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On the geodesic pre-hull number of a graph

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ABSTRACT

Given a convexity space X whose structure is induced by an interval operator I, we define a parameter, called the pre-hull number of X, which measures the intrinsic non-convexity of X in terms of the number of iterations of the pre-hull operator associated with I which are necessary in the worst case to reach the canonical extension of copoints of X when they are being extended by the adjunction of an attaching point. We consider primarily the geodesic convexity structure of connected graphs in the case where the pre-hull number is at most 1, with emphasis on bipartite graphs, in particular, partial cubes.

1. Introduction

How non-convex is a convexity space? We wish to assign a numerical measure to convexity spaces which is intended to provide a meaningful answer to this question. Suppose X is a finite set endowed with a convexity structure. Let A be a proper convex subset of X, and B a minimal convex extension of A, i.e. the convex hull of $A \cup \{x\}$ for a suitably chosen point x in $X - A$. How big is the gap between A and B—not in terms of cardinality but in terms of convexity? Assuming the convexity structure of X to arise from an interval operator, what we are asking for is the minimum number $r(x; A, B)$ of iterations of the associated pre-hull operator (which builds up the convex hull of a set "from below") that is necessary to get from $A \cup \{x\}$ to its convex hull B. The more iterations it takes, the "less convex" $A \cup \{x\}$ was to begin with. The measure of the gap between A and B that we have in mind then is $r(A, B) := \max_{x \in B-A} r(x; A, B)$.

In using this (local) measure to define a global measure for the whole space X we will, however, not consider arbitrary convex subsets of X and their minimal convex extensions, but will limit ourselves to copoints K of X as these are more tightly tied to the structure of X in the sense that they possess a canonical minimal convex extension K^+, the convex hull of $K \cup \{x\}$, where x is any attaching point of

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K. The number $r(K, K^+)$ is what we call the pre-hull number of K in X, and the pre-hull number of the whole space X, denoted by $ph(X)$, is defined to be $\max r(K, K^+)$, the maximum taken over all copoints K of X. This is the measure alluded to at the beginning of this section. Intuitively, the smaller the value of $ph(X)$, the "less non-convex" X may be considered to be. Formal definitions of these concepts with suitable modifications allowing for infinite convexity spaces are given in Section 2. All spaces considered in this paper will be interval spaces.

What can be said about the structure of an interval space when its pre-hull number is known? The paper deals with the easiest case of this question, that is, when $ph(X) \leq 1$. Moreover, except in Section 3, our concern will be almost exclusively with graph convexities (the convexity structures on the vertex-sets of connected graphs induced by the metric of the graph).

General interval spaces with $ph(X) = 0$ are easily seen to be precisely the convex geometries ($=$ anti-matroids); in the graph case they can be characterized as arising from a special type of chordal graphs (Section 3). The remaining sections of the paper deal primarily with the convex structure of bipartite graphs, in particular with partial cubes. An important class of partial cubes, the median graphs, are shown to have pre-hull number ≤ 1; indeed they are precisely the modular graphs (i.e. bipartite graphs satisfying the so-called quadrangle condition) with pre-hull number ≤ 1 (Sections 4 and 7). On the other hand, there are partial cubes whose pre-hull number can be arbitrarily large (Section 5). We have also considered the question whether any connected bipartite graph with pre-hull number at most 1 is a partial cube. We have only been able to obtain a partial answer to this question which we consider less than satisfactory as it involves a technical condition that may well be satisfied by all bipartite graphs with $ph \leq 1$ (Sections 6 and 7).

2. Definitions and notation

2.1. Graphs

The graphs we consider are undirected, without loops or multiple edges, and may be finite or infinite. If $x \in V(G)$, the set $N_G(x) := \{y \in V(G) : xy \in E(G)\}$ is the *neighborhood* of x in G. For a set S of vertices of a graph G we put $N_G(S) := \bigcup_{x \in S} N_G(x) - S$, and we denote by $\partial_G(S)$ the *edge-boundary* of S in G, that is the set of all edges of G having exactly one end-vertex in S. Moreover, $G[S]$ is the subgraph of G induced by S, and $G - S := G[V(G) - S]$.

Paths are considered as subgraphs rather than as sequences of vertices. Thus an (x, y)-path is also a (y, x)-path. If u and v are two vertices of a path P, then we denote by $P[u, v]$ the segment of P whose end-vertices are u and v.

Let G be a connected graph. The usual *distance* between two vertices x and y, that is, the length of any (x, y)-*geodesic* (= shortest (x, y)-path) in G, is denoted by $d_G(x, y)$. The *diameter* of a finite graph G will be denoted by diam G. A connected subgraph H of G is *isometric* in G if $d_H(x, y) = d_G(x, y)$ for all vertices x and y of H. The *(geodesic) interval* $I_G(x, y)$ between two vertices x and y of G consists of the vertices of all (x, y)-geodesics in G.

2.2. Convexities

A *convexity* on a set X is an algebraic closure system $\mathcal{C}$ on X. The elements of $\mathcal{C}$ are the *convex sets* and the pair $(X, \mathcal{C})$ is called a *convex structure* or a *convexity space*. The *convex hull* $co_{\mathcal{C}}(A)$ of a subset A of X is the smallest convex set which contains A. The convex hull of a finite set is called a *polytope*. An element x of a convex set C is an *extreme point* of C if $C - \{x\}$ is convex. Note that if C is the convex hull of a set A, then every extreme point of C belongs to A. A *minimal convex extension* of a convex set C is a convex set which properly contains C and which is minimal with respect to inclusion. A *copoint* at a point $x \in X$ is a convex set K which is maximal with respect to the property that $x \notin K$; x is an *attaching point* of K. Note that $K^+ := co_{\mathcal{C}}(K \cup \{x\}) = co_{\mathcal{C}}(K \cup \{y\})$ for any two attaching points x, y of K. K^+ is a minimal convex extension of K, called the *canonical extension* of K. We denote by $\mathrm{Att}(K)$ the set of all attaching points of K, i.e.,

$$\mathrm{Att}(K) := K^+ - K. \tag{1}$$

For an extensive study of abstract convex structures, see van de Vel [11].

We will be concerned with convexities $\mathcal{C}$ on X that can be induced by an *interval operator* on X, that is, a map $I : X \times X \to \mathcal{P}(X)$ such that $x, y \in I(x, y)$ and $I(x, y) = I(y, x)$ for all $x, y \in X$. In this case the pair (X, I) is called an *interval space*, and a subset C of X is convex provided $I(x, y) \subseteq C$ for all $x, y \in C$. Let $\mathcal{I}$ be the self-map of $\mathcal{P}(X)$ defined by

$$\mathcal{I}(A) := \bigcup_{x,y \in A} I(x, y)$$

for each $A \subseteq X$. Then $\mathcal{I}$ is a *pre-hull operator* of the convex structure $(X, \mathcal{C})$, that is, $\mathcal{I}$ is extensive, isotone and such that a set $C \subseteq X$ is convex if and only if $\mathcal{I}(F) \subseteq C$ for each finite set $F \subseteq C$.

In terms of a pre-hull operator the convex hull of a set $A \subseteq X$ is

$$co_{\mathcal{C}}(A) = \bigcup_{n \in \mathbb{N}} \mathcal{I}^n(A).$$

Let K be a copoint of X and $x \in \mathrm{Att}(K)$. If $\mathcal{I}^n(K \cup \{x\})$ is convex for some $n \in \mathbb{N}$ (i.e. if $\mathcal{I}^n(K \cup \{x\}) = K^+$) define $r(x; K)$ to be the smallest such n; if no such n exists put $r(x; K) = \infty$. Note that since $K^+ = K \cup \mathrm{Att}(K)$, the condition that $\mathcal{I}^n(K \cup \{x\}) = K^+$ is equivalent to

$$\mathrm{Att}(K) \subseteq \mathcal{I}^n(K \cup \{x\}). \tag{2}$$

The following concept is the principal subject of this paper.

Definition 2.1. Let $\mathcal{I}$ be a pre-hull operator of a convex structure $(X, \mathcal{C})$.

(i) Given a copoint K of X, the *pre-hull number* of K in X is

$$ph(X; K) := \sup\{r(x; K) : x \in \mathrm{Att}(K)\}. \tag{3}$$

(ii) The *pre-hull number* of X is

$$ph(X) := \sup ph(X; K) \tag{4}$$

the supremum taken over all copoints K of X.

Several kinds of graph convexities, that is, convexities on the vertex sets of connected graphs, have been investigated in the literature. The two most natural ones are the *geodesic convexity* and the *induced path* (or *monophonic*) *convexity*. Both are induced by interval operators: the geodesic convexity by the geodesic interval operator I_G, and the induced path convexity by the induced path interval operator (the induced path interval between two vertices x and y of a graph G is the set of vertices of all induced (x, y)-paths in G). In the following we denote by $\mathcal{I}_G$ the pre-hull operator of the geodesic convex structure of a graph G, and by $ph(G)$ the pre-hull number of the geodesic space $V(G)$.

