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Tracking chains of Σ_2-elementarity[☆]

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ABSTRACT

We apply the ordinal arithmetical tools that were developed in Wilken (2007) [10] and Carlson and Wilken (in press) [4] in order to introduce *tracking chains* as the crucial means in the arithmetical analysis of (pure) elementary patterns of resemblance of order 2; see Carlson (2001) [2], Carlson (2009) [3], and Carlson and Wilken (in preparation) [5]. Although underlying heuristics for an analysis of Σ_2-elementarity within the structure $\mathcal{R}_2$ is given in [5], this article is independent of [5] and provides a complete arithmetical analysis of the structure $\mathcal{R}_2$ below the least ordinal α such that any pure pattern of order 2 has a covering below α. α is shown to be the proof-theoretic ordinal of $\mathrm{KP}\ell_0$.

1. Introduction

In order to integrate and locate the present article into the general program on patterns of embeddings (cf. [2,3]) we commence with the central definition of structures of ordinals in which elementary patterns of resemblance arise.

Definition 1.1. Let $\mathcal{R}_0 := (\mathrm{Ord}; \le)$ be the structure of all ordinals with the usual ordering $\le$. Setting $\le_0 := \le$, for $n < \omega$ the structure $\mathcal{R}_0$ is extended to

$$\mathcal{R}_n := \left(\mathrm{Ord}; (\le_i)_{i\le n}\right)$$

where the relations $\le_i$ for $i = 1, \ldots, n$ are defined simultaneously by the (in β) inductive definition

$$\alpha \le_i \beta :\Leftrightarrow \left(\alpha; (\le_j)_{j\le n}\right) \preceq_{\Sigma_i} \left(\beta; (\le_j)_{j\le n}\right)$$

and $\preceq_{\Sigma_i}$ is the usual notion of Σ_i-elementary substructure. Similarly,

$$\mathcal{R}_\omega := (\mathrm{Ord}; (\le_i)_{i<\omega})$$

is defined by

$$\alpha \le_i \beta :\Leftrightarrow \left(\alpha; (\le_j)_{j<\omega}\right) \preceq_{\Sigma_i} \left(\beta; (\le_j)_{j<\omega}\right).$$

Note that we have a more liberal notion of structure in that we consider partial (possibly empty) class structures, identifying an ordinal α with the set $\{\gamma \mid \gamma < \alpha\}$.

Immediate consequences of the notion of Σ_i-elementary substructure are summarized in the following.

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Lemma 1.2. *For the first four assertions below fix any of the $\mathcal{R}$-structures introduced above.*

1. $\le_i$ *is a forest, i.e. a partial ordering in which the sets of* $\le_i$*-predecessors of an element are linearly ordered by* $\le_i$.
2. $\le_{i+1}$ *respects* $\le_i$:
   - $\le_{i+1} \subseteq \le_i$ *and*
   - $\alpha \le_i \beta \le_i \gamma \;\&\; \alpha \le_{i+1} \gamma \quad \Rightarrow \quad \alpha \le_{i+1} \beta$.
3. $\{\beta \mid \alpha \le_1 \beta\}$ *is a closed interval.*
4. $\{\beta \mid \alpha \le_2 \beta\}$ *is closed.*
5. *For* $n < m \le \omega$ *and the least* β *such that there exists an* α *with* $\mathcal{R}_m \models \alpha <_n \beta$ *we have*

$$\mathcal{R}_n \cap \beta = \mathcal{R}_m \restriction_{(\le_i)_{i\le n}} \cap \beta.$$

Proof. Straightforward. □

Interpreting 0 and + as graphs of the ordinal 0 and ordinal addition restricted to the respective universe, the above definition can be modified to extend the additive structure $\mathcal{R}_0^+ := (\mathrm{Ord}; 0, +, \le)$ to structures $\mathcal{R}_n^+$ and $\mathcal{R}_\omega^+$ where the notion of Σ_i-elementary substructure is based on the respective language including 0, +. The structure $\mathcal{R}_1^+$ and variants were first studied in [2] and later analyzed using familiar ordinal arithmetic in [9,11]. The general setting of underlying Ehrenfeucht–Mostowski structures was introduced in [3] and taken as a basis to investigate a variant of $\mathcal{R}_2^{\mathrm{EM}}$, where EM indicates the underlying Ehrenfeucht–Mostowski structure. The above lemma generalizes to $\mathcal{R}$-structures with underlying arithmetic structure.

