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Optimality and Duality for Minmax Problems Involving Arcwise Connected and Generalized Arcwise Connected Functions

Department of Mathematics, Acharya Narendra Dev College, Govindpuri, Kalkaji, New Delhi 110019, India

and

Davinder Bhatia

Department of Operations Research, Faculty of Mathematical Sciences, University of Delhi, Delhi 110007, India

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In this paper, we establish necessary optimality conditions for a static minmax programming problem of the form:

$$\min \max_{y \in Y} \phi(x, y) \quad \text{subject to } g(x) \leqq 0,$$

in terms of the right derivatives of the functions with respect to the same arc. Various theorems giving sufficient optimality conditions are proved. A Mond-Weir type dual is proposed and duality results are established under arcwise connectedness and generalized arcwise connectedness assumptions. © 1999 Academic Press

1. INTRODUCTION

Schmitendorf [14] generalized Gordan's theorem of alternatives [12] and used it to derive necessary optimality conditions for the following static minmax problem:

$$\text{(P)} \qquad \text{Minimize } f(x)$$
$$\text{subject to } g_j(x) \leqq 0,\ 1 \leqq j \leqq m,$$
$$x \in X,$$

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where

$$f(x) = \sup_{y \in Y} \phi(x, y)$$

and Y is a compact subset of $\mathbb{R}^m$, ϕ is a real-valued function defined on $X \times Y$ and $\phi(x, \cdot)$ continuous on Y for every $x \in X$.

Minmax problems of this type were investigated by Bram [4] and Danskin [6, 7]. These authors obtained the necessary optimality conditions in the form of Lagrange multiplier rule which was an inequality. However, the necessary optimality conditions obtained by Schmitendorf [14] were different from those presented in [4, 6, 7]. He obtained a Lagrange multiplier rule that was an equality rather than an inequality. Later, Tanimoto [16] formulated two duals to problem (P) and established various duality results under convexity assumptions.

Subsequently, necessary and sufficient optimality criteria similar to the well-known Fritz–John and Karush–Kuhn–Tucker optimality criteria of nonlinear programming were presented for various minmax problems in [2, 3, 5, 8, 9, 17] under different setups, and duality results were also studied. In all the above mentioned references the authors worked under differentiability assumptions.

Avriel and Zang [1] defined the right derivative of a real-valued function with respect to a continuous vector-valued function called an arc. Making use of arcwise connected set, as defined by Ortega and Rheinboldt [13], they extended the concept of convex functions to arcwise connected functions and generalized arcwise connected functions on an arcwise connected set. Some properties of these functions were also investigated by Singh [15].

In this paper, we obtain necessary optimality conditions for a static minmax programming problem (P) in terms of the right derivative of the functions involved with respect to the same arc. In doing so, we invoke the alternative theorem for the system of inequalities involving nonconvex functions in an infinite dimensional space as established by Jeyakumar and Gwinner [10]. Various theorems giving sufficient optimality conditions are proved. A Mond–Weir type dual is presented and duality results are developed under arcwise connectedness and generalized arcwise connectedness assumptions on the functions involved.

2. PRELIMINARIES

For a nonempty set Q in a topological vector space E, $\overline{Q}$ denote the closure of Q and

$$Q^* = \{v \in E^* : v(q) \geqq 0, \forall q \in Q\}$$

denotes the dual cone of Q, where E^* is the dual space of E.

For some nonempty set Y, let $\mathbb{R}^Y = \pi_Y \mathbb{R}$ denote the product space in a product topology. Then the topological dual space of $\mathbb{R}^Y$ is the generalized finite sequence space consisting of all the functions $u : Y \to \mathbb{R}$ with finite support [11]. The set $\mathbb{R}_+^Y = \pi_Y \mathbb{R}_+$ denotes the convex cone of all nonnegative functions on Y. Then $(\mathbb{R}_+^Y)^* = \Lambda = \{\lambda = (\lambda_y)_{y \in Y} : \exists$ a finite set $Y_0 \subseteq Y$ such that $\lambda_y = 0, \forall y \in Y \setminus Y_0$ and $\lambda_y \geqq 0, \forall y \in Y_0\}$.

DEFINITION 2.1 [1]. A set $X \subseteq \mathbb{R}^n$ is said to be an arcwise connected (AC) set if for every pair of points $x^1, x^2 \in X$, there exists a continuous vector-valued function $H_{x^1, x^2} : [0,1] \to X$, called an arc, such that

$$H_{x^1, x^2}(0) = x^1, \qquad H_{x^1, x^2}(1) = x^2.$$

DEFINITION 2.2 [1]. Let $\varphi : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^n$ is an AC set. Then the function φ is called

(a) arcwise connected (CN) function if for every $x^1, x^2 \in X$, there exists an arc H_{x^1, x^2} in X such that

$$\varphi\big(H_{x^1, x^2}(\theta)\big) \leqq (1-\theta)\varphi(x^1) + \theta\varphi(x^2), \qquad \forall \theta \in [0,1].$$

(b) Q-connected (QCN) function if for every $x^1, x^2 \in X$, there exists an arc H_{x^1, x^2} in X such that

$$\varphi(x^1) \leqq \varphi(x^2) \Rightarrow \varphi\big(H_{x^1, x^2}(\theta)\big) \leqq \varphi(x^1), \qquad \forall \theta \in [0,1].$$

(c) P-connected (PCN) function if for every $x^1, x^2 \in X$, there exist an arc H_{x^1, x^2} in X and a positive number β_{x^1, x^2} such that $\varphi(x^2) < \varphi(x^1) \Rightarrow \varphi(H_{x^1, x^2}(\theta)) \leqq \varphi(x^1) - \theta\beta_{x^1, x^2}, \forall \theta \in (0,1)$.

(d) Strictly P-connected (STPCN) function if for every $x^1, x^2 \in X$, there exist an arc H_{x^1, x^2} in X and a positive number β_{x^1, x^2} such that

$$\varphi(x^2) \leqq \varphi(x^1),\ x^1 \neq x^2$$
$$\Rightarrow \varphi\big(H_{x^1, x^2}(\theta)\big) \leqq \varphi(x^1) - \theta\beta_{x^1, x^2}, \qquad \forall \theta \in (0,1).$$

DEFINITION 2.3 [1]. Let $\varphi : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^n$ is an AC set. Let $x^1, x^2 \in X$ and H_{x^1, x^2} be the arc connecting x^1 and x^2 in X. The function φ is said to possess a right derivative, denoted by $\varphi^+(H_{x^1, x^2}(0))$, with respect to an arc H_{x^1, x^2} at $\theta = 0$ if

$$\lim_{\theta \to 0+} \frac{\varphi\big(H_{x^1, x^2}(\theta)\big) - \varphi(x^1)}{\theta}$$

exists. In that case

$$\varphi\big(H_{x^1, x^2}(\theta)\big) = \varphi(x^1) + \theta\varphi^+\big(H_{x^1, x^2}(0)\big) + \theta\alpha(\theta), \qquad (2.1)$$

where $\theta \in [0,1]$ and $\alpha : [0,1] \to \mathbb{R}$ satisfies

$$\lim_{\theta \to 0+} \alpha(\theta) = 0$$

The following theorem is an easy consequence of the above definitions.