By way of an example, if G is the Petersen graph then the copoints are the vertex-sets of the pentagons and the 3-claws, and $ph(G; C_5) = 2$, $ph(G; K_{1,3}) = 3$. Hence the Petersen graph has pre-hull number 3.

Throughout this paper, when working with graphs, by an interval we always mean a geodesic interval, and the terms convex, convex hull, polytope, copoint, etc., will always apply to the geodesic convexity.

3. Graphs with pre-hull number zero

By Definition 2.1, if a convexity on a set X is induced by an interval operator, then $ph(X) = 0$ if and only if for each point $x \in X$ and each copoint K at x, the canonical extension K^+ of K is obtained simply by adding x to K, $K^+ = K \cup \{x\}$; that is, K^+ is a "one-point" extension of K. The following proposition shows that more generally this one-point extension property holds for all minimal convex extensions of convex sets:

Proposition 3.1. *Let $\mathcal{C}$ be a convexity on a set X which is induced by an interval operator. Then $ph(X) = 0$ if and only if, for any convex set $C \subseteq X$ and any minimal convex extension C' of C, there exists $x \in X - C$ such that $C' = C \cup \{x\}$.*

Proof. We only have to prove the necessity. Suppose that $ph(X) = 0$. Let C be a convex set, C' a minimal convex extension of C, and $x \in C' - C$. Assume that $C' \neq C \cup \{x\}$, and let $y \in C' - (C \cup \{x\})$. Let K be a copoint at y containing C. If $x \in K$, then $K \cap C'$ is a convex extension of C which is properly contained in C', contrary to the minimality of C'. Hence $x \notin K$.

Because $ph(X) = 0$ by hypothesis, it follows that $K^+ = K \cup \{y\}$. Hence $K^+ \cap C'$ is a convex extension of C which is properly contained in C', contrary to the minimality of C'.

Therefore $C' = C \cup \{x\}$. □

We recall that a convex structure $(X, \mathcal{C})$ is a *convex geometry* (or *anti-matroid*) if it has the following equivalent properties:

Anti-Exchange Property: *For any $A \subseteq X$ and any two distinct points $x, y \notin co(A)$, $x \in co(A \cup \{y\})$ implies $y \notin co(A \cup \{x\})$.*

Minkowski–Krein–Milman Property: *Each polytope is the convex hull of its extreme points.*

A third equivalent condition is given by the following:

Proposition 3.2 (*Jamison [7]*). *A convex structure $(X, \mathcal{C})$ is a convex geometry if and only if, for each point $x \in X$ and for each copoint K at x, the set $K \cup \{x\}$ is convex.*

Hence:

Proposition 3.3. *If a convexity $\mathcal{C}$ on a set X is induced by an interval operator, then $ph(X) = 0$ if and only if $(X, \mathcal{C})$ is a convex geometry.*

We now specialize to graphs. Using results of Farber and Jamison we give a characterization of those graphs whose geodesic convexity is a convex geometry. Recall that a graph is *chordal* if it contains no induced cycle of length greater than 3. A vertex of a graph is *simplicial* if its neighborhood induces a simplex (= complete graph). Note that *a vertex x in a convex set K of a graph G is an extreme point of K if and only if x is a simplicial vertex of the subgraph $G[K]$.*

Proposition 3.4 (*Farber and Jamison [4, Theorem 3.2]*). *In a finite chordal graph, every non-simplicial vertex lies on an induced path joining two simplicial vertices.*

Proposition 3.5. *Let G be a chordal graph all of whose induced paths are geodesics, and let A be a finite subset of $V(G)$. Then $co_G(A) = \mathcal{I}_G(A)$. More precisely, $co_G(A) = \mathcal{I}_G(A^*)$, where $A^* \subseteq A$ is the set of all extreme points of $co_G(A)$.*

Proof. We will first show that $\mathcal{I}^2(A) = \mathcal{I}(A)$. Let $x \in \mathcal{I}^2(A) - \mathcal{I}(A)$. Then x lies on a (u, v)-geodesic P for some $u, v \in \mathcal{I}_G(A)$. Hence there are $a_0, a_1, a_2, a_3 \in A$ such that u lies on an (a_0, a_1)-geodesic Q_u and v lies on an (a_2, a_3)-geodesic Q_v. The graph $H := G[V(P) \cup V(Q_u) \cup V(Q_v)]$ is a finite induced chordal subgraph of $G[\mathcal{I}^2(A)]$ such that $V(H) = co_H(A \cap V(H))$. The vertex x is not simplicial in H since it is an inner vertex of P, which is a geodesic in H. Hence by Proposition 3.4, x lies on an induced path W of H joining two simplicial vertices b, b' of H, i.e. two extreme points of the convex set $V(H)$. Moreover $b, b' \in A \cap V(H)$ because $V(H) = co_H(A \cap V(H))$. W is also an induced path of G, and thus a geodesic of G by the hypothesis on G. It follows that $x \in I_G(b, b') \subseteq \mathcal{I}_G(A)$, contrary to assumption. Therefore $co_G(A) = \mathcal{I}_G(A)$.

We now show that, if u is a non-simplicial vertex of $K := G[co_G(A)]$, then u lies on a geodesic joining two vertices in A which are simplicial in K. Suppose that this is not true, and let A' be the set of end-vertices of all geodesics which join two elements of A and pass through u. By our assumption, each of these geodesics has at least one end-vertex which is not simplicial. Let A'' be the set of all these non-simplicial end-vertices. Each $a \in A''$ has two neighbors x_a and y_a in K which are not adjacent. For every pair (a, b) of distinct elements of A, let Δ_{ab} be the union of a finite set of (a, b)-geodesics such that

$$\{u\} \cup \bigcup_{c \in A'} \{x_c, y_c\} \subseteq \bigcup_{a,b \in A} V(\Delta_{ab}).$$

Then $K' := G[\bigcup_{a,b\in A} V(\Delta_{ab})]$ is a finite induced chordal subgraph of K. By Proposition 3.4, the vertex u lies on an induced path P of K' joining two simplicial vertices v, w. The path P is also an induced path of G, and thus is a (v, w)-geodesic. Hence the vertices v and w belong to A', and at least one of them, say v, belongs to A''. It follows in particular that the two neighbors x_v, y_v of v in K' must be adjacent since v is simplicial in K', contrary to the choice of these two neighbors. This proves the last part of the proposition. □

Let G be a graph, A a non-empty proper subset of $V(G)$ and $x \in V(G) - A$. We call *entrance* of x in A a vertex $a \in A$ which is joined to x by a geodesic having no inner vertices in A.

In the proofs of the next result and also of Theorem 3.7 we repeatedly use the following property of a chordal graph G: if C is a cycle in G of length ≥ 4 and $x \in V(C)$, then either x is incident with a chord of C or the two neighbors of x on C are adjacent to each other. Denote this property by $(*)$.

Lemma 3.6. *Let G be a chordal graph such that each of its 5-cycles has at least three chords, and let A be a non-empty proper subset of $V(G)$. Then A is convex if and only if for any $x \in V(G) - A$ the entrances of x in A are pairwise adjacent.*

Proof. (a) Suppose that A is not convex. Then there is a geodesic $\langle x_0, \dots, x_n\rangle$ joining two vertices x_0 and x_n of A such that $x_i \notin A$ for $0 < i < n$ and $n \geq 2$. It follows that x_0 and x_n are two entrances of x_1, and that they are not adjacent.

(b) Conversely suppose that A is convex, and that there exist vertices $x \in V(G) - A$ that have two distinct non-adjacent entrances $a, b \in A$. Choose a "bad" triple (x, a, b) such that among all bad triples its distance sum $s = d_G(a, b) + d_G(a, x) + d_G(b, x)$ is minimal. Let $P = \langle a, a_1, \dots, a_{p-1}, x\rangle$, $Q = \langle b, b_1, \dots, b_{q-1}, x\rangle$ and $R = \langle a, c_1, \dots, c_{r-1}, b\rangle$ be geodesics, where $r \geq 2$ and without loss of generality $p \leq q$. Because of the minimality of $s = p + q + r$ the three geodesics form a cycle.

Since a and c_2 are on the geodesic R they are non-adjacent, hence by $(*)$, C has a chord issuing from c_1. Its other end-vertex, say d, is either some a_i or b_j. In either case, if $r \geq 3$, then (d, b, c_1) is a bad triple violating the minimality of s. Hence $r = 2$. A similar minimality argument shows that C has no chords issuing from a or b. By $(*)$ this implies that a_1c_1 and b_1c_1 are edges. We now distinguish two cases (denoting c_1 by c from now on).

