On the existence and nonexistence of positive solutions for nonlinear Sturm–Liouville boundary value problems

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Abstract

In this paper the existence and nonexistence results of positive solutions are obtained for Sturm–Liouville boundary value problem

\[-(p(x)u')' + q(x)u = f(x, u), \quad x \in (0, 1),
\]

\[au(0) - bp(0)u'(0) = 0, \quad cu(1) + dp(1)u'(1) = 0,\]

where \(p \in C^1[0, 1], q \in C[0, 1], p(x) > 0, q(x) \geq 0\) for \(x \in [0, 1], f \in C([0, 1] \times \mathbb{R}^+), a, b, c, d \geq 0\) are constants and satisfy \((a + b)(c + d) > 0\). The discussion is based on the positivity estimation for the Green’s function of associated linear boundary value problem and the fixed point index theory in cones.

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1. Introduction and main results

In this paper we deal with the existence of positive solution for Sturm–Liouville boundary value problem (BVP)

\[-(p(x)u')' + q(x)u = f(x, u), \quad x \in (0, 1), \tag{1}\]

\[au(0) - bp(0)u'(0) = 0, \quad cu(1) + dp(1)u'(1) = 0, \tag{2}\]

where \(p, q, a, b, c, d\) and \(f\) satisfy the following conditions:

(H1) \(p \in C^1(I), q \in C(I), I = [0, 1]\) and \(p(x) > 0, q(x) \geq 0\) for \(x \in I\).

(H2) \(a, b, c, d \geq 0\) and \((a + b)(c + d) > 0\).

(H3) \(f \in C(I \times \mathbb{R}^+), \mathbb{R}^+ = [0, +\infty)\).

The BVP (1)–(2) arises in many different areas of applied mathematics and physics, and only its positive solution is significant in some practice. For the special case as follows with \(p(x) \equiv 1\) and \(q(x) \equiv 0\),

\[-u''(x) = f(x, u(x)), \quad 0 < x < 1, \tag{3}\]

\[u(0) = 0, \quad u(1) = 0, \tag{4}\]

the existence of positive solution of this problem has been studied by many authors in the assumption of \(f \geq 0\), see [1–5]. To be convenient, we introduce the notations

\[f_0 = \lim \inf_{v \to 0^+} \min_{x \in I} \left(\frac{f(x, v)}{v}\right), \quad \tilde{f}_0 = \lim \sup_{v \to 0^+} \max_{x \in I} \left(\frac{f(x, v)}{v}\right), \tag{5}\]

\[f_\infty = \lim \inf_{v \to +\infty} \min_{x \in I} \left(\frac{f(x, v)}{v}\right), \quad \tilde{f}_\infty = \lim \sup_{v \to +\infty} \max_{x \in I} \left(\frac{f(x, v)}{v}\right). \tag{6}\]

One well-known result is that BVP (3)–(4) has at least one positive solution if \(f\) satisfies the following condition (P1)\(^\circ\) or (P2)\(^\circ\):

(P1)\(^\circ\) \(f_0 < 0, \quad f_\infty = +\infty\) (superlinear case),

(P2)\(^\circ\) \(f_0 = +\infty, \quad f_\infty = 0\) (sublinear case).

This result can be proved by employing the well-known Krasnoselskii’s fixed-point theorem of cone mapping, see [1–4].

Recently, the present author [6] has omitted the assumption that \(f \geq 0\), and has improved the conditions (P1)\(^\circ\) and (P2)\(^\circ\) to (P1) and (P2), respectively:

(P1) \(-\infty < f_0, \quad \tilde{f}_0 < \pi^2 < f_\infty\),

(P2) \(-\infty < f_\infty, \quad \tilde{f}_\infty < \pi^2 < f_0\).

The present author has proved that if (P1) or (P2) is satisfied, then BVP (3)–(4) has at least one positive solution. In this case, the Krasnoselskii’s fixed-point theorem of cone mapping cannot be applied, and the argument is based upon the counting of the fixed point index theory in cones. Noting that \(\pi^2\) is the first eigenvalue of the associated linear eigenvalue
problem, the conditions (P1) and (P2) cannot be improved again, otherwise the existence of solution to BVP (3)–(4) cannot be guaranteed.

