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Some Properties of Character Products

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Very little is known about the nonprincipal irreducible constituents of a product $\chi\xi$ of characters of a finite group G. We prove

THEOREM 1. *Let χ be a character of G, and let χ^* denote its complex conjugate. Let $u, v \in G$ and assume $\chi(uv) \neq \chi(u'v)$ for some G conjugate u' of u. Then $\chi\chi^*$ has a nonprincipal irreducible constituent ξ such that $(\xi, 1)_{C_G(u)} \neq 0$ and $(\xi, 1)_{C_G(v)} \neq 0$.*

In practice the hypothesis that $\chi(uv) \neq \chi(u'v)$ for some conjugate u' of u is almost always satisfied. When χ is faithful and $u = v^{-1}$, then this just asserts that u does not belong to the center of G (though it is an easy exercise to give a direct proof of the theorem in this case).

We shall derive Theorem 1 from a general result (Theorem 2) about common constituents of permutation characters. This result is formulated in terms of characters of centralizer rings, and its proof is based upon the consideration of certain triple products of idempotents in centralizer rings.

We have taken this opportunity to give a proof of the Krein condition announced in [9], in order to make the additional observation that the coefficients c_{rst} which appear there also carry information on constituents of character products (Theorem 3). Again, this is in terms of characters of centralizer rings. As an application (Corollary 2) we prove a conjecture of Smith [10, p. 23]: If χ, ξ are the nonprincipal of a primitive rank 3 group G of even order, then the restrictions of χ, ξ to a point-stablilizer G_α have a nonprincipal irreducible constituent in common. Also, we obtain under the same hypothesis a further condition on the Higman parameters (Corollary 1), namely, the inequality $\xi(1) < \frac{1}{2}\chi(1)(\chi(1) + 1)$.

Since the appearance of [9], an excellent treatment of the Krein condition has been given by Higman in [5, 6]. We have wholly adopted his point of view here, and our Theorem 3 should be regarded as a footnote to Higman's theorem [5, Theorem (6.4)]. Finally it is worth noting for historical purposes that Higman's approach is based on a theorem of Schur [8], which is even much older than Krein's [7].

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Statement and Proof of Theorem 2

If G acts on a finite set Ω we let $O_1, \ldots, O_r$ denote the orbits of G on $\Omega \times \Omega$, and $A_1, \ldots, A_r$ denote the associated standard basis matrices for the (complex) centralizer ring $\mathbb{C}V = \mathbb{C}V(G, \Omega)$. We recall that A_i is a matrix of zeros and ones with a nonzero entry in the α, β position for exactly those pairs (α, β) which belong to the orbit O_i. If G is transitive, the subdegree n_i associated with O_i is the cardinality of $O_i(\alpha) = \{\beta \in \Omega \mid (\alpha, \beta) \in O_i\}$, where α is any element of Ω. The degree n is the cardinality of Ω.

THEOREM 2. *Suppose G is transitive on Ω with permutation character θ. Let χ be an irreducible constituent of θ, and ζ the associated irreducible character of $\mathbb{C}V$. Let L be a subgroup of G and Γ_1, Γ_2 orbits of L in Ω (not necessarily distinct) with permutation characters Π_1, Π_2 for L.*

Assume there exist orbits O_i and O_j for G on $\Omega \times \Omega$ such that

$$O_i \cap \Gamma_1 \times \Gamma_2 \neq \phi \neq O_j \cap \Gamma_1 \times \Gamma_2, \tag{1}$$

and

$$\zeta(A_i)/n_i \neq \zeta(A_j)/n_j. \tag{2}$$

Then the restriction $\chi|_L$ has a nonprincipal irreducible constituent in common with both Π_1 and Π_2.

