Bott Periodicity via Simplicial Spaces

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TO NATHAN JACOBSON ON HIS 70TH BIRTHDAY

1. INTRODUCTION

We give a proof of complex Bott periodicity that uses some of the general ideas of Segal and Quillen on classifying spaces [1, 2, 4]. We attempt to understand which essential properties of the complex numbers and the unitary group are responsible for the periodicity, as contrasted with general facts on classifying spaces and linear groups over general topological rings (i.e., we compare complex K-theory and general algebraic K-theory). From this point of view the only essential properties of the complex numbers are: the "spectral theorem" ($n \times n$ unitary matrices can be diagonalized) and the fact that the Stiefel manifold of $k$-frames in $\mathbb{C}^N$ is highly connected for large $N$.

2. The periodicity theorem to be proved is that $AU = \mathbb{Z} \times BU$ or equivalently $B(\mathbb{Z} \times BU) = U$ for a suitable monoid structure on $\mathbb{Z} \times BU$. Since this last monoid structure is not easy to describe directly we replace $\mathbb{Z} \times BU$, the space representing $K(X)$, by the space representing the monoid Vect($X$) of vector bundles and Whitney sum, namely, the disjoint union $\bigsqcup_{n \geq 0} BU_n (BU_0 = \text{point})$. The maps $BU_m \times BU_n \to BU_{m+n}$ induced by $U_m \times U_n \to U_{m+n}$ give $\bigsqcup_n BU_n$ its monoid structure. The Group Completion Theorem [1; 4, Section 7, 6] implies that the loops on the classifying space of the monoid $\bigsqcup_n BU_n$ are $\mathbb{Z} \times BU$, i.e., $B(\bigsqcup_n BU_n) = B(\mathbb{Z} \times BU)$: this is a general theorem valid for linear groups over arbitrary topological rings. Periodicity is then the statement

$$B\left(\bigsqcup_n BU_n\right) = U.$$  

We shall first prove that $\bigsqcup_n BU_n$ can be replaced by the disjoint union of Grassmann manifolds of $n$-planes in $\mathbb{R}_0$ dimensional space: $Gr = \bigsqcup_n Gr_n$; further in $Gr$ we will have a partially defined multiplication and consequently
a classifying space $B\ Gr$ (made up out of flag manifolds) which we will show to be homotopy equivalent to $B(\coprod_n BU_n)$; finally $B\ Gr$ will be shown to be homeomorphic to $U$. Let $C^\infty$ denote a vector space of countable infinite dimension over $\mathbb{C}$ and with hermitian inner product. $U$ will denote the unitary transformations of $C^\infty$ which are the identity on a subspace of finite codimension. $Gr$ is the set of hermitian projections $E$ of finite rank $\geq 0$: $E = E^* = E^2$. $B\ Gr$ is defined as the geometric realization $|X.|$ of the following simplicial space

$X_n = \{[E_1, \ldots, E_n] \in (Gr)^n \mid E_i E_j = 0 \text{ for } i \neq j\}$.

Thus $X_n$ is a flag manifold.

Face maps $|X.| \to |X.|$ are given by

$[E_1, \ldots, E_n] \mapsto [E_1, \ldots, E_i + E_{i+1}, E_{i+2}, \ldots, E_n]$ (1 $\leq i < n$)

or by omitting $E_1$ or $E_n$ (same as in the bar construction). Degeneracies consist of inserting the zero projection. We will show $|X.| \cong B(\coprod_n BU_n)$; granting this for the moment the main step is:

**Theorem** $|X.|$ is homeomorphic to $U$.

**Proof.** $|X.| = \coprod_n X_n \times \Delta^n$ modulo an equivalence relation. $\Delta^n$ is the standard $n$-simplex with points $t = (t_1, \ldots, t_n)$ where $1 \geq t_1 \geq t_2 \geq \cdots \geq t_n \geq 0$. Map $X_n \times \Delta^n$ to $U$ by

$([E_1, \ldots, E_n], (t_1, \ldots, t_n)) \mapsto e^{2\pi i (t_1 E_1 + \cdots + t_n E_n)}$.

The equivalence relation is preserved and the map is a homeomorphism by the spectral theorem.

3. Let $Y$ be the simplicial space associated to the monoid $\coprod_n BU_n$, i.e., $Y_0 =$ point, $Y_n = (\coprod_n BU_n)^\times$ so that $|Y.| = B(\coprod_n BU_n)$. We want to show $|X.|$ is homotopy equivalent to $|Y.|$; along the way we will prove $BU_n$ is homotopy equivalent to $Gr(n) = \{E \mid \text{tr}E = n\}$. To do this we construct an intermediate simplicial space $Z$, and maps

$X \leftarrow^r Z \rightarrow^d Y$.