The purpose of this paper is to extend the results on BVP (3)–(4) in [6] to more general BVP (1)–(2), and to obtain nonexistence results of positive solution for BVP (1)–(2). Let \( \lambda_1 \) denote the first eigenvalue of the linear eigenvalue problem

\[-\left(p(x)u'ight)' + q(x)u = \lambda u, \quad x \in (0, 1),\]

with the boundary condition (2). It is well known that \( \lambda_1 \geq 0 \) is simple eigenvalue and it has a positive eigenfunction \( e_1(x) \) such that \( e_1(x) > 0 \) for \( x \in (0, 1) \) and \( \|e_1\| = \max_{x \in I} |e_1(x)| = 1 \). \( \lambda_1 \) can be given by

\[
\lambda_1 = \inf_{u \in D(L), \|u\|_2 \neq 0} \frac{(Lu, u)}{\|u\|_2^2},
\]

where \((\cdot, \cdot)\) and \(\|\cdot\|_2\) denote the inner product and norm of the Hilbert space \( H = L^2(I) \), respectively, and \( L : D(L) \rightarrow H \) is the linear Sturm–Liouville operator defined by

\[
D(L) = \{ u \in H^2(I): u \text{ satisfies boundary conditions (2)} \},
\]

\[
Lu = -\left(p(x)u'ight)' + q(x)u.
\]

It is clear that \( L : D(L) \rightarrow H \) is a positive semi-definite operator.

The main results of this paper are as follows.

**Theorem 1.** Suppose (H1)–(H3) hold. If \( f \) satisfies one of the following conditions:

(F1) \( -\infty < f_0 < \lambda_1 < f_\infty \),

(F2) \( -\infty < f_\infty < \lambda_1 < f_0 \),

then the BVP (1)–(2) has at least one positive solution.

If we add an assumption that \( q(x) \neq 0 \) if \( a = c = 0 \) to (H1) and (H2), then \( \lambda_1 > 0 \). In fact, since \( Le_1 = \lambda_1 e_1 \), making inner product for the equation with \( e_1 \), we obtain that

\[
\lambda_1 = \frac{\int_0^1 p(x)e_1^2(x) \, dx + \int_0^1 q(x)e_1^2(x) \, dx}{\int_0^1 e_1^2(x) \, dx} > 0.
\]

Therefore from Theorem 1 we immediately obtain the following

**Corollary 1.** Suppose (H1)–(H3) hold, moreover \( q(x) \neq 0 \) if \( a = c = 0 \). Let \( f \geq 0 \) and satisfy the condition (P1) or (P2). Then the BVP (1)–(2) has at least one positive solution.

Applying Theorem 1 to the equation

\[-\left(p(x)u'ight)' + q(x)u = f(u(x)), \quad x \in (0, 1),\]

we have
Corollary 2. Assume (H1)–(H2) hold. Let $f \in C^1(\mathbb{R}^+)$ and $f(0) = 0$. Then in each case of the following:

\[(F1)^* \quad f'(0) < \lambda_1, \quad f'(+\infty) > \lambda_1, \]
\[(F2)^* \quad f'(0) > \lambda_1, \quad f'(+\infty) < \lambda_1, \]

where $f'(+\infty) = \lim_{v \to +\infty} (f(v)/v)$, the BVP (5)–(2) has at least one positive solution.

Particularly, we obtain the following nonexistence result of positive solutions for BVP (1)–(2).

Theorem 2. Suppose (H1)–(H3) hold. Then in each case of the following:

\[(F3) \quad \inf_{v>0, x \in I} (f(x,v)/v) > \lambda_1, \]
\[(F4) \quad \sup_{v>0, x \in I} (f(x,v)/v) < \lambda_1, \]

the BVP (1)–(2) has no positive solution.

From Theorem 2 we can obtain the following

Corollary 3. Assume (H1)–(H2) hold. Let $f \in C^1(\mathbb{R}^+)$ and $f(0) = 0$. Then in each case of the following:

\[(F3)^* \quad f'(v) < \lambda_1, \quad \forall v \geq 0, \]
\[(F4)^* \quad f'(v) > \lambda_1, \quad \forall v \geq 0, \]

the BVP (5)–(2) has no positive solution.

