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Constant tolerance intersection graphs of subtrees of a tree

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Abstract

A chordal graph is the intersection graph of a family of subtrees of a host tree. In this paper we generalize this. A graph $G=(V,E)$ has an (h,s,t)-representation if there exists a host tree T of maximum degree at most h, and a family of subtrees $\{S_v\}_{v\in V}$ of T, all of maximum degree at most s, such that $uv\in E$ if and only if $|S_u\cap S_v|\geqslant t$. For given h, s, and t, there exist infinitely many forbidden induced subgraphs for the class of (h,s,t)-graphs. On the other hand, for fixed $h\geqslant s\geqslant 3$, every graph is an (h,s,t)-graph provided that we take t large enough. Under certain conditions representations of larger graphs can be obtained from those of smaller ones by amalgamation procedures. Other representability and non-representability results are presented as well.

Keywords: Intersection graph; Tolerance; Chordal graph; *k*-simplicial vertex; Subtree representation; Regular tree

1. Introduction

An intriguing theme in graph theory is that of the intersection graph of a family of subsets of a set: the vertices of the graph are represented by the subsets of the family and adjacency is defined by a non-empty intersection of the corresponding subsets. Prime examples are interval graphs and chordal graphs. An interval graph is the intersection graph of a family of closed intervals on the real line. A classical result is the characterization of interval graphs by forbidden subgraphs by Lekkerkerker and Boland [18]. A chordal graph is a graph without

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induced cycles of length at least four. They were proven to be the intersection graphs of a family of subtrees of a tree [2,8,24]. In [21] McMorris and Scheinerman observed that this result may be sharpened in the following way: a graph G is chordal if and only if it is the intersection graph of a family of leaf-generated subtrees of a full binary tree. Special classes of chordal graphs are the vertex-intersection graphs or edge-intersection graphs of subpaths of a tree, see [10,11,22]. For a survey on intersection graphs the reader is referred to [20].

Golumbic and Monma [12] introduced a generalization of interval graphs using tolerances: each representing interval is assigned a positive real number, its tolerance, and two vertices are adjacent if the length of the intersection of their corresponding intervals exceeds the minimum of the two tolerances, see also [13]. This idea of tolerance was used in [14] to formulate a broad Master Plan on tolerance intersection graphs. Inspired by this setup we study the case where a graph $G = (V, E)$ may be represented by a *host tree* T together with a family $\{S_v\}_{v\in V}$ of subtrees of T and a tolerance t such that $uv \in E$ if and only if $|S_u \cap S_v| \geqslant t$. If we do not put extra conditions on the host tree, the representing subtrees, or the tolerance, then any graph can be represented, see Proposition 5 below. The extra conditions we put on the trees (host tree and representing subtrees) will be bounds on the maximum degree. Golumbic and Jamison [10,11] studied the case of vertex and edge intersection graphs of paths in a tree, that is, the tolerance is either 1 or 2, the host tree has unbounded degree, and the subtrees all have maximum degree 2. In [17] we focused on the case where the maximum degree of the host tree and the subtrees as well as the constant tolerance all are 3.

In Section 2 we present the basic definitions and rephrase a number of results from the literature in our terminology. In Section 3 we discuss in what ways chordal graphs can be represented as tolerance subtree graphs. Our main results are presented in Sections 4–6. First, if we fix the maximum degree of the trees in the representations, then we can still represent any graph, provided we choose our tolerance high enough (Section 4). Second, if we fix both the tolerance and the maximum degree of the trees in the representation, then there are infinitely many minimal graphs that do not have a representation of the required type (Section 6). In Section 5 we discuss how to produce representations of larger graphs from smaller ones using amalgamation. Finally, it turns out that complete bipartite graphs are crucial with respect to the question of representability or non-representability. This is the topic of Section 7. We close the paper with some concluding remarks. Note that so-called *p-intersection graphs* have been extensively studied, cf. e.g. [4–7]. Contrary to our approach, in the case of p-intersection graphs there is no structure presupposed on the host set of the representing subsets.

2. Preliminaries

In this paper all graphs will be connected, finite, simple, and loopless. For a graph $G = (V, E)$ with vertex set V and edge set E, the *order* $|G| = |V|$ of G is the number of vertices in G. The *neighborhood* $N(u)$ of a vertex u is the set consisting of all *neighbors* of u, i.e. vertices adjacent to u. The *closed neighborhood* $N[v]$ of v consists of v and all its neighbors. For two graphs G_1 and G_2, the *intersection* $G_1 \cap G_2$ is the graph with vertex set

$V_1 \cap V_2$ and edge set $E_1 \cap E_2$. If G_1 and G_2 and are two graphs with nonempty intersection $G_1 \cap G_2$, then the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ is the *amalgamation* of G_1 and G_2 along the common subgraph $G_1 \cap G_2$, or we just say that $G_1 \cup G_2$ is the amalgamation along the set of common vertices $V_1 \cap V_2$. If G_1 and G_2 contain isomorphic copies of the same graph as induced subgraph, then we can relabel the vertex sets such that this common subgraph is of the form $G_1 \cap G_2$. Thus, we can amalgamate two graphs along isomorphic copies of the same subgraph. Otherwise stated, we identify the two isomorphic subgraphs. Recall that a *k-clique* in a graph is a subset of *k* vertices inducing a complete graph. If the amalgamation is performed along a *k*-clique, then the graph $G_1 \cup G_2$ is the *k-sum* of G_1 and G_2. The 0-*sum* of two graphs is just the disjoint union of the two graphs.

Let h, s and t be positive integers with $h \geqslant s$. A graph $G = (V, E)$ is an (h, s, t)*-graph* if there exists a *host tree T* of maximum degree at most *h*, and a family of subtrees $\{S_v\}_{v \in V}$ of *T*, all of maximum degree at most *s*, such that

$$uv \quad E \text{ if and only if } |S_u \cap S_v| \geqslant t.$$

We call $(T, \{S_v\}_{v \in V})$ an (h, s, t)*-representation* of *G*. We can think of this representation also in terms of a mapping $v \mapsto S_v$. To distinguish between a graph *G* and its representation, we will call the elements of *V* the *vertices* of *G*, whereas we will speak of the *nodes* of the host tree *T* and its subtrees. Clearly, every induced subgraph of an (h, s, t)-graph, is itself an (h, s, t)-graph. Hence being an (h, s, t)-graph is a *hereditary* property, which raises the problem of characterizing these graphs by forbidden induced subgraphs.

We denote the class of (h, s, t)-graphs by $[h, s, t]$. If we do not impose restrictions on the maximum degree of the host tree, we write $h = \infty$. Similarly, we write $s = \infty$, if there is no restriction on the degree of the representing subtrees. By definition, we have

$$[h, s, t] \subseteq [h, s^*, t], \quad \text{for any } s^* \text{ with } s \leqslant s^* \leqslant h,$$

$$[h, s, t] \subseteq [h^*, s, t], \quad \text{for any } h^* \geqslant h.$$

In other words, the class $[h, s, t]$ is monotone in the first two parameters *h* and *s*. Whether it is monotone in the third parameter *t* is a non-trivial question. We deal with this problem in Section 4.