THEOREM 2.1. *Let $\varphi : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^n$ is an* AC *set. Then*

(a1) *φ is a* CN *function if for every $x^1, x^2 \in X$,*

$$\varphi(x^2) - \varphi(x^1) \geqq \varphi^+(H_{x^1,x^2}(0))$$

(b1) *φ is a* QCN *function if for every $x^1, x^2 \in X$,*

$$\varphi(x^2) \leqq \varphi(x^1) \Rightarrow \varphi^+(H_{x^1,x^2}(0)) \leqq 0.$$

(c1) *φ is a* PCN *function if for every $x^1, x^2 \in X$,*

$$\varphi^+(H_{x^1,x^2}(0)) \geqq 0 \Rightarrow \varphi(x^2) \geqq \varphi(x^1).$$

(d1) *φ is a* STPCN *function if for every $x^1, x^2 \in X$,*

$$\varphi^+(H_{x^1,x^2}(0)) \geqq 0 \Rightarrow \varphi(x^2) > \varphi(x^1).$$

DEFINITION 2.4 [10]. Let $\varphi : X \to \mathbb{R}$ and $G : X \times Y \to \mathbb{R}$, where X and Y are arbitrary sets. The pair (φ, G) is called convexlike on X if for every $x^1, x^2 \in X$ there exist $x^3 \in X$ and $\theta \in (0,1)$ such that

$$\varphi(x^3) \leqq (1-\theta)\varphi(x^1) + \theta\varphi(x^2)$$
$$\text{and} \quad G(x^3, y) \leqq (1-\theta)G(x^1, y) + \theta G(x^2, y), \forall y \in Y.$$

THEOREM 2.2 [10]. *Let $\varphi : X \to \mathbb{R}$ and $G : X \times Y \to \mathbb{R}$, where X and Y are arbitrary nonempty sets. Let the pair (φ, G) be convexlike on X. Assume that for some neighborhood U of "0" in $\mathbb{R}^Y$ and a constant $\nu > 0$, the set $\Omega_0 \cap \overline{U}x(product)(-\infty, \nu]$ is a nonempty closed subset of $\mathbb{R}^Y \times \mathbb{R}$, where*

$$\Omega_0 = \{(u, r) : \exists x \in X \text{ such that } \varphi(x) \leqq r, G(x, y) \leqq u(y), \forall y \in Y\}$$

Then exactly one of the following systems is solvable:

(I) $\varphi(x) < 0, G(x, y) \leqq 0, \forall y \in Y$ (2.2a)

(II) $\forall \varepsilon > 0, \exists 0 \neq (\lambda, \mu) \in \Lambda \times \mathbb{R}_+$ *such that*

$$\mu(\varphi(x) + \varepsilon) + \sum_{y \in Y} \lambda_y G(x, y) > 0 \qquad (2.2b)$$

Remark 2.1. If system (II) is solvable, then we have

$$\mu(\varphi(x) + \varepsilon) + \sum_{y \in Y} \lambda_y G(x, y) > 0.$$

Letting $\varepsilon \to 0+$, we get

$$\mu\varphi(x) + \sum_{y \in Y} \lambda_y G(x, y) \geqq 0. \tag{2.3}$$

Moreover, $\lambda_y \in \Lambda$; therefore, there exists a finite set $Y_0 \subseteq Y$ such that

$$\lambda_y = 0 \qquad \forall y \in Y \setminus Y_0, \tag{2.4a}$$

$$\lambda_y \geqq 0 \qquad \forall y \in Y_0 \tag{2.4b}$$

(2.3) together with (2.4a) and (2.4b) yield that there exist an integer $\alpha > 0$ and vectors $y^i \in Y$, $1 \leqq i \leqq \alpha$, such that

$$\mu\varphi(x) + \sum_{i=1}^{\alpha} \lambda_i G(x, y^i) \geqq 0.$$

To obtain the necessary conditions, we need the following conclusion of Theorem 2.2.

THEOREM 2.3. *Let all the conditions of Theorem* 2.2 *hold. Then exactly one of the following systems is solvable*

(I) $\varphi(x) < 0$, $G(x, y) \leqq 0$, $\forall y \in Y$

(II) $\exists$ *an integer* $\alpha > 0$, *scalars* $\lambda_i \geqq 0$, $1 \leqq i \leqq \alpha$, $\mu \geqq 0$ *and vectors* $y^i \in Y$, $1 \leqq i \leqq \alpha$, *such that* $(\lambda_1, \ldots, \lambda_\alpha, \mu) \neq 0$ *and*

$$\mu\varphi(x) + \sum_{i=1}^{\alpha} \lambda_i G(x, y^i) \geqq 0$$

3. NECESSARY OPTIMALITY CONDITIONS

In order to obtain the necessary optimality conditions for the minmax problem (P), we assume the following

(A-1) X is an open AC subset of $\mathbb{R}^n$.

(A-2) The right derivative of the functions $\phi(\cdot, y)$ and $g(\cdot)$ with respect to an arc H_{x^1, x^2} at $\theta = 0$ exist $\forall x^1, x^2 \in X$, $\forall y \in Y$.

(A-3) $\phi^+(H_{x^1, x^2}(0), \cdot)$ is continuous on Y, $\forall x^1, x^2 \in X$.

For $x \in X$, we define

$$I(x) = \{j : g_j(x) = 0\}$$

$$J(x) = \{1, 2, \ldots, m\} \setminus I(x)$$

$$Y(x) = \left\{y \in Y : \phi(x, y) = \sup_{z \in Y} \phi(x, z)\right\}$$

In view of the continuity of $\phi(x, \cdot)$ on Y and compactness of Y, it is clear that $Y(x)$ is nonempty compact subset of $Y, \forall x \in X$.

LEMMA 3.1. *Let x^* be a solution of minmax problem* (P). *Then the system*

$$\left.\begin{array}{ll} \phi^+\big(H_{x^*, x}(0), y\big) < 0, & \forall y \in Y(x^*) \\ g_j^+\big(H_{x^*, x}(0)\big) < 0, & \forall j \in I(x^*) \end{array}\right\} \tag{A}$$

has no solution $x \in X$.

