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Fixed Points of Asymptotically Linear Maps in Ordered Banach Spaces

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1. INTRODUCTION AND MAIN RESULTS

The purpose of this paper is to extend some fixed point theorems of M. A. Krasnosel'skii for asymptotically linear completely continuous maps leaving invariant a cone in a Banach space to strict set-contractions. The proofs will be based on the fixed point index and, as a by-product, we obtain new and simplified proofs of Krasnosel'skii's theorems also. Moreover, even in the case of completely continuous maps, some of our results improve the known theorems.

Throughout this paper all vector spaces will be over the reals. Let E be a normed vector space. A subset P of E is said to be a *cone* if it is closed, convex, invariant under multiplication by nonnegative real numbers, and if $P \cap (-P) = \{0\}$. Each cone P induces an antisymmetric, reflexive, transitive ordering in E by defining: $x \geqslant y$ if and only if $x - y \in P$. This ordering is compatible with the linear structure, i.e., $\alpha \in \mathbb{R}_+ := [0, \infty)$ and $x \geqslant 0$ imply $\alpha x \geqslant 0$ and, for every $z \in E$, $x \geqslant y$ implies $x + z \geqslant y + z$, and it is compatible with the topology, i.e., $x_j \geqslant 0$ and $x_j \to x$ imply $x \geqslant 0$. Let E be a normed vector space (resp. a Banach space), and let P be a cone in E. Then the pair (E, P) is called an *ordered normed vector space* (resp. *ordered Banach space*) *with positive cone* P if E is given the ordering induced by P. The elements $x \in \dot{P} := P \backslash \{0\}$ are said to be *positive* and we write $x > 0$ if $x \in \dot{P}$.

A cone is said to be *total* if $E = \overline{P - P}$ and *generating* if $E = P - P$. It is said to be *normal* if there exists $\delta \geqslant 1$ such that, for every pair $x, y \in P$, $\| x \| \leqslant \delta \| x + y \|$.

Let E, F be normed vector spaces. Then we denote by $L(E, F)$ the normed vector space of all continuous linear operators $u: E \to F$.

For every $u \in L(E) := L(E, E)$ we denote by $r(u)$ its spectral radius, i.e., $r(u) := \lim_{k\to\infty} \| u^k \|^{1/k}$.

Let P be a total cone in E. A map $f\colon P \to F$ is said to be *differentiable at $x \in P$ along P* if there exists $f'(x) \in L(E, F)$ such that

$$\lim_{\substack{h\in P\\ h\to 0}} \frac{\| f(x+h) - f(x) - f'(x)h \|}{\| h \|} = 0.$$

It is easily seen that $f'(x)$, the *derivative at x along P*, is uniquely determined. The map f is said to be *asymptotically linear along P* if there exists $f'(\infty) \in L(E, F)$ such that

$$\lim_{\substack{x\in P\\ \|x\|\to\infty}} \frac{\| f(x) - f'(\infty)x \|}{\| x \|} = 0.$$

Again, $f'(\infty)$ is uniquely determined and called the *derivative at infinity along P*.

Let X be a metric space, and let A be a bounded subset of X. Then we define $\gamma(A)$, the *measure of noncompactness of A*, to be the infimum of all positive numbers δ such that A can be covered by finitely many sets of diameter less than δ. Clearly, A is totally bounded if and only if $\gamma(A) = 0$.

Let X and Y be metric spaces with measures of noncompactness γ_X and γ_Y, respectively, and let D be a subset of X. A map $f\colon D \to Y$ is said to be an *α-set-contraction* if it is continuous and if there exists $\alpha \in \mathbb{R}_+$ such that, for every bounded subset A of D, $\gamma_Y(f(A)) \leqslant \alpha\gamma_X(A)$. It is said to be a *strict-set-contraction* if it is an *α-set-contraction* for some $\alpha < 1$. A map $f\colon D \to Y$ is said to be *completely continuous* if it is continuous and maps bounded subsets of D onto relatively compact subsets. Hence, every completely continuous map is a strict set-contraction.