Combining Theorems 1 and 2, we obtain an interesting conclusion: if the value of $f(x,v)/v$ crosses the first eigenvalue $\lambda_1$ from its one side to the other as $v$ is from 0 to $+\infty$, BVP (1)–(2) has a positive solution; and if the value of $f(x,v)/v$ stays at one side of $\lambda_1$ as $v$ is from 0 to $+\infty$, BVP (1)–(2) has no positive solution.

2. Preliminaries

If (F1) or (F2) is satisfied, it is easy to prove that $f(x,v)/v$ is lower-bounded for $x \in I$ and $v > 0$. Thus there exists $M > 0$ such that

$$f(x,v) \geq -Mv, \quad \forall x \in I, \quad v \geq 0.$$ 

Let $f_1(x,v) = f(x,v) +Mvc, \quad then \quad f_1(x,v) \geq 0$ for $x \in I, \quad v \geq 0, \quad and \quad Eq. (1)$ is equivalent to

$$L_1u := -\left(p(x)u'\right)' + (q(x) + M)u = f_1(x,u), \quad x \in (0,1). \quad (6)$$

We shall consider the existence of positive solutions of BVP (6)–(2).
Given \( h \in C(I) \), we consider the linear boundary value problem (LBVP) corresponding to Eq. (6),

\[
-\left( p(x)u' \right)' + (q(x) + M)u = h(x), \quad x \in (0, 1),
\]

with the boundary condition (2). We first structure the Green’s function of (LBVP) (7)–(2). To do this, we introduce the boundary operators

\[
B_1(u) := au(0) - bp(0)u'(0), \\
B_2(u) := cu(1) + dp(1)u'(1),
\]

where \( u \in C^1(I) \).

Let \( \varphi(x) \in C^2(I) \) be the unique solution of the linear boundary value problem

\[
-\left( p(x)\varphi'(x) \right)' + (q(x) + M)\varphi(x) = 0, \quad x \in I, \\
B_1(\varphi) = 0, \quad B_2(\varphi) = 1,
\]

and \( \psi(x) \in C^2(I) \) be the unique solution of the linear boundary value problem

\[
-\left( p(x)\psi'(x) \right)' + (q(x) + M)\psi(x) = 0, \quad x \in I, \\
B_1(\psi) = 1, \quad B_2(\psi) = 0.
\]

Then by maximum principle of elliptic operator, \( \varphi, \psi \geq 0 \), moreover \( \varphi(x), \psi(x) > 0 \) for \( x \in (0, 1) \). Furthermore, we have

**Lemma 1.** \( \varphi'(x) > 0, \forall x \in (0, 1], \psi'(x) < 0, \forall x \in [0, 1) \), and

\[
p(x)(\varphi'(x)\psi(x) - \varphi(x)\psi'(x)) \equiv \rho > 0, \quad x \in I,
\]

where \( \rho \) is a positive constant.

**Proof.** We first prove \( \varphi'(0) \geq 0 \). If \( b = 0 \), the boundary condition \( B_1(\varphi) = 0 \) implies \( \varphi(0) = 0 \), and therefore \( \varphi'(0) = \lim_{x \to 0+} \varphi(x)/x \geq 0 \). If \( b > 0 \), from the boundary condition \( B_1(\varphi) = 0 \) it follows that \( \varphi'(0) = a\varphi(0)/b \geq 0 \). Now, from (8) we obtain that

\[
\varphi'(x) = \frac{1}{p(x)} \left( p(0)\varphi'(0) + \int_0^x (q(y) + M)\varphi(y) \, dy \right), \quad x \in I,
\]

which implies that \( \varphi'(x) > 0, \forall x \in (0, 1] \).

In a similar way, we can prove that \( \psi'(1) \leq 0 \). From (9) we get that

\[
\psi'(x) = \frac{1}{p(x)} \left( p(1)\psi'(1) - \int_x^1 (q(y) + M)\psi(y) \, dy \right), \quad x \in I.
\]

Therefore, \( \psi'(x) < 0, \forall x \in [0, 1) \).

By (8) and (9), it is easy to verify that

\[
\left( p(x)(\varphi'(x)\psi(x) - \varphi(x)\psi'(x)) \right)' \equiv 0, \quad x \in I,
\]

from which it follows that
Thus (10) holds. The proof is completed.  