Proof. Suppose, on the contrary, that there exists $\hat{x} \in X$ such that

$$\phi^+\big(H_{x^*, \hat{x}}(0), y\big) < 0, \qquad \forall y \in Y(x^*) \tag{3.1a}$$

$$g_j^+\big(H_{x^*, \hat{x}}(0)\big) < 0, \qquad \forall j \in I(x^*) \tag{3.1b}$$

Since X is an open AC set and $x^* \in X$, there exists $\delta_0 > 0$ such that $H_{x^*, \hat{x}}(\theta) \in X$, $\forall \theta \in (0, \delta_0)$. In view of (A-2) and (2.1), there exist functions $\alpha_j : [0, 1] \to \mathbb{R}$ such that

$$g_j\big(H_{x^*, \hat{x}}(\theta)\big) = g_j(x^*) + \theta g_j^+\big(H_{x^*, \hat{x}}(0)\big) + \theta \bar{\alpha}_j(\theta), \qquad \forall j \in I(x^*)$$

where

$$\lim_{\theta \to 0+} \bar{\alpha}_j(\theta) = 0.$$

Using (3.1b) and the fact that $j \in I(x^*)$, it follows that there exist $\delta_j > 0$ such that

$$g_j\big(H_{x^*, \hat{x}}(\theta)\big) < 0, \qquad \forall \theta \in \big(0, \delta_j\big), \ j \in I(x^*). \tag{3.2}$$

Further, since x^* is feasible for (P), we have

$$g_j(x^*) \leqq 0, \qquad \forall j \in J(x^*)$$

which in view of continuity of $g_j(\cdot)$ at x^* implies that there exist $\delta_j > 0$ such that

$$g_j\big(H_{x^*, \hat{x}}(\theta)\big) \leqq 0, \qquad \forall \theta \in \big(0, \delta_j\big), \ j \in J(x^*) \tag{3.3}$$

Letting $\bar{\delta} = \min\{\delta_0, \delta_1, \dots, \delta_m\}$, we conclude, from (3.2) and (3.3), that $H_{x^*,\hat{x}}(\theta)$, $\forall\theta \in (0, \bar{\delta})$, are feasible points for (P).

We next prove that these points yield better objective function values than x^*.

Using (A-3), (3.1a) and the fact that $Y(x^*)$ is compact, we can find an $\varepsilon > 0$ and $\delta' > 0$ such that

$$\phi^+\big(H_{x^*,\hat{x}}(0), y\big) < -\varepsilon < 0, \qquad \forall y \in Y'_\delta \tag{3.4}$$

where $Y'_\delta = \{y \in Y\colon \|y - \hat{y}\| < \delta' \text{ for some } \hat{y} \in Y(x^*)\}$.

Now, for any $y \in Y$, we can write

$$\phi\big(H_{x^*,\hat{x}}(\theta), y\big) = \phi(x^*, y) + \theta\phi^+\big(H_{x^*,\hat{x}}(0), y\big) + \theta\alpha_0(\theta) \tag{3.5}$$

where $\alpha_0 : [0,1] \to \mathbb{R}$ satisfies

$$\lim_{\theta \to 0+} \alpha_0(\theta) = 0$$

Hence, for a given $\varepsilon > 0$, there exists $\delta'' > 0$, such that

$$\alpha_0(\theta) < \varepsilon, \qquad \forall\theta \in (0, \delta'') \tag{3.6}$$

Choosing $\hat{\delta} = \min\{\delta', \delta''\}$ and using (3.4) and (3.6) in (3.5), we obtain

$$\phi\big(H_{x^*,\hat{x}}(\theta), y\big) < \phi(x^*, y), \qquad \forall\theta \in (0, \hat{\delta}), \forall y \in Y'_{\hat{\delta}}.$$

Also, we have

$$\phi(x^*, y) \leqq \sup_{z \in Y} \phi(x^*, z), \qquad \forall y \in Y.$$

Therefore, we get

$$\phi\big(H_{x^*,\hat{x}}(\theta), y\big) < \sup_{z \in Y} \phi(x^*, z), \qquad \forall\theta \in (0, \hat{\delta}), \forall y \in Y'_{\hat{\delta}}. \tag{3.7}$$

Also, for any $y \in \overline{Y \setminus Y'_{\hat{\delta}}}$, we have

$$\phi(x^*, y) < \sup_{z \in Y} \phi(x^*, z)$$

Then, because of compactness of the set $\overline{Y \setminus Y'_{\hat{\delta}}}$ and continuity of $\phi(x^*, \cdot)$ on Y, we can find $\varepsilon_1 > 0$ such that

$$\phi(x^*, y) < \sup_{z \in Y} \phi(x^*, z) - \varepsilon_1, \qquad \forall y \in \overline{Y \setminus Y'_{\hat{\delta}}}. \tag{3.8}$$

Using (3.8) in (3.5), we get

$$\phi\big(H_{x^*,\hat{x}}(\theta), y\big) < \sup_{z\in Y} \phi(x^*, z) - \varepsilon_1 + \theta\big(\phi^+\big(H_{x^*,\hat{x}}(0), y\big) + \alpha_0(\theta)\big). \tag{3.9}$$

Since, for any $y \in \overline{Y \setminus Y_{\hat{\delta}}}$, we have

$$\lim_{\theta\to 0+} \theta\big(\phi^+\big(H_{x^*,\hat{x}}(0), y\big) + \alpha_0(\theta)\big) = 0$$

Hence, there exist $\tilde{\delta} > 0$ such that

$$\theta\big(\phi^+\big(H_{x^*,\hat{x}}(0), y\big) + \alpha_0(\theta)\big) < \varepsilon_1, \qquad \forall\theta \in (0, \tilde{\delta}), \forall y \in \overline{Y \setminus Y_{\hat{\delta}}}. \tag{3.10}$$

(3.9) and (3.10) together yield

$$\phi\big(H_{x^*,\hat{x}}(\theta), y\big) < \sup_{z\in Y} \phi(x^*, z), \qquad \forall\theta \in (0, \tilde{\delta}), y \in \overline{Y \setminus Y_{\hat{\delta}}}. \tag{3.11}$$

Letting $\delta = \min\{\hat{\delta}, \tilde{\delta}\}$ and noting that $Y = \overline{Y \setminus Y_{\hat{\delta}}} \cup Y_{\hat{\delta}}$, it follows from (3.7) and (3.11) that

$$\phi\big(H_{x^*,\hat{x}}(\theta), y\big) < \sup_{z\in Y} \phi(x^*, z), \qquad \forall\theta \in (0, \delta), \forall y \in Y,$$

which implies

$$\sup_{z\in Y} \phi\big(H_{x^*,\hat{x}}(\theta), z\big) < \sup_{z\in Y} \phi(x^*, z), \qquad \forall\theta \in (0, \delta),$$

that is,

$$f\big(H_{x^*,\hat{x}}(\theta)\big) < f(x^*), \qquad \forall\theta \in (0, \delta).$$

Finally, we choose $\delta^* = \min\{\delta, \bar{\delta}\}$; then it follows that $H_{x^*,\hat{x}}(\theta)$, with $\theta \in (0, \delta^*)$, are feasible points for (P) and yield better objective values than x^*, thus contradicting the optimality of x^*. Therefore, system (A) cannot have a solution in X.

For any $t > 0$ and $x \in X$, define

$$\varphi_0(x) = -t$$

$$\varphi_j(x) = g_j^+\big(H_{x^*,x}(0)\big) + t, \qquad j \in I(x^*)$$

$$G(x, y) = \phi^+\big(H_{x^*,x}(0), y\big) + t, \qquad y \in Y(x^*)$$

where x^* is a solution of the minmax problem (P). Then in view of Lemma 3.1, we have that, the system

$$\left.\begin{aligned} &\varphi_0(x) < 0 \\ &\varphi_j(x) < 0, \qquad \forall j \in I(x^*) \\ &G(x,y) \leqq 0, \quad \forall y \in Y(x^*) \end{aligned}\right\} \tag{B}$$

has no solution $x \in X$.