We review some terminology on trees. The nodes of degree 1 are the *leaves* of a tree, the other nodes being *internal nodes*. A subtree *S* of a tree *T* is *leaf-generated* if all endnodes of *S* are leaves in *T*. An *h-regular tree* is a tree, in which all internal nodes have degree *h*. A *cubic tree* is a 3-regular tree. If we have an (h, s, t)-representation of *G* with host tree *T* of maximum degree *h*, then we can add pendant nodes at internal nodes of *T* to make it *h*-regular without destroying any other properties of the representation. Hence we may assume without loss of generality that the host tree is regular. Note that we cannot apply this procedure to the representing subtrees, for then we may increase the intersections. In many of the proofs it is convenient to consider the host tree as being rooted. A rooted tree *T* can be viewed as a partially ordered set (poset) with its root as universal lower bound. Note that this poset is a *meet-semilattice*. The *outdegree* $d^+(v)$ of a node v is the number of nodes covering v. A *leaf* is a node of outdegree 0, the other nodes are *internal nodes*. Any subtree *S* of *T* has a minimal node in the ordering, its *meet*, which is the node of *S* closest to the root of *T*. A rooted tree is *d-ary* if all its internal nodes have outdegree *d*. If all leaves have the same distance $\eta = \eta(F)$ to the root, then we call T a *full d-ary tree* of $height\ \eta$. A full tree is also known as a *complete balanced tree*. The *jth level* is the set of nodes at distance j from the root. A *full binary tree* is a full 2-ary rooted tree F. Thus a full binary tree F is a poset, in which the maximal elements (nodes) are the leaves of F. Any other node of F is covered by exactly two nodes: its *children*, one of which is the *left child* and the other is the *right child*. Each node of F distinct from the root covers exactly one node: its *parent*. The nodes above node v are the *descendants* of v. Note that a leaf-generated subtree of F contains both children of its meet. The notions parent and descendant have their obvious analogues in arbitrary tree posets. We can convert a full d-ary tree simply into a $(d+1)$-regular tree by adding an extra node pending at the root. Furthermore, we can convert the host tree T of maximum degree at most h of a representation into an h-regular tree by adding the right amount of pendant nodes to each internal node of T. So, if necessary, we can assume the host tree to be regular.

Let $G = (V, E)$ be a graph, and let $(T, \{S_v\}_{v \in V})$ be a representation of G. The following properties emerge as important in the study of these representations. The representation is *leaf-generated* if all the representing subtrees S_v are leaf-generated subtrees of the host tree T. We denote the class of graphs having a leaf-generated (h, s, t)-representation by LG$[h, s, t]$.

The representation is *faithful* if representing subtrees that share a leaf of the host tree necessarily represent adjacent vertices in the graph. Note that we do not require the representation to be leaf-generated (contrary to what we did in [17]). Any representation can be easily turned into a faithful one by pending a new leaf at each leaf of the host tree. The importance of faithfulness only emerges when combined with other properties, as we will see below. We denote the class of graphs having a faithful, leaf-generated (h, s, t)-representation by FLG$[h, s, t]$.

The representation is *orthodox* if it is leaf-generated and representing subtrees share a leaf if and only if they represent adjacent vertices in the graph. We denote the class of graphs having an orthodox (h, s, t)-representation by ORTH$[h, s, t]$.

It was shown in [17] that LG[3, 3, 3] = FLG[3, 3, 3]. Furthermore, it was shown that the class ORTH[3, 3, 3] is properly contained in FLG[3, 3, 3] and FLG[3, 3, 3] is properly contained in [3, 3, 3].

Next we rephrase results from the literature in the terminology and notation developed above. Finite interval graphs, by definition, are the graphs in class [2, 2, 1]. The result on constant tolerances in [12] is equivalent to the statement that $[2, 2, t] = [2, 2, 1]$, for any $t \geqslant 1$. The classical result on chordal graphs, as being the intersection graphs of subtrees of a tree, reads as follows: a connected graph G is chordal if and only if G is in $[\infty, \infty, 1]$. The result of McMorris and Scheinerman mentioned above then essentially reads as

$$[\infty, \infty, 1] = \mathrm{ORTH}[3, 3, 1].$$

This result was mentioned in [21] without proof. In Section 3 we present a proof.

The path-graphs and EPT-graphs sensu Golumbic and Jamison [10,11] are the classes $[\infty, 2, 1]$ and $[\infty, 2, 2]$, respectively. Theorem 2 in [11] asserts that $[3, 2, 1] = [3, 2, 2] = [\infty, 2, 1] \cap [\infty, 2, 2]$. Theorem 3 of Sysło in [23] asserts that $[\infty, \infty, 1] \cap [\infty, 2, 2] \subseteq [\infty, 2, 1]$. Actually, Sysło showed even that $[\infty, \infty, 1] \cap [\infty, 2, 2] \subseteq [3, 2, 1]$. In fact, by combining this result with Theorem 2 from [11] one may deduce that

$$[3, 2, 1] = [3, 2, 2] = [\infty, \infty, 1] \cap [\infty, 2, 2].$$

Thus this result characterizes the chordal EPT-graphs as being the path-graphs (or equivalently, the EPT-graphs) with cubic host tree.

Theorem 3 of [11] sheds light on complexity questions: it is NP-complete to decide whether a $(\infty, 2, 1)$-representable graph is $(\infty, 2, 2)$-representable, whence it is NP-complete to decide whether a graph is in $[\infty, 2, 2]$. On the other hand, we know that the recognition of $(\infty, 2, 1)$-graphs is polynomial, cf. [9].

3. Chordal graphs

The following result is essentially due to McMorris and Scheinerman [21]. We present a proof here, so that we can use this proof in the sequel. A sketch of this proof was already given in [17].

Theorem 1. *A graph $G = (V, E)$ is chordal if and only if it has an orthodox* (3, 3, 1)-*representation.*

Proof. It suffices to prove the only-if part. So let $G = (V, E)$ be a chordal graph with $(T, \{S_v\}_{v\in V})$ as $(\infty, \infty, 1)$-representation. First we convert this representation into an orthodox one. For each node x of T, we add a new node p_x pending at x, so that the new nodes are the leaves of the extended host tree T^*. If a subtree S_v contains a node y in T, then we add the new node p_y to S_v pending at y in T^*. The extended host tree and the extended subtrees clearly are an orthodox $(\infty, \infty, 1)$-representation of G.

If T^* contains an internal node x with neighbors $z_1, z_2, \ldots, z_k$, for some $k > 3$, then we replace x by a path P_x with vertices $x_1, x_2, \ldots, x_k$ and we join x_i to z, for $i = 1, \ldots, k$. In the extended representing subtrees we make the according adjustments. Now we have an orthodox (3, 3, 1)-representation. This is easily transformed into an representation with a full binary tree as host tree, without changing the required characteristics of the representation. □

Theorem 2. *Let $G = (V, E)$ be an $(h, s, 2)$-graph with $h \geqslant 3$. Then any induced cycle in G has length at most h.*

Proof. Let $(T, \{S_v\}_{v\in V})$ be an $(h, s, 2)$-representation of G, and let $C = v_1 \to v_2 \to \cdots \to v_k \to v_1$ be an induced cycle in G of length $k \geqslant 3$. We may assume that T is a rooted tree. Among the subtrees representing the vertices of C, choose one with maximal meet m in T. We will show that there is a path of length at least $k - 2$ in C, all of whose vertices are represented by subtrees with meet m. Assume to the contrary that, say, the subtrees representing $v_2, v_3, \ldots, v_j$ all have meet m, whereas v_1 and v_{j+1} do not have m as meet, with $2 \leqslant j < k - 1$. Since v_1 and v_{i+1} are not adjacent in G, their subtrees share at most one node in T. On the other hand, since v_1 is adjacent to v_2 and v_{j+1} is adjacent to v_j, both their subtrees contain node m. Hence their meets are strictly below m, so that both contain the parent of m as well. This contradiction settles our claim.

Without loss of generality, let the subtrees representing $v_2, v_3, \ldots, v_{k-1}$ all have the same meet m, and let the subtrees representing v_1 and v_k have either m as meet or some other node not above m. In any case, these subtrees also contain m, so that their meets are m or strictly below m. Now the representing subtrees of any two consecutive vertices on C share an edge, whence share an edge incident with node m. On the other hand, the representing subtrees of any two non-consecutive vertices (being nonadjacent in G) share node m but no other node of T. This means that we can assign to any edge of C a unique edge in T incident with node m, which is contained in the two subtrees representing the end vertices of that edge of C. Hence, since T is of maximum degree h, there are at most h edges on C in G, which completes the proof. □

Corollary 3. *A* (3, 3, 2)-*graph is chordal*.