We now define the function \( G : I \times I \rightarrow \mathbb{R}^+ \) by

\[
G(x, y) = \begin{cases} 
\frac{1}{\rho} \varphi(x) \psi(y), & 0 \leq x \leq y \leq 1, \\
\frac{1}{\rho} \varphi(y) \psi(x), & 0 \leq y \leq x \leq 1.
\end{cases}
\]

Then \( G \in C(I \times I) \). We show that \( G(x, y) \) is the Green’s function of the LBVP (7)–(2), namely

**Lemma 2.** Let \( h \in C(I) \). The LBVP (7)–(2) has a unique solution \( u(x) \) which is given by

\[
u(x) = \int_0^1 G(x, y) h(y) \, dy, \quad x \in I.
\]

**Proof.** We directly verify that \( u(x) \) defined by (12) is a solution of LBVP (7)–(2). From the definition of \( G(x, y) \), we have that

\[
u(x) = \frac{1}{\rho} \int_0^x \varphi(y) \psi(x) h(y) \, dy + \frac{1}{\rho} \int_x^1 \varphi(y) \psi(y) h(y) \, dy.
\]

Making derivation, we get that

\[
 p(x) u'(x) = \frac{1}{\rho} \int_0^x \left( p(x) \psi'(x) \right) \psi(y) h(y) \, dy + \frac{1}{\rho} \int_x^1 \left( p(x) \varphi'(x) \right) \psi(y) h(y) \, dy.
\]

Making derivation to Eq. (14) and then using (10), (8) and (9), we have that

\[
 \left( p(x) u'(x) \right)' = \frac{1}{\rho} p(x) \left( \varphi(x) \psi'(x) - \varphi'(x) \psi(x) \right) h(x)
 + \frac{1}{\rho} \int_0^x \left( p(x) \psi'(x) \right) \psi(y) h(y) \, dy
 + \frac{1}{\rho} \int_x^1 \left( p(x) \varphi'(x) \right) \psi(y) h(y) \, dy
 = -h(x) + (q(x) + M) u(x).
\]

Therefore, \( u(x) \) satisfies Eq. (7). From (13) and (14) we get that
\[ B_1(u) = B_1(\psi) \cdot \frac{1}{\rho} \int_0^1 \psi(y)h(y) \, dy = 0, \]
\[ B_2(u) = B_2(\psi) \cdot \frac{1}{\rho} \int_0^1 \psi(y)h(y) \, dy = 0. \]

Thus \( u(x) \) is a solution of LBVP (7)–(2). By maximum principle, LBVP (7)–(2) has only one solution. The proof is completed.

**Lemma 3.** The Green’s function \( G(x, y) \) has the following properties:

(i) \( G(x, y) = G(y, x) \), \( \forall x, y \in I \).
(ii) \( G(x, y) > 0, \forall x, y \in (0, 1) \).
(iii) \( G(x, y) \leq G(y, y), \forall x, y \in I \).
(iv) \( G(x, y) \geq \delta G(x, x) G(y, y), \forall x, y \in I \), where \( \delta > 0 \) is a constant.

**Proof.** From the expression of \( G(x, y) \) we see that (i) and (ii) hold. By Lemma 1, \( \psi(x) \) is strictly monotone increasing in \( I \), \( \psi(x) \) is strictly monotone decreasing in \( I \), and hence (iii) holds. From (11), we have
\[
\frac{G(x, y)}{G(x, x) G(y, y)} \geq \delta \frac{\rho}{\psi(\psi(0))} := \delta > 0, \quad \forall x, y \in (0, 1).
\]
Therefore, (iv) holds. The proof is completed.

We denote the maximum norm of \( C(I) \) by \( \|u\| \). Let \( C^+(I) \) be the cone of all nonnegative functions in \( C(I) \). We have

**Lemma 4.** Let \( h \in C^+(I) \), then the solution of LBVP (7)–(2) satisfies
\[ u(x) \geq \delta G(x, x) \|u\|, \quad \forall x \in I. \]

**Proof.** From (12) and (iii) of Lemma 3 it is easy to see that
\[ u(x) \leq \int_0^1 G(y, y)h(y) \, dy, \quad \forall x \in I, \]
and therefore
\[ \|u\| \leq \int_0^1 G(y, y)h(y) \, dy. \]
Using (iv) of Lemma 3 and the above inequality, we have
\[ u(x) = \int_0^1 G(x, y)h(y) \, dy \geq \delta G(x, x) \int_0^1 G(y, y)h(y) \, dy \]
\[ \geq \delta G(x, x) \|u\|, \quad \forall x \in I. \]