Proof. Let e be the centrally primitive idempotent in $\mathbb{C}V$ corresponding to ζ. We have

$$e = \frac{\chi(1)}{n} \sum_k \frac{\zeta(A_k)^*}{n_k} A_k.$$

(Compare, for example, [1].) Now regard $\mathbb{C}V(G, \Omega) \subseteq \mathbb{C}V(L, \Omega)$ and let e_1, $e_2 \in \mathbb{C}V(L, \Omega)$ be the projections $\mathbb{C}\Omega \to \mathbb{C}\Gamma_1$, $\mathbb{C}\Omega \to \mathbb{C}\Gamma_2$. Then the space of L-invariants in $\mathbb{C}\Omega e_2 = \mathbb{C}\Gamma_2$ is precisely $\mathbb{C}\underline{\Gamma}_2$ where $\underline{\Gamma}_2 \in \mathbb{C}\Gamma_2$ denotes the sum of all elements of Γ_2. We shall show that

$$\mathbb{C}\Omega e_1 e e_2 \not\subseteq \mathbb{C}\underline{\Gamma}_2. \tag{*}$$

Thus the space $\mathbb{C}\Omega e_1 e e_2$ is L-stable and contains a nonprincipal constituent. On the other hand we have $\mathbb{C}\Omega e_1 e e_2 \subseteq \mathbb{C}\Omega e_2$; $\mathbb{C}\Omega e_1 e e_2$ is a homomorphic image of $\mathbb{C}\Omega e_1 = \mathbb{C}\Gamma_1$ via multiplication by $e e_2$, and $\mathbb{C}\Omega e_1 e e_2$ is contained in the homomorphic image $\mathbb{C}\Omega e e_2$ of $\mathbb{C}\Omega e$. Since the latter affords a multiple of $\chi|_L$, it follows that Π_1, Π_2, $\chi|_L$ have a common constituent $\neq 1_L$.

To prove (*) we recall our hypothesis that there exist orbits O_i, O_j of G on $\Omega \times \Omega$ such that 1 and 2 are satisfied. Let $(\alpha, \beta) \in O_i$. By a direct (and trivial) matrix computation, the α, β entry of $e_1 e e_2$ is $(\chi(1)/n)(\zeta(A_i)^*/n_i)$; similarly if

$(\alpha', \beta') \in O_j \cap \Gamma_1 \times \Gamma_2$ the (α', β') entry in $e_1 e e_2$ is $(\chi(1)/n)(\zeta(A_j)^*/n_j)$. Since L is transitive on Γ_1 we can choose $\alpha = \alpha'$. Now $\alpha e_1 e e_2 \in \mathbb{C}\Omega e_1 e e_2$ but $\alpha e_1 e e_2 \notin \mathbb{C}\underline{\Gamma}_2$ since $\zeta(A_i)/n_i \neq \zeta(A_j)/n_j$. This proves (*) and completes the proof of Theorem 2.

Remark. In case $\Gamma_1 = \Gamma_2 = \Gamma$ and $|\Gamma| > 1$ then $\Gamma \times \Gamma$ intersects the diagonal orbit $O_1 = \{(\alpha, \alpha) \mid \alpha \in \Omega\}$ as well as nondiagonal orbits $O_2, \ldots, O_k$. If L, together with G_α for $\alpha \in \Gamma$, generate G, then the Perron–Frobenius theory guarantees (cf. [4]) that $\sum_2^k n_i$ has multiplicity one as an eigenvalue for $\sum_2^k A_i$; also, each eigenvalue for a given A_i has absolute value at most n_i. Thus for χ, ζ as in Theorem 2 and $\chi \neq 1$, we have

$$\zeta(A_1)/n_1 = \zeta(1) \neq \zeta(A_i)/n_i$$

for some i with $2 \leqslant i \leqslant k$. Thus $\chi|_L$ must have a nontrivial irreducible in common with Π_1, according to Theorem 2. However, the same result can be obtained directly via the "Brauer trick" [2, p. 438; 11]. In this way Theorem 2 may be viewed as a refinement of Brauer's result.

Proof of Theorem 1. We apply Theorem 2 to the case where G plays the role of L as a subgroup of $G \times G$ (embedded diagonally) which in turn plays the role of G. The set Ω on which $G \times G$ acts will be the original group G itself, with action $g^{(x,y)} = x^{-1}gy$ for $x, y, g \in G$. The orbits Γ_1 and Γ_2 for L are the conjugacy classes to which u^{-1} and v belong.

