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Kings in quasi-transitive digraphs[1]

Jørgen Bang-Jensen[a,*], Jing Huang[b]

[a] *Department of Mathematics and Computer Science, Odense University, Campusvej 55, DK-5230, Odense M, Denmark*

[b] *Department of Mathematics and Statistics, University of Victoria, P.O. Box 3045, Victoria, B. C., Canada V8W 3P4*

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Abstract

A k-king in a digraph D is a vertex which can reach every other vertex by a directed path of length at most k. This definition generalizes the definition of a king in a tournament. We consider quasi-transitive digraphs — a generalization of tournaments recently investigated by the authors (Bang-Jensen and Huang, 1995). We prove that a quasi-transitive digraph has a 3-king if and only if it has an out-branching. We give several results on 3-kings in quasi-transitive digraphs which are analogous to well-known results about kings in tournaments. © 1998 Elsevier Science B.V. All rights reserved

1. Introduction

All digraphs considered in this paper may contain opposite arcs with the same end-vertices, i.e., a directed cycle of length 2, but contain no multiple arcs or loops. We use $U(D)$ to denote the underlying undirected graph of the digraph D. For an undireced graph G we use $\overline{G}$ to denote the complement graph of G.

A *strong component* of a digraph D is a maximal subset $S \subseteq V(D)$ such that $d(x,y) < \infty$ for every choice of vertices $x, y \in S$. Here $d(x,y)$ denotes the length of a shortest directed path from x to y (if there is no such path $d(x,y) = \infty$). A digraph is *strong* if it has only one strong component. An *initial* (strong) component is one which has no arcs comming in from any other strong components. (Note that when D is strong the whole graph D is the initial component.)

A *semicomplete* digraph is a digraph in which every pair of distinct vertices is joined by at least one arc. A *tournament* is a semicomplete digraph with no cycles of length two.

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* Corresponding author. E-mail: jbj@imada.ou.dk.

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A digraph is *transitive* if it is acyclic and, for every pair of arcs $x \to y, y \to z$ on distinct vertices x, y, z, the arc $x \to z$ is also an arc of the digraph. A digraph is *quasi-transitive* if, for every pair of arcs $x \to y, y \to z$ on distinct vertices x, y, z, the vertices x and z are adjacent. It is easy to see that every semicomplete digraph, and hence every tournament is quasi-transitive. Clearly, every transitive digraph is quasi-transitive. Quasi-transitive digraphs are related to comparability graphs and share a lot of structure with tournaments, see e.g. [1,2,6] and Theorem 1.1. The relation to comparability graphs was shown by Ghouilà-Houri who proved that a graph can be oriented as a quasi-transitive digraph if and only if it can be oriented as a transitive digraph [4].

Let D be a digraph on n vertices and let $S_1, S_2, \ldots, S_n$ be distinct digraphs. The digraph $D[S_1, S_2, \ldots, S_n]$ is the digraph obtained from D by replacing the ith vertex of D by the digraph S_i in such a way that for every arc $i \to j$ in D, $D[S_1, S_2, \ldots, S_n]$ contains all possible arcs from $V(S_i)$ to $V(S_j)$. Furthermore, all the original arcs of an S_i are also in $D[S_1, S_2, \ldots, S_n]$. In the case when no S_i contains an arc, i.e. the underlying graphs are independent sets, we call the digraph $D[S_1, S_2, \ldots, S_n]$ an *extension* of D.

A *k-king* in a digraph D is a vertex x such that $d(x, y) \leqslant k$ for all $y \in V(D) - x$. The definition of a k-king generalizes the definition of a king in tournament, in which, a king is defined to be a vertex x such that $d(x, y) \leqslant 2$ for every $y \in V(D) - x$, cf. [3,7,12] (i.e. a 2-king according to our definition).

2-kings in tournaments were first introduced by Maurer [9] in his delightful exposition on the use of tournaments to model dominance in flocks of chickens. The idea to use 2-kings in the study of dominance in tournaments emerged from an earlier work by the mathematical sociologist Landau [8] who proved that every vertex of maximum outdegree in a tournament is a 2-king. For further work on 2-kings in tournaments we refer the reader to [3,7,12].