Theorem 4. $[\infty, \infty, 1] = [3, 3, 1] = \mathrm{ORTH}[3, 3, 1] = [3, 3, 2] = \mathrm{ORTH}[3, 3, 2]$.

Proof. By definition, we have $\mathrm{ORTH}[3, 3, 1] \subseteq [3, 3, 1] \subseteq [\infty, \infty, 1]$, and by Theorem 1, we have equality. Take the (3, 3, 1)-representation constructed in the proof of Theorem 1. For any leaf x of the host tree, we add an extra node y_x to the host tree T pending at x, and add y_x to all the representing subtrees containing x. Thus we get an orthodox (3, 3, 2)-representation. Thus we have shown that $[\infty, \infty, 1] \subseteq \mathrm{ORTH}[3, 3, 2]$. By Corollary 3, we have $[3, 3, 2] \subseteq [\infty, \infty, 1]$. This gives the remaining equalities. □

The result in Theorem 4 is quite special in the sense that, in general, we cannot reduce the degree of the host tree or the subtrees without destroying representability. Here are a few examples. A cycle of length n, with $n \geqslant 4$, is in $[n, 2, 2]$, but it is not chordal, whence it is not in $[3, 2, 2] \subseteq [3, 3, 2]$. The graph $\Theta(2, 2, 2)$ consisting of two nonadjacent vertices u and v joined by three internally disjoint induced paths of length 3 has a (4, 3, 3)-representation, and the graph $K_{2,6}$ has a (4, 4, 3)-representation. Both representations are relatively simple to construct, see [17]. But, by Theorem 7 in [17], neither of them is in [3, 3, 3].

4. Representability

In this section we study the monotonicity of the various subclasses of $[h, s, t]$ with respect to the parameter t. Proposition 5 is the tolerance-analogue of the classical result of Marczewski that each graph is an intersection graph [19].

Proposition 5. *Let* $G = (V, E)$ *be a connected graph. Then G is in* $[\infty, \infty, t]$, *for any* $t \geqslant 2$.

Proof. We construct stars as follows. A fixed node z is the central node of the host star and all representing substars. For an edge $e = uv$ of G, we introduce $t - 1$ extra nodes adjacent to z in the host star and add these nodes to the substars representing u and v. Now substars representing adjacent vertices of G have t nodes in common, whereas substars representing nonadjacent vertices of G only have z in common. □

Note that we could have used a similar construction in the case of non-constant tolerances. Also note that, if we use representations involving stars only, then the *split graphs* are precisely the graphs representable with constant tolerance 1.

We will call two subtrees *r-intersecting* whenever they share at least r nodes. Now we address the question what can be said about representability if we enlarge the tolerance t.

Proposition 6. *Let $G = (V, E)$ be an orthodox (h, s, t)-graph. Then G is an orthodox $(h, s, t + 1)$-graph.*

Proof. Let $(T, \{S_v\}_{v \in V})$ be an orthodox (h, s, t)-representation of G. Take any leaf x of T. We add a pendant node adjacent to x to T and to all representing subtrees containing x. Since the representation is orthodox all subtrees in $\{S_v\}_{v \in V}$ containing x represent adjacent vertices of G, and hence are mutually t-intersecting. In the extended representation they are now $(t+1)$-intersecting. On the other hand, subtrees that were not t-intersecting still aren't. Clearly the new representation is orthodox. □

Proposition 7. *Let $G = (V, E)$ be an (h, s, t)-graph with $t \geqslant 2$, and let k be a positive integer. Then G is an (h, s, r)-graph, for any integer r with $k(t - 2) + t \leqslant r \leqslant k(t - 1) + t$.*

Proof. Let $(T, \{S_v\}_{v \in V})$ be an (h, s, t)-representation of G, and let r be an integer with $k(t - 2) + t \leqslant r \leqslant k(t - 1) + t$. Subdivide each edge in T and all S_v by inserting k new nodes. If u and v are adjacent vertices of G, then S_u and S_v share at least t nodes and at least $t - 1$ edges of T. Hence, in the subdivided situation, they share at least $k(t - 1) + t$ nodes and thus are r-intersecting. If u and v are nonadjacent vertices of G, then S_u and S_v share at most $t - 1$ nodes and $t - 2$ edges of T. Hence, in the subdivided situation they share at most $k(t - 2) + t - 1$ nodes of T, so that they are not r-intersecting. □

Corollary 8. *Let $G = (V, E)$ be an (h, s, t)-graph with $t \geqslant 2$. If $r \geqslant (t - 3)(t - 2) + t$, then G is also an (h, s, r)-graph.*

Proof. Consider the intervals $I_k = \{r | k(t - 2) + t \leqslant r \leqslant k(t - 1) + t\}$, for $k \geqslant 0$. If we want to avoid gaps between two consecutive intervals I_k and I_{k+1}, then the inequality $k(t - 1) + t + 1 \geqslant (k + 1)(t - 2) + t$ should hold. Straightforward calculation tells us that this holds whenever $k \geqslant t - 3$. Since the left-hand endpoint of interval I_{t-3} is $(t-3)(t-2)+t$, the assertion follows. □

Note that we have $[h, s, 2] \subseteq [h, s, 3] \subseteq [h, s, r]$, for all $r \geqslant 4$. In Section 6 we will see that $K_{4,4}$ has a faithful, leaf-generated (3, 3, 4)-representation but not an orthodox (3, 3, 4)-representation. So, from Corollary 8, we can deduce that $K_{4,4}$ is a (3, 3, 6)-graph, but we cannot deduce from Proposition 6 that $K_{4,4}$ is a (3, 3, 5)-graph. On the other hand, by subdividing some special edges in the (3, 3, 4)-representation of $K_{4,4}$ in Fig. 3, we can construct a (3, 3, 5)-representation. But as yet we do not have such constructions for arbitrary graphs.

Conjecture. *Let $G = (V, E)$ be an (h, s, t)-graph with $t \geqslant 2$. Then G is an $(h, s, t + 1)$-graph.*

Now we present the main result of this section. Recall that the *bandwidth* of a connected graph $G=(V,E)$ of order $|V|=n$ is the minimum, over all the numbering of the vertices with the numbers $1,2,\ldots,n$, of the maximum difference between labels of adjacent vertices (cf. [25]).

Theorem 9. *Let $G=(V,E)$ be a graph with bandwidth b and maximum degree Δ. Let Δ_2 be the maximum number of common neighbors of pairs of nonadjacent vertices in G, and let $l=\lceil \log_2 \Delta \rceil$. Then G has an orthodox $(3,3,t)$-representation, for every $t \geqslant b+\Delta_2 l+1$.*

Proof. Let $v_1, v_2, \ldots, v_n$ denote an ordering of the vertices of G realizing the bandwidth b of G.

We construct a binary host tree T starting from a path $P = x_1 \to x_2 \to \cdots \to x_n$ rooted at x_1. Adjoin a new child z_i to each x_i in P. Now at each z_i, adjoin a full binary tree D_i of height t with z_i as its root. The host tree consists of P and the descendant binary trees $D_1, D_2, \ldots, D_n$, and its root is x_1.

The idea is to represent vertex v_i by subtree D_i together with a specially chosen path in D_j, for each v_j adjacent to v_i with $j<i$, and a subpath of P connecting D_i to all these paths in the other D_j.

Let v_k be a vertex of G, and let $N^+(v_k)$ denote the set of neighbors v_i of v_k with $i>k$. Since $d(v_k) \leqslant \Delta \leqslant 2^l$, we can assign distinct 0,1-strings $\sigma(i,k)$ of length l to the vertices v_i in $N^+(v_k)$. Thus, for each fixed k, the strings $\sigma(i,k)$ are all different.