This article is motivated by the ordinal arithmetical analysis of a special case of ordinal structures accommodating elementary patterns of resemblance, namely $\mathcal{R}_2$. As will be shown elsewhere, many ideas applied in this article will generalize to the higher levels. Unless explicitly stated otherwise, in this article we are going to specifically refer to the structure $\mathcal{R}_2 = (\mathrm{Ord}; \le, \le_1, \le_2)$. In the light of part 5 of the above lemma we will be able to naturally build upon the approach and results of [1], however, we will obtain alternative proofs. Heuristics of the analysis of $\mathcal{R}_2$ is exposed in [5], however, in this article we directly deal with elementary substructurehood.

The article is organized as follows: In the preliminaries section we are going to introduce those basic arithmetical notions that will be used throughout the entire paper. For the more sophisticated means of ordinal arithmetic used in this article the reader is referred to [10,4]. In Section 3 we establish a structure on the additive principal numbers below the proof-theoretic ordinal of $\mathrm{KP}\ell_0$ which will frequently be denoted by 1^∞. The structure introduced in Section 3 will serve as the backbone for the central Definition 4.4 in Section 4 which introduces functions on the ordinals below 1^∞ that will turn out to characterize the enumeration of (suitably relativized) connectivity components of the relations $\le_1$ and $\le_2$ in Section 7, as can be read off from Theorem 7.9 and its corollaries, on the basis of Section 6. Though specifically motivated by the analysis of $\mathcal{R}_2$, the "facet" structure of ordinals that is introduced in Section 4 and revealed in Sections 5 and 6 is of interest in its own right. In Section 7 we finally apply the apparatus built up in the earlier sections in order to analyze the structure $\mathcal{R}_2$ up to the least ordinal α such that any pure pattern of order 2 is covered below α; see Theorem 7.9 and Corollary 7.13. This ordinal α is shown in this article to be 1^∞; see Corollary 7.14. The present article provides the means for work in progress which will show that 1^∞ is equal to the core of $\mathcal{R}_2$, the union of all isominimal substructures of $\mathcal{R}_2$; see [3]. There we are going to show that $1^\infty \le_1 \beta$ for any $\beta \ge 1^\infty$. Moreover, the relations $\le_1$ and $\le_2$ will be arithmetically characterized within the entire $\mathcal{R}_2$, and the isominimal substructures of $\mathcal{R}_2$ will be characterized arithmetically.

2. Preliminaries

We presume familiarity with basics of ordinal arithmetic (see e.g. [6] for a comprehensive introduction) and the ordinal arithmetical tools developed in [10] and Section 5 of [4]. See the index at the end of [10] for quick access to its terminology, which is not included in this article's index.

For α with additive normal form $\alpha_1 + \cdots + \alpha_n$, according to the terminology in [10] also written as $\alpha =_{\mathrm{ANF}} \alpha_1 + \cdots + \alpha_n$, we define $\mathrm{mc}(\alpha) := \alpha_1$ and $\mathrm{end}(\alpha) := \alpha_n$. We set $\mathrm{end}(0) := 0$. As usual let $\alpha \dot{-} \beta$ be 0 if $\beta \ge \alpha$, γ if $\beta < \alpha$ and there exists the minimal γ s.t. $\alpha = \gamma + \beta$, and α otherwise.