such that $|Z.| \to |X.|$ and $|Z.| \to |Y.|$ are homotopy equivalences. The following general criterion will be used (see appendixes in [1, 2, 5]: a map of simplicial spaces $Z \to X$ (satisfying a very mild condition on each space) such that for each $n$, $Z_n \to X_n$ is a homotopy equivalence, induces a homotopy equivalence $|Z.| \to |X.|$. To construct $Y$, $Z$, $X$, we begin with certain categories $C$, $F$, $G$ each of which will have a "composition": if $C$ denotes the
category then we will have in each case a subcategory $\mathcal{C}_n$ of the product category $\mathcal{C}^n$ and functors $\mathcal{C}_n \rightarrow \mathcal{C}_m$ defining in fact a simplicial category $\mathcal{C}$. Then the sequence of classifying spaces $\{ BC_n ; n = 0, 1, \ldots \}$ will be the simplicial space $Y$, or $X$, or $Z$. The first category, $C$, has objects the integers $n \geq 0$; the morphisms go from $n$ to $n$ and consist of the unitary automorphisms of $\mathbb{C}^n$. $C_n$ is just the cartesian product $\mathbb{C}^n (= \text{point for } n = 0)$ and $Y = \{ Y_n \}$ is defined as $Y_0 = \text{point}$, $Y_1 = BC = BC_1 = \coprod_k BU_k$, $Y_n = (\coprod_k BU_k)^n$ so that $| Y_n | = B(\coprod_k BU_k)$. The multiplication on $\coprod_k BU_k$ defines $Y_n \rightarrow Y_m$ for $m < n$ in the usual way.

The category $F$ has as objects $n$-frames in $\mathbb{C}^\infty$, i.e., injective isometric maps $\theta: \mathbb{C}^n \rightarrow \mathbb{C}^\infty$ ($n = 0, 2, \ldots$) a morphism $\alpha: \theta \rightarrow \theta'$ is defined if and only if $\theta, \theta'$ are both $n$-frames with the same image in $\mathbb{C}^\infty$ and consists of the unique isometry $\alpha$ of $\mathbb{C}^n$ such that $\theta' \circ \alpha = \theta$. The category $F_n$ is the subcategory of $F^n$ with objects the $n$-tuples $(\theta_1, \ldots, \theta_n)$ of objects of $F$ such that the images of $\theta_i$ and $\theta_j$ are mutually orthogonal for $i \neq j$; functors $F_n \rightarrow F_{n-1}$ are obtained by adding $\theta$, and $\theta_{i+1}$ ($i = 1, \ldots, n - 1$) or omitting $\theta_1$ or $\theta_n$ (as in the bar construction). Degeneracies $F_n \rightarrow F_{n+1}$ are given by inserting the frame $\theta = 0$. A functor $d: F \rightarrow C$ is given by $d(\theta) = n$ if $\theta$ has domain $\mathbb{C}^n$, and $d(\alpha) = \alpha$ (as a morphism) and similarly for $F_n \rightarrow C_n$. Thus $F, C$ differ only in their objects: for each $n$, $C$ has one object whereas $F$ has as objects the Stiefel manifold $V_{n, \infty}$ of all $n$-frames in $\mathbb{C}^\infty$, which is a contractible space. The classifying space $BF$ is constructed from the simplicial space whose $q$-simplices are the sequences $(\theta, \alpha_1, \ldots, \alpha_q)$ where $\theta: \mathbb{C}^n \rightarrow \mathbb{C}^\infty$ and $\alpha_i \in U_n$; the corresponding simplex for $BC$ (under the map $d$) is $(\alpha_1, \ldots, \alpha_q)$; thus the spaces of $q$-simplices for $BF, BC$ are: disjoint union over all $n$ of $V_{n, \infty} \times (U_n)^q$, respectively $(U_n)^q$, and the map given by $d$ is projection on the second factor, thus a homotopy equivalence for each $q$. It follows that $BF = Z_1 \rightarrow BC = X_1$ is a homotopy equivalence. In exactly the same way, $BF_q = Z_q \rightarrow BC_q = X_q$ is a homotopy equivalence so that finally $| Z | \rightarrow | X | = B(\coprod_n BU_n)$ is a homotopy equivalence.

The category $G$ with $BG = Gr$ has objects the elements of $Gr$ and morphisms only the identity morphisms (one for each object), so that $BG$ = space of objects $= Gr$. The category $G_n$ = point, $G_1 = G$; $G_n$ has objects the $n$-tuples $(E_1, \ldots, E_n)$ of objects of $G$ which are mutually orthogonal and morphisms only the identities. $BG_n = X_n$; the face and degeneracy functors $G_n \rightarrow G_n$ were previously described.