Now let v_i be any vertex of G. For each neighbor v_k of v_i with $k<i$, construct a path Q_{ik} as follows. Start at z_i, go down to x_i, walk via P from x_i down to x_k, then go up to z_k. Now read the 0,1-string $\sigma(i,k)$ from left to right while moving upwards in D_k, where 0 means moving to the left child and 1 means moving to the right child. This takes us $l<t$ levels up into D_k. Finish Q_{ik} by moving up the remaining $t-l$ levels in an arbitrary way to a leaf of D_k.

Denote by R_i the union of the paths Q_{ik} with $k<i$ and v_k adjacent to v_i. We now represent vertex v_i by the leaf-generated subtree $S_i = D_i \cup R_i$ of T. Note that, b being the bandwidth of G, we have

$$|S_i \cap P| = |R_i \cap P| \leqslant b+1.$$

Let v_i and v_j be adjacent vertices in G. Suppose that $j<i$. Then $Q_{ij} \subset S_i$ contains a subpath with t nodes in D_j, so $|S_i \cap S_j| \geqslant t$. Moreover, Q_{ij} contains a leaf of D_j, so $S_i \cap S_j$ contains a leaf of the host tree T.

Now let v_i and v_j be non-adjacent vertices of G. Again suppose that $j<i$. Note that R_j does not contain x_i because the paths Q_{jk} go down from x_j and never up to x_i. Hence $|R_i \cap R_j| < |R_i \cap P| \leqslant b+1$.

For any common neighbor v_k of v_i and v_j, with $k<j<i$, the subtrees S_i and S_j will have common nodes in D_k. These start at z_k where Q_{ik} and Q_{jk} both enter D_k and continue as long as the two 0,1-strings $\sigma(i,k)$ and $\sigma(j,k)$ are the same. But $\sigma(i,k)$ and $\sigma(j,k)$ were chosen to be different, so the paths Q_{ik} and Q_{jk} must diverge in the first l steps into D_k. Hence $|Q_{ik} \cap Q_{jk} \cap D_k| \leqslant l$. There are at most Δ_2 common neighbors v_k of v_i and v_j. Hence we conclude that $S_i \cap S_j$ has less than $b+1$ nodes on P and at most $\Delta_2 l$ nodes in the various D_k. Thus, $|S_i \cap S_j| < (b+1)+\Delta_2 l \leqslant t$ as desired. Since the paths Q_{ik} and Q_{jk}

diverge before reaching the level of the leaves in D_k, the subtrees S_i and S_j cannot share a common leaf of the host tree. Therefore, the representation is orthodox. $\square$

Given a graph G, it seems to be a non-trivial problem to determine the smallest value of t for which G is in $[3, 3, t]$. We do not address this question here, and leave it as an open problem.

5. Amalgamation

It sometimes happens that the larger members of a graph class can be built by amalgamating smaller members of the class together. For example, a graph is chordal if and only if each of its blocks (maximal 2-connected subgraphs) is chordal. This allows a reduction in the study of chordal graphs to the 2-connected case. Since the tree-tolerance classes we are considering are hereditary, one direction generalizes trivially: every block of a graph in the class is again in the class. However, as the interval graphs illustrate, the gluing required for the converse can be problematic. A triangle with a pendant edge attached at each vertex is a classical forbidden subgraph for interval graphs, although each of its blocks is in a trivial way an interval graph. Two pendant edges can be attached to the triangle, but the third creates a problem.

Generally speaking, we would like to amalgamate two representations by gluing their host trees together at suitable nodes. Endnodes are the natural candidates because of the degree restrictions that we are imposing. If a representing subtree is in the interior of the host tree, then as in the case of interval graphs, there is no way to attach another block at the corresponding vertex of the graph and still guarantee representability. The extra conditions of leaf-generated, faithful, and orthodox, introduced earlier in the paper, provide a means of overcoming this difficulty. The goal is to find suitably large classes in which gluing can occur indefinitely. The arguments are easy but the details are somewhat subtle, and it is for that reason that we will treat them carefully here.

A clique C of an (h, s, t)-graph G is an *orthodox clique* if, with respect to some (h, s, t)-representation of G, the subtrees representing the vertices of C all share a common leaf of the host tree. A k-sum of two (h, s, t)-graphs G and H is an *orthodox k-sum* if the k-cliques being identified are orthodox in G and H. It is easy to represent an orthodox k-sum. Simply join the two leaves involved by an edge, thus joining the host trees. Also use this new edge to join the representing subtrees of corresponding vertices in the k-cliques that are identified. The drawback of this procedure is that the identified clique may lose its orthodoxy in the gluing process, so the operation cannot be repeated. Worse, the process may destroy the orthodoxy of other cliques also ending at the same leaves, making future gluing at these cliques impossible. To overcome these obstacles, we need to modify the construction and, to be sure that the modifications are allowed, we need to invoke additional properties of the representation. Roughly speaking, orthodoxy allows the amalgamation of two cliques and faithfulness preserves orthodoxy after amalgamation.

An (h, s, t)-representation of a graph G is *vertex orthodox* if every representing subtree contains at least one leaf of the host tree. That is, each vertex of G is an orthodox 1-clique. Similarly, a representation is *edge orthodox* if the representing subtrees of any two adjacent

Table 1
Properties of representations

Symbol	Name	Description
VO	*Vertex orthodox*	Every S_v contains a leaf of T
LG	*Leaf-generated*	Every S_v is generated by leaves of T
EO	*Edge orthodox*	$uv \in E \Rightarrow S_u \cap S_v$ contains a leaf of T
LGEO		LG and EO
FAITH	*Faithful*	$uv \in E \Leftarrow S_u \cap S_v$ contains a leaf of T
FVO		FAITH and VO
FLG		FAITH and LG
FEO		FAITH and EO
ORTH	*Orthodox*	FAITH and LG and EO

vertices share a leaf in common. That is, all edges of G are orthodox 2-cliques. We will use the symbols VO, EO, FVO, and FEO in conjunction with the parameter list (h, s, t) to denote the classes of graphs which have (h, s, t)-representations with the specified property. For example, FVO$[h, s, t]$ denotes the class of graphs with a faithful, vertex orthodox (h, s, t)-representation.

Table 1 contains a quick-reference list of the various properties of representations introduced here. The conditions VO, LG, EO, and LGEO will be referred to collectively as orthodoxy conditions. Note that LG is a "global" version of VO. The first two results below illustrate these ideas in the simple case of disjoint unions. We allow ∞ as a possible value of the parameters h and s.

Theorem 10. *For $t \geqslant 1$ and $h \geqslant s \geqslant 2$, the following closure results hold.*

(i) *The class $[h, s, t]$ is closed under 0-sums.*
(ii) *The classes FVO$[h, s, t]$, FLG$[h, s, t]$, FEO$[h, s, t]$, and ORTH$[h, s, t]$ are closed under 0-sums, for $h \geqslant 3$.*
(iii) *The class $[h, s, t]$ is closed under orthodox k-sums.*
(iv) *The class FLG$[h, s, t]$ is closed under orthodox k-sums, for $h \geqslant 3$.*
(v) *The classes FVO$[h, s, t]$, FEO$[h, s, t]$, and ORTH$[h, s, t]$ are closed under orthodox k-sums, for $h \geqslant s \geqslant 3$.*

Proof. Let G_1 and G_2 be two graphs in the given class. Let the k-sum be performed along the orthodox cliques C_1 in G_1 and C_2 in G_2, with $|C_1| = |C_2| = k$. Take (h, s, t)-representations of the given type of G_1 and G_2 with host trees T_1 and T_2, respectively. Let x_i be a leaf in T_i of the orthodox clique C_i, for $i = 1, 2$. In the case of 0-sums in (i) and (ii), let x_i just be any leaf of T_i, for $i = 1, 2$. The host tree T of G in (i) and (iii) is obtained from T_1 and T_2 by joining x_1 and x_2 by a new edge. In the cases (ii), (iv), and (v), the host tree is obtained by extending this tree T with two extra nodes y_1 and y_2, where y_i is adjacent to x, for $i = 1, 2$.