The proof is completed. \( \square \)

We now define a mapping \( A : C^+(I) \to C^+(I) \)
by
\[
Au(x) = \int_0^1 G(x, y) f_1(y, u(y)) \, dy, \quad x \in I. \tag{15}
\]

It is clear that \( A : C^+(I) \to C^+(I) \) is completely continuous. By Lemmas 2 and 4, positive solution of BVP (1)–(2) is equivalent to nontrivial fixed point of \( A \). We will find the nonzero fixed point of \( A \) by using the fixed point index theory in cones. For this, choosing the sub-cone \( K \) of \( C^+(I) \) by
\[
K = \{ u \in C^+(I) \mid u(x) \geq \sigma \|u\|, \quad \forall x \in [1/4, 3/4] \},
\]
where \( \sigma = \delta \varphi(\frac{1}{4}) \psi(\frac{3}{4}) / \rho > 0 \), we have

**Lemma 5.** \( A(K) \subset K, \) and \( A : K \to K \) is completely continuous.

**Proof.** For \( u \in K \), let \( h(x) = f_1(x, u(x)) \), then \( Au(x) \) is the solution of LBVP (7)–(2). By Lemma 4 and (11), for \( x \in [1/4, 3/4] \), we have
\[
Au(x) \geq \delta G(x, x) \|Au\| \geq \frac{\delta}{\rho} \varphi\left(\frac{1}{4}\right) \psi\left(\frac{3}{4}\right) \|Au\| = \sigma \|Au\|, \]
namely \( Au \in K \). Therefore \( A(K) \subset K \). The complete continuity of \( A : K \to K \) is obvious. \( \square \)

We recall some concepts and conclusions on the fixed point index in [7,8], which will be used in the proof of Theorem 1. Let \( E \) be a Banach space and let \( K \subset E \) be a closed convex cone in \( E \). Assume \( \Omega \) is a bounded open subset of \( E \) with boundary \( \partial \Omega \), and \( K \cap \Omega \neq \emptyset \). Let \( A : K \cap \Omega \to K \) be a completely continuous operator. If \( Au \neq u \) for any \( u \in K \cap \partial \Omega \), then the fixed point index \( i(A, K \cap \Omega) \) has definition. One important fact is that if
\[
\inf_{u \in \partial K_r} \|Au\| > 0
\]
and
\[
\mu Au \neq u \text{ for any } u \in \partial K_r \text{ and } \mu \geq 1,
\]
then \( i(A, K_r, K) = 0 \).

**Lemma 6** [7]. Let \( A : K \to K \) be completely continuous mapping. If \( \mu Au \neq u \) for any \( u \in \partial K_r \) and \( 0 < \mu \leq 1 \), then \( i(A, K_r, K) = 1 \).

**Lemma 7** [7]. Let \( A : K \to K \) be completely continuous mapping. Suppose that the following two conditions are satisfied:

(i) \( \inf_{u \in \partial K_r} \|Au\| > 0 \).

(ii) \( \mu Au \neq u \) for any \( u \in \partial K_r \) and \( \mu \geq 1 \).

Then, \( i(A, K_r, K) = 0 \).
3. Proof of the main results

Proof of Theorem 1. We show respectively that the operator \( A \) defined by (15) has a nonzero fixed point in two cases that (F1) is satisfied and (F2) is satisfied.

Case (i). Assume (F1) is satisfied. Since \( \bar{f}_0 < \lambda_1 \), by the definition of \( \bar{f}_0 \), we may choose \( \varepsilon \in (0, M + \lambda_1) \) and \( r_0 > 0 \) so that
\[
f(x, v) \leq (\lambda_1 - \varepsilon)v, \quad \forall x \in I, \ 0 \leq v \leq r_0.
\]
(16)