Let (E, P) be an ordered normed vector space. Then a map $f\colon P \to E$ is said to be *increasing* if $x \leqslant y$ implies $f(x) \leqslant f(y)$.

After these preparations we are ready for the statement of our main results.

THEOREM 1. *Let (E, P) be an ordered Banach space with total positive cone and let $f\colon P \to P$ be a strict set-contraction which is asymptotically linear along P. Suppose that $f'(\infty)$ does not have a positive eigenfunction belonging to an eigenvalue greater or equal to* 1. *Then f has a fixed point. If, in addition, P is normal and f is increasing, then there exists a minimal fixed point $\bar{x}$ in the sense that, for every* *fixed point x of f, $\bar{x} \leqslant x$. Moreover, $\bar{x}$ can be computed iteratively by the method*

$$x_0 = 0, \qquad x_{j+1} = f(x_j), \qquad j = 0, 1, 2,...,$$

and the sequence (x_j) converges from below to $\bar{x}$.

Clearly, $f'(\infty)$ does not have a positive eigenfunction belonging to an eigenvalue greater or equal to 1 if $r(f'(\infty)) < 1$.

The first part of Theorem 1 generalizes [4, Theorem 4.10] where it has been supposed that f is completely continuous (In this connection it should be observed that our assumption that P is total, which guarantees the uniqueness of the derivative along P, contains no loss in generality since, in a more general setting, we may always restrict our considerations to the closed subspace $\overline{P - P}$ of E). The fact that there exists a minimal fixed point which can be computed iteratively, seems to be new even in the case where f is supposed to be completely continuous.

In case $f(0) = 0$, Theorem 1 does not guarantee the existence of a nonzero fixed point. This is done by the next theorem for whose formulation we shall use the following notation. Let (E, P) be an ordered normed vector space. Then we set

$L_1^+(E) := \{u \in L(E) \mid u$ has a positive eigenfunction belonging to an eigenvalue greater than 1 and no positive eigenfunction with eigenvalue equal to $1\}$

and

$L_1^-(E) := \{u \in L(E) \mid u$ does not have a positive eigenfunction belonging to an eigenvalue greater or equal to $1\}$.

The condition for $u \in L(E)$ to belong to $L_1^+(E)$ simplifies if it is known that u has at most one eigenvalue having a positive eigenfunction. For example, this is the case if u is e-positive in the sense of [4, Chapter 2(u_0-positive)].

THEOREM 2. *Let (E, P) be an ordered Banach space with total positive cone and let $f: P \to E$ be a strict set-contraction which is asymptotically linear along P. Suppose that there exists a fixed point $\bar{x}$ of f such that, for all $x \geqslant \bar{x}$, $f(x) \geqslant \bar{x}$ and such that f is differentiable at $\bar{x}$ along P. Then f has a fixed point $x^* > \bar{x}$ provided one of the following conditions is satisfied:*

(i) *$f'(\bar{x}) \in L_1^-(E)$ and $f'(\infty) \in L_1^+(E)$,*

(ii) *$f'(\bar{x}) \in L_1^+(E)$ and $f'(\infty) \in L_1^-(E)$.*

Clearly $x \geqslant \bar{x}$ implies $f(x) \geqslant \bar{x}$ whenever f is increasing.

Theorem 2(i) generalizes [4, Theorem 4.16] and Theorem 2(ii) generalizes [4, Theorem 4.11]. Recently [4, Theorem 4.11] has also been extended to strict set-contractions by Edmunds, Potter, and Stuart [2]. However, these authors had to assume that f maps all of E into P and that P is normal.

In case of a generating cone and of a completely continuous map conditions (i) and (ii) can be stated more simply.