The functor $r: F \rightarrow G$ takes the object $\theta$ to the projection on the image of $\theta$; and similarly for functors $F_n \rightarrow G_n$. For each point $E$ of $BG_1 = BG$ the inverse image of $E$ under the map $B(r): BF \rightarrow BG$ consists of the classifying space of the subcategory of $F$ of all frames $\theta$ with $r(\theta) = E$ as objects, and all "changes of frames" $\alpha$ as morphisms; this category of "frames for $E$" has a unique morphism between each two objects and so its classifying space is contractible (it may be identified with the total space $E_{U_n}$ of the universal principal $U_n$ bundle, where $n = \text{tr } E$). For each $n$, if $BF(n)$ and $BG(n)$ denote the components corresponding to projections of rank $n$, then $B(r): BF(n) \rightarrow BG(n)$ is a locally
trivial fibration with contractible fiber; thus \( BF \rightarrow BG \) is a homotopy equivalence. In exactly the same way \( BF_q \rightarrow BG_q \) is a homotopy equivalence (\( F_q \), \( G_q \) are categories of \( q \)-tuples), and finally \(| r | : | Z, | \rightarrow | X, | \) is a homotopy equivalence. This concludes the proof that \( B(\coprod_n BU_n) \) is homotopy equivalent to \( U \).

4. We indicate briefly how the Group Completion Theorem can be avoided, or rather replaced by the simpler Proposition 1.4 of [1] which states, in the context of interest to us, that if \( \mathcal{C} \) is a simplicial category of the type considered above such that \( B\mathcal{C}_2 \rightarrow \mathcal{C}_1 \times B\mathcal{C}_1 \) is a homotopy equivalence (\( p_1 \), \( p_2 : B\mathcal{C}_2 \rightarrow B\mathcal{C}_1 \) being the “projections” induced by the face maps \([0, 1] \rightarrow [0, 1, 2] \) taking 0, 1 to 0, 1 respectively, to 1, 2) and if \( m : B\mathcal{C}_2 \rightarrow B\mathcal{C}_1 \) denotes the “composition” (induced by the face map \([0, 1] \rightarrow [0, 1, 2] \) taking 0 to 0, 1 to 2) and if the map

\[
B\mathcal{C}_1 \times B\mathcal{C}_1 \xrightarrow{(p_1 \times p_2)^{-1}} B\mathcal{C}_2 \xrightarrow{m} B\mathcal{C}_1
\]

makes \( B\mathcal{C}_2 \) an \( H \)-space with homotopy inverse, then \( B\mathcal{C}_1 \) is the loop space on the geometric realization \( B\mathcal{C}_1 \). In this way of proceeding we also work directly with \( \mathbb{Z} \times BU \) (rather than \( \prod_n BU_n \)) and construct a simplicial space whose realization is \( B(\mathbb{Z} \times BU) \) and is homeomorphic to \( U \times U/U \) (that is, the quotient of \( U \times U \) by the relation \((u_1, u_2) \sim (u_1u, u_2u) \) where \( u_1 - I, u_2 - I, \) and \( u - I \) have mutually orthogonal supports). To construct \( \mathbb{Z} \times BU \) we start with the category \( PG \) whose objects are arbitrary pairs \((E, E')\) of projections on \( \mathbb{C}^\infty \) and with unique morphism from \((E_1, E'_1)\) to \((E_2, E'_2)\) given by a projection \( F \) orthogonal to both \( E_1, E'_1 \) such that \( E_1 + F = E_2, E'_1 + F = E'_2 \). We could similarly define categories of pairs \( PF, PC \) to show that we obtain the same classifying space for \( PG \) and \( PC \) where \( PC \) has objects pairs \((m, m')\) and morphisms \((m, m')\) to \((n, n')\) pairs of injective isometries \( \alpha : \mathbb{C}^m \rightarrow \mathbb{C}^n, \alpha' : \mathbb{C}^{m'} \rightarrow \mathbb{C}^{n'} \) together with isometry \( \beta \) from the orthogonal complement of \( 1m \alpha \) to that of \( 1m \alpha' \) (a category used by Quillen for general rings). Instead we will just remark that the classifying space of \( PG \) at least intuitively classifies pairs of vector bundles modulo addition of a common bundle: i.e., we claim \( BPG = \mathbb{Z} \times BU \).

Further categories \( PG_n \) have objects \( n \)-tuples of pairs \([(E_1, E'_1), \ldots, (E_n, E'_n)]\) where \( E_1, \ldots, E_n \) are orthogonal, and \( E'_1, \ldots, E'_n \) are orthogonal, and morphisms \( n \)-tuples \((F_1, \ldots, F_n)\) \((F_i \) orthogonal to \( E_i, E'_i \) and \( E_i + F_i \) orthogonal for \( i = 1, \ldots, n \) and similarly for \( E'_i + F_i \)).

We can then describe a homeomorphism \( |BPG, | \) to \( U \times U/U, \) only the notation being cumbersome. The functor \( G \rightarrow PG, E \mapsto (E, 0) \) on objects, then corresponds to \( U \rightarrow U \times U/U, u \mapsto (u, 1) \).

5. Finally we note that if we replace the complex numbers throughout by reals, then we have a proof that \( B(\coprod_n BO_n) = U/O \). However the complete real periodicity requires something more.
REFERENCES