Define the sets

$$\Omega_0(x^*,t) = \Big\{(u,r,r_0) : r = (r_j)_{j \in I(x^*)} \text{ and } \exists x \in X \text{ such that } \varphi_0(x) \leqq r_0,$$
$$\varphi_j(x) \leqq r_j, j \in I(x^*), G(x,y) \leqq u(y), \forall y \in Y(x^*)\Big\}$$
$$\Omega(x^*,t) = \Big\{(u,r) : r = (r_j)_{j \in I(x^*)} \text{ and } \exists x \in X \text{ such that } \varphi_j(x) \leqq r_j,$$
$$j \in I(x^*), G(x,y) \leqq u(y), \forall y \in Y(x^*)\Big\}$$

We now prove the following theorem which gives the necessary optimality conditions for a minmax solution of problem (P).

THEOREM 3.1 (necessary optimality conditions). *Let x^* be a solution of minmax problem* (P). *Further, let $(\varphi_j, G)_{j \in I(x^*)}$ be convexlike on X and let there exist a neighborhood U of* ‘0’ *in $\mathbb{R}^{Y(x^*)}$ and constants $\nu = (\nu_j)_{j \in I(x^*)}$ such that the set $\Omega(x^*,t) \cap \overline{U} \times \pi_{j \in I(x^*)}(-\infty, \nu_j]$ is a nonempty closed set for every $t > 0$. Then there exist an integer $\alpha > 0$, scalars $\lambda_i \geqq 0$, $1 \leqq i \leqq \alpha$, $\mu \geqq 0$ and vectors $y^i \in Y(x^*)$, $1 \leqq i \leqq \alpha$, such that*

$$\sum_{i=1}^{\alpha} \lambda_i \phi^+\big(H_{x^*,x}(0), y^i\big) + \sum_{j=1}^{m} \mu_j g_j^+\big(H_{x^*,x}(0)\big) \geqq 0, \qquad \forall x \in X$$
$$\mu_j g_j(x^*) = 0, \qquad \forall 1 \leqq j \leqq m$$
$$\sum_{i=1}^{\alpha} \lambda_i + \sum_{j=1}^{m} \mu_j \neq 0.$$

Proof. Since x^* is a solution of minmax problem (P). Hence, by Lemma 3.1, system (B) has no solution $x \in X$, i.e., for any $t > 0$ the system

$$\begin{aligned} &\varphi_0(x) < 0 \\ &\varphi_j(x) < 0, \qquad \forall j \in I(x^*) \\ &G(x,y) \leqq 0, \qquad \forall y \in Y(x^*) \end{aligned}$$

has no solution $x \in X$.

By the assumptions, $(\varphi_0, \varphi_j, G)_{j \in I(x^*)}$ is convexlike on X and for any constant $\nu_0 > 0$,

$$\Omega_0(x^*, t) \cap \overline{U} \times \pi_{j \in I(x^*)}(-\infty, \nu_j] \times (-\infty, \nu_0]$$

is a nonempty closed set for any $t > 0$.

Since all the conditions of Theorem 2.3 are satisfied, there exists an integer $\alpha > 0$, scalars $\lambda_i \geqq 0$, $0 \leqq i \leqq \alpha$, $\mu_j \geqq 0$, $j \in I(x^*)$ such that

$$(\lambda_0, \lambda_1, \dots, \lambda_\alpha, \mu_j)_{j \in I(x^*)} \neq 0$$

and

$$\lambda_0 \varphi_0(x) + \sum_{j \in I(x^*)} \mu_j \varphi_j(x) + \sum_{i=1}^{\alpha} \lambda_i G(x, y^i) \geqq 0, \qquad \forall x \in X$$

which implies

$$\sum_{i=1}^{\alpha} \lambda_i \phi^+(H_{x^*,x}(0), y^i) + \sum_{j \in I(x^*)} \mu_j g_j^+(H_{x^*,x}(0))$$
$$\geqq \left(\lambda_0 - \sum_{j \in I(x^*)} \mu_j - \sum_{i=1}^{\alpha} \lambda_i \right) t, \qquad \forall x \in X, \forall t > 0.$$

giving

$$\inf_{x \in X} \left[\sum_{i=1}^{\alpha} \lambda_i \phi^+(H_{x^*,x}(0), y^i) + \sum_{j \in I(x^*)} \mu_j g_j^+(H_{x^*,x}(0)) \right]$$
$$\geqq \sup_{t > 0} \left[t \left(\lambda_0 - \sum_{j \in I(x^*)} \mu_j - \sum_{i=1}^{\alpha} \lambda_i \right) \right] \tag{3.13}$$

In order that inequality (3.13) always holds, we must have

$$\sum_{i=1}^{\alpha} \lambda_i + \sum_{j \in I(x^*)} \mu_j = \lambda_0 \tag{3.14}$$

because otherwise as $t > 0$, the right hand side of (3.13) can be made arbitrarily large.

On using (3.14) in (3.13), we get

$$\sum_{i=1}^{\alpha} \lambda_i \phi^+(H_{x^*,x}(0), y^i) + \sum_{j \in I(x^*)} \mu_j g_j^+(H_{x^*,x}(0)) \geqq 0, \qquad \forall x \in X.$$

Letting $\mu_j = 0\ \forall j \in J(x^*)$, we obtain

$$\sum_{i=1}^{\alpha} \lambda_i \phi^+\big(H_{x^*,x}(0), y^i\big) + \sum_{j=1}^{m} \mu_j g_j^+\big(H_{x^*,x}(0)\big) \geqq 0, \qquad \forall x \in X$$

$$\mu_j g_j(x^*) = 0, \qquad 1 \leqq j \leqq m.$$

$$\sum_{i=1}^{\alpha} \lambda_i + \sum_{j=1}^{m} \mu_j = \lambda_0 \tag{3.15}$$

$$(\lambda_1, \lambda_2, \dots, \lambda_\alpha, \mu_1, \mu_2, \dots, \mu_m) \neq 0 \tag{3.16}$$

$$\lambda_i \geqq 0, \qquad 0 \leqq i \leqq \alpha \tag{3.17}$$

$$\mu_j \geqq 0, \qquad 1 \leqq j \leqq m \tag{3.18}$$

Now, $\lambda_0 \neq 0$ because if $\lambda_0 = 0$ then (3.15) yields that

$$\sum_{i=1}^{\alpha} \lambda_i + \sum_{j=1}^{m} \mu_j = 0$$

which in view of (3.17) and (3.18) implies

$$\lambda_i = 0, \qquad 0 \leqq i \leqq \alpha$$

$$\mu_j = 0, \qquad 1 \leqq j \leqq m$$

But this contradicts (3.16). Hence $\lambda_0 \neq 0$. This completes the proof.

4. SUFFICIENT OPTIMALITY CONDITIONS

We establish sufficient optimality conditions for minmax problem (P) under arcwise connectedness and generalized arcwise connectedness assumptions on the functions involved.

Let $X^0 = \{x \in X : g_j(x) \leqq 0,\ \forall 1 \leqq j \leqq m\}$ denote the set of feasible solutions of (P).