The permutation character for $G \times G$ on G is $\sum \chi\chi^*$ where χ ranges over the irreducible characters of G. The centralizer ring $\mathbb{C}V(G \times G, G)$ is naturally isomorphic to the center $Z(\mathbb{C}G)$ of the group algebra $\mathbb{C}G$, with standard basis matrices corresponding to class sums. The correspondence is given as follows: if O is an orbit of $G \times G$ on $\Omega \times \Omega$ and $(u^{-1}, v) \in O$ then O corresponds to the class containing uv. The associated subdegree is just the cardinality of the class. Thinking of the centralizer ring as $Z(\mathbb{C}G)$, the character of the centralizer ring associated with a given character $\chi\chi^*$ of $G \times G$ is just the central character $\omega = \omega_\chi$ of $Z(\mathbb{C}G)$, defined on a class sum $\underline{K}$ by

$$\omega(\underline{K}) = |K| \chi(x)/\chi(1),$$

where $x \in K$. It is clear from this that the hypothesis of Theorem 1 is precisely what is needed to satisfy the hypothesis of Theorem 2.

It remains to interpret the conclusion of Theorem 2 in this case. The restriction $\chi\chi^*|_L$ is the product $\chi\chi^*$ on G. The characters Π_1, Π_2 are the permutation characters for the action of G on the classes to which u^{-1}, v belong; that is, they are the induced characters $1_{C_G(u^{-1})}|^G = 1_{C_G(u)}|^G$ and $1_{C_G(v)}|^G$. The conclusion of Theorem 1 follows immediately.

The Krein Condition

Suppose G is transitive on Ω, and regard $\Omega \subseteq \Omega \times \Omega$ via the embedding $\alpha \to (\alpha, \alpha)$. Then the pointwise product $A \circ B$ of matrices $A, B \in \mathbb{C}V$ (or any $|\Omega| \times |\Omega|$ matrices) can be written

$$\begin{bmatrix} A \circ B & 0 \\ 0 & 0 \end{bmatrix} = \Pi(A \dot{\times} B)\Pi,$$

where Π is the projection $\mathbb{C}(\Omega \times \Omega) \to \mathbb{C}\Omega \subseteq \mathbb{C}(\Omega \times \Omega)$ and $\dot{\times}$ denotes the usual Kronecker product. The results of this section are based entirely on this equation and the direct definition[1] of the pointwise product.

First, Schur's result, that if A, B are Hermitian positive semidefinite, then so is $A \circ B$, is an immediate consequence of the equation.

Next observe that the pointwise product of standard basis matrices is particularly easy to compute directly from the definition: We have $A_i \circ A_i = A_i$ and $A_i \circ A_j = 0$ for $i \neq j$. Thus, the formula

$$e_s = \frac{\chi_s(1)}{n} \sum_i \frac{\zeta_s(A_i)^*}{n_i} A_i$$

for the centrally primitive idempotent in $\mathbb{C}V$ associated with an irreducible constituent χ_s of the permutation character θ gives

$$e_r \circ e_s = \frac{\chi_r(1)}{n} \frac{\chi_s(1)}{n} \sum_i \frac{\zeta_r(A_i)^* \, \zeta_s(A_i)^*}{n_i^2} A_i .$$

Note that $e_s = e_s^*$, where $*$ denotes Hermitian transpose, since $e_s e_s^* \neq 0$. (The matrix A_i^* is the standard basis matrix A_{i*} associated with $O_{i*} = \{(\beta, \alpha) \mid (\alpha, \beta) \in O_i\}$. Also, the equation $e_s = e_s^*$ can be observed directly, since it is true that $\zeta_s(A_i)^* = \zeta_s(A_{i*})$ because of the unitary reduction of G on $\mathbb{C}\Omega$; alternately, our proof that $e_s = e_s^*$ may be taken as a proof of this equation.) Thus, $e_s = e_s^* = e_s e_s^*$ is Hermitian positive semidefinite, and so the previous paragraph applies to $e_r \circ e_s$.

Finally we observe that the $\mathbb{C}G$ module $\mathbb{C}\Omega(e_r \circ e_s)$ is contained in a homomorphic image of the $\mathbb{C}G$ module $\mathbb{C}\Omega e_r \otimes \mathbb{C}\Omega e_s$ (see the proof of Theorem 2) and the latter affords the character $\zeta_r(1)\,\zeta_s(1)\,\chi_r\chi_s$. In particular, if $(e_r \circ e_s)\, e_t \neq 0$ (or what amounts to the same thing here: if $\zeta_t(e_r \circ e_s) > 0$ then $(\chi_r\chi_s, \chi_t) > 0$.

