Note

Partitions of the set of natural numbers and their representation functions

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Abstract

For a given set $A$ of nonnegative integers the representation functions $R_2(A, n)$, $R_3(A, n)$ are defined as the number of solutions of the equation $n = a + a'$, $a, a' \in A$ with $a < a'$, $a \leq a'$, respectively. In this paper we give a simple proof to two results by Sándor. © 2007 Elsevier B.V. All rights reserved.

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1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. For a set $A \subset \mathbb{N}$, let $R_1(A, n)$, $R_2(A, n)$, $R_3(A, n)$ denote the numbers of solutions of

- $a + a' = n$, $a, a' \in A$,
- $a + a' = n$, $a, a' \in A$, $a < a'$,
- $a + a' = n$, $a, a' \in A$, $a \leq a'$,

respectively. For $i \in \{1, 2, 3\}$, Sárközy asked ever whether there are sets $A$ and $B$ with infinite symmetric difference such that $R_i(A, n) = R_i(B, n)$ for all sufficiently large integers $n$. As Dombi [3] has shown, the answer is negative for $i = 1$ by a simple observation that $R_1(A, n)$ is odd if and only if $n = 2a$ for some $a \in A$, and positive for $i = 2$. For $i = 3$, Chen and Wang [2] proved that the set of nonnegative integers can be partitioned into two subsets $A$ and $B$ such that $R_3(A, n) = R_3(B, n)$ for all $n \geq n_0$.

For a subset $A$ of $\mathbb{N}$ and any integer $n$ let $A(n) = \{a : 0 \leq a \leq n, a \in A\}$. Using generating functions, Lev [7] and independently Sándor [8] gave a simple common proof to the results of Dombi and of Chen and Wang. Sándor actually established the two following stronger results(which are also implicit in Lev’s paper):

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Theorem 1. Let $N$ be a positive integer. The equality $R_2(A, n) = R_2(\mathbb{N} \setminus A, n)$ holds for $n \geq 2N - 1$ if and only if $A(2N - 1) = N$ and $2m \in A \iff m \in A$, $2m + 1 \in A \iff m \notin A$ for $m \geq N$.

Theorem 2. Let $N$ be a positive integer. The equality $R_3(A, n) = R_3(\mathbb{N} \setminus A, n)$ holds for $n \geq 2N - 1$ if and only if $A(2N - 1) = N$ and $2m \in A \iff m \notin A$, $2m + 1 \in A \iff m \in A$ for $m \geq N$.

In this paper, we give a simple proof of the above two theorems. For other related results, the reader is referred to [1,4–6,9].

Currently we have no answer for the following problem.

**Problem.** Given a positive integer $k (k \geq 3)$. For $i \in \{2, 3\}$, does there exist a partition

$$\mathbb{N} = \bigcup_{m=1}^{k} A_m, \quad A_u \cap A_v = \emptyset, \quad u \neq v$$

such that $R_i(A_m, n) = R_i(A_{m'}, n) (1 \leq m < m' \leq k)$ for all sufficiently large integers $n$?

2. Proofs

**Proof of Theorem 1.** Write

$$\eta(i) = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{otherwise}. \end{cases}$$

Then

$$R_2(\mathbb{N} \setminus A, n) = |\{(a, a') : a, a' \in \mathbb{N} \setminus A, a < a', a + a' = n\}|$$

$$= \sum_{0 \leq i < n/2} (1 - \eta(i))(1 - \eta(n - i))$$

$$= \sum_{0 \leq i < n/2} 1 - |\{i : 0 \leq i \leq n, i \in A\}| + \eta(\frac{n}{2})$$

$$+ |\{(b, b') : b, b' \in A, b < b', b + b' = n\}|$$

$$= \sum_{0 \leq i < n/2} 1 - A(n) + \eta(\frac{n}{2}) + R_2(A, n).$$

Thus, $R_2(A, n) = R_2(\mathbb{N} \setminus A, n)$ holds for all $n \geq 2N - 1$ if and only if $\sum_{0 \leq i < n/2} 1 = A(n) - \eta(n/2)$ holds for all $n \geq 2N - 1$, that is,

$$\begin{cases} A(2m - 1) = m, & m \geq N, \\ A(2m) - \eta(m) = m, & m \geq N, \end{cases}$$

equivalently,

$$\begin{cases} A(2m - 1) = m, & m \geq N, \\ A(2m - 1) + \eta(2m) - \eta(m) = m, & m \geq N, \\ A(2m + 1) - \eta(2m + 1) - \eta(m) = m, & m \geq N, \end{cases}$$

equivalently,

$$\begin{cases} A(2m - 1) = m, & m \geq N, \\ \eta(2m) = \eta(m), & m \geq N, \\ \eta(2m + 1) + \eta(m) = 1, & m \geq N, \end{cases}$$
equivalently,
\[
\begin{align*}
A(2m - 1) &= m, \quad m \geq N, \\
2m \in A &\iff m \in A, \quad m \geq N, \\
2m + 1 \in A &\iff m \notin A, \quad m \geq N,
\end{align*}
\]
equivalently,
\[
\begin{align*}
A(2N - 1) &= N, \\
2m \in A &\iff m \in A, \quad m \geq N, \\
2m + 1 \in A &\iff m \notin A, \quad m \geq N.
\end{align*}
\]

This completes the proof of Theorem 1. \(\square\)

**Proof of Theorem 2.** Define \(\eta(i)\) as in the proof of Theorem 1. Then
\[
R_3(\mathbb{N}\setminus A, n) = \sum_{0 \leq i \leq n/2} (1 - \eta(i))(1 - \eta(n - i))
= \sum_{0 \leq i \leq n/2} 1 - A(n) - \eta \left(\frac{n}{2}\right) + R_3(A, n).
\]

Thus, \(R_3(A, n) = R_3(\mathbb{N}\setminus A, n)\) holds for all \(n \geq 2N - 1\) if and only if \(\sum_{0 \leq i \leq n/2} 1 = A(n) + \eta(n/2)\) holds for all \(n \geq 2N - 1\). \(\square\)

The remainder of the proof is very similar to that of the proof of Theorem 1. We omit it here.

**References**