THEOREM 4.1. *Let $x^* \in X^0$ and assume that there exist an integer $\alpha > 0$, scalars $\lambda_i \geqq 0$, $1 \leqq i \leqq \alpha$, $\sum_{i=1}^{\alpha} \lambda_i \neq 0$, $\mu_j \geqq 0$, $1 \leqq j \leqq m$ and vectors $y^i \in Y(x^*)$, $1 \leqq i \leqq \alpha$, such that*

$$\sum_{i=1}^{\alpha} \lambda_i \phi^+\big(H_{x^*,x}(0), y^i\big) + \sum_{j=1}^{m} \mu_j g_j^+\big(H_{x^*,x}(0)\big) \geqq 0, \qquad \forall x \in X \tag{4.1}$$

$$\mu_j g_j(x^*) = 0, \qquad \forall 1 \leqq j \leqq m. \tag{4.2}$$

Further assume that the functions $g_j(\cdot)$, $1 \leqq j \leqq m$ *and* $\phi(\cdot, y)$, $y \in Y$, *are* CN *with respect to the same arc on* X^0. *Then* x^* *is a minmax solution of* (P).

Proof. Suppose that x^* is not a minmax solution of (P). Then there exists an $\hat{x} \in X^0$ such that

$$\sup_{z \in Y} \phi(\hat{x}, z) < \sup_{z \in Y} \phi(x^*, z). \tag{4.3}$$

Also, since $\hat{x} \in X^0$, $\mu_j \geqq 0$, $\forall 1 \leqq j \leqq m$, we have

$$\mu_j g_j(\hat{x}) \leqq 0$$

which in view of (4.2) implies that

$$\sum_{j=1}^{m} \mu_j g_j(\hat{x}) \leqq \sum_{j=1}^{m} \mu_j g_j(x^*). \tag{4.4}$$

Further, as $y^i \in Y(x^*)$, we have

$$\sup_{z \in Y} \phi(x^*, z) = \phi(x^*, y^i), \qquad 1 \leqq i \leqq \alpha. \tag{4.5}$$

Also, $y^i \in Y$, $1 \leqq i \leqq \alpha$, we have

$$\phi(\hat{x}, y^i) \leqq \sup_{z \in Y} \phi(\hat{x}, z), \qquad 1 \leqq i \leqq \alpha. \tag{4.6}$$

(4.3), (4.5) and (4.6) imply

$$\phi(\hat{x}, y^i) < \phi(x^*, y^i), \qquad 1 \leqq i \leqq \alpha. \tag{4.7}$$

Since $\lambda_i \geqq 0$, $1 \leqq i \leqq \alpha$ and $\sum_{i=1}^{\alpha} \lambda_i \neq 0$, it follows from (4.7) and (4.4) that

$$\sum_{i=1}^{\alpha} \lambda_i \phi(\hat{x}, y^i) + \sum_{j=1}^{m} \mu_j g_j(\hat{x}) < \sum_{i=1}^{\alpha} \lambda_i \phi(x^*, y^i) + \sum_{j=1}^{m} \mu_j g_j(x^*) \tag{4.8}$$

Also, since $\phi(\cdot, y^i)$, $1 \leqq i \leqq \alpha$ and $g_j(\cdot)$, $1 \leqq j \leqq m$, are CN, we have

$$\phi(\hat{x}, y^i) - \phi(x^*, y^i) \geqq \phi^+(H_{x^*, \hat{x}}(0), y^i), \qquad 1 \leqq i \leqq \alpha \tag{4.9}$$

$$g_j(\hat{x}) - g_j(x^*) \geqq g_j^+(H_{x^*, \hat{x}}(0)), \qquad 1 \leqq j \leqq m \tag{4.10}$$

From (4.9) and (4.10) together with $\lambda_i \geqq 0$, $1 \leqq i \leqq \alpha$ and $\mu_j \geqq 0$, $1 \leqq j \leqq m$, we get

$$\left\{ \sum_{i=1}^{\alpha} \lambda_i \phi(\hat{x}, y^i) + \sum_{j=1}^{m} \mu_j g_j(\hat{x}) \right\} - \left\{ \sum_{i=1}^{\alpha} \lambda_i \phi(x^*, y^i) + \sum_{j=1}^{m} \mu_j g_j(x^*) \right\}$$

$$\geqq \sum_{i=1}^{\alpha} \lambda_i \phi^+(H_{x^*, \hat{x}}(0), y^i) + \sum_{j=1}^{m} \mu_j g_j^+(H_{x^*, \hat{x}}(0))$$

which on using (4.8) gives

$$\sum_{i=1}^{\alpha} \lambda_i \phi^+\left(H_{x^*,\hat{x}}(0), y^i\right) + \sum_{j=1}^{m} \mu_j g_j^+\left(H_{x^*,\hat{x}}(0)\right) < 0$$

which contradicts (4.1) for $x = \hat{x}$. Hence the result follows.

THEOREM 4.2. *Let $x^* \in X^0$ and assume that there exist an integer $\alpha > 0$, scalars $\lambda_i \geqq 0$, $1 \leqq i \leqq \alpha$, $\sum_{i=1}^{\alpha} \lambda_i \neq 0$, $\mu_j \geqq 0$, $1 \leqq j \leqq m$ and vectors $y^i \in Y(x^*)$, $1 \leqq i \leqq \alpha$, such that* (4.1) *and* (4.2) *are satisfied. If the function $\sum_{j=1}^{m} \mu_j g_j(\cdot)$ is* QCN *and the function $\sum_{i=1}^{\alpha} \lambda_i \phi(\cdot, y^i)$ is* PCN *with respect to the same arc on X^0, then x^* is a minmax solution of* (P).

Proof. From (4.2), we have

$$\sum_{j=1}^{m} \mu_j g_j(x^*) = 0.$$

Let $x \in X^0$. Then, since $\mu_j \geqq 0$, $1 \leqq j \leqq m$, we get

$$\sum_{j=1}^{m} \mu_j g_j(x) \leqq \sum_{j=1}^{m} \mu_j g_j(x^*)$$

which, by QCN-ness of $\sum_{j=1}^{m} \mu_j g_j(\cdot)$ at x^*, yields

$$\sum_{j=1}^{m} \mu_j g_j^+\left(H_{x^*,x}(0)\right) \leqq 0. \tag{4.11}$$

Using (4.11) in (4.1), we get

$$\sum_{i=1}^{\alpha} \lambda_i \phi^+\left(H_{x^*,x}(0), y^i\right) \geqq 0$$

which by PCN-ness of $\sum_{i=1}^{\alpha} \lambda_i \phi(\cdot, y^i)$ at x^* implies

$$\sum_{i=1}^{\alpha} \lambda_i \phi(x, y^i) \geqq \sum_{i=1}^{\alpha} \lambda_i \phi(x^*, y^i). \tag{4.12}$$

Since $y^i \in Y(x^*)$, $1 \leqq i \leqq \alpha$, we have

$$\phi(x^*, y^i) = \sup_{z \in Y} \phi(x^*, z). \tag{4.13}$$

Also, since $y^i \in Y$, $1 \leqq i \leqq \alpha$, we have

$$\phi(x, y^i) \leqq \sup_{z \in Y} \phi(x, z) \tag{4.14}$$

Since $\lambda_i \geqq 0$, $1 \leqq i \leqq \alpha$, $\sum_{i=1}^{\alpha} \lambda_i \neq 0$, (4.12)–(4.14) imply

$$\sup_{z \in Y} \phi(x, z) \geqq \sup_{z \in Y} \phi(x^*, z)$$

i.e., $f(x) \geqq f(x^*)$, $\forall x \in X^0$.