Let \( r \in (0, r_0) \), we now prove that \( \mu Au \neq u \) for \( u \in \partial K_r \) and \( 0 < \mu \leq 1 \). In fact, if there exist \( u_0 \in \partial K_r \) and \( 0 < \mu_0 \leq 1 \) such that \( \mu_0 Au_0 = u_0 \), then by the definition of \( A \), \( u_0(x) \) satisfies differential equation
\[
-(p(x)u_0(x))' + (q(x) + M)u_0(x) = \mu_0 f_1(x, u_0(x)), \quad x \in I,
\]
and boundary condition (2). Multiplying Eq. (17) by \( e_1(x) \) and integrating on \( I \), since \( 0 \leq u_0(x) \leq \|u_0\| \leq r_1 \), from (16) we have
\[
\int_0^1 (L_1 u_0(x)) e_1(x) \, dx = \mu_0 \int_0^1 f_1(x, u_0(x)) e_1(x) \, dx \leq \int_0^1 f_1(x, u_0(x)) e_1(x) \, dx
\]
\[
\leq (M + \lambda_1 - \varepsilon) \int_0^1 u_0(x) e_1(x) \, dx.
\]
(18)

For the left side of the above inequality using integration by parts, we have
\[
\int_0^1 (L_1 u_0(x)) e_1(x) \, dx = \int_0^1 u_0(x) (L_1 e_1)(x) \, dx = (M + \lambda_1) \int_0^1 u_0(x) e_1(x) \, dx.
\]
(17)

Consequently, we obtain that
\[
(M + \lambda_1) \int_0^1 u_0(x) e_1(x) \, dx \leq (M + \lambda_1 - \varepsilon) \int_0^1 u_0(x) e_1(x) \, dx.
\]

Since \( u_0 \in \partial K_r \), by the definition of \( K \),
\[
\int_0^1 u_0(x) e_1(x) \, dx \geq \int_{1/4}^{3/4} u(x) e_1(x) \, dx \geq \sigma \|u_0\| \int_{1/4}^{3/4} e_1(x) \, dx > 0.
\]
(19)

Therefore we conclude that \( M + \lambda_1 \leq M + \lambda_1 - \varepsilon \), which is a contradiction. Hence \( A \) satisfies the hypothesis of Lemma 6 in \( K_r \). By Lemma 6 we have
\[
i(A, K_r, K) = 1.
\]
(20)

On the other hand, since \( f_\infty > \lambda_1 \), there exist \( \varepsilon > 0 \) and \( H > 0 \) such that
\[
f(x, v) \geq (\lambda_1 + \varepsilon)v, \quad \forall x \in I, \ v \geq H.
\]
(21)
Setting \( C = \max_{0 \leq x \leq 1, 0 \leq v \leq H} |f(x, v) - (\lambda_1 + \varepsilon)v| + 1 \), then we have
\[
f(x, v) \geq (\lambda_1 + \varepsilon)v - C, \quad \forall x \in I, \ v \geq 0.
\] (22)

Choose \( R > R_0 := \max\{H/\sigma, r_0\}. \) Let \( u \in \partial K_R. \) Since \( u(x) \geq \sigma \|u\| > H \) for \( x \in [1/4, 3/4], \) from (21) we see that
\[
f(x, u(x)) \geq (\lambda_1 + \varepsilon)u(x) \geq (\lambda_1 + \varepsilon)\sigma \|u\|, \quad \forall x \in [1/4, 3/4].
\]

By (15) and (iv) of Lemma 3, we have
\[
\|Au\| \geq Au\left(\frac{1}{2}\right) = \int_{0}^{1} G\left(\frac{1}{2}, y\right) f_1(y, u(y)) \, dy
\]
\[
\geq \delta G\left(\frac{1}{2}, \frac{1}{2}\right) \int_{0}^{1} G(y, y) f_1(y, u(y)) \, dy
\]
\[
\geq \delta G\left(\frac{1}{2}, \frac{1}{2}\right) \int_{1/4}^{3/4} G(y, y) f_1(y, u(y)) \, dy
\]
\[
\geq G\left(\frac{1}{2}, \frac{1}{2}\right) \frac{\delta}{\rho} \psi\left(\frac{1}{4}\right) \psi\left(\frac{3}{4}\right) \int_{1/4}^{3/4} f_1(y, u(y)) \, dy
\]
\[
\geq \frac{1}{2} G\left(\frac{1}{2}, \frac{1}{2}\right) (M + \lambda_1 + \varepsilon)\sigma^2 \|u\|. \quad (23)
\]