THEOREM 3. *Let (E, P) be an ordered Banach space with generating positive cone, and let $f: P \to E$ be completely continuous and asymptotically linear along P. Suppose that there exists a fixed point $\bar{x}$ of f such that, for all $x \geqslant \bar{x}$, $f(x) \geqslant \bar{x}$ and such that f is differentiable at $\bar{x}$ along P. Then f has a fixed point $x^* > \bar{x}$ provided one of the following conditions is satisfied:*

(i) $r(f'(\bar{x})) < 1$, $r(f'(\infty)) > 1$, *and* 1 *is not an eigenvalue of* $f'(\infty)$ *having a positive eigenfunction,*

(ii) $r(f'(\infty)) < 1$, $r(f'(\bar{x})) > 1$, *and* 1 *is not an eigenvalue of* $f'(\bar{x})$ *having a positive eigenfunction.*

It is not difficult to apply these theorems to the study of maps depending on a parameter. Here we shall give two immediate applications only. A more detailed study of solutions of the equation $x = f(x, \lambda)$ as well as applications to nonlinear elliptic boundary value problems will be given in a forthcoming paper.

THEOREM 4. *Let (E, P) be an ordered Banach space with total positive cone and let $f: P \times \mathbb{R}_+ \to P$ be a map such that, for every $\lambda \in \mathbb{R}_+$, $f(\cdot, \lambda)$ is a strict set-contraction. Suppose that there exists $u \in L(E)$ such that*

$$f(x, \lambda) = \lambda u(x) + r(x, \lambda)$$

with $r(x, \lambda) = o(\| x \|)$ as $\| x \| \to \infty$ in P. Then there exists $\lambda^* \in \mathbb{R}_+ \cup \{\infty\}$ such that, for every $\lambda \in [0, \lambda^*)$, the map $f(\cdot, \lambda)$ has a fixed point. If, in addition, P is normal and $f(\cdot, \lambda)$ is increasing, then there exists a minimal fixed point $\bar{x}_\lambda$ of $f(\cdot, \lambda)$. Moreover, $\bar{x}_\lambda$ can be computed iteratively by means of the method.

$$x_0 = 0, \qquad x_{j+1} = f(x_j, \lambda), \qquad j = 0, 1, 2, \ldots,$$

and the sequence (x_j) converges to $\bar{x}_\lambda$ from below. Finally suppose that, in addition, $\lambda < \mu$ implies, for all $x \in P$, $f(x, \lambda) \leqslant f(x, \mu)$. Then $\bar{x}_\lambda \leqslant \bar{x}_\mu$, with strict inequality if, for all $x \in P$, $f(x, \lambda) < f(x, \mu)$.

Again, in the case where $f(0, \cdot) = 0$ the problem of the existence of nontrivial fixed points is more complicated. In the statement of the next theorem generalizing [4, Theorem 5.1] we shall use the following *convention*: Whenever $\alpha, \beta \in \mathbb{R}_+ \cup \{\infty\}$ are distinct, $\lambda \in (\alpha, \beta)$ is understood to imply $\alpha < \lambda < \beta$ if $\alpha < \beta$ and $\beta < \lambda < \alpha$ if $\beta < \alpha$.

THEOREM 5. *Let (E, P) be an ordered Banach space with total positive cone and let $f: P \times \mathbb{R}_+ \to P$ be a map such that $f(0, \cdot) = 0$ and such that, for every $\lambda \in \mathbb{R}_+$, $f(\cdot, \lambda)$ is a strict set-contraction. Suppose that there exist $u_0, u_\infty \in L(E)$ such that, for $m = 0, \infty$ and every $\lambda \in \mathbb{R}_+$,*

$$f(x, \lambda) = \lambda u_m(x) + r_m(x, \lambda)$$

with $r_m(x, \lambda) = o(\| x \|)$ as $\| x \| \to m$ in P. Moreover suppose that there exists a unique nonnegative eigenvalue μ_m of u_m having a positive eigenfunction. Then, if $\mu_0 \neq \mu_\infty$, for every

$$\lambda \in \left(\frac{1}{\mu_0}, \frac{1}{\mu_\infty}\right),$$

there exists a positive fixed point of $f(\cdot, \lambda)$.