A necessary condition for the existence of a k-king, $k < n$, in a digraph D on n vertices is that D has a vertex which can actually reach every other vertex by a directed path. This is equivalent to saying that D must have an out-branching. An *out-branching* rooted at v is a spanning tree such that each $x \neq v$ has exactly one arc coming in. Gutin [5] and independently Petrovic and Thomassen [11], proved that a multipartite tournament (i.e., an orientation of a complete r-partite graph for some $r \geqslant 2$) has a 4-king if and only if it has an out-branching. A multipartite tournament D is an *extended tournament* if all arcs between two partite classes have the same direction, i.e. D is of the form $D = T[U_1, U_2, \ldots, U_t]$ where T is a tournament on t vertices and each U_i is a digraph with no arcs and at least one vertex. Using the fact that every tournament has a 2-king, it is not difficult to prove that every extended tournament has a 3-king if and only if it has an out-branching. In Section 2 we shall prove that a quasi-transitive digraph has a 3-king if and only if it has an out-branching (Theorem 2.5). Since an extended tournament is also quasi-transitive, our result generalizes the result for extended tournaments above.

The following theorem characterizes quasi-transitive digraphs in a recursive fashion.

Theorem 1.1 (Bang-Jensen and Huang [1]). *Let D be a digraph. Then D is a quasi-transitive digraph if and only if the following holds.*

(1) *If D is not strong, then there exist a natural number $q \geqslant 2$, a transitive digraph Q on q vertices and strong quasi-transitive digraphs $W_1, W_2, \ldots, W_q$ such that $D = Q[W_1, W_2, \ldots, W_q]$.*

(2) *If D is strong, then there exist a natural number $q \geqslant 2$, a strong semicomplete digraph Q on q vertices and quasi-transitive digraphs $W_1, W_2, \ldots, W_q$, where each W_i is either a single vertex or a non-strong quasi-transitive digraph, such that $D = Q[W_1, W_2, \ldots, W_q]$. Furthermore, if Q has a cycle of length two induced by vertices v_i and v_j, then the corresponding digraphs W_i and W_j are trivial, i.e., each of them has only one vertex.*

Let D be a quasi-transitive digraph. By Theorem 1.1 we can decompose D as $D = Q[W_1, W_2, \ldots, W_q]$ as described in the theorem. It is easy to see that if D is not strong, then this decomposition is unique, since each W_i is a strong digraph and Q has no directed cycle. If D is strong, there may be several such decompositions, but it was shown in [1] that there is a unique such decomposition in which each W_i corresponds to a connected component in $\overline{U(D)}$. Hence, we can speak of the *first level* of the decomposition of D.

Since each of the digraphs $W_1, W_2, \ldots, W_q$ are also quasi-transitive, we may further decompose any W_i which has at least 2 vertices as $W_i = R_i[W_{i1}, \ldots, W_{ir_i}]$ according to Theorem 1.1. By doing this for all non-trivial W_i we obtain the *second level*, $D = Q^*[W_{11}, \ldots, W_{1r_1}, \ldots, W_{q1}, \ldots, W_{qr_q}]$ (where Q^* is the subdigraph induced by $V(R_1) \cup \cdots \cup V(R_q)$ in D), of the decomposition of D and so on. Each of these levels will be uniquely determined, provided that we proceed as above whenever we decompose a strong quasi-transitive subdigraph of D.

2. Existence of kings

In this section D is always a quasi-transitive digraph with at least two vertices. Recall that a necessary condition for the existence of a 3-king is that D has an out-branching. Secondly, if D has an out-branching, then it has a unique initial component. Finally, observe that by Theorem 1.1, if D has a unique initial component D', then every vertex in D' dominates every vertex outside D'. So if D has an out-branching, then D has a 3-king if and only if the unique initial component has a 3-king. Hence, it suffices to study 3-kings in strong quasi-transitive digraphs.

Assume that D is a quasi-transitive digraph. We shall always use $D = Q[W_1, \ldots, W_q]$ to denote the first level of the decomposition of D as prescribed in Theorem 1.1. Furthermore, we shall assume that vertices of Q are labeled $g_1, g_2, \ldots, g_q$ such that in forming D each g_i is substituted by W_i.

By a well-known result of Moon [10], every vertex of a strong semicomplete digraph on at least three vertices lies on a directed cycle of length three. Using this and Theorem 1.1, it is easy to prove the following lemma.

