The linear convergence of limit periodic continued fractions

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Abstract: The only linearly convergent continued fractions are the limit periodic ones.

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Let us consider the continued fraction
\[ \frac{a_1}{1 + \frac{a_2}{1 + \ldots}} \]
where the \( a_n \) are complex numbers.

Let \( C_n = \frac{A_n}{B_n} \) be its convergents. We set \( h_n = \frac{B_n}{B_{n-1}} \).

We assume that \( \lim_{n \to \infty} h_n = h \) is finite.

Theorem 1. If \( \exists a \in \mathbb{C}, \ a \neq -\frac{1}{4} + c \) with \( c \leq 0 \) such that \( \lim_{n \to \infty} a_n = a \) then \( \exists r \in \mathbb{C}, \ |r| < 1 \) such that \( \lim_{n \to \infty} (C_n - C)/(C_{n-1} - C) = r \).

Proof. If \( a \neq -\frac{1}{4} + c \) with \( c \leq 0 \) the two zeros of \( x^2 - x - a = 0 \) have distinct moduli. Since \( B_{n+1} = B_n + a_{n+1}B_{n-1} \) then, by Poincaré’s theorem [9], \( \exists h \in \mathbb{C} \) such that \( \lim_{n \to \infty} h_n = h \). Moreover it is easy to see that \( |h| \neq (\frac{1}{4} - c)^{1/2} \). But we have [10]
\[ \frac{\Delta C_n}{\Delta C_{n-1}} = -1 + 1/h_{n+1}. \]
Thus \( \exists r \in \mathbb{C} \) such that \( \lim_{n \to \infty} \Delta C_n/\Delta C_{n-1} = r = -1 + 1/h \). Moreover, since \( h \) is equal to the zero of greatest modulus of \( x^2 - x - a = 0 \), then \( |r| < 1 \). Then, by a result of Delahaye [2],
\[ \lim_{n \to \infty} (C_n - C)/(C_{n-1} - C) = r. \]

This theorem was given in [7] with a different proof. See also [8].

Let us now give the reciprocal of this result:

Theorem 2. If \( \exists r \in \mathbb{C}, \ r \neq -1 \) such that \( \lim_{n \to \infty} \Delta C_n/\Delta C_{n-1} = r \) then \( \exists a \in \mathbb{C}, \ such \ that \ lim_{n \to \infty} a_n = a = -r/(1 + r)^2 \).
Proof. From (1) we see that, if \( r \neq -1 \), \( \exists h \neq 0 \) and finite such that \( \lim_{n \to \infty} h_n = h \). But 
\[
h_{n+1} = 1 + a_{n+1}/h_n \quad \text{or} \quad h_n(h_{n+1} - 1) = a_{n+1}
\]
which shows that \( \exists a \in \mathbb{C} \) such that \( \lim_{n \to \infty} a_n = a \). \( \square \)

Let us study this reciprocal in more detail. As we saw before \( r \) and \( h \) are related by 
\[
h = \frac{1}{1 + r}.
\]
If \( r = e^{i\theta} \), that is if \( |r| = 1 \),
\[
h = \frac{1}{2} - i \frac{\sin \theta}{2(1 + \cos \theta)}.
\]
Hence \( |h|^2 = \frac{1}{4} + \sin^2 \theta / 4(1 + \cos \theta)^2 = \frac{1}{4} - c \) with \( c \leq 0 \). Thus, if \( |r| \neq 1 \), then \( |h| \neq (\frac{1}{4} - c)^{1/2} \) with \( c \leq 0 \). Let us now examine \( |a| \). If \( |r| = 1 \), then
\[
a = -\left( \frac{1}{2} - i \frac{\sin \theta}{2(1 + \cos \theta)} \right) \left( \frac{1}{2} + i \frac{\sin \theta}{2(1 + \cos \theta)} \right)
\]
\[
= -\frac{1}{4} - \frac{\sin^2 \theta}{4(1 + \cos \theta)^2} = -\frac{1}{4} + c.
\]
Finally, if \( |r| \neq 1 \) then \( a \neq -\frac{1}{4} + c \) with \( c \leq 0 \). This last result can be gathered with that of Theorem 1 and we get the:

