

Similarly, a functional

$$T : (\mathrm{DOM} \to \mathrm{P(DOM)}) \to (\mathrm{DOM} \to \mathrm{P(DOM)})$$

is called closely union continuous iff

$$T[\mathrm{CLOSE} \circ (\bigcup{}^* f_i)] = \mathrm{CLOSE} \circ (\bigcup{}^* T[f_i])$$

for every $\subseteq$-chain, $(f_i)_{i\in\mathbb{N}}$, $f_i : \mathrm{DOM} \to \mathrm{P(DOM)}$. Here, $\circ$ denotes the composition of functions and $\bigcup{}^* f_i$ denotes the elementwise union of the set-valued function f_i, i.e.,

$$(\bigcup{}^* f_i)(x) \stackrel{\mathrm{def}}{=} \bigcup f_i(x).$$

Based on these general definitions we are now going to give a general construction for a nondeterministic Herbrand-Kleene interpretation.

5. Nondeterministic interpretations

Let A be an algebraic $\Sigma^\perp$-algebra with least element $\perp_A$.

The nondeterministic interpretation of terms $t \in W^\infty_{\Sigma^+}$ in A is to be defined by the function

$$B^\infty_A : W^\infty_{\Sigma^+} \to \mathrm{P}(A)$$

fulfilling the equations

$$B^\infty_A[\mathrm{or}(t1, t2)] = B^\infty_A[t1] \cup B^\infty_A[t2],$$

$$B^\infty_A[f(t_1, \ldots, t_n)] = \{f^A(a_1, \ldots, a_n) : \forall i, 1 \leqslant i \leqslant n : a_i \in B^\infty_A[t_i]\}.$$

Note that we do not require that the images of B_A^∞ are in PD(A) but just in P(A), the powerset over A. It is not difficult to show that there are many distinct functions that fulfill these equations, since these equations do not characterize B_A^∞ uniquely. Such a situation is well known from classical fixed point theory. Besides the least fixed point, generally there are many other fixed points (optimal, maximal fixed points, etc.). What we would like to do is to proceed as in classical fixed point theory: we would like to find an appropriate ordering on P(A) such that B_A^∞ can be specified as the least fixed point. But according to our Lemma 3.1 such an ordering does not exist. So we have to use more refined techniques instead. In particular now we have

- to prove that the class of functions fulfilling the equations above is nonempty;
- to define uniquely which function from this class is to be taken.

We are going to define the function B_A^∞ in three steps:

Step 1. We define an approximation BP_A^∞ for the intended function B_A^∞ in the powerdomain of erratic nondeterminism, such that the following relation should hold:

$$\mathrm{BP}_A^\infty[t] = \mathrm{PDOM}(B_A^\infty[t]).$$

Define

$$\mathrm{BP}_A^\infty : W_{\Sigma^+}^\infty \to \mathrm{PD}(A)$$

by the least fixed point in the set of functions:

$$W_{\Sigma^+}^\infty \to \mathrm{PD}(A)$$

fulfilling the following equations:

$$\mathrm{BP}_A^\infty[\mathrm{or}(t1, t2)] = \mathrm{CONE}(\mathrm{BP}_A^\infty[t1] \cup \mathrm{BP}_A^\infty[t2]),$$

$$\mathrm{BP}_A^\infty[f(t_1, \ldots, t_n)] = \mathrm{CONE}(\{f^A(a_1, \ldots, a_n) : \forall i, 1 \leq i \leq n : a_i \in \mathrm{BP}_A^\infty[t_i]\}).$$

Now let

$$B_A : W_{\Sigma^+} \to \mathrm{P}(A)$$

be defined (inductively on finite terms) by

$$B_A[\mathrm{or}(t1, t2)] = B_A[t1] \cup B_A[t2],$$

$$B_A[f(t_1, \ldots, t_n)] = \{f^A(a_1, \ldots, a_n) : \forall i, 1 \leq i \leq n : a_i \in B_A[t_i]\}.$$

Note that because B_A works only on finite terms, it is characterized uniquely by these two equations.