Lemma 2.1. *Let $D = Q[W_1, \ldots, W_q]$ be a strong quasi-transitive digraph. For every $i = 1, 2, \ldots, q$: $d(x, y) \leqslant 3$ for every pair of vertices $x, y \in W_i$. Furthermore, if W_i is non-trivial, there exist $x, y \in W_i$ such that $d(x, y) = 3$.*

A *strict-3-king* is a vertex that is a 3-king, but not a 2-king. A *non-king* is a vertex which is not a 3-king.

Lemma 2.2. *Let $D = Q[W_1, \ldots, W_q]$ be a strong quasi-transitive digraph.*
1. *A vertex $h \in W_i$ is a 3-king if and only if the vertex g_i of Q is a 3-king in Q.*
2. *If g_i is a 3-king of Q and W_i is non-trivial, then W_i contains a vertex which is a strict-3-king.*

Proof. The first claim follows easily from Theorem 1.1 and Lemma 2.1: If g_i is a 3-king in Q, then h can reach every vertex outside W_i by a path of length at most three. By Lemma 2.1, g_i is on a 3-cycle in Q and from this we get that h can reach every vertex inside W_i by a path of length at most three. The other direction is trivial.

The second claim follows from the fact that W_i induces a non-strong quasi-transitive digraph. Hence, there is some vertex u which must use a path leaving W_i to reach some other vertex $v \in W_i$. Note that since W_i is non-trivial, the vertex g_i is not on any 2-cycle, by Theorem 1.1. □

It follows from Lemma 2.2 that in the case when $D = Q[W_1, \ldots, W_q]$ is strong we can divide $W_1, \ldots, W_q$ into those for which all $h \in W_i$ are 3-kings and those for which no $h \in W_i$ is a 3-king.

Corollary 2.3. *Suppose D has a 3-king. Then there is an arc between every 3-king and every non-king.*

Proof. If D is strong, then this follows easily from Theorem 1.1, Lemma 2.2, and the remark above. If D is not strong, then every 3-king x must belong to the unique initial strong component D', hence if $y \notin V(D')$ then $x \rightarrow y$. If $y \in V(D')$, then the claim follows from the case when D is strong, since x is also a 3-king in D'. □

Corollary 2.4. *Let $D = Q[W_1, \ldots, W_q]$. Suppose that D has a 3-king. Then, for every non-king u, there exists a 3-king v such that $d(u, v) > 3$ and v dominates u.*

Proof. Suppose that D is strong. Let u be a non-king of D and let W be the set of vertices w such that $d(u, w) > 3$. Then $W \neq \emptyset$ and, by Theorem 1.1, $W = W_{i_1} \cup \cdots \cup W_{i_k}$, for some $1 \leqslant i_1 < \cdots < i_k \leqslant q$. Clearly each vertex of W dominates u. Consider the subgraph Q' of Q induced by $\{g_{i_1}, \ldots, g_{i_k}\}$. Note that by Theorem 1.1, Q' is semicomplete and hence has a 2-king. Let g_{i_p} be a 2-king of Q'. We claim that each vertex of W_{i_p} is a 3-king of D. Indeed, let x be a vertex of W_{i_p}. Assume that x cannot reach some y in three steps, i.e., $d(x, y) > 3$. Then $y \notin W$ and hence u can reach y

in at most three steps. Since $x \to u$ and x cannot reach y in three steps, u cannot reach y in less than three steps. Let $u \to a \to b \to y$ be a path of length 3 from u to y. Since $x \to u$ and $u \to a$, there is an arc between x and a. However neither $x \to a$ nor $a \to x$ is possible: if $x \to a$ then x can reach y in three steps and if $a \to x$ then u can reach x in two steps. Hence, every $x \in W_{i_p}$ is a 3-king such that $x \to u$ and $d(u,x) > 3$.

Now suppose D is not strong. If u belongs to the unique initial component D', then we can apply the argument above, since every 3-king of D' is a 3-king in D. If u does not belong to D', then D' completely dominates u so we just take any 3-king from D'. □

As we mentioned above, every tournament has a 2-king. In general digraphs, even in a quasi-transitive digraph, there may not exist a vertex which can reach every other vertex. For example, an alternating path of length at least three does not have such a vertex ($x_1 \to x_2 \leftarrow x_3 \to x_4 \leftarrow x_5 \ldots$). Hence such a digraph contains no k-king for any finite k.