Therefore \( \inf_{u \in \partial K_R} \|Au\| > 0, \) namely the hypothesis (i) of Lemma 7 holds. Next we show that if \( R \) is large enough, then \( \mu Au \neq u \) for any \( u \in \partial K_R \) and \( \mu \geq 1. \) In fact, if there exist \( u_0 \in \partial K_R \) and \( \mu_0 \geq 1 \) such that \( \mu_0 Au_0 = u_0, \) then \( u_0(x) \) satisfies Eq. (17) and boundary condition (2). Multiplying Eq. (17) by \( e_1(x) \) and integrating, then using (18) and (22) we have
\[
(M + \lambda_1) \int_{0}^{1} u_0(x)e_1(x) \, dx = \mu_0 \int_{0}^{1} f_1(x, u_0(x))e_1(x) \, dx
\]
\[
\geq \int_{0}^{1} f_1(x, u_0(x))e_1(x) \, dx \geq (M + \lambda_1 + \varepsilon) \int_{0}^{1} u_0(x)e_1(x) \, dx - C \int_{0}^{1} e_1(x) \, dx.
\]
Consequently,
\[
\int_{0}^{1} u_0(x)e_1(x) \, dx \leq \frac{C}{\varepsilon} \int_{0}^{1} e_1(x) \, dx.
\]
Since (19) holds for \(u_0\), from the above inequality and (19) it follows that
\[
\|u_0\| \leq \frac{C}{\sigma \varepsilon} \int_0^1 e_1(x) \frac{\int_{1/4}^{3/4} e_1(x) \, dx}{\int_{1/4}^{3/4} e_1(x) \, dx}^{-1} := \bar{R}.
\] (24)

Let \(R > \max\{\bar{R}, R_0\}\), then for any \(u \in \partial KR\) and \(\mu \geq 1\), \(\mu Au \neq u\). Hence the hypothesis (ii) of Lemma 7 also holds. By Lemma 7,
\[
i(A, KR, K) = 0.
\] (25)

Now by the additivity of fixed point index, (20) and (25) we have
\[
i(A, KR \setminus \bar{K}_r, K) = i(A, KR, K) - i(A, K_r, K) = -1.
\]

Therefore \(A\) has a fixed point in \(KR \setminus \bar{K}_r\), which is the positive solution of BVP (1)–(2).

**Case (ii).** Assume (F2) is satisfied. Since \(\int_0^1 f(x, v) \, dv > \lambda_1\), there exist \(\varepsilon > 0\) and \(\eta > 0\) such that
\[
f(x, v) \geq (\lambda_1 + \varepsilon)v, \quad \forall x \in I, \ 0 \leq v \leq \eta.
\] (26)

Let \(r \in (0, \eta)\), then for every \(u \in \partial Kr\), through the same argument used in (23), we have
\[
\|Au\| \geq \frac{1}{2} G \left( \frac{1}{2}, \frac{1}{2} \right) (M + \lambda_1 + \varepsilon) \sigma^2 \|u\|.
\]

Hence \(\inf_{u \in \partial Kr} \|Au\| > 0\). Next we show that \(\mu Au \neq u\) for any \(u \in \partial KR\) and \(\mu \geq 1\). In fact, if there exist \(u_0 \in \partial K_r\) and \(\mu_0 \geq 1\) such that \(\mu_0 Au_0 = u_0\), then \(u_0(x)\) satisfies Eq. (17) and boundary condition (2). Multiplying Eq. (17) by \(e_1(x)\) and integrating, from (26) and (18) we have
\[
(M + \lambda_1) \int_0^1 u_0(x)e_1(x) \, dx \geq (M + \lambda_1 + \varepsilon) \int_0^1 u_0(x)e_1(x) \, dx.
\]

Since \(\int_0^1 u_0(x)e_1(x) \, dx > 0\), we see that \(M + \lambda_1 \geq M + \lambda_1 + \varepsilon\), which is a contradiction. Hence by Lemma 7, we have
\[
i(A, K_r, K) = 0.
\] (27)

Since \(\int_0^1 f(x, v) e_1(x) \, dx > 0\), there exist \(\varepsilon \in (0, M + \lambda_1)\) and \(H > 0\) such that
\[
f(x, v) \leq (\lambda_1 - \varepsilon)v, \quad \forall x \in I, \ v \geq H.
\]