Hence, x^* is a minmax solution of (P).

THEOREM 4.3. *Let $x^* \in X^0$ and assume that there exist an integer $\alpha > 0$, scalars $\lambda_i \geqq 0$, $1 \leqq i \leqq \alpha$, $\sum_{i=1}^{\alpha} \lambda_i \neq 0$, $\mu_j \geqq 0$, $1 \leqq j \leqq m$ and vectors $y^i \in Y(x^*)$, $1 \leqq i \leqq \alpha$, such that* (4.1) *and* (4.2) *are satisfied. If the function $\sum_{j=1}^{m} \mu_j g_j(\cdot)$ is* STPCN *and the function $\sum_{i=1}^{\alpha} \lambda_i \phi(\cdot, y^i)$ is* QCN *with respect to the same arc on X^0, then x^* is a minmax solution of* (P).

Proof. The proof follows along similar lines as the proof of Theorem 4.2 and hence is omitted.

THEOREM 4.4. *Let $x^* \in X^0$ and assume that there exist an integer $\alpha > 0$, scalars $\lambda_i \geqq 0$, $1 \leqq i \leqq \alpha$, $\sum_{i=1}^{\alpha} \lambda_i \neq 0$, $\mu_j \geqq 0$, $1 \leqq j \leqq m$ and vectors $y^i \in Y(x^*)$, $1 \leqq i \leqq \alpha$, such that* (4.1) *and* (4.2) *are satisfied. If the function $g_j(\cdot)$, $j \in I(x^*)$, $j \neq s$, is* QCN *and $g_s(\cdot)$ is* STPCN *with $\mu_s > 0$ and the function $\sum_{i=1}^{\alpha} \lambda_i \phi(\cdot, y^i)$ is* QCN *with respect to the same arc on X^0, then x^* is a minmax solution of* (P).

Proof. Since for any $x \in X^0$, we have

$$g_j(x) \leqq g_j(x^*), \qquad \forall j \in I(x^*) \tag{4.15}$$

which by means of the QCN-ness assumption on $g_j(\cdot)$, $j \in I(x^*)$, $j \neq s$, gives

$$g_j^+(H_{x^*,x}(0)) \leqq 0, \qquad j \in I(x^*), j \neq s \tag{4.16}$$

Also, strict PCN-ness of $g_s(\cdot)$, in view of (4.15), implies

$$g_s^+(H_{x^*,x}(0)) < 0 \tag{4.17}$$

Since $\mu_j \geqq 0$, $\forall j \in I(x^*)$, $\mu_s > 0$ and $\mu_j = 0$ for $j \in J(x^*)$, therefore, from (4.16) and (4.17), we have

$$\sum_{j=1}^{m} \mu_j g_j^+(H_{x^*,x}(0)) < 0. \tag{4.18}$$

Using (4.18) in (4.1), we obtain

$$\sum_{i=1}^{\alpha} \lambda_i \phi^+(H_{x^*,x}(0), y^i) > 0$$

which by QCN-ness of $\sum_{i=1}^{\alpha}\lambda_i\phi(\cdot, y^i)$ at x^* implies

$$\sum_{i=1}^{\alpha}\lambda_i\phi(x, y^i) \geqq \sum_{i=1}^{\alpha}\lambda_i\phi(x^*, y^i) \tag{4.19}$$

(4.19) is same as (4.12). The remainder of the proof is same as that of Theorem 4.2.

5. DUALITY

In this section, we formulate a Mond-Weir type dual (D) to minmax problem (P).

Let

$$\mathbb{G} = \Big\{(\alpha, \lambda, y) : \alpha \text{ is a positive integer, } \lambda \in \mathbb{R}_+^{\alpha},\ \sum_{i=1}^{\alpha}\lambda_i = 1,$$
$$y = (y^1, y^2, \dots, y^{\alpha}),\ y^i \in Y(x) \text{ for some } x \in X,\ 1 \leqq i \leqq \alpha\Big\}$$

$$\chi_0(\alpha, \lambda, y)$$
$$= \Big\{(x, \mu) \in X \times \mathbb{R}_+^m : (y^1, \dots, y^{\alpha}) \subset Y(x), \sum_{i=1}^{\alpha}\lambda_i\phi^+\big(H_{x,w}(0), y^i\big)$$
$$+ \sum_{j=1}^{m}\mu_j g_j^+(H_{x,w}(0)) \geqq 0, \forall w \in X \text{ and } \sum_{j=1}^{m}\mu_j g_j(x) \geqq 0\Big\}$$

With the above notations, we introduce a dual to problem (P) as follows:

$$\text{(D)} \qquad \max_{(\alpha,\lambda,y)\in\mathbb{G}}\ \sup_{(x,\mu)\in\chi_0(\alpha,\lambda,y)}\ \sum_{i=1}^{\alpha}\lambda_i\phi(x, y^i)$$

If for a triplet (α, λ, y) in $\mathbb{G}$ the set $\chi_0(\alpha, \lambda, y)$ is empty then we define the supremum over it to be $-\infty$.

We now establish duality relationship between problems (P) and (D).

THEOREM 5.1. *Let x^* be an optimal solution of problem* (P) *and let there exist $w^* \in X$ such that $g_j^+(H_{x^*,w^*}(0)) < 0$, $1 \leqq j \leqq m$. Assume that conditions of Theorem* 3.1 *are satisfied. Then there exist $(\alpha^*, \lambda^*, y^*) \in \mathbb{G}$ and $\mu^* \in \mathbb{R}_+^m$ such that $(x^*, \mu^*) \in \chi_0(\alpha^*, \lambda^*, y^*)$. Further, if for each fixed $x \in X^0$ and $(\hat{x}, \mu) \in \chi_0(\alpha, \lambda, y)$ any one of the following conditions hold*

(i) $\phi(\cdot, y^i)$, $1 \leqq i \leqq \alpha$, $g_j(\cdot)$, $1 \leqq j \leqq m$ *are* CN *with respect to the same arc*

(ii) $\sum_{i=1}^{\alpha} \lambda_i \phi(\cdot, y^i)$ *is* PCN, $\sum_{j=1}^{m} \mu_j g_j(\cdot)$ *is* QCN *with respect to the same arc*

(iii) $\sum_{i=1}^{\alpha} \lambda_i \phi(\cdot, y^i)$ *is* QCN, $\sum_{j=1}^{m} \mu_j g_j(\cdot)$ *is* STPCN *with respect to the same arc*