Case 1: *x is not adjacent to c*. In this case there are no chords of C issuing from x, hence by $(*)$, a_{p-1} and b_{q-1} are neighbors. We can therefore find an edge a_ib_j such that $i + j$ is minimal. Then $i, j > 0$ as no chords of C issue from a or b, and without loss of generality $i \leq j$. In the cycle C' formed by $P[a, a_i]$, R, $Q[b, b_j]$ and the edge a_ib_j, the two neighbors a_{i-1} and b_j of a_i are non-adjacent by the minimality of $i + j$. Hence by $(*)$, a_ic is an edge of G, and by the same argument so is b_jc. Consequently, $i, j \leq 2$, otherwise P and Q would not be geodesics.

If $i = 2$, then the pentagon $aa_1a_2b_2c$ has at most the two chords a_1c and a_2c by the minimality of $i + j$. If $i = 1, j = 2$ the same argument applies to the pentagon $bb_1b_2a_2c$. If $i = j = 1$ there is again a pentagon with at most two chords, viz. aa_1b_1bc. Hence we obtain a contradiction in all cases.

Case 2: *x and c are adjacent*. This implies that $p, q \leq 2$ and without loss of generality $p \leq q$. The case $p = q = 1$ is impossible by convexity of A, hence $q = 2$. But then the pentagons bb_1xa_1c (if $p = 2$) or bb_1xac (if $p = 1$) have only two chords. This is the final contradiction. □

Theorem 3.7. *Let G be a graph. Then the following are equivalent:*

(i) $ph(G) = 0$.
(ii) *G is chordal and each of its 5-cycles has at least three chords.*
(iii) *G is chordal and all induced paths of G are geodesics.*
(iv) *G is chordal and, for each non-empty convex subset A of $V(G)$ and each $x \in V(G) - A$, the entrances of x in A are pairwise adjacent.*
(v) *The geodesic convexity of G is a convex geometry.*

Note that the equivalence of conditions (ii), (iii) and (v) is an extension to arbitrary graphs (finite or infinite) of the equivalence of conditions (b), (d) and (e) of Theorem 4.1 of Farber and Jamison [4] for finite graphs.

Proof. (i) ⇔ (v) is a consequence of Proposition 3.3.

(ii) ⇒ (iii): Suppose that there is an induced path of G which is not a geodesic, and among all such paths let P be of minimal length. Let a and b be its end-vertices and let Q be an (a, b)-geodesic.

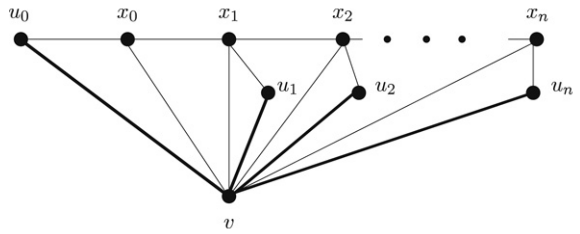

Fig. 1. G_n.

Suppose that $P = \langle x_0, \ldots, x_p \rangle$ and $Q = \langle y_0, \ldots, y_q \rangle$ with $x_0 = y_0 = a$ and $x_p = y_q = b$. Then $V(P) \cap V(Q) = \{a, b\}$ and $2 \le q < p$. Both P and Q being induced paths, the cycle $P \cup Q$ has no chord issuing from a, hence applying $(*)$, x_1 and y_1 are adjacent. By the minimality of P, $P[b, x_1]$ is a geodesic, hence $p - 1 = d_G(b, x_1) \le d_G(b, y_1) + 1 = q$, and therefore $p = q + 1$. Repeated application of $(*)$ then gives that y_i is adjacent to x_j if and only if $j \in \{i, i + 1\}$ for all i with $1 \le i \le q - 1$. Then $\langle a, x_1, x_2, y_2, y_1 \rangle$ is a 5-cycle of G which has only two chords.

(iii) $\Rightarrow$ (v) is a consequence of Proposition 3.5.

(v) $\Rightarrow$ (ii): Assume that the geodesic convexity of G is a convex geometry. Let C be a cycle of G and suppose that C has no chord. Then no vertex of C is simplicial in $G[co_G(V(C))]$. Hence the polytope $co_G(V(C))$ has no extreme point, contrary to it being the convex hull of its extreme points by (v). Therefore G is a chordal graph.

Now suppose that G contains a 5-cycle $C = \langle x_0, x_1, x_2, x_3, x_4 \rangle$ with only two chords x_0x_2 and x_0x_3. Since $d_G(x_1, x_4) = 2$, it follows that $K := co_G(\{x_1, x_4\})$ contains the set N of all common neighbors of x_1 and x_4. Because G is chordal and x_1 and x_4 are not adjacent, it follows that these common neighbors must be pairwise adjacent. Therefore $K = \{x_1, x_4\} \cup N$, and thus $x_2, x_3 \notin K$. On the other hand, $x_3 \in co_G(K \cup \{x_2\})$ and $x_2 \in co_G(K \cup \{x_3\})$, contrary to the fact that the geodesic convexity of G has the anti-exchange property. Consequently every 5-cycle of G has at least three chords.

(ii) $\Rightarrow$ (iv) is a consequence of Lemma 3.6.

(iv) $\Rightarrow$ (ii) Suppose that a chordal graph G has a 5-cycle $C = \langle x_0, \ldots, x_4 \rangle$ with exactly two chords. Then one of its vertices, say x_0, is adjacent to all other vertices of C. Note than in a chordal graph, any two distinct neighbors of two non-adjacent vertices are adjacent. Hence the interval of two vertices at distance 2 is convex. Therefore $I_G(x_1, x_4)$ is convex and does not contain x_2 and x_3. It follows that x_1 and x_4 are two entrances of x_2 in $I_G(x_1, x_4)$, and that they are not adjacent. Hence G does not satisfy (iv). $\square$

Because a connected bipartite chordal graph is a tree, it follows from Theorem 3.7 that:

Corollary 3.8. *The pre-hull number of a connected bipartite graph G is zero if and only if G is a tree.*

In connection with condition (ii) of Theorem 3.7 note that if some pentagons of a chordal graph have only two chords then the pre-hull number of the graph may be arbitrarily large. For the graph G_n of Fig. 1 one has:

- G_n: $K = \{v, u_0, \ldots, u_n\}$, $\mathrm{Att}(K) = \{x_0, \ldots, x_n\}$, $ph(G_n) = ph(G_n; K) = n$;
- G_∞: $K = \{v, u_0, u_1, \ldots\}$, $\mathrm{Att}(K) = \{x_0, x_1, \ldots\}$, $ph(G_\infty) = ph(G_\infty; K) = \infty$.

4. Pre-hull number of median-like graphs

Having characterized the graphs with pre-hull number zero, naturally it is the next step to try to do the same for graphs with pre-hull number equal to 1, or at most 1. We do not know the answer to this question, not even when restricting ourselves to bipartite graphs. In this section we show that it follows easily from results of Chepoi [2] that the so-called weakly median graphs have pre-hull number at most 1. In particular, this includes the median graphs. It will be seen later (Theorem 7.7) that within the class of *bipartite* weakly modular graphs the median graphs are the only ones with this property.

We recall the definitions (all graphs considered here are *connected*; they may be finite or infinite).

A graph G is *weakly modular* if it satisfies the following two conditions:

Triangle Condition: for any three vertices x_0, x_1, x_2 such that $1 = d_G(x_1, x_2) < d_G(x_0, x_1) = d_G(x_0, x_2)$, there exists a common neighbor u of x_1 and x_2 such that $d_G(x_0, u) = d_G(x_0, x_1) - 1$.

Quadrangle Condition: for any four vertices x_0, x_1, x_2, x_3 such that $d_G(x_1, x_3) = d_G(x_2, x_3) = 1$ and $d_G(x_0, x_1) = d_G(x_0, x_2) = d_G(x_0, x_3) - 1$, there exists a common neighbor u of x_1 and x_2 such that $d_G(x_0, u) = d_G(x_0, x_1) - 1$.

Note that chordal graphs are weakly modular.

A *quasi-median* of a triple (u_0, u_1, u_2) of vertices of a graph G is a triple (x_0, x_1, x_2) of vertices of G such that: $\{x_i, x_j\} \subseteq I_G(u_i, u_j)$ for all $i, j \in \{0, 1, 2\}$ with $i \neq j$, and $d_G(x_0, x_1) = d_G(x_1, x_2) = d_G(x_2, x_0) = k$, where k is minimal with respect to these conditions; k is called the *size* of the quasi-median. A quasi-median of size 0 consists of a single vertex which is called a *median* of the triple (u_0, u_1, u_2).