Set \(C = \max_{0 \leq x \leq 1, 0 \leq v \leq H} |f(x, v) - (\lambda_1 - \varepsilon)v| + 1\), it is clear that
\[
f(x, v) \leq (\lambda_1 - \varepsilon)v + C, \quad \forall x \in I, \ v \geq 0.
\] (28)

If there exist \(u_0 \in K\) and \(0 < \mu_0 \leq 1\) such that \(\mu_0 Au_0 = u_0\), then (17) is valid. Multiplying Eq. (17) by \(e_1(x)\) and integrating, from (28) and (18) it follows that
\[(M + \lambda_1) \int_0^1 u_0(x)e_1(x) \, dx = \mu_0 \int_0^1 f_1(x, u_0(x))e_1(x) \, dx \]
\[\leq \int_0^1 f_1(x, u_0(x))e_1(x) \, dx \leq (M + \lambda_1 - \varepsilon) \int_0^1 u_0(x)e_1(x) \, dx + C \int_0^1 e_1(x) \, dx.\]

By the proof of (24), we see that \[\|u_0\| \leq \bar{R}.\] Let \(R > \max\{\bar{R}, \eta\}\), then \(\mu Au \neq u\) for any \(u \in \partial K_R\) and \(0 < \mu \leq 1\). Therefore by Lemma 6,
\[i(A, K_R, K) = 1.\] (29)

From (27) and (29) it follows that
\[i(A, K_R \setminus \bar{K}_r, K) = i(A, K_R, K) - i(A, K_r, K) = 1.\]
Therefore \(A\) has a fixed point in \(K_R \setminus \bar{K}_r\), which is the positive solution of BVP (1)–(2).

The proof of Theorem 1 is completed. \(\square\)

**Proof of Theorem 2.** We consider two cases of which (F3) and (F4) hold, respectively.

**Case (i).** Assume (F3) holds. If BVP (1)–(2) has a nonzero solution \(u_0 \in C^+(I)\), then \(u_0\) satisfies
\[-(p(x)u_0'(x))' + q(x)u_0(x) = f(x, u_0(x)), \quad x \in I,\] (30)
and boundary condition (2). From assumption (F3), there exists \(\varepsilon > 0\) such that
\[f(x, u_0(x)) \geq (\lambda_1 + \varepsilon) u_0(x), \quad x \in I.\]
Combining this and (30), we have
\[-(p(x)u_0'(x))' + q(x)u_0(x) \geq (\lambda_1 + \varepsilon) u_0(x), \quad x \in I.\]
Multiplying both the sides of this inequality by \(e_1(x)\) and integrating on \(I\), and then using integration by parts in the left side we have that
\[\lambda_1 \int_0^1 u_0(x)e_1(x) \, dx \geq (\lambda_1 + \varepsilon) \int_0^1 u_0(x)e_1(x) \, dx.\]
Since \(\int_0^1 u_0(x)e_1(x) \, dx > 0\), we conclude that \(\lambda_1 \geq \lambda_1 + \varepsilon\), which is a contradiction. Therefore BVP (1)–(2) has no positive solution.

**Case (ii).** Assume (F4) holds. If BVP (1)–(2) has a nonzero solution \(u_0 \in C^+(I)\), then \(u_0\) satisfies Eq. (30) and boundary condition (2). From assumption (F4), there exists \(\varepsilon > 0\) such that
\[f(x, u_0(x)) \leq (\lambda_1 - \varepsilon) u_0(x), \quad x \in I.\]
Combining this and (30), we have
\[- \left( p(x)u'_0(x) \right)' + q(x)u_0(x) \leq (\lambda_1 - \varepsilon)u_0(x), \quad x \in I.\]

Multiplying the both sides of this inequality by \(e_1(x)\) and integrating on \(I\), we can obtain that
\[
\lambda_1 \int_0^1 u_0(x)e_1(x) \, dx \leq (\lambda_1 - \varepsilon) \int_0^1 u_0(x)e_1(x) \, dx,
\]
from which we get that \(\lambda_1 \leq \lambda_1 - \varepsilon\). This is a contradiction. Therefore BVP (1)–(2) has no positive solution.

The proof of Theorem 2 is completed. \(\square\)

References