We close this section with a remark concerning the method of proof used in this paper. The main tool we shall employ is the fixed point index $i(f, U, P)$ for a strict set-contraction f defined on the closure of the relatively open subset U of the cone P with values in P. In fact we shall use subsets of the form $\rho B \cap P$ only where B denotes the open unit ball in E. Now, in the special case where completely continuous maps are considered only, it is easily seen that $i(f, \rho B \cap P, P)$ can be replaced everywhere by $d(id - f \circ p, \rho B, 0)$, where d denotes the well known Leray–Schauder degree and where $p: E \to P$ denotes an arbitrary retraction. Hence, by this way we obtain new and simple proofs of Krasnosel'skii's theorems. Furthermore, since the Leray–Schauder degree for completely continuous maps is defined in normed vector spaces, we obtain the following result: *If f is completely continuous, then Theorems* 1, 2, 4, and 5 *remain valid if the hypothesis that E is complete is omitted.*

2. PROOF OF THE THEOREMS

Let E be a normed vector space with open unit ball B, denote by γ the measure of noncompactness of E, and let A, A_1, A_2 be bounded subsets of E. Then it is easy to see that $\gamma(B) \leqslant 2$, $\gamma(A_1 \cup A_2) \leqslant$

$\max\{\gamma(A_1), \gamma(A_2)\}$, $\gamma(A_1 + A_2) \leqslant \gamma(A_1) + \gamma(A_2)$, that $A_1 \subset A_2$ implies $\gamma(A_1) \leqslant \gamma(A_2)$, and that, for every $\lambda \in \mathbb{R}_+$, $\gamma(\lambda A) = \lambda\gamma(A)$.

LEMMA 1. *Let E and F be normed vector spaces and let P be a total cone in E. Let $f: P \to F$ be an α-set-contraction which is asymptotically linear along P and, for some $x \in P$, differentiable at x along P. Then $f'(\infty) \mid P$ and $f'(x) \mid P$ are α-set-contractions.*

Proof. The proof that $f'(x) \mid P$ is an α-set-contraction is, with the obvious modifications, literally the same as the proof of Lemma 4 in [7]. Hence, we shall consider the derivative at infinity along P only.

By assumption there exists $r: P \to F$ with $f = f'(\infty) + r$ and $r(x) = o(\| x \|)$ as $\| x \| \to \infty$ in P. We set $u := f'(\infty)$, and we denote by γ simultaneously the measures of noncompactness in E and in F.

Let $A \subset P$ be a bounded subset such that, for some $\rho > 0$ and all $x \in A$, $\| x \| \geqslant \rho$. Set $\sigma := \sup\{\| x \| \mid x \in A\}$, and let $\epsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that, for all $x \in P$ with $\| x \| \geqslant \delta$, $\| r(x)\| \leqslant \epsilon \| x \|/2\sigma$. Hence, for every $\lambda \geqslant \delta/\rho$, $r(\lambda A) \subset (\lambda\epsilon/2)B$ and

$$u(\lambda A) \subset f(\lambda A) - r(\lambda A) \subset f(\lambda A) + (\lambda\epsilon/2)B.$$

These inclusions imply

$$\begin{aligned}\lambda\gamma(u(A)) &= \gamma(u(\lambda A)) \leqslant \gamma(f(\lambda A)) + (\lambda\epsilon/2)\,\gamma(B)\\ &\leqslant \alpha\gamma(\lambda A) + \lambda\epsilon = \lambda(\alpha\gamma(A) + \epsilon).\end{aligned}$$

Hence $\gamma(u(A)) \leqslant \alpha\gamma(A) + \epsilon$ and, $\epsilon > 0$ being arbitrary, $\gamma(u(A)) \leqslant \alpha\gamma(A)$.