Lemma 5.1. BP_A^∞ *is consistently defined,* $\sqsubseteq_{\mathrm{EM}}$*-monotonic, and* $\sqsubseteq_{\mathrm{EM}}$*-continuous.*

We have, for finite terms $t \in W_{\Sigma^+}$,

$$\mathrm{BP}_A^\infty[t] = \mathrm{CONE}(B_A[t]).$$

For arbitrary terms $t \in W^{\infty}_{\Sigma^+}$ *we have*

$$\mathrm{BP}^{\infty}_{A}[t] = \sqsubseteq_{\mathrm{EM}}\text{-lub}\{\mathrm{CONE}(B_A[t0]) : t0 \in W_{\Sigma^+} \wedge t0 \sqsubseteq t\}.$$

Proof. Since both right-hand sides of the equations for BP^{∞}_{A} are monotonic and continuous for BP^{∞}_{A} in the $\sqsubseteq_{\mathrm{EM}}$-ordering, the least fixed point of the equations is identical to the lub of the iterated application of the right-hand side of the equations starting with

$$\mathrm{BP}^{0}_{A}[t] = \{\perp\},$$

$$\mathrm{BP}^{i+1}_{A}[\mathrm{or}(t1, t2)] = \mathrm{CONE}(\mathrm{BP}^{i}_{A}[t1] \cup \mathrm{BP}^{i}_{A}[t2]),$$

$$\mathrm{BP}^{i+1}_{A}[f(t_1, \ldots, t_n)] =$$
$$= \mathrm{CONE}(\{f^A(a_1, \ldots, a_n) : \forall i, 1 \leq i \leq n : a_i \in \mathrm{BP}^{i}_{A}[t_i]\}).$$

However, this coincides with our definition of BP^{∞}_{A} if we take the lub on both sides of the equations.

According to its definition, BP^{∞}_{A} is the continuous extension of $\mathrm{CONE} \circ B_A$ to infinite terms on the powerdomain $\mathrm{PD}(A)$.

In particular, for every $t0 \in W_{\Sigma^+}$ with $t0 \sqsubseteq t$ there exists some $i \in \mathbb{N}$ such that

$$\mathrm{CONE}(B_A[t0]) \sqsubseteq_{\mathrm{EM}} \mathrm{BP}^{i}_{A}[t].$$

Vice versa for every $i \in \mathbb{N}$ there exist $t0 \in W_{\Sigma^+}$ with $t0 \sqsubseteq t$ such that

$$\mathrm{BP}^{i}_{A}[t] \sqsubseteq_{\mathrm{EM}} \mathrm{CONE}(B_A[t0]).$$

This proves the claim of the lemma. □

Step 2. We define an approximation for B^{∞}_{A} in the powerdomain of demonic nondeterminism. For $t \in W^{\infty}_{\Sigma^{\perp}}$ we have according to the results of Nivat [3]:

$$\mathrm{BP}^{\infty}_{A}[t] = \{I^{\infty}_{A}(t)\}.$$

According to Lemma 4.3,

$$\mathrm{MIN} \circ \mathrm{BP}^{\infty}_{A} : W^{\infty}_{\Sigma^+} \to \mathrm{PM}(W^{\infty}_{\Sigma^{\perp}})$$

defines a monotonic and continuous mapping, too.

Lemma 5.2. $\mathrm{MIN} \circ \mathrm{BP}^{\infty}_{A}$ *is consistently defined*, $\sqsubseteq_{\mathrm{M}}$-monotonic, and $\sqsubseteq_{\mathrm{M}}$-*continuous. It is the least fixed point in the set*

$$W^{\infty}_{\Sigma^+} \to \mathrm{PM}(A)$$

fulfilling the following equations:

$$\mathrm{MIN}(\mathrm{BP}^{\infty}_{A}[\mathrm{or}(t1, t2)]) = \mathrm{MIN}(\mathrm{BP}^{\infty}_{A}[t1] \cup \mathrm{BP}^{\infty}_{A}[t2]),$$

$$\mathrm{MIN}(\mathrm{BP}^{\infty}_{A}[f(t_1, \ldots, t_n)]) =$$
$$= \mathrm{MIN}(\{f^A(a_1, \ldots, a_n) : \forall i, 1 \leq i \leq n : a_i \in \mathrm{BP}^{\infty}_{A}[t_i]\}).$$

Proof. MIN is the continuous function mapping PD(A) into PM(A). □

Step 3. We define B_A^∞ as the $\subseteq^*$-least closed function with

$$\mathrm{MIN} \circ \mathrm{BP}_A^\infty \subseteq^* B_A^\infty \subseteq^* \mathrm{BP}_A^\infty,$$

that fulfills the specifying equations.