**Theorem 3.** A necessary and sufficient condition that \( \exists r \in \mathbb{C}, \ |r| < 1 \) such that \( \lim_{n \to \infty} (C_n - C)/(C_{n-1} - C) = r \) is that \( \exists a \in \mathbb{C}, \ a \neq -\frac{1}{4} + c \) with \( c \leq 0 \) such that \( \lim_{n \to \infty} a_n = a \). Moreover \( a \) and \( r \) are related by \( a = -r/(1 + r)^2 \).

**Proof.** If \( \exists C \in \mathbb{C}, \exists r \in \mathbb{C}, \ r \neq 1 \) such that \( \lim_{n \to \infty} C_n = C \) and \( \lim_{n \to \infty} (C_n - C)/(C_{n-1} - C) = r \) then, by a result due to Delahaye [2], the ratio \( \Delta C_n/\Delta C_{n-1} \) has a limit and this limit is equal to \( r \).

By Theorem 2, if \( r \neq -1 \), the continued fraction is limit periodic. Moreover, as we saw above, if \( |r| \neq 1 \) then \( a \neq -\frac{1}{4} + c \) with \( c \leq 0 \) and the first part of the result follows from Theorem 1. The reciprocal is Theorem 1. \( \square \)

**Remarks.** Let us make some remarks on the respective values of \( a \) and \( r \):

(i) \( r = 0 \) if and only if \( a = 0 \). Since \( r = -1 + 1/h \), \( r \) is zero if and only if \( h \) equals 1. If \( h = 1 \) then \( h(h-1) = a = 0 \). Reciprocally if \( a = 0 \), the zeros of \( x^2 - x - a = 0 \) are 0 and 1 and thus, by Poincaré’s theorem, \( h \) is 0 or 1. If \( h = 0 \) then \( r \) is infinite which is impossible. Thus \( h = 1 \) which gives \( r = 0 \). Thus limit periodic continued fractions converge super linearly if and only if \( \lim_{n \to \infty} a_n = 0 \). In that case it is less crucial to be able to accelerate the convergence.

(ii) If \( r = 1 \) then \( a = -\frac{1}{4} \). This is the worst case since the convergence, when it occurs, is very slow (logarithmic convergence). Reciprocally if \( a = -\frac{1}{4} \), the two zeros of \( x^2 - x + \frac{1}{4} = 0 \) are equal to \( \frac{1}{2} \) and Poincaré’s theorem does not allow to conclude.

(iii) Another proof of Theorem 3 by using properties of linear functional transformations and the Koebe function was given to us by Waadeland.

Theorem 3 has important consequences concerning the convergence acceleration of limit periodic continued fractions. Since such fractions are linearly converging if \( a \neq 0 \) (if \( a = 0 \), the
continued fraction converges super linearly and, thus, is less important to accelerate) they can be accelerated in many different ways such as modifications, see [5] for a review, or various sequence transformations, [1,6].

On the other hand, continued fractions which are not 1-limit periodic will be difficult to accelerate. This follows from the theory of remanence of a set of sequences introduced by Delahaye and Germain Bonne [4]. It means that a universal algorithmic method for transforming \((C_n)\) into another sequence converging faster cannot exist for all continued fractions which are not 1-limit periodic (by algorithmic method it is meant a method which does not depend on asymptotic properties of \((C_n)\) but only on a finite number of its terms). Subsets of such continued fractions will have to be considered. Even in the case where the ratios \((C_n - C)/(C_{n-1} - C)\) remain bounded from below and above such a universal transformation cannot exist [3].

Finally let us mention that some similar results seem to exist for limit \(k\)-periodic continued fractions. For example it is easy to see that the even and odd parts of a limit 2-periodic continued fraction are limit periodic with the same asymptotic error coefficient. Obviously, by our Theorem 3, the converse is false.

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References