In a weakly modular graph every triple of vertices has a quasi-median. Of particular interest in the present context are:

• the *weakly median* graphs, i.e. the weakly modular graphs in which every triple of vertices has a unique quasi-median, or equivalently weakly modular graphs that do not contain any pair of vertices with an unconnected triple of common neighbors; and

• the *median* graphs, i.e. the weakly median graphs in which every triple of vertices has a unique median.

Bipartite weakly modular graphs are called *modular* graphs. These are the graphs for which every triple of vertices has at least one median. Thus the median graphs, which are the bipartite weakly median graphs, are particular modular graphs.

The following properties of the geodesic convexity of weakly median graphs are relevant in the present context.

Proposition 4.1 (*Chepoi [2]*). *Every interval of a weakly median graph is convex.*

Recall that an abstract convex structure $(X, \mathcal{C})$ is *join-hull commutative* if for any convex set $C \subseteq X$ and any $u \in X$, the convex hull of $\{u\} \cup C$ equals the union of the convex hulls $co_G(\{u, v\})$ for all $v \in C$. Clearly if X is a join-hull commutative interval space whose intervals are convex, then $co_{\mathcal{C}}(\{u\} \cup C) = \mathcal{I}(\{u\} \cup C)$, whence $ph(X) \leq 1$. But note that to have pre-hull number at most 1 is in general a strictly weaker property than join-hull commutativity (cf. Remark 8.1).

Proposition 4.2 (*Chepoi [2]*). *The geodesic structure of a weakly median graph is join-hull commutative.*

The two preceding propositions imply:

Corollary 4.3. *Let G be a weakly median graph, C a convex subset of $V(G)$ and $u \in V(G) - C$. Then*

$$co_G(\{u\} \cup C) = \bigcup_{c \in C} I_G(u, c) = \mathcal{I}_G(\{u\} \cup C).$$

Hence:

Theorem 4.4. *The pre-hull number of any weakly median graph (hence in particular of any median graph) is at most 1.*

5. Pre-hull number of partial cubes

Median graphs are particular *partial cubes*, i.e. isometric subgraphs of hypercubes. It is tempting to think that – as for median graphs – the pre-hull number of arbitrary partial cubes is always at most 1. That this is not necessarily the case is shown by the example of the partial cube Q_n^- (the n-cube minus a vertex). It is easy to check that $ph(Q_n^-) = 2$ (see Theorem 5.8 for a more general result; in Fig. 2, $ph(Q_3^-; W_{ab}) = 2$). Modular partial cubes being median (cf. [9, Theorem 4]), it follows that Q_n^-

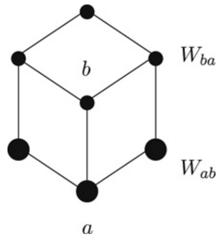

Fig. 2. $ph(Q_3^-) = 2$.

is not modular (also easily seen directly). However, failure of a partial cube to be modular does not in itself force its pre-hull number to be greater than 1, as instanced by the example of the even cycles of length at least 6.

All graphs considered in this section are *connected* and *finite*.

We will need the well-known characterizations of partial cubes due to Djoković [3] and Winkler [12]. For an edge ab of a graph G let

$$W_{ab} := \{x \in V(G) : d_G(a, x) < d_G(b, x)\},$$
$$U_{ab} := W_{ab} \cap N_G(W_{ba}).$$

Note that the sets W_{ab} and W_{ba} are disjoint and that $V(G) = W_{ab} \cup W_{ba}$ if G is bipartite.

Whenever necessary, we will indicate that the sets W_{ab}, U_{ab} are taken within a given graph G by writing W_{ab}^G and U_{ab}^G.

Proposition 5.1 (*Djoković [3, Theorem 1]*). *A connected and bipartite graph G is a partial cube if and only if for every edge ab of G the sets W_{ab} and W_{ba} are convex.*

Recall the definition of the Djoković–Winkler relation Θ for bipartite graphs (see [3,6,12]): two edges $e = ab$ and $e' = a'b'$ are in relation Θ if $d_G(a, a') = d_G(b, b')$ and $d_G(a, b') = d_G(b, a')$. The relation Θ is clearly reflexive and symmetric.

Proposition 5.2 (*Winkler [12]*). *A connected bipartite graph is a partial cube if and only if the relation Θ is transitive.*

Note that in any bipartite graph G if $e = ab$ and $e' = a'b'$ are edges such that $a' \in W_{ab}$ and $b' \in W_{ba}$ then $e\Theta e'$, $b \in I_G(a, b')$ and $b' \in I_G(a', b)$.

5.1. Copoints in partial cubes

For use in Section 5.2 (and also in Section 6) we derive here some straightforward technical results concerning copoints and their sets of attaching vertices in general bipartite graphs and, more specifically, in partial cubes.

Lemma 5.3. *Let K be a convex set of a bipartite graph G. Then any vertex of G not belonging to K has at most one neighbor in K.*

Proof. Suppose $x \in V(G)$ has two distinct neighbors $x', x'' \in K$. Since G is bipartite, $\langle x', x, x'' \rangle$ is a geodesic, hence $x \in K$. $\square$

If $x \in N_G(K)$, the unique neighbor of x in K will be called the *projection* of x in K and will be denoted by $\mathrm{pr}_K(x)$.

Lemma 5.4. (i) *Let K be a convex set in a bipartite graph G, and $e = ab \in \partial_G(K)$, where $a \in K$, $b \notin K$. Then $a = \mathrm{pr}_K(b) \in I_G(b, x)$ for every $x \in K$; in particular $K \subseteq W_{ab}$.*
(ii) *Moreover, if K is a copoint at b such that $\mathrm{Att}(K)$ is convex, then $\mathrm{Att}(K) \subseteq W_{ba}$.*

Proof. (i) Let $x \in K$ and suppose that $x \notin W_{ab}$. Then $x \in W_{ba}$, i.e. $d_G(b, x) = d_G(a, x) - 1$. Therefore $b \in I_G(a, x)$ and hence $b \in K$ because K is convex, a contradiction.

(ii) Now suppose that K is a copoint such that $\mathrm{Att}(K)$ is convex and $b \in \mathrm{Att}(K)$. Let $x \in \mathrm{Att}(K)$ and suppose that $x \in W_{ab}$. Then $d_G(a, x) = d_G(b, x) - 1$ and hence $a \in I_G(b, x)$. Therefore $a \in \mathrm{Att}(K)$ because $\mathrm{Att}(K)$ is convex, contradicting the hypothesis that $a \in K$. □

A short way of restating part (i) of the lemma is to say that *any proper convex subset of $V(G)$ is contained in some W_{ab}*.

Lemma 5.5. *Let $e = ab$ be an edge of a bipartite graph G. If W_{ab} is convex, then it is a copoint of G and $\mathrm{Att}(W_{ab}) \supseteq U_{ab}$.*

Moreover, if both W_{ab} and W_{ba} are convex, then

$$\mathrm{Att}(W_{ab}) = co_G(U_{ba}) \tag{5}$$

or equivalently,

$$co_G(W_{ab} \cup \{b\}) = W_{ab} \cup co_G(U_{ba}). \tag{6}$$

Proof. Let K be a convex set such that $K \supseteq W_{ab}$ and $b \notin K$. Then $a \in K$, i.e. $e \in \partial_G(K)$, hence $K \subseteq W_{ab}$ by Lemma 5.4, proving that if W_{ab} is convex, then it is a copoint at b.

Let $u \in U_{ba}$. Then $u \in I_G(b, v)$, where v is the projection of u in W_{ab}. If $u \notin \mathrm{Att}(W_{ab})$ there is a convex set K properly containing W_{ab} such that $u \notin K$. Then $b \in K$ and therefore $u \in I_G(b, v) \subseteq K$, a contradiction. Hence $u \in \mathrm{Att}(W_{ab})$.

For the second part of the lemma observe that in a bipartite graph $U_{ba} \subseteq \mathcal{I}_G(W_{ab} \cup \{b\})$ for any edge $ab \in E(G)$, and therefore

$$W_{ab} \cup U_{ba} \subseteq \mathcal{I}_G(W_{ab} \cup \{b\}). \tag{7}$$

Moreover, one easily sees that if W_{ab} is convex and $U_{ba} \subseteq S \subseteq W_{ba}$, then $\mathcal{I}_G(W_{ab} \cup S) = W_{ab} \cup \mathcal{I}_G(S)$. Hence in particular

$$\mathcal{I}_G^k(W_{ab} \cup U_{ba}) = W_{ab} \cup \mathcal{I}_G^k(U_{ba}) \subseteq \mathcal{I}_G^{k+1}(W_{ab} \cup \{b\}) \tag{8}$$

for all $k \in \mathbf{N}$, and therefore

$$W_{ab} \cup co_G(U_{ba}) = co_G(W_{ab} \cup \{b\}). \tag{9}$$

Furthermore, if (7) is an equality then so also is (8) for all k.