We summarize our remarks in a theorem.

Theorem 3. *For irreducible characters ζ_r, ζ_s of $\mathbb{C}V(G, \Omega)$ put*

$$\sigma_{rs} = \sum_i \frac{\zeta_r(A_i)\, \zeta_s(A_i)}{n_i^2} A_i^*.$$

[1] Namely, $A \circ B = (a_{\alpha\beta} b_{\alpha\beta})$ for $A = (a_{\alpha\beta})$, $B = (b_{\alpha\beta})$.

Then:

(a) *The matrix* σ_{rs} *is Hermitian positive semidefinite.*[2]

(b) *We have*

$$\sum_i \frac{\zeta_r(A_i)\,\zeta_s(A_i)\,\zeta_t(A_i)^*}{n_i^2} = \zeta_t(\sigma_{rs}) \geqslant 0.$$

(c) *If* $\zeta_t(\sigma_{rs}) \neq 0$, *then* $(\chi_r\chi_s, \chi_t) \neq 0$.

The proof follows from our preceding discussion. We remark that (b) is a version of the Krein condition as stated in [9], since $\zeta_t(\sigma_{rs})$ can be shown to be a positive multiple of the coefficient c_{rst} in the multiplicity-free case.

PROPOSITION 1. *If* $r = s$ *and* $\zeta_r(1) = 1$, *then the expression* $\chi_r\chi_s$ *in* (c) *of Theorem* 3 *may be replaced by* $\mathrm{Sym}^2\,\chi_r$, *the character of the symmetric part of the tensor product of a representation affording* χ_r *with itself.*

Proof. Viewing $\mathbb{C}(\Omega \times \Omega) = \mathbb{C}\Omega \otimes \mathbb{C}\Omega$ we have $\mathbb{C}(\Omega \times \Omega)\,\Pi(e_r \dot{\times} e_r) = \{\sum_{\alpha\in\Omega} c_\alpha(\alpha e_r \otimes \alpha e_r) \mid c_\alpha \in \mathbb{C} \text{ for } \alpha \in \Omega\}$, which is contained entirely in the symmetric part of $\mathbb{C}\Omega e_r \otimes \mathbb{C}\Omega e_r$. Hence $\mathbb{C}\Omega(e_r \circ e_r)\,e_t$ is a homomorphic image of a submodule of the symmetric part of the tensor product of a representation affording χ_r with itself.

G. E. Keller has observed in conversations with the author that (c) in Theorem 3 may be extended from products $\chi_r\chi_s$ to arbitrary products $\chi_{r_1}\chi_{r_2}\cdots\chi_{r_k}$. The analog of the above proposition then gives a sufficient condition for χ_r to have a symmetric invariant of degree k, if $\zeta_r(1) = 1$. Also we note that N. Biggs has made a similar observation with respect to (b) [6, (4.2)].

We now give some applications of these results to rank 3 groups. Corollary 2 had been conjectured by Margaret Smith and proved by her in the case when the two transitive constituents $\neq \alpha$ of a point-stabilizer G_α afford multiplicity-free characters [10].

[2] In [5] Higman notes an estimate for the spectral radius of a quantity like σ_{rs} (but defined in terms of idempotents which are primitive in all $\mathbb{C}V$). The estimate is made from properties of the pointwise product; however, it turns out that a direct estimate of the spectral radius of σ_{rs} (summing absolute values in a column) is better. The bound is $n(\zeta_r(1)\,\zeta_s(1)/\chi_r(1)\,\chi_s(1))^{1/2}$. (The corresponding quantity for [5, (6.4)] is $1/n(\chi_r(1)\,\chi_s(1))^{1/2}$, as opposed to $1/\chi_r(1)\,\chi_s(1)$.) Such an estimate is relevant if we are interested in obtaining a lower bound on $(\chi_r\chi_s, \chi_t)$ from $\zeta_t(\sigma_{rs})$. If the actual representations of $\mathbb{C}V$ are known, then I suspect the best lower bounds would be obtained by following Higman's more detailed analysis, and computing the *ranks* of matrices representing pointwise products of primitive idempotents in an irreducible representation of $\mathbb{C}V$ affording ζ_t. Finally it is worth pointing out in this connection that results analogous to Proposition 1 can be formulated even if $\zeta_r(1) \neq 1$ if one is able to deal with actual primitive idempotents.