Now let $A \subset P$ be an arbitrary bounded subset, let $\epsilon > 0$ be arbitrary, set $\rho := \epsilon/2 \| u \|$, and define $A_1 := A \cap \rho B$ and $A := A \backslash A_1$. Then $u(A_1) \subset \| u \| \rho B = (\epsilon/2)B$, and, hence, $\gamma(u(A_1)) \leqslant \epsilon$. By the considerations above, $\gamma(u(A_2)) \leqslant \alpha\gamma(A_2) \leqslant \alpha\gamma(A)$. Hence,

$$\begin{aligned}\gamma(u(A)) &= \gamma(u(A_1) \cup u(A_2))\\ &\leqslant \max\{\gamma(u(A_1)), \gamma(u(A_2))\} \leqslant \max\{\epsilon, \alpha\gamma(A)\}\end{aligned}$$

and, $\epsilon > 0$ being arbitrary, $\gamma(u(A)) \leqslant \alpha\gamma(A)$. This proves the statement.

Remark 1. Lemma 1 generalizes Krasnosel'skii's result [4, Lemma 3.1] for completely continuous maps. In [4, Chapter 3] it is shown that, if f is completely continuous, E is complete, and P is generating, the derivative along P at x (resp. at infinity) is a completely continuous linear operator. This result is based on the fact that, in case of an ordered Banach space (E, P) with generating cone P, there exists $\beta \geqslant 1$ such that, for every $x \in E$, there are $y, z \in P$ with $x = y - z$ and $\max(\| y \|, \| z \|) \leqslant \beta \| x \|$ (compare [3, Theorem 3.5.2] and [8, p. 222]). However, in the general case, where $f: P \to F$ is supposed to be an α-set-contraction, this implies only that $f'(x)$ (resp. $f'(\infty)$) is a $2\,\alpha\beta$-set-contraction where the value of β is not known in general.

Let X be a closed convex subset of some Banach space, and let U be a bounded open subset of X. Let $f: \bar{U} \to X$ be a strict set-contraction with no fixed points on $\partial U := \bar{U} \backslash U$. Then it has been shown by Nussbaum [6] that there exists an integer $i(f, U, X)$, *the fixed point index* of f having the following properties:

(i) If f is a constant map with $f(\bar{U}) \subset U$, then $i(f, U, X) = 1$.

(ii) If U_1 and U_2 are disjoint open subsets of U containing all the fixed points of f, then $i(f, U, X) = i(f, U_1, X) + i(f, U_2, X)$. In particular, if f has no fixed points, then $i(f, U, X) = 0$.

(iii) Let $f_0, f_1: \bar{U} \to X$ be strict set-contractions and suppose that, for every $\lambda \in [0, 1]$, $(1 - \lambda) f_0 + \lambda f_1$ has no fixed points on ∂U. Then $i(f_0, U, X) = i(f_1, U, X)$.

These properties are the special cases we shall use of more general properties given in [6] (for a proof of (i) see [1]).

Let D be a closed bounded subset of some Banach space E and let $f: D \to E$ be a strict set-contraction. Then it is known (e.g. [6]) that $id - f$ is proper, i.e., for every compact $K \subset E$, the preimage $(id - f)^{-1}(K)$ is compact in D. Hence, $id - f$ is *closed*, i.e., it maps closed subsets of D onto closed subsets of E.

In the following we shall denote by B the open unit ball in E, by S its boundary, and we shall set $i(f, U) := i(f, U, P)$ where U is a bounded open subset of P.

LEMMA 2. *Let (E, P) be an ordered Banach space with total positive cone and let $f: P \to P$ be a strict set-contraction.*

(a) *Suppose that f is asymptotically linear along P. Then there exists $\rho_\infty > 0$ such that, for all $\rho \geqslant \rho_\infty$,*

(i) $i(f, \rho B \cap P) = 1$ *if* $f'(\infty) \in L_1^-(E)$,

(ii) $i(f, \rho B \cap P) = 0$ *if* $f'(\infty) \in L_1^+(E)$.