Now assume that W_{ba} is convex. Then $co_G(U_{ba}) \subseteq W_{ba}$, hence by (9),

$$\mathrm{Att}(W_{ab}) = co_G(W_{ab} \cup \{b\}) - W_{ab} = co_G(U_{ba}). \quad \square$$

Applying the preceding lemmas to partial cubes, and using the fact that in this case the sets W_{ab}, $ab \in E(G)$, are convex, we obtain:

Proposition 5.6. *The copoints of a partial cube G are precisely the sets W_{ab}, $ab \in E(G)$, and $\mathrm{Att}(W_{ab}) = co_G(U_{ba})$.*

Proof. Let K be a copoint of G. Given $z \in \mathrm{Att}(W_{ab})$ let a be a vertex of K closest to z, and b the neighbor of a on some (a, z)-geodesic. Then $ab \in \partial_G(K)$, whence $K \subseteq W_{ab}$ by Lemma 5.4. Since W_{ab} is convex and $b \notin W_{ab}$ it follows that $K = W_{ab}$. Thus every copoint of G is of the form W_{ab}.

The converse – that every W_{ab} is a copoint – is part of Lemma 5.5, as is the statement that $\mathrm{Att}(W_{ab}) = co_G(U_{ba})$. □

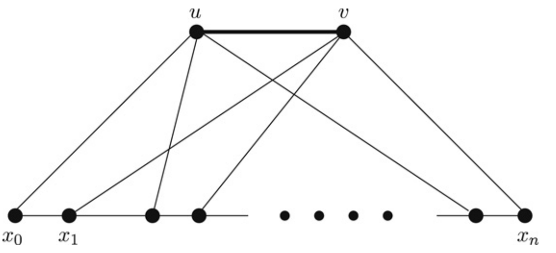

Fig. 3. G for n odd.

5.2. Pre-hull number and diameter

For arbitrary bipartite graphs there is no relationship between the pre-hull number and the diameter. A simple example showing the independence of the two graph parameters is the complete bipartite join of K_2 with a path of length $n \geq 3$ (Fig. 3). G has diameter 3, $K = \{u, v\}$ is a copoint, $\mathrm{Att}(K) = \{x_0, \ldots, x_n\}$, and $r(x_0; K) = n$ (notation of Definition 2.1). Hence $ph(G) \geq n$ (in fact $ph(G) = n$).

The situation changes drastically when one considers partial cubes. For these, the pre-hull number does not exceed the diameter. In proving this we use the fact that the diameter of a partial cube G is bounded above by the dimension of G. Recall that the dimension of G, i.e. the least n such that G is an isometric subgraph of an n-cube, coincides with the number of Θ-classes of $E(G)$.

Theorem 5.7. *If G is a partial cube of dimension n, then*

$$ph(G) \leq \min\{n-1, \operatorname{diam} G\}. \tag{10}$$

Moreover, for any positive integer n, there exists a partial cube G whose diameter and dimension are equal to n and whose pre-hull number is $n-1$.

Proof. For the proof of (10) let $ab \in E(G)$ be such that $ph(G) = ph(G; W_{ba})$. By Proposition 5.6, $A := \mathrm{Att}(W_{ba}) = co_G(U_{ab}) \subseteq W_{ab}$. We will show that $ph(G; W_{ba}) \leq \operatorname{diam} G[A] =: d$. A being convex, $G[A]$ is isometric in G, hence the relation Θ of $G[A]$ is the restriction of the relation Θ on G to $E(G[A])$. Since $G[A]$ contains no edges belonging to the Θ-class of ab, its dimension is at most $n-1$, hence $d \leq n-1$; also, $d \leq \operatorname{diam} G$.

For any $x \in A$ put $S_x := W_{ba} \cup \{x\}$. It will suffice to show that

$$A \subseteq \mathcal{I}_G^d(S_x) \quad \text{for all } x \in A.$$

Let $x, y \in A$ be arbitrary and let $\langle x_0, \ldots, x_r\rangle$ be an (x, y)-geodesic ($x_0 = x, x_r = y$). Then $r \leq d$, and $x_i \in A$ by the convexity of A. We now show by induction that $x_i \in \mathcal{I}_G^i(S_x)$, $i = 0, \ldots, r$. This is obvious for $i = 0$. For the induction step assume the statement true for $i-1$, and note that $U_{ab} \cap W_{x_i x_{i-1}} \neq \emptyset$, for if not then $U_{ab} \subseteq W_{x_{i-1}x_i}$, and hence $x_i \in A = co_G(U_{ab}) \subseteq W_{x_{i-1}x_1}$, which is absurd. Let $u \in U_{ab} \cap W_{x_i x_{i-1}}$ and v its projection in W_{ba}. Then uv is in relation Θ with ab but not with $x_{i-1}x_i$. This in turn implies that $v \in W_{x_i x_{i-1}}$. It follows that any (v, x_i)-geodesic of G can be extended at x_i to a (v, x_{i-1})-geodesic. Since $v \in S_x$ and $x_{i-1} \in \mathcal{I}_G^{i-1}(S_x)$ by the induction hypothesis, we therefore have $x_i \in \mathcal{I}_G^i(S_x)$. In particular, for $i = r$, $y \in \mathcal{I}_G^r(S_x) \subseteq \mathcal{I}_G^d(S_x)$, completing the proof of (10).

To prove the second part of the theorem let Q be an n-cube and choose an arbitrary edge $ab \in E(G)$. Let $P = \langle v_0, v_1, \ldots, v_{n-2}\rangle$ be a geodesic in Q with $v_0 = a$, $v_1 = b$. Define G to be the subgraph of Q induced by $N_Q[V(P)] \cup \{z\}$, where z is the antipode of a in Q. This is an isometric subgraph of Q with the property that if x, y are two distinct vertices at the same distance from a, then $d_G(x, y) = 2$ (for $n = 2$, G is a 4-cycle; for $n = 3$ it is the graph of Fig. 2). Furthermore, $W_{ab}^G \cup \{b\} = N_G[a] = N_Q[a]$. Hence for $k = 0, \ldots, n-1$, $\mathcal{I}_G^k(W_{ab}^G \cup \{b\})$ consists precisely of the vertices of G at distance $\leq k+1$ from a. Thus $ph(G; W_{ab}^G) \geq n-1$, which, together with (10) gives $ph(G) = n-1$. $\square$

Among the isometric subgraphs of a finite hypercube Q there are those which are obtained by removing a subhypercube from Q, e.g. the graphs Q_n^- mentioned at the beginning of this section. For these particular partial cubes, the pre-hull number is small.

Theorem 5.8. *Let Q be an n-cube, and Q' a p-subcube of Q with $0 \leq p < n$. If $n \leq 2$, then $ph(Q - Q') = 0$; and if $n \geq 3$, then*

$$ph(Q - Q') = \begin{cases} 1 & \text{if } p \geq n-2 \\ 2 & \text{if } p \leq n-3. \end{cases}$$

Proof. If $n \leq 2$, then $Q - Q'$ is a tree and thus $ph(Q - Q') = 0$ by Corollary 3.8. Assume that $n \geq 3$, and let $G := Q - Q'$. If $p = n - 1$, then G is an $(n-1)$-cube; if $p = n-2$, then G consists of two $(n-1)$-cubes attached to each other along an $(n-2)$-face. In both cases, G is median, and hence $ph(G) = 1$ by Theorem 4.4.

Suppose that $p \leq n - 3$, and let $e = ab$ be an edge of G.

(a) If e is Θ-equivalent in Q to some edge $e' \in E(Q')$, then $U^G_{ba} = W^G_{ba}$, hence by (5), $\mathrm{Att}(W^G_{ab}) = U^G_{ba}$ and $W^G_{ba} \subseteq I_G(W^G_{ab} \cup \{u\})$ for any $u \in \mathrm{Att}(W^G_{ab})$. Thus $ph(G; W^G_{ab}) = 1$.

(b) Now suppose that e is not Θ-equivalent to any edge e' of Q'. Then $V(Q')$ is contained in W^Q_{ab} or in W^Q_{ba}, say $V(Q') \subseteq W^Q_{ab}$. Whence $U^G_{ab} = W^G_{ab} = W^Q_{ab} - V(Q')$. It follows in particular that $ph(G; W^G_{ba}) = 1$.