COROLLARY 1. *Suppose G is a primitive rank 3 group of even order with permutation character $1 + \chi + \xi$. Then ξ is a constituent of* $\mathrm{Sym}^2 \chi$. *In particular* $\xi(1) < \chi(1)(\chi(1) + 1)/2$.

Remark. The inequality is best possible. The McLaughlin group has $\chi(1) = 22$ and $\xi(1) = 252$ in its rank 3 representation of degree 275. Still, it would be interesting to have more detailed information on the case when $\xi(1) \geqslant \frac{1}{2}\chi(1)(\chi(1) - 1)$; if this occurs, then $\chi \not\subseteq \mathrm{Sym}^2 \chi$, and we have $\zeta_2(\sigma_{22}) = 0$ for $\chi = \chi_2$ (see the proof below).

Proof of Corollary 1. Since G is a primitive rank 3 group of even order we have $A_i = A_i^*$, $\chi_r = \chi_r^*$, $\zeta_r = \zeta_r^*$, and $\zeta_r(A_i) \neq 0$ for each i and r; also $\zeta_r(A_i) \neq n_i$ for $r \neq 1$, $i \neq 1$ [3].

If the result is false, then by the proposition above we have $\zeta_3(\sigma_{22}) = 0$, choosing notation $\xi = \chi_3$, $\chi = \chi_2$. Thus,

$$\zeta_2(\sigma_{23}) = \sum \frac{\zeta_2(A_i)\, \zeta_3(A_i)\, \zeta_2(A_i)}{n_i^2} = \zeta_3(\sigma_{22}) = 0.$$

Of course $\zeta_1(\sigma_{23}) = 0$ by a standard orthogonality relation for centralizer rings (cf., for example, [5]); this can also be seen from Theorem 3 and the fact $(\chi\xi, 1) = 0$.

Since the three idempotents e_1, e_2, e_3 span $\mathbb{C}V$ and $\zeta_1(\sigma_{23}) = \zeta_2(\sigma_{23}) = 0$ we must have $\sigma_{23} = ce_3$ for some $c \in \mathbb{C}$. But

$$\sigma_{23} = 1 + \frac{\zeta_2(A_2)\, \zeta_3(A_2)}{n_2^2} A_2 + \frac{\zeta_2(A_3)\, \zeta_3(A_3)}{n_3^2} A_3,$$

and

$$e_3 = \frac{\chi_3(1)}{n} \left(1 + \frac{\zeta_3(A_2)}{n_2} A_2 + \frac{\zeta_3(A_3)}{n_3}\right).$$

Comparing coefficients of $1 = A_1$ we determine that the above two expressions must be identical if $\chi_3(1)/n$ is removed from the bottom one. In particular we have

$$\frac{\zeta_2(A_2)\, \zeta_3(A_2)}{n_2^2} = \frac{\zeta_3(A_2)}{n_2},$$

so either $\zeta_3(A_2) = 0$ or $\zeta_2(A_2) = n_2$. Both of these possibilities are prohibited, and the corollary is proved. (For the degree inequality, note that 1 is a constituent of $\mathrm{Sym}^2 \chi$ by the Frobenius–Schur theory.)

COROLLARY 2. *Suppose G is a primitive rank 3 group of even order with permutation character $\theta = 1 + \chi + \xi$. Then the restrictions $\chi|_{G_\alpha}$ and $\xi|_{G_\alpha}$ of χ and ξ to a point-stabilizer G_α have a nonprincipal irreducible constituent in common.*

Proof. Set $H = G_\alpha$. Since $(\chi, 1)_H = 1 = (\xi, 1)_H$ by Frobenius reciprocity, it suffices to show $(\chi, \xi)_H > 1$. But $(\chi, \xi)_H = (\chi\xi, 1)_H$ since $\xi^* = \xi$, and $(\chi\xi, 1)_H = (\chi\xi, \theta) \geqslant (\chi\xi, \chi) + (\chi\xi, \xi) = (\chi^2, \xi) + (\xi^2, \chi) \geqslant 2$ by Corollary 1. Q.E.D.

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