(i) This is trivial: take as representation host tree T and the representing subtrees of G_1 and G_2 in T_1 and T_2, respectively. Notice that this construction does not increase the maximum degree of the host tree, but it does destroy the endnodes x_1 and x_2.

(ii) In addition to the construction in (i), we have to recover lost endnodes. We use the new endnodes y_1 and y_2 for this purpose. Extend the representing subtrees for G_1 that formerly terminated at x_1 to contain y_1 as well. This will preserve whatever orthodoxy G_1 initially possessed. Note that the subtrees that contained x_1 now have an additional node y_1 in common. This increases the cardinality oftheir intersections. However, faithfulness ensures that the subtrees ending at x_1 represented adjacent vertices in G_1, so no unwanted new adjacencies appear. Since the representing subtrees that end in y_1 correspond to those that previously ended in x_1, faithfulness is also preserved. An analogous procedure may be applied at x_2 and y_2 for G_2. The addition of y_1 and y_2 makes x_1 and x_2 into nodes of degree three in the joined host. This is allowed since $h \geqslant 3$. Since the edge x_1x_2 is unused by representing subtrees, no new vertices of degree three appear in the representing subtrees, so their maximum degrees are unchanged.

(iii) In addition to the construction in (i), we perform the following operation: if vertex v_1 of C_1 is identified with vertex v_2 of C_2, then we join their representing subtrees by the new edge x_1x_2 to obtain the subtree representing the amalgamated vertex of G. The maximum degrees of the host tree and the representing subtrees are not increased.

(iv) Loosely speaking, we combine the constructions in (ii) and (iii). The host tree also contains y_1 and y_2. If vertex v_1 of C_1 is identified with vertex v_2 of C_2, then we join the representing subtrees by the new edge x_1x_2 to obtain the subtree representing the amalgamated vertex $v_1 = v_2$ of G. As in case (ii), this introduces new nodes of degree three in the host, but not in the representing subtrees. Any subtree terminating at x_1 that represents a vertex in $G_1 - C_1$ is extended to y_1. Similarly, any subtree terminating at x_2 that represents a vertex in $G_2 - C_2$ is extended to y_2. Thus, only new vertices of degree two are introduced in these representing subtrees. Representing subtrees of vertices not identified in the amalgamation remain leaf-generated since the roles of x_1 and x_2 are taken over by y_1 and y_2. Each joined subtree is still leaf-generated since its two halves contain leaves of their hosts. The representation is still faithful, because representing subtrees that share a leaf shared a leaf before and hence corresponded to adjacent vertices. The faithfulness of the original representations guarantees that adding y_1 and y_2, while increasing certain overlaps, will not introduce unwanted adjacencies.

(v) In the previous construction orthodoxy might be destroyed. We can cover this by extending *all* subtrees terminating at x_i in the representation of G_i to y_i, for $i = 1, 2$, whence also those of the identified vertices. This preserves orthodoxy, but it introduces two nodes of degree three into the representing subtrees of the identified vertices. Hence the condition $s \geqslant 3$ is necessary in this case. □

Corollary 11. *For $t \geqslant 1$, the following closure results hold.*

(i) *The class* FLG$[h, s, t]$ *is closed under* 1*-sums, for $h \geqslant s \geqslant 2$, with $h \geqslant 3$.*
(ii) *The classes* FVO$[h, s, t]$, FEO$[h, s, t]$, *and* ORTH$[h, s, t]$ *are closed under* 1*-sums, for $h \geqslant s \geqslant 3$.*

Proof. Each of the orthodoxy conditions VO, LG, and ORTH imply that all vertices are orthodox. EO implies that all non-isolated vertices are orthodox. But amalgamating along an isolated vertex can be rephrased as a 0-sum, which is covered by Theorem 10(ii). □

In certain cases, a representation may be made faithful without losing its orthodoxy. A result of this kind was proved in [17], which in our current language may be stated as follows: every LG[3, 3, 3]-graph has an FLG[3,3,3]-representation. Thus, LG[3, 3, 3] $=$ FLG[3, 3, 3], so LG[3, 3, 3] is also closed under 1-sums by the above corollary. This kind of "faithful for free" result holds in three others cases ($h=\infty$, $s=2$, and $t=3$) which are presented below, but it seems unlikely that it holds in general.

Proposition 12. *For $t \geqslant 1$ and $s \geqslant 2$, the class $[\infty, s, t]$ coincides with the class* FLG $[\infty, s, t]$ *and hence is closed under* 1*-sums.*

Proof. Consider any (h, s, t)-representation of a graph G. At each endnode of a representing subtree, append a new leaf corresponding to that subtree and that endnode. This may increase the maximum degree of the host, but as that is not bounded, it is allowed. Every representing subtree becomes leaf generated, and since each new leaf lies in a unique representing subtree, the representation is trivially faithful. □

As we will see later in Theorem 21, the class $[\infty, s, t]$ does not contain all graphs, so the above result does have content. A similar approach could be applied to ORTH, but that would require allowing $s=\infty$ as well. Such a result, however, is trivial, since $[\infty, \infty, t]$ is the class of all graphs if $t \geqslant 2$, as we saw in Proposition 5.

Let us say that the orthodoxy conditions *extend faithfully* for a parameter list (h, s, t) provided, for each orthodoxy condition X $\in$ {VO, LG, EO, LGEO}, every graph in X$[h, s, t]$ has a faithful (h, s, t)-representation satisfying condition X.

Proposition 13. *For $h \geqslant s \geqslant 2$ with $h \geqslant 3$, the orthodoxy conditions extend faithfully for* $(h, s, 3)$.

Proof. Consider an $(h, s, 3)$-representation of G in a host tree T. We will show that, if G has no isolated vertices, then we can extend this to a faithful $(h, s, 3)$-representation with the same orthodox vertices, orthodox edges, and leaf-generated representing subtrees as existed for G. This will establish the faithful extension result when G has no isolates, and the case of isolated vertices can then be handled by 0-sums using Theorem 10(ii).

So assume that G has no isolated vertices. Consider a leaf p of the host T with q as its unique neighbor. Let $x_1, x_2, \ldots, x_d$ be the other neighbors of q, so $d \leqslant h-1$. Attach d new leaves $y_1, y_2, \ldots, y_d$ to p and enlarge each representing subtree S through p by adding y_i to S if and only if S contains x_i. Since G has no isolated vertices, each such S has at least three nodes and hence must contain at least one x_i. The extension of S thus contains a leaf y_i, and hence remains leaf-generated if S was. Moreover, the represented vertex remains orthodox. Notice that any two representing subtrees S_v and S_w through p must contain p and q. Hence they are 3-intersecting if and only if they also share another neighbor x_i of q. This happens if and only if their extensions share a leaf y_i. This establishes faithfulness at p and preserves the orthodoxy of any edges at p. Notice that the degree of p in the extension of a representing subtree is the same as the degree of q in the subtree. Hence the maximum degree s is maintained. □

This construction may destroy the orthodoxy of some cliques with more than two vertices. Indeed, $\{p, q, x_1, x_2\}$, $\{p, q, x_2, x_3\}$, and $\{p, q, x_1, x_3\}$ would represent an orthodox triangle, of which the orthodoxy would be destroyed by the construction. Notice that this cannot happen when $h = 3$. Notice also that it is precisely the isolated vertices which form an obstacle to the above construction being applied in the case $h = s = 2$.