Proof of the existence of B_A^∞ (constructive)

(1) Define

$$B_A^i : W_{\Sigma^+}^\infty \to \mathrm{PC}(A)$$

by

$$B_A^0 = \mathrm{MIN} \circ \mathrm{BP}_A^\infty,$$

$$B_A^{i+1}[\mathrm{or}(t1, t2)] = B_A^i[t1] \cup B_A^i[t2],$$

$$B_A^{i+1}[f(t_1, \dots, t_n)] = \{f^A(a_1, \dots, a_n) : \forall i, 1 \leq i \leq n : a_i \in \mathrm{BP}_A^i[t_i]\}).$$

(2) We have

$$B_A^i \subseteq^* B_A^{i+1}$$

because the above definitions of B_i are $\subseteq^*$-monotonic in B and

$$\begin{aligned} B_A^0[t] &= \mathrm{MIN}(\mathrm{BP}_A^\infty[t]) \\ &= \begin{cases} \mathrm{MIN}(\mathrm{BP}_A^\infty[t1] \cup \mathrm{BP}_A^\infty[t2]) \\ \mathrm{MIN}(\{f(a_1, \dots, a_n) : a_i \in \mathrm{BP}_A^\infty[t_i]\}) \end{cases} \\ &\qquad\qquad \left.\begin{aligned} &\subseteq \mathrm{MIN}(\mathrm{BP}_A^\infty[t1]) \cup \mathrm{MIN}(\mathrm{BP}_A^\infty[t2]) \\ &\subseteq \{f(a_1, \dots, a_n) : a_i \in \mathrm{MIN}(\mathrm{BP}_A^\infty[t_i])\} \end{aligned}\right\} = B_A^1[t]. \end{aligned}$$

(3) Define

$$B_A^\infty = \mathrm{CLOSE} \circ \bigcup{}^* B_A^i;$$

since all language constructs are closely union-continuous, B_A^∞ is a fixed point of the defining equations. □

Corollary 5.3. *With the definitions of the previous lemma we have for all terms t:*

$$\begin{aligned} \mathrm{PM\ DOM}(B_A^\infty[t]) &= \mathrm{MIN}(\mathrm{PDOM}(B_A^\infty[t])) \\ &= \mathrm{MIN}(\mathrm{BP}_A^\infty[t]) \\ &= \mathrm{MIN}(B_A^\infty[t]) \subseteq B_A^\infty[t] \\ &\subseteq \mathrm{CONE}(B_A^\infty[t]) \\ &\subseteq \mathrm{PDOM}(B_A^\infty[t]) = \mathrm{BP}_A^\infty[t]. \end{aligned}$$

For $A = W^{\infty}_{\Sigma^{\perp}}$ the function B^{∞}_{A} will be denoted by

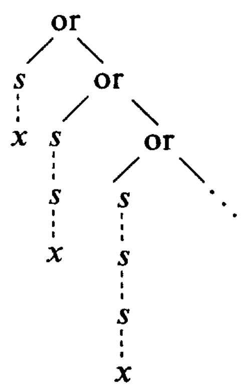
$$B^{\infty}_{\mathrm{HK}} : W^{\infty}_{\Sigma^{+}} \to \mathrm{P}(W^{\infty}_{\Sigma^{\perp}})$$

and called the nondeterministic *Herbrand-Kleene interpretation.* Now let us consider a basic example.

Example 5.4 (à la Maurice Nivat, taken from [4])

$$\Phi(x) = \mathrm{or}(s(x), \Phi(s(x))).$$

The infinite tree T (which is the least fixed point of the equation above in $W^{\infty}_{\Sigma^{+}}$) associated with $\Phi(x)$ can be represented as follows:



Following our construction we obtain

$$\mathrm{BP}^{\infty}_{\mathrm{HK}}[T] = \{\perp, s(\perp), s(s(\perp)), \ldots, s(x), s(s(x)), \ldots\},$$

$$\mathrm{MIN}(\mathrm{BP}^{\infty}_{\mathrm{HK}}[T]) = \{\perp\} = B^{0}_{\mathrm{HK}}[T],$$

$$B^{1}_{\mathrm{HK}}[T] = \{\perp, s(x)\},$$

$$B^{2}_{\mathrm{HK}}[T] = \{\perp, s(x), s(s(x))\},$$

$$\vdots$$

$$B^{\infty}_{\mathrm{HK}}[T] = \{\perp, s(x), s(s(x)), \ldots\}.$$

As a second example for recursive equations, Nivat considers

$$\Phi(x) = \mathrm{or}(s(\sigma(x)), \Phi(s(x))), \qquad \sigma(x) = x,$$

with the infinite trees $T1$ and $T2$. Obviously, both definitions of Φ should be equivalent. One obtains

$$\mathrm{BP}^{\infty}_{\mathrm{HK}}[T1] = \{\perp, s(\perp), s(s(\perp)), \ldots, s(x), s(s(x)), \ldots\},$$