While $\mathbb{P}$, $\mathbb{L}$, and $\mathbb{E}$ denote the classes of additive principal numbers, their limits, and epsilon numbers, respectively, let $\mathbb{M}$ denote the class of *multiplicative principal numbers*, i.e. the positive ordinals which are closed under ordinal multiplication. For any class $\mathbb{X}$ of ordinals and any ordinal α we sometimes use the abbreviation $\mathbb{X}^{>\alpha}$ for the class of ordinals in $\mathbb{X}$ which are strictly greater than α. Expressions such as $\mathbb{X}^{\le\alpha}$ are defined likewise. Note that

$$\mathbb{M} = \{1\} \cup \{\omega^{\omega^\eta} \mid \eta \in \mathrm{Ord}\}.$$

For $\alpha \in \mathrm{Ord}$ we denote the least multiplicative principal number greater than α by $\alpha^{\mathbb{M}}$. Notice that if $\alpha \in \mathbb{P}$, $\alpha > 1$, say $\alpha = \omega^{\alpha'}$, we have $\alpha^{\mathbb{M}} = \alpha^\omega = \omega^{\alpha'\cdot\omega}$.

For $\alpha \in \mathbb{P}$ we use the following notations for the notion of *multiplicative normal form*:

- $\alpha =_{\mathrm{NF}} \eta \cdot \xi$ if and only if $\xi = \omega^{\xi_0} \in \mathbb{M}$ (i.e. $\xi_0 \in \{0\} \cup \mathbb{P}$) and either $\eta = 1$ or $\eta = \omega^{\eta_1 + \cdots + \eta_n}$ such that $\eta_1 + \cdots + \eta_n + \xi_0$ is in additive normal form (i.e. $\eta_1, \ldots, \eta_n, \xi_0 \in \mathbb{P}$ and $\eta_1 \ge \cdots \ge \eta_n \ge \xi_0$).
- $\alpha =_{\mathrm{MNF}} \alpha_1 \cdots \alpha_k$ if and only if $\alpha_1, \ldots, \alpha_k$ is the unique decreasing sequence of multiplicative principal numbers whose product is equal to α.

For $\alpha \in \mathbb{P}$, $\alpha =_{\mathrm{MNF}} \alpha_1 \cdots \alpha_k$, we write $\mathrm{lf}(\alpha)$ for α_k. Note that if $\alpha \in \mathbb{P} - \mathbb{M}$ then $\mathrm{lf}(\alpha) \in \mathbb{M}^{>1}$ and $\alpha =_{\mathrm{NF}} \bar{\alpha} \cdot \mathrm{lf}(\alpha)$ where the definition of $\bar{\alpha}$ given in [10] for limits of additive principal numbers is extended to ordinals α of a form $\alpha = \omega^{\alpha'+1}$ by $\bar{\alpha} := \omega^{\alpha'}$; see Section 5 of [4].

Given ordinals α, β with $\alpha \leq \beta$ we write $-\alpha + \beta$ for the unique γ such that $\alpha + \gamma = \beta$. Given $\alpha, \beta \in \mathbb{P}$ with $\alpha \leq \beta$ we write $(1/\alpha) \cdot \beta$ for the uniquely determined ordinal $\gamma \leq \beta$ such that $\alpha \cdot \gamma = \beta$. Note that with the representations $\alpha = \omega^{\alpha'}$ and $\beta = \omega^{\beta'}$ we have

$$(1/\alpha) \cdot \beta = \omega^{-\alpha'+\beta'}.$$

If α is of a form $\omega^{\alpha'}$ then we write $\log(\alpha)$ for α' and set $\log(0) := 0$, so that for an arbitrary ordinal β we have $\mathrm{logend}(\beta) = \log(\mathrm{end}(\beta))$.

We denote the (strict) lexicographic ordering on sequences of objects (ordered by $<$) by $\leq_{\mathrm{lex}}$ ($<_{\mathrm{lex}}$).

Settings of relativization are given by ordinals from $\mathbb{E}_1 := \{1\} \cup \mathbb{E}$ and frequently indicated by Greek letters, preferably σ or τ. Clearly, in this context $\tau = 1$ denotes the trivial setting of relativization. For a setting τ of relativization we define $\tau^\infty := \mathrm{T}^\tau \cap \Omega_1$. In order to avoid confusion, in the present article we will *not* use the notation $\tau_0 := 1^\infty$, $\tau_{\xi+1} := \tau_\xi^\infty$, and $\tau_\lambda := \sup\{\tau_\xi \mid \xi < \lambda\}$ for $\lambda \in \mathrm{Lim}$ as defined in [10]. Indeed, our considerations will mostly be confined to the segment 1^∞.