(iv) $\sum_{i=1}^{\alpha} \lambda_i \phi(\cdot, y^i)$ *is* QCN, $g_j(\cdot)$, $j \in I(x^*)$, $j \neq s$ *is* QCN, $g_s(\cdot)$ *is* STPCN *with* $\mu_s > 0$, *with respect to the same arc*

Then (x^*, μ^*) *and* $(\alpha^*, \lambda^*, y^*)$ *give an optimal solution to* (D). *Furthermore, the two problems* (P) *and* (D) *have the same extremal values.*

Proof. Since x^* is an optimal solution of (P) at which the conditions of Theorem 3.1 are true, therefore there exist an integer $\alpha^* > 0$, scalars $\lambda^0 \in \mathbb{R}_+^{\alpha^*}$, $\mu^0 \in \mathbb{R}_+^m$ and vectors $y^{*^i} \in Y(x^*)$, $1 \leqq i \leqq \alpha^*$ such that

$$\sum_{i=1}^{\alpha^*} \lambda_i^0 \phi^+ \left(H_{x^*,w}(0), y^{*^i}\right) + \sum_{j=1}^{m} \mu_j^0 g_j^+ \left(H_{x^*,w}(0)\right) \geqq 0, \qquad \forall w \in X \quad (5.1)$$

$$\mu_j^0 g_j(x^*) = 0, \qquad 1 \leqq j \leqq m \quad (5.2)$$

$$\sum_{i=1}^{\alpha^*} \lambda_i^0 + \sum_{j=1}^{m} \mu_j^0 \neq 0 \quad (5.3)$$

Now, if $\lambda_i^0 = 0$, $1 \leqq i \leqq \alpha^*$ then, in view of (5.3), $\mu^0 = (\mu_1^0, \dots, \mu_m^0) \neq 0$. Also, (5.1) then reduces to

$$\sum_{j=1}^{m} \mu_j^0 g_j^+ \left(H_{x^*,w}(0)\right) \geqq 0, \qquad \forall w \in X. \quad (5.4)$$

By assumption, there exists $w^* \in X$ such that

$$g_j^+ \left(H_{x^*,w^*}(0)\right) < 0, \qquad 1 \leqq j \leqq m$$

which together with $\mu_j^0 \geqq 0$, $1 \leqq j \leqq m$, $\mu^0 = (\mu_1^0, \mu_2^0, \dots, \mu_m^0) \neq 0$ implies

$$\sum_{j=1}^{m} \mu_j^0 g_j^+ \left(H_{x^*,w^*}(0)\right) < 0$$

But this contradicts (5.4) for $w = w^*$. Therefore, we have

$$\sum_{i=1}^{\alpha^*} \lambda_i^0 \neq 0$$

Let $\tau = \sum_{i=1}^{\alpha^*} \lambda_i^0$, $\lambda^* = \tau^{-1}\lambda^0$, $\mu^* = \tau^{-1}\mu^0$. Then $(\alpha^*, \lambda^*, y^*) \in \mathbb{G}$ and $(x^*, \mu^*) \in \chi_0(\alpha^*, \lambda^*, y^*)$.

(1) Let hypothesis (i) hold, and let (x, μ) be any element of $\chi_0(\alpha^*, \lambda^*, y^*)$. Then we have

$$\sum_{i=1}^{\alpha^*} \lambda_i^* \phi^+\left(H_{x,x^*}(0), y^{*^i}\right) + \sum_{j=1}^{m} \mu_j g_j^+\left(H_{x,x^*}(0)\right) \geqq 0 \tag{5.5}$$

$$\sum_{j=1}^{m} \mu_j g_j(x) \geqq 0 \tag{5.6}$$

Since $\phi(\cdot, y^{*^i})$, $1 \leqq i \leqq \alpha^*$, $g_j(\cdot)$, $1 \leqq j \leqq m$ are CN with respect to the same arc, we have

$$\phi\left(x^*, y^{*^i}\right) - \phi\left(x, y^{*^i}\right) \geqq \phi^+\left(H_{x,x^*}(0), y^{*^i}\right), \qquad 1 \leqq i \leqq \alpha^* \tag{5.7}$$

$$g_j(x^*) - g_j(x) \geqq g_j^+\left(H_{x,x^*}(0)\right), \qquad 1 \leqq j \leqq m \tag{5.8}$$

Further, $\lambda_i^* \geqq 0$, $1 \leqq i \leqq \alpha^*$, and $\mu_j \geqq 0$, $1 \leqq j \leqq m$, it follows from (5.7) and (5.8) that

$$\sum_{i=1}^{\alpha^*} \lambda_i^* \phi\left(x^*, y^{*^i}\right) - \sum_{i=1}^{\alpha^*} \lambda_i^* \phi\left(x, y^{*^i}\right) + \sum_{j=1}^{m} \mu_j g_j(x^*) - \sum_{j=1}^{m} \mu_j g_j(x)$$
$$\geqq \sum_{i=1}^{\alpha^*} \lambda_i^* \phi^+\left(H_{x,x^*}(0), y^{*^i}\right) + \sum_{j=1}^{m} \mu_j g_j^+\left(H_{x,x^*}(0)\right)$$

Using (5.5) and (5.6) in the above inequality, we get

$$\sum_{i=1}^{\alpha^*} \lambda_i^* \phi\left(x^*, y^{*^i}\right) - \sum_{i=1}^{\alpha^*} \lambda_i^* \phi\left(x, y^{*^i}\right) \geqq -\sum_{j=1}^{m} \mu_j g_j(x^*)$$

which implies

$$\sum_{i=1}^{\alpha^*} \lambda_i^* \phi\left(x^*, y^{*^i}\right) \geqq \sum_{i=1}^{\alpha^*} \lambda_i^* \phi\left(x, y^{*^i}\right)$$

as $g_j(x^*) \leqq 0$, $\mu_j \geqq 0$, $1 \leqq j \leqq m$. Hence, (x^*, μ^*) attains the maximum of the following problem

$$(\mathrm{D}_{\eta^*}) \qquad \max \sum_{i=1}^{\alpha^*} \lambda_i^* \phi\left(x, y^{*^i}\right)$$
$$\text{subject to } (x, \mu) \in \chi_0(\alpha^*, \lambda^*, y^*)$$

where $\eta^* = (\alpha^*, \lambda^*, y^*)$.