Theorem 2.5. *Let D be a quasi-transitive digraph and let $D = K[H_1, \ldots, H_k]$ denote the first (respectively, second) level of the decomposition of D if D is strong (respectively, not strong). Then D has a 3-king if and only if it has an out-branching. Furthermore, if D has a 3-king, then every vertex x of maximum out-degree is a 3-king. Finally, D has a 2-king if and only if $|H_i| = 1$ for some i such that g_i is a 2-king in K.*

Proof. Clearly, the existence of an out-branching is necessary. Below we shall prove the other direction, assuming that D has an out-branching.

By the remark in the begining of this section, we may assume that D is strong. Let g_i be a 2-king of K. By Lemma 2.2 and Theorem 1.1, every vertex of W_i is a 3-king in D.

To prove the second claim suppose D has a 3-king. Let x be a vertex of maximum out-degree in D. By Theorem 1.1, x must belong to the initial strong component of D, so again we may assume that D is strong. Note that by Theorem 1.1 and Lemma 2.1, $d(x, y) = 3$ for every non-neighbour of x (each of these belong to the same W_i as x). Suppose now that some in-neighbour y of x cannot be reached from x by a path of length two. Then y dominates all the out-neighbours of x and x, contradicting the choice of x. This shows that x is a 3-king.

The last claim follows easily from Lemma 2.2. □

Let T denote the unique strong tournament on three vertices and let I be the digraph with two vertices and no arcs. It is easy to see that the quasi-transitive digraph $T[I, I, I]$ has an out-branching, but no 2-king.

In the case when the quasi-transitive digraph D has no transmitter (a vertex of in-degree zero), the above theorem can be strengthened to.

Proposition 2.6. *Let D be a quasi-transitive digraph with a 3-king. If D has no transmitter, then D has at least two 3-kings. Furthermore, if D has a unique initial component and this has at least three vertices, then D has at least three 3-kings.*

Proof. Suppose first D is strong. Then, by Theorem 1.1, Q has no transmitter and hence has at least two 2-kings and if Q is not just a 2-cycle, then it has at least three 2-kings (this is a well-known result for tournaments [9] which is easy to extend to semicomplete digraphs). The claim now follows from Lemma 2.2.

If D is not strong, then the initial component (which must be unique since D has a 3-king) is non-trivial (since D has no transmitter), so by the result above we find the desired kings. □

Proposition 2.7. *Let D be a quasi-transitive digraph which contains a 3-king but no transmitter. Every non-king is dominated by at least three 3-kings, unless the initial component is a 2-cycle, in which case every non-king is dominated by exactly two 3-kings.*

Proof. Let x be a non-king. If x is not in the initial component of D, then the claim follows from Proposition 2.6. So we may assume that D is strong. Let i be chosen such that $x \in W_i$. Let $g_{i_1}, \dots, g_{i_r}$, $r \geqslant 1$ be the vertices dominating g_i in Q.

Observe that if a vertex z is a 3-king in $D' = D(W_{i_1} \cup \cdots \cup W_{i_r})$, then z is a 3-king in D. Here we used that Q is semicomplete and hence z has a path of length 2 to every vertex $u \notin W_{i_1} \cup \cdots \cup W_{i_r}$ (via x). Hence, we can assume that D' has at most two 3-kings. By Proposition 2.6, this means that D' either has a transmitter, is not strong, or it is just a 2-cycle. In the first two cases the reader can easily verify that x is a 3-king, contradicting the assumption. □

3. Establishing kings

In this section we let D be a quasi-transitive digraph and let $D = Q[W_1, \dots, W_q]$, be the first level of the decomposition of D according to Theorem 1.1. As in Section 2 we associate W_i with the ith vertex of Q. We consider the problem of whether there exists a quasi-transitive digraph D^* which contains D as an induced subgraph such that set of 3-kings of D^* is precisely the vertex set of D. If such a D^* exists, then we say that D can *be established.*

A similar problem for tournaments has been studied by Reid; cf. [12]. He proved that a tournament T is contained in a tournament whose 2-kings are the vertices of T if and only if T contains no transmitter. A more general result can be found in [7].