As in [11], by $\mathrm{lh}(\alpha)$ we denote the maximum ordinal $\beta \geq \alpha$ such that $\alpha \leq_1 \beta$ if that exists, and ∞ otherwise. We say that α is τ-$\leq_i$-minimal if there does not exist any $\beta \in (\tau, \alpha)$ such that $\beta \leq_i \alpha$.

3. Tracking sequences

The notion of tracking sequence introduced in this section will provide us with a coarse-grained raster which operates on the additive principal numbers below 1^∞. It will turn out in the end that this already admits a rough orientation within the structure $\mathcal{R}_2$. More precisely, it will be shown that any additive principal number α in the core of $\mathcal{R}_2$ is the last element of a finite increasing $\leq_1$-chain $\alpha_1, \ldots, \alpha_n$ that starts with a $\leq_1$-minimal ordinal and continues (in case of $n > 1$) with $\leq_2$-connected ordinals $\alpha_2, \ldots, \alpha_n$ where α_2 is $\leq_2$-minimal with $\mathrm{lh}(\alpha_1) = \mathrm{lh}(\alpha_2)$ and α_{i+1} is α_i-$\leq_2$-minimal for $i = 2, \ldots, n-1$. The *tracking sequence* for α yields the indices of appropriate enumeration functions of (relativized) $\leq_1$- and $\leq_2$-connectivity components that will be characterized in purely arithmetical terms in the next section. The semantical correctness of this arithmetical characterization will be shown at the end of this article.

It will be shown that the indicator functions defined below are crucial in characterizing those (relativized) $\leq_2$-connectivity components which $\leq_1$-connect back to the $\leq_1$-component they started from (cf. also the comment preceding Definition 4.4).

Definition 3.1. For $\tau \in \mathbb{E}$ the **indicator function** $\chi^\tau : \mathrm{T}^\tau \to \{0, 1\}$ is defined by

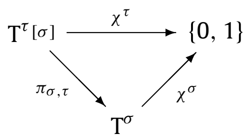
- $\chi^\tau(\xi) := 0$ for parameters $\xi < \tau$
- $\chi^\tau(\tau) := 1$
- $\chi^\tau(\eta + \xi) := \chi^\tau(\xi)$ if $\eta + \xi > \tau$ is in normal form
- Let $i < \omega$ and $\xi = \Delta + \eta \in \mathrm{dom}(\vartheta_i)$ where $\eta < \Omega_{i+1} \mid \Delta$ with $\xi > 0$ in case of $i = 0$.
  - $\chi^\tau(\vartheta_i(\xi)) := \chi^\tau(\Delta)$ if $\eta = \sup_{\sigma<\eta} \vartheta_i(\Delta + \sigma)$ or $\mathrm{logend}(\eta) = 0$
  - $\chi^\tau(\vartheta_i(\xi)) := \chi^\tau(\xi)$ otherwise.

Let $\check{\chi}^\tau : \mathrm{T}^\tau \to \{0, 1\}$ be the dual indicator function, i.e. $\check{\chi}^\tau := 1 - \chi^\tau$.

Remark. An essential property of the χ-indicator will be revealed in Corollary 5.6.

The next lemma shows an important uniformity property of the χ-operator, namely that it commutes with respect to base transformation. Recall the concept of base transformation from Section 5 of [10].

Lemma 3.2. *Let $\sigma, \tau \in \mathbb{E}$, $\sigma < \tau$, and $\alpha \in \mathrm{T}^{\tau[\sigma]}$. Then $\chi^\sigma(\pi_{\sigma,\tau}(\alpha)) = \chi^\tau(\alpha)$, i.e. the following diagram is commutative:*



The analogue statement holds for $\check{\chi}^\tau$.