Because $Q[W^Q_{ab}]$ is an $(n-1)$-cube and $p \leq n-3$, it follows that there are a subset S of U^G_{ab} which induces in Q a $(n-2)$-cube, and a vertex $s \in U^G_{ab} - S$. Then $\mathcal{I}_Q(\{s\} \cup S) = W^Q_{ab}$. Whence $\mathcal{I}_Q(U^G_{ab}) = W^Q_{ab}$. It follows that $\mathcal{I}_G(U^G_{ba}) = W^G_{ba}$ since $W^G_{ba} = W^Q_{ba}$. Therefore $\mathrm{Att}(W^G_{ab}) = \mathcal{I}_G(U^G_{ba})$, and thus $ph(G; W^G_{ab}) \leq 2$.

On the other hand, let $x \in W^G_{ba} - U^G_{ba}$ (i.e. x is the projection on W^G_{ba} of some vertex of Q'). Then, since $Q[W^G_{ba}]$ is an $(n-1)$-cube, there is exactly one $y \in W^G_{ba}$ such that $d_G(x, y) = d_Q(x, y) = n - 1$. Moreover, because $W^G_{ba} - U^G_{ba}$, which is the projection of $V(Q')$ on W^G_{ba}, induces a hypercube of dimension $p \leq n - 3$, it follows that $y \in U^G_{ba}$. Hence $x \notin I_G(y, z)$ for any $z \in U^G_{ba}$. It follows that $ph(G; W^G_{ab}) \geq 2$. Therefore $ph(G; W^G_{ab}) = 2$.

From this result and (a), we infer that $ph(G) = 2$. $\square$

There are straightforward extensions of this result to infinite hypercubes. In particular, with the part $p \leq n - 3$ of the preceding proof, we obtain the following theorem.

Theorem 5.9. *Let Q be an infinite hypercube, and Q' a subhypercube of Q of finite dimension. Then $ph(Q - Q') = 2$.*

6. A characterization of partial cubes

For application in the next section we derive here a (somewhat technical) characterization of partial cubes based on properties of the sets of attaching vertices of copoints. As always, all graphs under consideration are connected. They may be infinite except where stated otherwise.

Lemma 6.1 (*Imrich and Klavžar [5, Lemma 2.6]*). *The vertex-set of an induced connected subgraph H of a bipartite graph G is convex if and only if no edge in $\partial_G(V(H))$ is in relation Θ with an edge of H.*

Although proved in [5] only for finite graphs, the proof of the lemma is equally valid in the infinite case.

The following lemma lists some simple properties of the set of attaching vertices of copoints in graphs. Given a copoint K of G we denote by X_K the set of attaching vertices of K which have a neighbor in K, i.e.

$$X_K := \mathrm{Att}(K) \cap N_G(K). \tag{11}$$

This notation will also be used in Lemma 7.3. Recall that we denote the canonical extension of K by K^+.

Lemma 6.2. *Let K be a copoint of G. Then:*

(i) *If $u \in K$ and $v \in \mathrm{Att}(K)$, then any (u, v)-geodesic of G meets X_K.*
(ii) *$K^+ = K \cup co_G(X_K)$.*
(iii) *If $\mathrm{Att}(K)$ is convex, then $\mathrm{Att}(K) = co_G(X_K)$.*

Proof. (i) Let $\langle u, x_1, \ldots, x_{r-1}, v\rangle$ be a (u, v)-geodesic in G, where $u \in K$ and $v \in \mathrm{Att}(K)$. Denote by k the least subscript such that $x_k \notin K$. Then $x_k \in N_G(x_{k-1}) \subseteq N_G(K)$. Furthermore, $u, v \in K^+$, hence $x_k \in K^+$ by convexity of K^+, so that $x_k \in K^+ - K = \mathrm{Att}(K)$ by (i). Therefore, $x_k \in X_K$.

(ii) Clearly $K' := K \cup co_G(X_K) \subseteq K^+$ because X_K is contained in K^+. It therefore suffices to show that K' is convex; the minimality of the extension K^+ of K then implies that $K' = K^+$.

Let $u, v \in K$ and $P = \langle u, x_1, \ldots, x_{r-1}, v\rangle$ a (u, v)-geodesic. If $u, v \in K$ or $u, v \in co_G(X_K)$, then obviously all vertices of P are in K', so the only case to check is $u \in K$, $v \in co_G(X_K)$. Without loss of generality we may assume that u is the only vertex of P in K, whence by the proof of (i), $x_1 \in X_K$. Therefore $I_G(x_1, v) \subseteq co_G(X_K)$, so that again all vertices of P are in K'.

(iii) By (ii), $K^+ = K \cup co_G(X_K) = K \cup \mathrm{Att}(K)$, hence by (1), $\mathrm{Att}(K) = K^+ - K \subseteq co_G(X_K)$. On the other hand, if $\mathrm{Att}(K)$ is convex, then $co_G(X_K) \subseteq \mathrm{Att}(K)$. □

Definition 6.3. A bipartite graph G is said to be *Att-convex* (resp. *Att-full*) if for each copoint K of G, $\mathrm{Att}(K)$ is convex (resp. $N_G(K) \subseteq \mathrm{Att}(K)$).

Proposition 6.4. *A bipartite graph G is Att-convex if and only if $X_K \subseteq \mathcal{I}_G(\{a\} \cup K)$ for each copoint K of G and each vertex $a \in X_K$.*

Proof. Suppose that G is Att-convex. Let K be a copoint of G, and let $a, x \in X_K$. Then, by Lemma 5.4 and since $\mathrm{Att}(K)$ is convex, $x \in I_G(a, \mathrm{pr}_K(x))$. Hence $X_K \subseteq \mathcal{I}_G(\{a\} \cup K)$.

Conversely let K be a copoint at a vertex a. Suppose that $\mathrm{Att}(K)$ is not convex. Then there exist two non-adjacent vertices $a, b \in X_K$ which are the end-vertices of a geodesic whose inner vertices belong to K. Then $b \notin I_G(a, \mathrm{pr}_K(b))$. Let $u \in K$. Because $\mathrm{pr}_K(b) \in I_G(b, u)$ by Lemma 5.4 and since K is convex, it follows that $b \notin I_G(a, u)$. Therefore $X_K \not\subseteq \mathcal{I}_G(\{a\} \cup K)$. □

Proposition 6.5. *A bipartite graph G is Att-convex if and only if G has the following two properties:*

(i) *The restriction of the relation Θ to the edge-boundary of any convex set of G is transitive.*
(ii) *For each copoint K of G there exists a vertex $a \in X_K$ such that $X_K \subseteq \mathcal{I}_G(\{a\} \cup K)$.*

Proof. (a) Suppose that G is Att-convex. Condition (ii) is a consequence of Proposition 6.4.

Let C be a convex set of G. Let $ab \in \partial_G(C)$ with $b \in C$, and let K be a copoint at a containing C. Then $K \subseteq W_{ba}$. Furthermore every edge in $\partial_G(C)$ which is in relation Θ with ab belongs to $\partial_G(K)$ since by Lemma 6.1 no edge in $G[K]$ is in relation Θ with ab. If $uv \in \partial_G(K)$ with $v \in K$ is in relation Θ with ab, then $u \in W_{ab}$, and thus $u \in I_G(a, v)$. Hence $u \in \mathrm{Att}(K)$. Therefore all edges of $\partial_G(K)$ in relation Θ with ab are edges joining vertices of K with vertices of $\mathrm{Att}(K)$. Then any two of these edges are in relation Θ because K and $\mathrm{Att}(K)$ are two disjoint convex sets.

(b) Conversely suppose that G has properties (i) and (ii). Let K be a copoint of G. By property (ii), $X_K \subseteq \mathcal{I}_G(\{a\} \cup K)$ for some vertex $a \in X_K$. Let b be the neighbor of a in K, and let u be any vertex in X_K. Then there exists a vertex $v \in X_K$ such that $u, v \in I_G(a, v')$, where v' is the neighbor of v in K. Then the edges ab and vv' are in relation Θ. Let u' be the neighbor of u in K. Then, because K is convex and G is bipartite, $d_G(b, u') \leq d_G(a, u)$ and $d_G(u', v') \leq d_G(u, v)$. Therefore, since $u, v \in I_G(a, v')$, it follows that $d_G(b, u') = d_G(a, u)$ and $d_G(u', v') = d_G(u, v)$, and hence the edges ab and uu' are in relation Θ.