Theorem 3.1. *Let D be a quasi-transitive digraph with a non-king. Let D be decomposed as $D = K[H_1, \dots, H_k]$, where this is the first level of the decomposition of D if D is strong and the second level of the decomposition of D if D is not strong. Then*

D can be established if and only if the following conditions are met
(1) *K has a strict-3-king.*
(2) *Every 2-king of K is dominated by some strict 3-king of K.*
In particular if D can be established, then it has a 3-king.

Proof. Suppose that D satisfies (1) and (2) . Let $g_{i_1},\dots,g_{i_s}$, respectively, $g_{j_1},\dots,g_{j_t}$ denote the 2-kings, respectively, the strict-3-kings of K. Add t new vertices $v_1,\dots,v_t$ and the following arcs to D: Each v_i completely dominates all vertices in H_{j_i} and is completely dominated by all other vertices of D, and v_a dominates v_b if and only if g_{j_a} dominates g_{j_b} and g_{j_b} does not dominate g_{j_a}. It is easy to check that this new digraph D^* is quasi-transitive. Furthermore, no new vertex v_r can be a king, because, in D the vertices of H_{j_r} cannot reach the vertices of some H_p in two steps, hence v_r cannot reach the vertices of H_p in three steps in D^*. It remains to prove that all vertices of D will be 3-kings in D^*. First observe that in K no non-king dominates a 2-king and hence, by Corollary 2.3, if g_i is a non-king and g_j is a 2-king of K, then every vertex of H_j dominates every vertex of H_i. It is clear that every 3-king of D belonging to $H_{g_{i_1}}\cup\cdots\cup H_{g_{i_s}}$ will become a 3-king of D^* since it dominates all the new vertices $v_1,\dots,v_t$. Similarly, it is easy to see that every 3-king of D belonging to $H_{g_{j_1}}\cup\cdots\cup H_{g_{j_t}}$ will be a 3-king in D^*: if $z\in H_{j_i}$, then z dominates all the vertices $v_1,\dots v_t$, except v_i. Furthermore, if g is an out-neighbour of g_{j_i} in K, then z dominates every vertex in H_g and each of these vertices dominate v_i.

Finally, let u be a non-king of D. To reach a vertex $w_{i_h}\in H_{i_h}$, $1\leqslant h\leqslant s$, u can use a path of the form $u\to v_r\to w_{j_r}\to w_{i_h}$ where $w_{j_r}\in H_{j_r}$. Such a path exists because, by (2), every 2-king of K is dominated by some strict 3-king of K. To reach a vertex $w_{j_f}\in H_{j_f}$, $1\leqslant f\leqslant t$, we use a path of the form $u\to v_f\to w_{j_f}$. Finally, to reach another non-king v, u uses a path of the form $u\to v_r\to w_{j_r}\to v$ where $w_{j_r}\in H_{j_r}$. To see why such a path exists, it suffices to notice that every non-king of K must be dominated by some strict 3-king of K (otherwise it would be a 3-king, by assumption (2) and Corollary 2.4). Thus, we have shown that D can be established.

To prove the other direction suppose that D can be established and let D^* be a quasi-transitive digraph whose set of 3-kings is precisely $V(D)$. First observe that we may assume that D^* is strong, since if it is not strong, then all vertices of D must belong to the initial strong component of D^* and this dominates all other components, by Theorem 1.1. Hence, the graph D^{**} obtained by deleting all other components from D^* other than the initial one will also have precisely the vertices of D as 3-kings.

Let $D^*=Q^*[W_1^*,\dots,W_{q^*}^*]$ denote the first level of the strong decomposition of D^*. Since no vertex of $V(D^*)\backslash V(D)$ is a 3-king, it follows from Lemma 2.2 that no W_i^* contains vertices from both of the sets $V(D)$ and $V(D^*)\backslash V(D)$. Secondly, by the choice of the decomposition of D as $K[H_1,\dots,H_k]$, we have ensured that each H_i is part of a connected component in $\overline{U(D)}$ and hence $H_i\subseteq W_j^*$ for some $j\in\{1,\dots,q^*\}$ (it may be a proper subset, but only in the case when D is not strong). Finally, note that by Corollary 2.3 there is an arc between every vertex of $V(D^*)\backslash V(D)$ and every vertex of $V(D)$.