(b) *Suppose that $f(0) = 0$ and that f is differentiable at 0 along P. Then there exists $\rho_0 > 0$ such that, for all $\rho \in (0, \rho_0]$,*

(iii) $i(f, \rho B \cap P) = 1$ *if* $f'(0) \in L_1^-(E)$,

(iv) $i(f, \rho B \cap P) = 0$ *if* $f'(0) \in L_1^+(E)$.

Proof. Let $u \in L_1^-(E) \cup L_1^+(E)$ be given and suppose that $u \mid P$ is a strict set-contraction. Then, by the closedness of $(id - u) \mid P$ on closed bounded sets, $(id - u)(S \cap P)$ is closed. Hence, since $0 \notin (id - u)(S \cap P)$, there exists $\alpha > 0$ such that, for all $x \in P$, $\| x - u(x)\| \geqslant \alpha \| x \|$. Hence, under the hypotheses of Lemma 2, for $m = 0, \infty$, there exists $\alpha_m > 0$ such that, for all $x \in P$, $\| x - f'(m)x \| \geqslant \alpha_m \| x \|$. Moreover, it is easily seen (compare [4]) that $f'(m)P \subset P$, i.e., the operators $f'(m)$ are positive.

Choose $\rho_m > 0$ such that, for all $x \in P$ with $\| x \| \geqslant \rho_\infty$, respectively, $\| x \| \leqslant \rho_0$, $\|f(x) - f'(m)x \| < (\alpha_m/2) \| x \|$. Then we claim that, for every $\rho \geqslant \rho_\infty$, respectively $\rho \leqslant \rho_0$, every $y \in P$ with $\| y \|/\rho < \alpha_m/2$, and every $\lambda \in [0, 1]$,

$$(1 - \lambda)(f'(m) + y) + \lambda f$$

has no fixed point on $\rho S \cap P$. Indeed, for $x \in \rho S \cap P$,

$$\begin{aligned} \| x - (1 - \lambda)(f'(m) + y) - \lambda f(x)\| &\geqslant \| x - f'(m)x \| \\ &\quad - \| f(x) - f'(m)x \| - \| y \| \\ &\geqslant \rho(\alpha_m - \alpha_m/2 - \| y \|/\rho) > 0. \end{aligned}$$

Hence, for every $y \in P$ with $\| y \| < \rho\alpha_m/2$,

$$i(f, \rho B \cap P) = i(f'(m) + y, \rho B \cap P).$$

Cases (i) *and* (iii). In this case we set $y = 0$ and we observe that $f'(m) \in L_1^-(E)$ implies that, for every $\lambda \in [0, 1]$, $x - \lambda f'(m)x = 0$ does not have a positive solution. Hence,

$$i(f, \rho B \cap P) = i(f'(m), \rho B \cap P) = i(0, \rho B \cap P) = 1$$

by property (i) of the fixed point index.

Cases (ii) *and* (iv). In this case we denote by h_m a positive eigenfunction of $f'(m)$ belonging to an eigenvalue $\lambda_m > 1$. Then, for every $\alpha > 0$, the equation $x - f'(m)x = \alpha h_m$ does not have a solution in P. Indeed, suppose that there exists $x_m \in P$ with $x_m - f'(m)\, x_m = \alpha h_m$. Then $x_m > 0$ and there exists $\tau_m \geqslant 0$ such that $x_m \geqslant \tau_m h_m$ and, for all $\tau > \tau_m$, $x_m \not\geqslant \tau h_m$. But then

$$x_m = f'(m)\, x_m + \alpha h_m \geqslant f'(m)(\tau_m h_m) + \alpha h_m > (\tau_m + \alpha)\, h_m$$

which contradicts the maximality of τ_m.

Now, by setting $y = \alpha h_m$ with $\alpha > 0$ sufficiently small, we find

$$i(f, \rho B \cap P) = i(f'(m) + \alpha h_m, \rho B \cap P) = 0$$

by condition (ii) of the fixed point index. Q.E.D.