Consequently, each edge between $\mathrm{Att}(K)$ and K is in relation Θ with ab. Hence, by property (i), every pair of edges between $\mathrm{Att}(K)$ and K are in relation Θ. It follows that $\mathcal{I}_G(\mathrm{Att}(K)) \cap K = \emptyset$, and thus that $\mathrm{Att}(K)$ is convex because if some (x, y)-geodesic P with $x, y \in \mathrm{Att}(K)$ were to pass through K, then P would contain two boundary edges of K, i.e. two edges in relation Θ, which is impossible. □

We do not know if property (ii) is really necessary in Proposition 6.5. Concerning condition (i), note that we have the following equivalence:

Proposition 6.6. *Let G be a bipartite graph. The following properties are equivalent:*

(i) *The restriction of the relation Θ to the edge-boundary of any copoint of G is transitive.*
(ii) *The restriction of the relation Θ to the edge-boundary of any convex set of G is transitive.*

Proof. Suppose that the restriction of the relation Θ to the edge-boundary of any copoint of G is transitive. Let C be a convex set of G. Let $ab \in \partial_G(C)$ with $b \in C$, and let K be a copoint at a containing C. Then $K \subseteq W_{ba}$. Furthermore every edge in $\partial_G(C)$ which is in relation Θ with ab belongs to $\partial_G(K)$ since by Lemma 6.1 no edge in $G[K]$ is in relation Θ with ab. Hence, because by hypothesis the restriction of Θ to $\partial_G(K)$ is transitive, it follows that any two edges of $\partial_G(C)$ which are in relation Θ with ab are also in relation Θ with each other. Therefore the restriction of the relation Θ to the edge-boundary of C is transitive.

The converse is trivial. □

We obtain the following characterizations of partial cubes.

Theorem 6.7. *Let G be a finite, connected bipartite graph. The following properties are equivalent:*

(i) *G is a partial cube.*
(ii) *G is Att-convex and Att-full.*
(iii) *$N_G(K) \subseteq \mathcal{I}_G(\{a\} \cup K)$ for each copoint K of G and each vertex $a \in N_G(K)$.*
(iv) *The relation Θ on $E(G)$ is conditionally transitive in the sense that if e_0, e_1, e_2 are three edges of G such that*

- *$e_1 \Theta e_0$, $e_0 \Theta e_2$, and*
- *the shortest intervals between end-vertices of e_0 and e_1, e_1 and e_2, and e_2 and e_0 are convex,*

then $e_1 \Theta e_2$.

By a "shortest interval" between two edges e and e' we mean an interval formed by the geodesics connecting a pair of end-vertices of e and e', respectively, which are at minimum distance from each other.

Proof. (i) ⇒ (ii): Suppose that G is a partial cube. Let K be a copoint of G at a vertex $b \in N_G(K)$, and let $a := \mathrm{pr}_K(b)$. Then W_{ab} and W_{ba} are convex by Proposition 5.1. It follows that $K = W_{ab}$, and that $\mathrm{Att}(K)$ is contained in W_{ba} and thus is convex. Let $x \in N_G(K)$. Then $x \in I_G(b, \mathrm{pr}_K(x))$ by Lemma 5.4 and since W_{ba} is convex, and hence $x \in co_G(\{b\} \cup K) \cap W_{ba} = \mathrm{Att}(K)$. Therefore $N_G(K) \subseteq \mathrm{Att}(K)$.

(ii) ⇒ (i): Suppose that G is Att-convex and Att-full. Let K be a copoint of G, and $ab \in \partial_G(K)$ with $a \in K$. Note that $K \subseteq W_{ab}$. Let $x \in N_G(K)$. Then $x \in \mathrm{Att}(K)$ since G is Att-full. Hence $b \in I_G(x, a)$ by Lemma 5.4 and since $\mathrm{Att}(K)$ is convex. It follows that $K = W_{ab}$, and thus W_{ab} is convex and $\mathrm{Att}(K) \subseteq W_{ba}$. Then W_{ba} is also convex because the boundary of W_{ba} is equal to $N_G(K)$, and thus is contained in $\mathrm{Att}(K)$ which is convex since G is Att-convex by hypothesis. Therefore G is a partial cube by Djoković's Theorem (Proposition 5.1).

(i) ⇒ (iii) is a consequence of Proposition 6.4.

(iii) ⇒ (ii): Let K be a copoint of G, and $a \in X_K$. Because $N_G(K) \subseteq \mathcal{I}_G(\{a\} \cup K)$ by (iii), it follows that $N_G(K) \subseteq \mathrm{Att}(K)$, and thus that $\mathrm{Att}(K)$ is convex by Proposition 6.4. Therefore G is Att-convex and Att-full.

(i) ⇒ (iv) follows from the fact that Θ is transitive if G is a partial cube by Winkler's Theorem (Proposition 5.2).

(iv) ⇒ (i): Suppose Θ satisfies condition (iv). Clearly it suffices to show that all intervals in G are convex because it then follows from (iv) that Θ is transitive, whence by Winkler's Theorem G is a partial cube. To show that any interval $I_G(u, v)$ is convex we use induction on $d_G(u, v)$. Trivially $I_G(u, v)$ is convex if $d_G(u, v) = 1$. Assume the convexity of any interval whose end-vertices are at distance $\leq n$ from each other.

Let $u, v \in V(G)$ be such that $d_G(u, v) = n + 1$ (hence $d_G(x, y) \leq n + 1$ for any $x, y \in I_G(u, v)$). Suppose that $I_G(u, v)$ is not convex. Then there is a pair of vertices $w_0, w_1 \in I_G(u, v)$ and a (w_0, w_1)-geodesic not all of whose vertices belong to $I_G(u, v)$. Among all such pairs choose w_0, w_1 so as to have minimum distance from each other. Then no (w_0, w_1)-geodesic has any inner vertex in $I_G(u, v)$. Note that $2 \le d_G(w_0, w_1) \le n + 1$. Let P_i be a (u, v)-geodesic such that $w_i \in V(P_i)$, $i = 0, 1$, and let Q be any (w_0, w_1)-geodesic. Denote by e_0 the edge of Q incident with w_0. Then $P_0[u, w_0] \cup Q \cup P_1[w_1, u]$ and $P_0[v, w_0] \cup Q \cup P_1[w_1, v]$ contain cycles C_1 and C_2, respectively, both of which pass through e_0. Hence there is an edge $e_i = x_iy_i \in E(C_i)$, $e_i \neq e_0$, such that $e_i \Theta e_0$, $i = 1, 2$. Clearly $e_1 \notin E(Q)$, Q being a geodesic; likewise, $e_1 \notin E(P_0[u, w_0])$ because by the minimality of $d_G(w_0, w_1)$, $P_0[u, w_0] \cup e_0$ is a geodesic. Hence e_1 is in $P_1[u, w_1]$. In the same way one obtains that e_2 is in $P_1[w_1, v]$. Let the notation of the end-vertices of e_i be so chosen that $d_G(w_0, x_i) < d_G(w_0, y_i)$, $i = 1, 2$. Then $d_G(w_0, x_i) \le n$, $i = 1, 2$, and $d_G(y_1, y_2) \le n$. By the induction hypothesis the intervals $I_G(w_0, x_1)$, $I_G(y_1, y_2)$, $I_G(x_2, w_0)$ are convex, hence $e_1 \Theta e_2$ by (iv). But this is impossible because e_1 and e_2 are two distinct edges of the geodesic P_1. □

7. Bipartite graphs with pre-hull number at most 1

Definition 7.1. Call a set A of vertices of a graph G *ph-stable* if any two vertices $u, v \in \mathcal{I}_G(A)$ lie on a geodesic joining two vertices in A.

For example any convex set and any set A such that $\mathcal{I}_G(A)$ induces a finite or infinite path are obviously ph-stable. Note that if a set $A \subseteq V(G)$ is ph-stable, then any edge of $G[\mathcal{I}_G(A)]$ is an edge of an (a, b)-geodesic for some $a, b \in A$.

Remark 7.2. (i) The condition of Definition 7.1, which is symmetric in u and v, can be replaced by the formally "one-sided" condition: *for any two vertices* $u, v \in \mathcal{I}_G(A)$ *there is a* $w \in A$ *such that* $v \subseteq I_G(u, w)$. (To get a geodesic which joins two vertices in A and contains u and v, apply the one-sided condition first to u and v, and then to w and u.)

(ii) It is immediate from the definition that if A is ph-stable, then $\mathcal{I}_G^2(A) = \mathcal{I}_G(A)$ and hence $co_G(A) = \mathcal{I}_G(A)$. The converse is not true. For example, in the partial cube Q_3^- of Fig. 2, the set U_{ba} is not ph-stable but $co_G(U_{ba}) = W_{ba} = \mathcal{I}_G(U_{ba})$.