Let us first prove that D must have a 3-king if it can be established. Suppose D has no 3-king. Then it follows from Theorem 1.1, Lemma 2.2 and the fact that every semicomplete digraph has a 2-king that D is not strong and has at least two initial strong components. Thus, all initial components will belong to the same W_i^* in the above decomposition of D^*. Furthermore, since every vertex of D is a 3-king in D^* there must be some vertex $v \in W_j^*$ for some $j \neq i$ such that v dominates all the vertices in W_i^*. However, this means that v can reach all vertices in D by a directed path of length two, contradicting Corollary 2.4. Hence D must have a 3-king if it can be established.

Suppose that K has no strict 3-king. Let $g_{i_1}, \dots, g_{i_s}$, $s \geqslant 1$, denote the 2-kings of K. Let $H = H_{i_1} \cup \cdots \cup H_{i_s}$. Then it is easy to see that there is no arc from $V \backslash H$ to H in D. Hence it follows from the remarks above and the fact that every vertex of D is a 3-king in D^* that $V(D^*) \backslash V(D)$ contains a vertex $v \in W_r^*$, for some r, such that every vertex of W_r^* dominates all the vertices of some H_{i_j}. However, now v can reach all vertices in $V(D)$ by a directed path of length at most 3, contradicting Corollary 2.4. Hence (1) must hold if D can be established.

Suppose that K has a 2-king which is not dominated by any strict 3-king. Then, using the fact that each H_i is either equal to some W_j^* or a proper subset of some W_j^* we can argue as above that $V(D^*) \backslash V(D)$ contains a vertex v such that v dominates all the vertices of some H_{i_j}, where the vertex g_{i_j} is a 2-king in K and again we obtain a contradiction to Corollary 2.4. Hence (2) must also hold if D can be established. □

Note that in the case when D is not strong we must consider the second and not just the first level of the decomposition. This can be seen from the example in Fig. 1. Here D is not strong and the first level of the decomposition is $D = Q[\{x_1, x_2, x_3, y_1, y_2,$

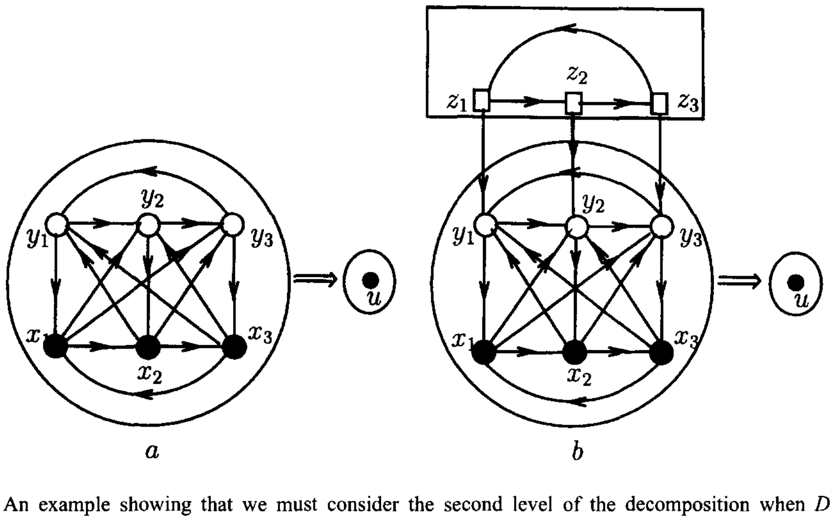

Fig. 1. An example showing that we must consider the second level of the decomposition when D is not strong. The 2-kings of K are $\{x_1, x_2, x_3\}$ and the strict 3-kings are $\{y_1, y_2, y_3\}$. The vertices z_1, z_2, z_3 are the new vertices added to D. All arcs not shown go from $V(D)$ to $\{z_1, z_2, z_3\}$.

$y_3\}, \{u\}]$, where Q is just the edge $q_1 \rightarrow q_2$ and hence has no strict 3-king. Still D can be established as shown in (b). Note that the new digraph is made by following the rule in the first part of the proof of Theorem 3.1 (the second level of the decomposition is $D = K[\{x_1\}, \{x_2\}, \{x_3\}, \{y_1\}, \{y_2\}, \{y_3\}, \{u\}]$, thus $K = D$).

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