$$\mathrm{BP}^{\infty}_{\mathrm{HK}}[T2] = \{x\};$$

hence

$$\mathrm{MIN}(\mathrm{BP}^{\infty}_{\mathrm{HK}}[T1]) = \{\perp\} = B^{0}_{\mathrm{HK}}[T1],$$

$$\mathrm{MIN}(\mathrm{BP}^{\infty}_{\mathrm{HK}}[T2]) = \{x\} = B^{0}_{\mathrm{HK}}[T2],$$

$$B^1_{\mathrm{HK}}[T1]=\{\perp, s(x)\}, \qquad B^1_{\mathrm{HK}}[T2]=\{x\},$$

$$B^2_{\mathrm{HK}}[T1]=\{\perp, s(x), s(s(x))\}, \qquad B^2_{\mathrm{HK}}[T2]=\{x\},$$

$$\vdots \qquad\qquad \vdots$$

Finally we obtain again

$$B^\infty_{\mathrm{HK}}[T1]=\{\perp, s(x), s(s(x)), \ldots\}.$$

So we get the same fixed point for both equations for Φ.

The problem found in [3] simply stems from the fact that $B^\infty_{\mathrm{HK}}[T]$ is not convex. As Nivat demonstrates, if the powerdomain idea is applied to nonconvex sets S in a naive way, however, arbitrary sets $S0$ with $\mathrm{CONE}(S)=\mathrm{CONE}(S0)$ can be obtained as limits by taking the lubs pointwise.

B^∞_A has the following properties:

- $\mathrm{PDOM}\circ B^\infty_A$ is $\sqsubseteq_{\mathrm{EM}}$-continuous and $\sqsubseteq_{\mathrm{EM}}$-least fixed point of the defining equation in $\mathrm{PD}(A)$,
- $B^\infty_A[t]=\{I^\infty_A(t)\}$ for $t\in W_{\Sigma^\perp}$,
- B^∞_A is the $\subseteq^*$-least closed function that is $\sqsubseteq_{\mathrm{EM}}$-equivalent to $\mathrm{PDOM}\circ B_A$.

The basic results of the definitions above can be condensed into the following theorem.

Theorem 5.5. *For every algebraic Σ-algebra A the following diagram commutes*:

$$\begin{tikzcd}
 & \mathrm{P}(W^\infty_{\Sigma^\perp}) \arrow[r, "\mathrm{PDOM}"] \arrow[d, "\mathrm{CLOSE}\circ I^\infty_A"'] & \mathrm{PD}(W^\infty_{\Sigma^\perp}) \arrow[d, "\mathrm{CLOSE}\circ I^\infty_A"'] \\
W^\infty_{\Sigma^+} \arrow[ur, "B^\infty_{\mathrm{HK}}"] \arrow[r, "B^\infty_A"'] & \mathrm{P}(A) \arrow[r, "\mathrm{PDOM}"'] & \mathrm{PD}(A)
\end{tikzcd}$$

Note that here I^∞_A is to be taken elementwise.

Proof. According to the monotonicity of I^∞_A we have

$$\mathrm{MIN}\circ B^\infty_A \subseteq^* \mathrm{CLOSE}\circ I^\infty_A\circ \mathrm{MIN}\circ B^\infty_{\mathrm{HK}}.$$

Now for arbitrary algebras C we define

$$\mathrm{BP}^0_C[t]=\{\perp\},$$

$$\mathrm{BP}^{i+1}_C[\mathrm{or}(t1, t2)]=\mathrm{CONE}(\mathrm{BP}^i_C[t1]\cup \mathrm{BP}^i_C[t2]),$$

$$\mathrm{BP}^{i+1}_C[f(t_1,\ldots,t_n)]=\{f(a_1,\ldots,a_n): a_i\in \mathrm{BP}^i_C[t_i]\}.$$

Let B^i_A and B^i_{HK} be defined as above; we can prove by induction on i:

$$(*) \qquad \forall t \in W^\infty_{\Sigma^+}:$$

$$\forall t1 \in \mathrm{MIN} \circ \mathrm{BP}^i_A[t],\ t2 \in \mathrm{CLOSE} \circ I^\infty_A \circ \mathrm{MIN} \circ B^\infty_{\mathrm{HK}}[t] \text{ with } t1 \sqsubseteq t2:$$

$$\exists a \in B^i_A[t]: I_A[t1] \sqsubseteq a \sqsubseteq I^\infty_A[t2].$$

For $i=0$, $(*)$ is trivial. Now let $(*)$ be valid for all $i \leqslant n$; we consider three cases:

- if $t1 = \perp$, then $(*)$ is trivial,
- if $t1 = \mathrm{or}(t11, t12)$, then $t2 = \mathrm{or}(t21, t22)$ with $t11 \sqsubseteq t21$, $t12 \sqsubseteq t22$, and

$$t11, t12 \in \mathrm{MIN} \circ \mathrm{BP}^i_{\mathrm{HK}}[t],$$

so there exist according to our induction hypothesis $a1, a2 \in B^i_A[t]$ with

$$I^\infty_A[t11] \sqsubseteq a1 \sqsubseteq I^\infty_A[t21], \qquad I^\infty_A[t12] \sqsubseteq a2 \sqsubseteq I^\infty_A[t22];$$

therefore, we obtain

$$I^\infty_A[t1] \sqsubseteq a1 \sqsubseteq I^\infty_A[t2] \quad \text{or} \quad I^\infty_A[t1] \sqsubseteq a2 \sqsubseteq I^\infty_A[t2],$$

- if $t1 = f(t_1, \ldots, t_n)$, the proof can be done analogously.

So we have

$$a \in \mathrm{CLOSE} \circ I^\infty_A \circ \mathrm{MIN} \circ B^\infty_{\mathrm{HK}}[t] \Leftrightarrow a \in \mathrm{CLOSE}(\textstyle\bigcup^* B^i_A[t])$$

since for every $x \in \mathrm{MIN} \circ B^\infty_{\mathrm{HK}}[t]$ there exist

$$x_i \in \mathrm{MIN} \circ \mathrm{BP}^i_{\mathrm{HK}}[t] \quad \text{with lub } x_i = x,$$

and according to our lemma above there exist

$$y_i \in B^i_A[t] \quad \text{with } x_i \sqsubseteq y_i \sqsubseteq x.$$

So

$$\mathrm{CLOSE} \circ I^\infty_A \circ \mathrm{MIN} \circ B_{\mathrm{HK}} \subseteq^* B^\infty_A,$$

and since both B^∞_A and $\mathrm{CLOSE} \circ I_A \circ B_{\mathrm{HK}}$ are the uniquely determined $\subseteq$-least fixed points containing

$$\mathrm{CLOSE} \circ I^\infty_A \circ \mathrm{MIN} \circ B^\infty_{\mathrm{HK}},$$

they are identical.

The equality

$$\mathrm{CLOSE} \circ I_A \circ \mathrm{PDOM} \circ B^\infty_{\mathrm{HK}} = \mathrm{PDOM} \circ B^\infty_A$$

simply follows by continuity arguments. □

This means that it is possible either to interpret a term $t \in W^\infty_{\Sigma^+}$ immediately in $\mathrm{P}(A)$ as representing a set of elements of A or at first interpret these term elementwise to obtain a set over A. The results of both ways of proceeding coincide, and the diagram commutes.

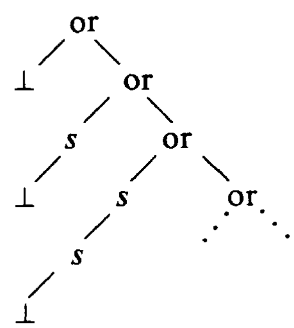
6. Concluding remarks

The basic idea that was leading to the approach presented above comes from a particular view of nondeterminism: a nondeterministic program does not specify just *one* computation (for every input), but rather a *class* of computations. This is why the classical domain theory and fixed point theory (which is tailored for treating classical notions of computability) cannot work in a straightforward way in the case of nondeterminism. If ideas from domain theory and fixed point theory are applied in a more sohpisticated way, however, one can cope with nondeterminism as well.

The technique applied in this paper was used the first time in [1] to define a denotational semantics for concurrent, communicating, nondeterministic programs. In the preceding section it is demonstrated that this technique is general in the sense that it can be applied to nondeterministic computations over arbitrary domains.

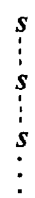
Note that one problem remains: All sets $B^{\infty}_{\mathrm{HK}}[t]$ and $B^{\infty}_{\mathrm{A}}[t]$ are closed. So if we consider the tree

t:



one may expect that B^{∞}_{HK} does not contain the infinite tree s^{∞} defined by

s^{∞}:



However, this tree s^{∞} is an element of B^{∞}_{HK} since $B^{\infty}_{\mathrm{HK}}[t]$ is always closed. Only at first sight this seems a drawback, but it is a consequence of the notion of finite observability and a particular interpretation of $\perp$. If all finite approximations of an element x are members of $B^{\infty}_{\mathrm{HK}}[t]$, then so is x. Note that this has nothing to do with fairness, since fairness rather corresponds to the tree T associated with Φ of the problem above. Note that $B^{\infty}_{\mathrm{HK}}[T]$ does not contain s^{∞}.

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