Corollary 14. *For $h \geqslant s \geqslant 2$, with $h \geqslant 3$, the class* $\mathrm{LG}[h, s, 3]$ *is closed under* 1*-sums. For $h \geqslant s \geqslant 3$, the classes* $\mathrm{VO}[h, s, 3]$ *and* $\mathrm{EO}[h, s, 3]$ *are closed under* 1*-sums.*

Proof. By the above proposition, these classes coincide with $\mathrm{FLG}[h, s, 3]$, $\mathrm{FVO}[h, s, 3]$, and $\mathrm{FEO}[h, s, 3]$, respectively. So the closure under 1-sums follows from Corollary 11. □

Proposition 15. *For $h \geqslant 3$ and $t \geqslant 2$, the orthodoxy conditions extend faithfully for $(h, 2, t)$.*

Proof. Consider an $(h, 2, t)$-representation of G in a host tree T. We will show that, if G has no isolated vertices, then we can extend this to a faithful $(h, 2, t)$-representation with the same orthodox vertices, orthodox edges, and leaf-generated representing subtrees as existed for G. This will establish the faithful extension result when G has no isolates, and the case of isolated vertices can then be handled by 0-sums using Theorem 10(ii).

So assume that G has no isolated vertices. Consider a leaf p of the host T with q as its unique neighbor. Let $d = h - 1$. We root T at p. Note that the first level consists of q only. For each node z in the levels 1 up to $t - 2$, we number the children of z by $1, 2, \ldots, d^+(z)$. Note that we have $d^+(z) \leqslant d$. Let P be any representing path containing p. Since G does not contain isolated vertices, P must contain at least t nodes, whence P grows from p up to the $(t - 1)$th level (and maybe even further). We can describe the way that P grows to the $(t - 1)$th level by a list π of $t - 2$ entries, each between 1 and d, where the entry in the ith position (from the left) gives the number of the child of the vertex P in the ith level.

Now we enlarge the host tree T as follows. Let T_1 be a full d-ary tree of height $t - 2$ with root r. The new host tree is obtained from T by identifying the root r of T_1 with p. For each node y in the levels 0 up to $t - 3$, we number the children of y by $1, 2, \ldots, d$. To recover the required properties of the representation, each path P containing p in the original representation must grow up to some leaf of T_1. Use π, read in reverse order, to describe the extension of P through T_1.

Now let P and R be two representing paths through p, with lists π and ρ, respectively. The intersection of P and R is a path containing p. The paths P and R represent adjacent vertices if and only if in the original representation this common path has at least t nodes. And that happens if and only if the two lists π and ρ are the same. This means that in the extended representation, P and R end up in the same leaf of T_1. Thus, any clique of vertices of G represented by paths containing leaf p in the original representation still contain the same leaf in T_1. This means that all the orthodox cliques will be preserved, as will leaf-generation.

It remains to check that this representation is now faithful. If P and R represent non-adjacent vertices in the original representation, then their respective lists π and ρ differ somewhere. Say they differ for the first time from the left at the kth position and for the first time from the right at the nth position. Then of course, $k \leqslant n$. Thus, the extended paths have $1 + (k - 1)$ nodes in common reaching up into the original host T. And they have

$t-2-(n-1) \leqslant t-1-k$ nodes in common reaching up into the extension T_1. Adding up we see that they have at most $t-1$ nodes in common altogether. Hence no new unwanted adjacencies are created. Also since π and ρ differ in the nth position from the right, in extending P and R using the reversals of π and ρ, we find that their extensions must differ at the $(t-n)$th step from p. Thus, P and R end up at different new leaves when extended in the new host. Hence the representation is faithful. □

Corollary 16. *For all h and t big enough*, $\mathrm{LG}[h, 2, t]$ *is closed under* 1-*sums*.

Proof. The above Proposition says that $\mathrm{LG}[h, 2, t] = \mathrm{FLG}[h, 2, t]$, so Corollary 11(i) gives us the closure result. □

Corollary 17. *Let* $G = (V, E)$ *be a tree*. *Then G is in* $[3, 3, t]$, *for any* $t \geqslant 1$.

Proof. By Theorem 1, G is an orthodox (3, 3, 1)-graph. This also follows easily by induction on the number of vertices from Corollary 11(ii). Then, by Proposition 6, the result follows. □

For trees we can prove an even “stronger” result.

Proposition 18. *Let* $G = (V, E)$ *be a tree*. *Then G has a* $(3, 2, t)$-*representation such that every node of the host tree is contained in at most two representing paths*, *for any* $t \geqslant 1$.

Proof. By induction on $n = |V|$. For $n = 1$, we take as host tree and representing subtree a path on t vertices. So let $n \geqslant 2$. Let x be any vertex of degree 1 in G, and assume that $(T, \{S_v\}_{v \in V-x})$ is a representation of the tree $G - x$, where host tree T is a cubic tree and all S_v are paths of length at least t such that every node of the host tree T is contained in at most two paths S_v. Let y be the neighbor of x in G. We may assume that S_y contains an edge pq that is not on any other representing path. For, otherwise, suppose that each edge of the path S_y is on some other representing path. Since each node of S_y is on at most one other representing path, the only way that this is possible is that there is another representing path S_z with $S_y \subseteq S_z$. But now there can be no other vertices in $G - x$ than y and z, whence $T = S_y = S_z$. Let w be a leaf of T. Add two new nodes r_y and r_z to T, add r_y to S_y, and r_z to S_z. Then S_y contains the edge wr_y that is not on S_z, and we take this edge to be pq. Now we subdivide pq in the host tree T as well as in the subtree S_y by inserting t new nodes. Let S_x be the subpath on the t new nodes. Then we have the required $(3, 2, t)$-representation of the tree G. □

6. Non-representability

Above we noted that if no restrictions are placed on h and s, then already for $t \geqslant 2$, all graphs are (h, s, t)-representable. In the previous section we showed that for any fixed h and s (both at least 3), every graph is (h, s, t)-representable if t is allowed to be arbitrarily large. In this section we will show in Theorem 21 that, when both the maximum degree s of the representing subtrees and the tolerance t are fixed, then $[\infty, s, t]$ is a nontrivial class in the sense that there are infinitely many minimal forbidden subgraphs. The main tool is

Theorem 19, which shows that (h, s, t)-graphs must have vertices which are analogous to the simplicial vertices of chordal graphs.

A vertex v in a graph G is *k-simplicial* if the closed neighborhood $N[v]$ in G can be vertex-covered by at most k-cliques. The *simpliciality* of a vertex v is the smallest k such that v is k-simplicial. Note that the simpliciality of a vertex is just the chromatic number of the complement of its neighborhood. Thus, the fact that the local independence number (sensu [17]) is an obvious lower bound on the simpliciality of a vertex is just the complementary fact that the clique number is a lower bound for the chromatic number. Evidently, 1-simplicial is just simplicial in the classical sense. As is well-known [3], cf. [9], every nontrivial chordal graph has at least *two* 1-simplicial vertices. We show below that, for each s and t, there is a constant k depending on s and t but not on h and not on the graph, such that every (∞, s, t)-graph has at least *two* k-simplicial vertices. For this we consider a special tree whose structure includes all possibilities for given s and t. Let $R(s, t)$ denote the rooted tree whose root has s children, all other internal nodes have $s - 1$ children, and all leaves are at distance $t - 1$ from the root. Since each node has degree either s or 1, this is an s-regular tree of radius $t - 1$. Let $\gamma(s, t)$ denote the number of subtrees of $R(s, t)$ which have exactly t nodes and which contain the root. These numbers have appeared in previous studies of the lattice of subtrees of a tree [15,16]. Note that $R(s, t)$ has exactly $s(s - 1)^{t-2}$ leaves and less than s^t nodes in all. Thus, $\gamma(s, t)$ is at most the binomial coefficient $C(s^t, t)$.