LEMMA 3. *Let (E, P) be an ordered Banach space with a normal positive cone and let $f\colon P \to P$ be an increasing strict set-contraction. Suppose that f has a fixed point. Then f has a minimal fixed point $\bar{x}$ and $\bar{x}$ can be computed iteratively by the method*

$$x_0 = 0, \qquad x_{j+1} = f(x_j), \qquad j = 0, 1, 2,\ldots .$$

Lastly, the sequence (x_j) converges to $\bar{x}$ from below.

Proof. Let $x \in P$ be a fixed point. Then, P being normal, $[0, x] := \{y \in E \mid 0 \leqslant y \leqslant x\}$ is a bounded order interval (e.g. [3, 4, 8]) with $f(0) \geqslant 0$ and $f(x) \leqslant x$. Hence, by Theorem 3 of [1], there exists a minimal fixed point $\bar{x}$ in $[0, x]$, and $\bar{x}$ can be computed iteratively as stated. This being true for every fixed point x of f, the statement follows. Q.E.D.

Proof of Theorem 1. By Lemma 2, for $\rho > 0$ sufficiently large, $i(f, \rho B \cap P) = 1$. Hence, by property (ii) of the fixed point index, f has a fixed point in P. The rest of the statement follows from Lemma 3. Q.E.D.

Proof of Theorem 2. Define $g\colon P \to P$ by

$$g(x) := f(x + \bar{x}) - f(\bar{x}) = f(x + \bar{x}) - \bar{x}.$$

Then g is a strict set-contraction which is asymptotically linear along P with $g'(\infty) = f'(\infty)$ (compare [4, p. 105]), and g is differentiable at 0 along P with $g'(0) = f'(\bar{x})$. Hence, without loss of generality we may assume that $\bar{x} = 0$.

By Lemma 2, there exist $\rho_0 > 0$ and $\rho_\infty > 0$ with $\rho_0 < \rho_\infty$ such that $i(f, \rho_m B \cap P)$, $m = 0, \infty$, is well defined. Hence, by property (ii) of the fixed point index,

$$i(f, (\rho_\infty B \backslash \rho_0 \bar{B}) \cap P) = i(f, \rho_\infty B \cap P) - i(f, \rho_0 B \cap P)$$

and, by Lemma 2, this expression is equal to -1 in case (i) and equal to 1 in case (ii). Therefore, in each case f has a fixed point in $\rho_\infty B \backslash \rho_0 \bar{B}$, i.e., a nonzero fixed point. Q.E.D.

Proof of Theorem 3. Since P is generating, E is complete, and f is completely continuous, $f'(m)$, $m = 0, \infty$, is completely continuous (compare Remark 1). Hence, $f'(m)$ being positive, by a well known theorem of Krein and Rutman [5] (see also [3, 4, 8]), $\lambda_m := r(f'(m)) > 1$ implies that λ_m is an eigenvalue of $f'(m)$ having a positive eigenfunction. Hence, the hypotheses of Theorem 2 are satisfied. Q.E.D.

Proof of Theorem 4. Clearly, for every $\lambda \in \mathbb{R}_+$ with $\lambda r(u) < 1$, the hypotheses of Theorem 1 are satisfied. Hence everything save the last part of the statement follows.

Now suppose that $\lambda < \mu$ and let $\bar{x}_\mu$ be the minimal fixed point of $f(\cdot, \mu)$. Then

$$\bar{x}_\mu = f(\bar{x}_\mu, \mu) \geqslant f(\bar{x}_\mu, \lambda)$$

and $f(0, \lambda) \geqslant 0$. Hence, by [1, Theorem 3], $f(\cdot, \lambda)$ has a fixed point in $[0, \bar{x}_\mu]$. Now the statement follows. Q.E.D.

Proof of Theorem 5. It suffices to observe that the stated inequalities for λ imply that $f(\cdot, \lambda)$ satisfies the hypotheses of Theorem 2. Q.E.D.

Remark 2. It is easily seen that for the convergence of the iteration method for the computation of the minimal fixed point the completeness of E is not needed if f is completely continuous. This justifies the statement at the end of the preceeding paragraph.

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