Pre-hull stability plays a role in the characterization of bipartite graphs with pre-hull number at most 1 (recall the notation introduced in (11): $X_K = \mathrm{Att}(K) \cap N_G(K)$):

Lemma 7.3. *Let K be a copoint of a graph G. If $ph(G; K) \le 1$, then X_K is ph-stable. Conversely, if G is bipartite, $\mathrm{Att}(K)$ is convex and X_K is ph-stable, then $ph(G; K) \le 1$.*

Proof. Suppose that $ph(G; K) \le 1$. Recall from (2) that this means that $\mathrm{Att}(K) \subseteq \mathcal{I}_G(K \cup \{x\})$ for every $x \in \mathrm{Att}(K)$. Hence in particular

$$X_K \subseteq \mathcal{I}_G(K \cup \{x\}) \quad \text{for every } x \in X_K.$$

Let $u, v \in \mathrm{Att}(K)$. Then $v \in \mathcal{I}_G(K \cup \{u\})$, i.e. v is on some (u, w)-geodesic P, where $w \in K$. As K is convex and $v \notin K$, there is a unique vertex of K on P which is closest to v, and without loss of generality we may assume w to be that vertex. Therefore $v \in I_G(u, z)$, where z is the neighbor of w on P not belonging to K. Thus $z \in N_G(K)$. On the other hand, since both u and w are in K^+ it follows that $z \in K^+ - K = \mathrm{Att}(K)$, hence $z \in X_K$. By Remark 7.2(i) this means that X_K is ph-stable.

For the converse suppose that G is bipartite, $\mathrm{Att}(K)$ is convex, and X_K is ph-stable. We have to show that $\mathrm{Att}(K) \subseteq \mathcal{I}_G(K \cup \{u\})$ for every $u \in \mathrm{Att}(K)$. By Lemma 6.2(iv), $\mathrm{Att}(K) = co_G(X_K)$; and by Remark 7.2(ii), $co_G(X_K) = \mathcal{I}_G(X_K)$. Hence what has to be shown is that $\mathcal{I}_G(X_K) \subseteq \mathcal{I}_G(K \cup \{u\})$ for every $u \in \mathcal{I}_G(X_K)$.

Let $u, v \in \mathcal{I}_G(X_K)$. X_K being ph-stable there is a vertex $w \in X_K$ such that $v \in I_G(u, w)$. Let w' be the projection of w in K. As $\mathcal{I}_G(X_K) = \mathrm{Att}(K)$ and $\mathrm{Att}(K)$ is convex, it follows that $I_G(u, w) \subseteq \mathrm{Att}(K)$, i.e. $I_G(u, w)$ is disjoint from K. Therefore $w' \notin I_G(u, w)$ and consequently $v \in I_G(u, w') \subseteq \mathcal{I}_G(K \cup \{u\})$. □

Note that the full strength of bipartiteness is not used here; all that is needed is that given any $w \in N_G(K)$ there is a neighbor w' of w in K such that $I_G(u, w) \in I_G(u, w')$ for any $u \in \mathrm{Att}(K)$. One may call this property *quasi-bipartiteness* at K. Examples of quasi-bipartite graphs with $ph = 1$ that are non-bipartite are the odd cycles C_{2k+1} (the copoints of C_{2k+1} are the vertex-sets of the paths of length k, the corresponding sets of attaching vertices are the complementary paths of length $k+1$).

Theorem 7.4. *Let G be a bipartite graph. Then $ph(G) \leq 1$ if and only if, for every copoint K of G, the set $\mathrm{Att}(K)$ is convex and X_K is ph-stable.*

Proof. By Lemma 7.3 we only have to prove the necessity. Suppose that $ph(G) \leq 1$, and let K be a copoint of G. To prove that $\mathrm{Att}(K)$ is convex it is sufficient to show that $I_G(u, v) \subseteq \mathrm{Att}(K)$ for any two vertices $u, v \in X_K$. By Lemma 5.4 and the fact that $ph(G) \leq 1$, we have $\mathrm{pr}_K(u) \in I_G(u, \mathrm{pr}_K(v))$ and $v \in I_G(u, \mathrm{pr}_K(v))$, respectively. It follows that $d_G(u, v) = d_G(\mathrm{pr}_K(u), \mathrm{pr}_K(v))$, and thus that $I_G(u, v) \cap K = \emptyset$.

Now X_K is ph-stable by Lemma 7.3. □

Since any edge is a convex set, it follows in particular that a bipartite graph G with $ph(G) \leq 1$ contains no $K_{2,3}$ as a subgraph. The first of the following two results follows immediately from Theorem 7.4, and the second from Theorems 6.7 and 7.4.

Theorem 7.5. *Let G be a partial cube. Then $ph(G) \leq 1$ if and only if U_{ab} and U_{ba} are ph-stable for every edge ab of G.*

Theorem 7.6. *A connected bipartite graph G such that $ph(G) \leq 1$ is a partial cube if and only if $N_G(K) \subseteq \mathrm{Att}(K)$ for each copoint K of G.*

For modular graphs we have the following theorem which adds three new ones to the long list of characterizations of the world's most characterized graphs (see [9]):

Theorem 7.7. *Let G be a connected modular graph. Then the following assertions are equivalent:*

(i) *$ph(G) \leq 1$.*
(ii) *The set of all attaching points of any copoint of G is convex.*
(iii) *The restriction of the relation Θ to the edge-boundary of any convex set of G is transitive.*
(iv) *G is a median graph.*

Proof. (i) ⇒ (ii) and (ii) ⇒ (iii) are parts of Theorem 7.4 and Proposition 6.5, respectively.

(iii) ⇒ (iv): Suppose that G is not a median graph. Then, by [1, Theorem 4.6] (also see [9]), G contains an induced subgraph H which is isomorphic to $K_{2,3}$. Let $V(H) = \{a, b, c, u, v\}$ be such that a, b and c are adjacent to u and v. The set $\{a, u\}$ is convex. The edges bu and cu are in relation Θ with the edge av, while bu and cu are not in relation Θ. Hence the restriction of the relation Θ to $\partial_G(\{a, u\})$ is not transitive.

(iv) ⇒ (i) is a consequence of Theorem 4.4. □

8. Antipodal partial cubes

Besides the median graphs there is another – albeit much smaller – interesting class of partial cubes whose pre-hull number is 1, namely the antipodal partial cubes. Among these graphs are some of the regular partial cubes mentioned by Klavžar and Lipovec [8].

A connected graph G is called *antipodal* if for any vertex $x \in V(G)$ there is a (necessarily unique) vertex $\bar{x}$ (the *antipode* of x) such that $I_G(x, \bar{x}) = V(G)$.[1] In such a graph one obviously has that

$$d_G(x, y) + d_G(y, \bar{x}) = d_G(x, \bar{x}) = r \quad \text{for any } x, y \in V(G), \tag{12}$$

where r is the diameter of G. If, in addition to being antipodal, G is a partial cube, then it follows immediately from (12) and the definition of the sets W_{ab} that for any edge $ab \in E(G)$,

$$W_{ba} = \overline{W}_{ab} := \{\bar{x} : x \in W_{ab}\}.$$

[1] This concept of antipodality was introduced by Kotzig [10]; it is a special case of the general concept of antipodality commonly used in algebraic graph theory.

Hence given any $u \in W_{ba}$, then $u, \bar{u} \in W_{ab} \cup \{u\}$, hence $V(G) = I_G(u, \bar{u}) \subseteq \mathcal{I}_G(W_{ab} \cup \{u\})$ so that $ph(G) \leq 1$. In fact, $ph(G) = 1$ unless $G = K_2$.

Examples of antipodal partial cubes are the *generalized middle-levels graphs* in the lattice of the subset representation of Q_n. Given any $n \geq 3$ consider Q_n as the Hasse diagram of the lattice of subsets of $N = \{1, \dots, n\}$, and for $0 \leq k < n/2$ let $M_{n,k}$ be the induced subgraph of Q_n with vertex-set $\{x \subseteq N : k \leq |x| \leq n - k\}$. Clearly $M_{n,k}$ is a partial cube, and it is easy to see that it is antipodal (the antipode of $x \in V(M_{n,k})$ is $\bar{x} = N - x$). $M_{n,0}$ is Q_n itself; other well-known examples are $M_{3,1}$ (the hexagon), and $M_{5,2}$ (the Desargues graph = generalized Petersen graph $P_{10,3}$). The graphs $M_{2k+1,k}$ are the *middle-levels graphs* of odd-dimensional hypercubes in the usual sense of the term; they are regular of degree $k + 1$.

Remark 8.1. Recall that join-hull commutativity of an interval space X with convex intervals implies that $ph(X) \leq 1$. The converse fails to hold, even within such a restrictive class of spaces as the partial cubes. An example is the graph $M_{n,1}$, $n \geq 4$, i.e. the cube Q_n from which a pair of antipodal vertices has been removed. Clearly $M_{n,1}$ contains copies of Q^-_{n-1} (the cube Q_{n-1} with only one vertex deleted) as convex subgraphs. Since join-hull commutativity is a convex-hereditary property and Q^-_{n-1} is not join-hull commutative (cf. the beginning of Section 5), it follows that $M_{n,1}$ is not join-hull commutative. On the other hand, $ph(M_{n,1}) = 1$. This example also shows that the pre-hull number is not convex-isotone, i.e. H convex in G does not in general imply that $ph(H) \leq ph(G)$.

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