Theorem 19. *If G is a* (∞, s, t)*-graph of order at least* 2, *then G has at least two vertices that are* $\gamma(s, t)$*-simplicial*.

Proof. Consider an (h, s, t)-representation in a host tree T. Root T at any node r. For each representing subtree S, let inf(S) denote the meet of S in this meet-semilattice. Now choose a representing subtree S so that inf(S) is maximal. Choosing inf(S) at maximum distance to r will accomplish this, but other choices may also be possible. We now show that such a subtree S has the desired simpliciality.

Indeed, let $m = \inf(S)$, and let S^* denote the subtree of S consisting of all nodes of S at distance at most $t - 1$ from m. Since S has maximum degree s, it is clear that S^* is (abstractly) a subtree of the full s-regular tree $R(s, t)$. Thus, the number of t-node subtrees of S^* which contain m is at most $\gamma(s, t)$. Any representing subtree R that is adjacent to S must intersect S in at least t nodes. Since inf(R) is not above inf(S), it follows that R must contain m. Thus, each such R must contain some t-node subtree Q of S through m. Since Q has t nodes, the family of all representing subtrees containing Q is a clique. Hence the neighborhood of S is covered by at most $\gamma(s, t)$ cliques, so S represents a $\gamma(s, t)$-simplicial vertex of G.

To obtain two $\gamma(s, t)$-simplicial vertices of G, it suffices to show that there are two rootings of T which necessarily lead to different choices of S. For this we need some non-degeneracy assumptions. If all but at most one vertex of G is universal (i.e., adjacent to all other vertices), then G is in fact complete, and the final result is trivial. Thus we may suppose G has at least two vertices that are not universal. Since universal vertices cannot increase the simpliciality of any other vertex, we may remove the universal vertices without loss of generality and still have a nontrivial graph (i.e. a graph of order at least 2). Similarly, we may assume that G has no isolated vertices.

Since G has no isolates, every representing subtree must have at least t nodes. Since G has no universal vertices, it follows that for any representing subtree A, there must be a representing subtree B with $|A \cap B| < t \leqslant |A|$. Hence B does not contain A. We will use this non-degeneracy condition to get the second $\gamma(s, t)$-simplicial vertex of G.

Let A represent a $\gamma(s, t)$-simplicial vertex of G. By the above argument, there is a representing subtree B that does not contain A. Root the host T at any node r of A that is not in B. Then $r = \inf(A)$ and $\inf(B)$ is strictly above r. Now select a representing subtree S such that $\inf(S)$ is maximal and above $\inf(B)$ or equal to $\inf(B)$. Then $\inf(S) \neq r$, so S is different from A. But by the above argument, S also represents a $\gamma(s, t)$-simplicial vertex of G. $\square$

Let $\xi(h, s, t)$ denote the smallest k such that every (h, s, t)-graph has a k-simplicial vertex. The bound on $\xi(h, s, t)$ implied by the above result is very crude. For example, it was shown in [17] by exhaustive case analysis that $\xi(3, 3, 3) = 3$. It would be of interest to know more accurate bounds for $\xi(h, s, t)$, but at this stage they seem to be difficult to obtain except through rather tedious case analyses.

Note that if G is triangle-free, then the simpliciality of any vertex is just its degree. Thus, we have the following corollary.

Corollary 20. *If G is a triangle-free (h, s, t)-graph, then the minimum degree $\delta(G)$ of G is at most $\gamma(s, t)$.*

Extending the ideas of Corollary 20, we see that the class $[h, s, t]$ is a non-trivial class of graphs. Recall that the *girth* of a graph G is the length of a shortest cycle in G.

Theorem 21. *Let h, s, and t be integers with $h \geqslant s \geqslant 2$. Then the class $[h, s, t]$ has infinitely many minimal forbidden induced subgraphs.*

Proof. For $h = 2$, the class $[h, s, t]$ is just the class of the interval graphs. In this case, the assertion follows from [18]. So we may assume that $h \geqslant 3$.

For every p and q, there exists a graph G of minimum degree $\delta(G) \geqslant p$ and of girth $g(G) \geqslant q$ (see [1]). We choose a sequence of graphs $G_1, G_2, \ldots$ as follows. Let G_1 be a graph with $\delta(G_1) > \gamma(s, t)$ and $g(G_1) \geqslant 4$. Then, for $i \geqslant 1$, let G_{i+1} be a graph with $\delta(G_{i+1}) > \gamma(s, t)$ and $g(G_{i+1}) > |G_i|$. The girth condition tells us that, for $i < j$, any connected induced proper subgraph of G_i, which is also an induced proper subgraph of G_j, must be a tree. The degree condition tells us that none of our graphs G_i is in $[h, s, t]$. Hence every G_i contains a non-representable induced subgraph B_i of minimal order. Since a tree is always representable for $h \geqslant 3$ by Corollary 17, it follows that B_i is not a subgraph of any B_j, for $j \neq i$. Hence the graphs $B_1, B_2, \ldots$ form an infinite class of minimally non-representable graphs. $\square$

Theorem 21 raises the problem of characterizing the class $[h, s, t]$ by forbidden subgraphs. But this seems to be a very tough problem in general. So far, only the classical characterizations of the interval graphs and the chordal graphs are available. In [17], we have only first attempts at producing candidates for the list of forbidden subgraphs for the class [3, 3, 3]. We pursue some of those ideas in the next section.

7. The case of complete bipartite graphs

The complete bipartite graphs $K_{m,n}$ are triangle-free graphs with relatively few vertices, which still have relatively large minimum degree. Therefore, we may expect that these graphs are critical with respect to representability and non-representability. The aim of this section is to explore this idea. First we consider the case of $K_{2,n}$.

Theorem 22. *Let* $t = \lceil \log_2 n \rceil + 2$. *Then* $K_{2,n}$ *has an orthodox* $(3, 3, t)$*-representation.*

Proof. Let a and b be the two vertices on the 2-side of $K_{2,n}$ and $1, 2, \ldots, n$ the n vertices on the n-side. Let A and B be two full binary trees of height $L = t - 2 = \lceil \log_2 n \rceil$, and let r_a be the root of A and r_b the root of B. Since $n \leqslant 2^L$, we can assign each vertex i on the n-side a distinct 0,1-string σ_i of length L. We use these strings σ_i to construct paths in A and B, where we interpret a 0 as 'going to the left child' and 1 as 'going to the right child'. For each i, we construct a path P_i on t nodes in A and a path Q_i on t nodes in B. In A we start in the root r_a, and reading σ_i from left to right we move upwards following the instructions given by σ_i until we reach a leaf in level L. In B we start at root r_b, and reading σ_i from right to left we move upwards following the instructions given by σ_i until we reach a leaf in level L. Now we join r_a and r_b by an edge, thus obtaining a cubic tree T. We represent vertex a by subtree A, vertex b by subtree B, and vertex i by the path R_i consisting of $P_i \cup Q_i$ together with edge $r_a r_b$, for $i = 1, 2, \ldots, n$.

Clearly path R_i has t nodes in common with A as well as with B, for $i = 1, 2, \ldots, n$. So these adjacencies are represented correctly. Subtrees A and B are disjoint, reflecting the fact that vertices a and b are nonadjacent in $K_{2,n}$. Take any two distinct paths R_i and R_j. Their 0,1-strings σ_i and σ_j differ in at least one place, say in place k from the left. Then, in A, the paths P_i and P_j differ from level k upwards. So they contain at most k common nodes (including root r_a). In B, the paths Q_i and Q_j differ from level $L - k + 1$ upwards. So they contain at most $L - k + 1$ common nodes (including root r_b). Hence the paths R_i and R_j contain at most $L + 1 = t - 1$ common nodes. By construction, all representing subtrees are leaf-generated and the representation is orthodox. □

Although the two vertices on the 2-side in $K_{2,n}$ have large degree, the vertices on the n-side have only degree 2. So Theorem 19 is not relevant for $K_{2,n}$.