In order to prove the result, we shall show that for any $(\alpha, \lambda, y) \in \mathbb{G}$,

$$\sup_{(x,\mu) \in \chi_0(\alpha,\lambda,y)} \sum_{i=1}^{\alpha} \lambda_i \phi(x, y^i) \leqq \sum_{i=1}^{\alpha^*} \lambda_i^* \phi(x^*, y^{*^i})$$

Let (x, μ) be any element of $\chi_0(\alpha, \lambda, y)$, where $(\alpha, \lambda, y) \in \mathbb{G}$. Then we have

$$\sum_{i=1}^{\alpha} \lambda_i \phi^+(H_{x,x^*}(0), y^i) + \sum_{j=1}^{m} \mu_j g_j^+(H_{x,x^*}(0)) \geqq 0 \tag{5.9}$$

$$\sum_{j=1}^{m} \mu_j g_j(x) \geqq 0 \tag{5.10}$$

Since $\phi(\cdot, y^i)$, $1 \leqq i \leqq \alpha$, $g_j(\cdot)$, $1 \leqq j \leqq m$ are CN with respect to the same arc, we have

$$\phi(x^*, y^i) - \phi(x, y^i) \geqq \phi^+(H_{x,x^*}(0), y^i), \qquad 1 \leqq i \leqq \alpha \tag{5.11}$$

$$g_j(x^*) - g_j(x) \geqq g_j^+(H_{x,x^*}(0)), \qquad 1 \leqq j \leqq m \tag{5.12}$$

Further, $\lambda_i \geqq 0$, $1 \leqq i \leqq \alpha$ and $\mu_j \geqq 0$, $1 \leqq j \leqq m$; hence, (5.11) and (5.12) on using (5.9) and (5.10) implies

$$\sum_{i=1}^{\alpha} \lambda_i \phi(x^*, y^i) - \sum_{i=1}^{\alpha} \lambda_i \phi(x, y^i) \geqq -\sum_{j=1}^{m} \mu_j g_j(x^*)$$

which implies

$$\sum_{i=1}^{\alpha} \lambda_i \phi(x^*, y^i) \geqq \sum_{i=1}^{\alpha} \lambda_i \phi(x, y^i) \tag{5.13}$$

as $g_j(x^*) \leqq 0$, $\mu_j \geqq 0$, $1 \leqq j \leqq m$. Now, as $y^i \in Y(x) \subset Y$, $1 \leqq i \leqq \alpha$, we have

$$\phi(x^*, y^{*^k}) = \sup_{z \in Y} \phi(x^*, z) \geqq \phi(x^*, y^i), \qquad 1 \leqq i \leqq \alpha, 1 \leqq k \leqq \alpha^*$$

which implies

$$\sum_{i=1}^{\alpha} \lambda_i \phi(x^*, y^{*^k}) \geqq \sum_{i=1}^{\alpha} \lambda_i \phi(x^*, y^i), \quad \text{for each } k, 1 \leqq k \leqq \alpha^*.$$

Since $\sum_{i=1}^{\alpha}\lambda_i = 1$, therefore, we get

$$\phi\left(x^*, y^{*^k}\right) \geqq \sum_{i=1}^{\alpha} \lambda_i \phi(x^*, y^i), \quad \text{for each } k, 1 \leqq k \leqq \alpha^*. \quad (5.14)$$

(5.14) together with the fact that $\lambda_k^* \geqq 0$, $1 \leqq k \leqq \alpha^*$ and $\sum_{i=1}^{\alpha^*}\lambda_i^* = 1$, imply

$$\sum_{i=1}^{\alpha^*} \lambda_i^* \phi\left(x^*, y^{*^i}\right) \geqq \sum_{i=1}^{\alpha} \lambda_i \phi(x^*, y^i). \quad (5.15)$$

It follows from (5.13) and (5.15) that

$$\sum_{i=1}^{\alpha^*} \lambda_i^* \phi\left(x^*, y^{*^i}\right) \geqq \sum_{i=1}^{\alpha} \lambda_i \phi(x, y^i), \qquad \forall (x, \mu) \in \chi_0(\alpha, \lambda, y).$$

Therefore, we obtain

$$\sup_{(x, \mu) \in \chi_0(\alpha, \lambda, y)} \sum_{i=1}^{\alpha} \lambda_i \phi(x, y^i) \leqq \sum_{i=1}^{\alpha^*} \lambda_i^* \phi\left(x^*, y^{*^i}\right).$$

Thus $(\alpha^*, \lambda^*, y^*)$ and (x^*, μ^*) give an optimal solution to (D). Moreover, the optimal value of the (D) objective is $\sum_{i=1}^{\alpha^*}\lambda_i^*\phi(x^*, y^{*^i})$ which on using the facts that $y^{*^i} \in Y(x^*)$, $1 \leqq i \leqq \alpha^*$ and $\sum_{i=1}^{\alpha^*}\lambda_i^* = 1$ implies

$$\sum_{i=1}^{\alpha^*} \lambda_i^* \phi\left(x^*, y^{*^i}\right) = f(x^*).$$

Hence, the two problems (P) and (D) have the same extremal values.

(2) Let hypothesis (ii) hold, and let (x, μ) be any element of $\chi_0(\alpha^*, \lambda^*, y^*)$. Then (5.5) and (5.6) are satisfied. For $\mu \in \mathbb{R}_+^m$, $g_j(x^*) \leqq 0$, $1 \leqq j \leqq m$ and $\sum_{j=1}^{m} \mu_j g_j(x) \geqq 0$, we obtain

$$\sum_{j=1}^{m} \mu_j g_j(x^*) \leqq \sum_{j=1}^{m} \mu_j g_j(x)$$

which on using QCN-ness of $\sum_{j=1}^{m} \mu_j g_j(\cdot)$ implies

$$\sum_{j=1}^{m} \mu_j g_j^+(H_{x, x^*}(0)) \leqq 0$$

which along with (5.5) yields

$$\sum_{i=1}^{\alpha^*} \lambda_i^* \phi^+\left(H_{x, x^*}(0), y^{*^i}\right) \geqq 0.$$

On using PCN-ness of $\sum_{i=1}^{\alpha^*} \lambda_i^* \phi(\cdot, y^{*^i})$, we get

$$\sum_{i=1}^{\alpha^*} \lambda_i^* \phi\left(x^*, y^{*^i}\right) \geqq \sum_{i=1}^{\alpha^*} \lambda_i^* \phi\left(x, y^{*^i}\right)$$

Hence, (x^*, μ^*) attains the maximum of (D_{η^*}) where $\eta^* = (\alpha^*, \lambda^*, y^*)$.

Next, let $(x, \mu) \in \chi_0(\alpha, \lambda, y)$ for any $(\alpha, \lambda, y) \in \mathbb{G}$. Then (5.9) and (5.10) are satisfied. Therefore, we have

$$\sum_{j=1}^{m} \mu_j g_j(x^*) \leqq 0 \leqq \sum_{j=1}^{m} \mu_j g_j(x)$$

which in view of the QCN-ness of $\sum_{j=1}^{m} \mu_j g_j(\cdot)$ implies

$$\sum_{j=1}^{m} \mu_j g_j^+ \left(H_{x, x^*}(0)\right) \geqq 0.$$

The above relation along with (5.9) yields

$$\sum_{i=1}^{\alpha} \lambda_i \phi^+ \left(H_{x, x^*}(0), y^i\right) \geqq 0.$$

On using the PCN-ness of $\sum_{i=1}^{\alpha} \lambda_i \phi(\cdot, y^i)$, we have

$$\sum_{i=1}^{\alpha} \lambda_i \phi\left(x^*, y^i\right) \geqq \sum_{i=1}^{\alpha} \lambda_i \phi\left(x, y^i\right)$$

which is the same as (5.13). The remaining part of the proof is the same as that under hypothesis (i).

In case hypothesis (iii) or hypothesis (iv) hold, then the proof also runs on similar lines and hence is omitted.

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