Theorem 23. *Let* h, s, *and* t *be integers with* $h \geqslant s$, *and let* n *be an integer with* $n > \gamma(s, t)(t + 1)$. *Then* $K_{2,n}$ *is not in* $[h, s, t]$.

Proof. Since $K_{2,n}$ is not chordal, we may assume that $t > 1$. Let us write $\gamma = \gamma(s, t)$. Assume to the contrary that $K_{2,n}$ has a (h, s, t)-representation with host tree T. Without loss of generality, T is a full h-regular tree with root r. Let a and b be the vertices on the 2-side of $K_{2,n}$ and $1, 2, \ldots, n$ the vertices on the n-side. Let A be the subtree representing a with meet r_a in T, and let B be the subtree representing b with meet r_b in T. For $i = 1, 2, \ldots, n$, let S_i be the subtree representing vertex i with meet r_i in T. Then r_a is comparable with all r_i. If any meet r_i is below r_a, then S_i grows into A and contains a subtree of order t rooted at

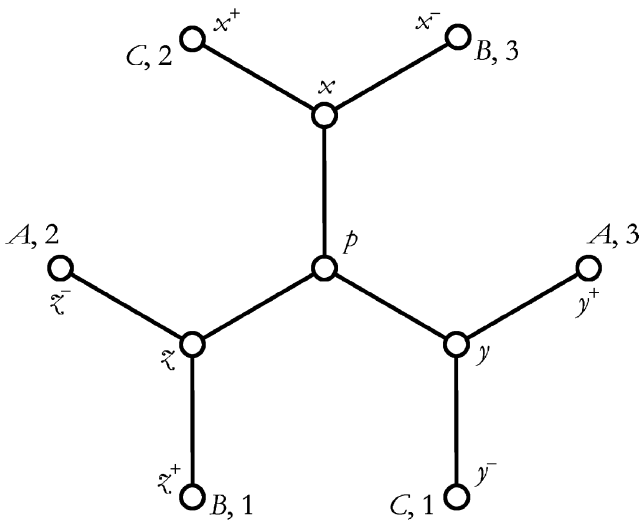

Fig. 1. The canonical (3,2,3)-representation of $K_{3,3}$.

r_a. Hence at most γ meets of the meets $r_1, r_2, \ldots, r_n$ are below r_a, so that at least $n-\gamma$ of these meets are strictly above r_a, which are all contained in A. The same holds with respect to r_b. Since $n > 2\gamma$, there exists a meet r_i, which is above r_a as well as r_b. This implies that r_a and r_b are comparable, say r_a is below r_b. There may not be more than γ meets of $r_1, r_2, \ldots, r_n$ at each node of B. Since there are at least $n-\gamma$ of $r_1, r_2, \ldots, r_n$ strictly above r_b, there are at least $(n-\gamma)\gamma^{-1}$ different nodes in B that are the meet of some representing subtree S_i. Subtree A must contain all these meets, so that A contains at least $(n-\gamma)\gamma^{-1}$ nodes of B. But this is impossible, since $(n-\gamma)\gamma^{-1} > t$. This contradiction settles the proof. □

In [17] it was shown that $K_{2,4}$ is in [3, 2, 3], but $K_{2,n}$ is not in [3, 3, 3], for $n \geqslant 5$. This shows that the bounds in Theorems 22 and 23 are not sharp. Obviously, the one in Theorem 23 is not very good. This raises the question of determining the value of $\rho(t)$ such that $K_{2,n}$ is in [3, 3, t] if $n \leqslant \rho(t)$ and $K_{2,n}$ is not in [3, 3, t] if $n > \rho(t)$. Note that in the (3, 2, 3)-representation of $K_{2,4}$ in [17] the vertices A and B on the 2-side are represented by subtrees having an edge in common, instead of being disjoint as in the proof of Theorem 22. So one might gain a lot by searching for subtrees A and B sharing as many nodes as possible.

The complete bipartite graph $K_{n,n}$ is the smallest triangle-free graph of minimum degree n. So, in view of Theorem 22, it is interesting to determine the smallest value of t such that $K_{n,n}$ is in [3, 3, t]. In [17] a (3, 2, 3)-representation of $K_{3,3}$ was constructed. This representation is given in Fig. 1. The $K_{3,3}$ has A, B, C as the three independent vertices on the one side and 1, 2, 3 as the vertices on the other side. Each of the vertices is represented by a path between the leaves labeled with the name of the vertex. The representation is faithful but not orthodox. Up to the labeling of the vertices (and extensions beyond the given figure)

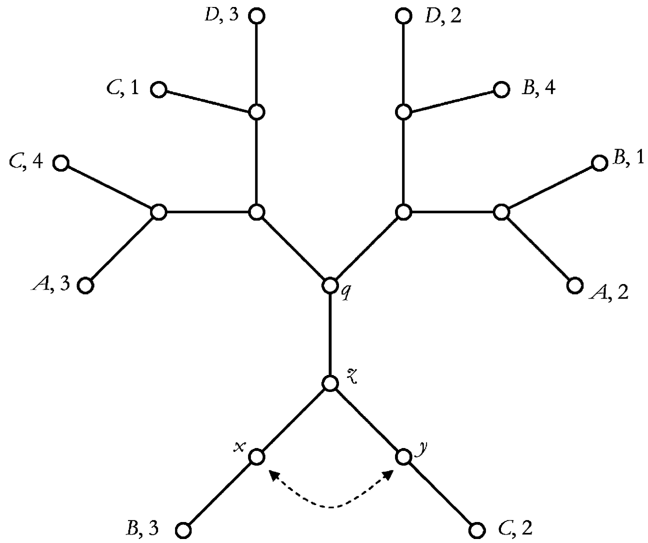

Fig. 2. The canonical (3,3,4)-representation of $K_{4,4}$.

it is unique. From this unicity one easily deduces that $K_{3,4}$ is not in [3, 3, 3]. The proof of these facts is still quite straightforward.

By similar arguments we can show that $K_{4,4}$ is in [3, 3, 4]. But in this case the arguments are much more tedious and involve a lot of case checking. Therefore we omit the proof here, and just present our (3, 3, 4)-representation in Fig. 2. The four independent vertices of the $K_{4,4}$ on the one side are labeled A, B, C, D, and the vertices on the other side are labeled 1, 2, 3, 4. The representing subtrees are leaf-generated by the leaves bearing the name of the represented vertex. Of course, we can relabel the vertices of the $K_{4,4}$. Moreover, we can interchange the roles of 1 and 4, obtaining a non-isomorphic representation. Finally, we may identify the two nodes x and y and the edges xz and yz of the host tree. But apart from these operations and extensions beyond the given figure, the representation is unique. The representation is faithful but not orthodox. Again, one easily deduces from this unicity that $K_{4,5}$ is not in [3, 3, 4]. The examples of $K_{3,3}$ and $K_{4,4}$ suggest the following conjecture.

Conjecture. *For* $n \geqslant 3$, *the complete bipartite graph* $K_{n,n}$ *has a faithful* $(3, 3, n)$-*representation, but not an orthodox* $(3, 3, n)$-*representation or any* $(3, 3, t)$-*representation with* $t < n$.

We just state the conjecture for what it is worth. Maybe we should rephrase it into a question: what is the smallest t such that $K_{n,n}$ is in [3, 3, t]?

Note that, if we insert a new node in the six edges, which are incident with the neighbors of q but not with q itself, then we obtain a (3, 3, 5)-representation of $K_{4,4}$, see the observations and our conjecture after Corollary 8.

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