Proof. Straightforward, cf. the proof of Lemma 5.6 in [10]. □

Recall the operator ζ^τ_α which indicates the degree of thinning out limit points as well as the cofinality operators $\iota_{\tau,\alpha}$ and λ^τ_α from [10]. We show a crucial property of the indicator functions defined above.

Lemma 3.3. *Let $\tau \in \mathbb{E}$ and $\alpha = \vartheta^\tau(\Delta + \eta) > \tau$.*

(a) *$\chi^\tau(\alpha)$ is equal to each of the following: $\chi^\tau(\beta + \alpha)$ for all $\beta < \tau^\infty$, $\chi^\tau(\mathrm{logend}(\alpha))$, $\chi^\tau(\omega^\alpha)$, $\chi^\tau(\beta \cdot \alpha)$ for all $\beta \in (0, \tau^\infty)$, $\chi^\tau((1/\beta) \cdot \alpha)$ for all $\beta \in \mathbb{P}^{<\alpha}$, and $\chi^\tau(\lambda^\tau_\alpha)$.*
(b) *If $\alpha \in \mathbb{E}$ then for all $\xi \in \mathrm{T}^\tau_{\restriction\alpha}$ such that $\chi^\alpha(\iota_{\tau,\alpha}(\xi)) = 1$ we have $\chi^\tau(\xi) = 0$.*

Proof. Part (a) follows once we show the claims concerning logend and λ^τ. In order to verify the equality $\chi^\tau(\mathrm{logend}(\alpha)) = \chi^\tau(\alpha)$ we use Lemma 4.10 of [10]. The equality concerning λ^τ_α is clear if $\zeta^\tau_\alpha > 0$ whence $\chi^\tau(\lambda^\tau_\alpha) = \chi^\tau(\mathrm{logend}(\alpha))$. Let us now assume that $\zeta^\tau_\alpha = 0$ which implies that $\eta = \sup_{\sigma<\eta} \vartheta^\tau(\Delta + \sigma)$ and hence $\chi^\tau(\alpha) = \chi^\tau(\Delta)$. Since the case $\Delta = 0$ is trivial we may assume $\Delta > 0$. We prove the following

Claim. *For every* $\xi \in \mathrm{Sub}^\tau_0(\Delta)$ *we have*

$$\chi^\tau(\xi) = \chi^\tau\left(\iota_{\tau,\alpha}(\xi)^{\mathrm{t}^\alpha_\tau}\right).$$

The claim implies the desired equality since $\Delta \in \mathrm{Sub}^\tau_0(\Delta)$ and is shown by induction on the buildup of T^τ-terms (cf. Lemma 7.9 of [10]). The interesting cases are ϑ-terms, in particular the case $\xi = \vartheta_1(\rho)$ where $\rho = \Xi + \nu > 0$ with $\nu < \Omega_2 \mid \Xi$. Setting $\nu' := \iota_{\tau,\alpha}(\nu)^{\mathrm{t}^\alpha_\tau}$ and $\Xi' := \iota_{\tau,\alpha}(\Xi)^{\mathrm{t}^\alpha_\tau}$ notice that

$$\nu = \sup_{\sigma<\nu} \vartheta_1(\Xi + \sigma) \vee \mathrm{logend}(\nu) = 0 \quad \Leftrightarrow \quad \nu' = \sup_{\sigma<\nu'} \vartheta^\tau(\Xi' + \sigma) \vee \mathrm{logend}(\nu') = 0.$$

Notice also that in case of $\nu = \sup_{\sigma<\nu} \vartheta_1(\Xi + \sigma)$ we have the equality $\nu' = \sup_{\sigma<\nu'} \vartheta^\tau(\Xi' + \alpha + \sigma)$. We now have a closer look at the term $\xi' := \iota_{\tau,\alpha}(\xi)^{\mathrm{t}^\alpha_\tau}$:

Case 1: $\Xi = 0$.

Then $\xi' = \vartheta^\tau(\alpha + \iota_{\tau,\alpha}(\nu_0)^{\mathrm{t}^\alpha_\tau})$ where $\nu_0 := -1 + \nu$. If in this case $\nu \in \mathbb{E}^{>\Omega_1}$ or ν is a successor ordinal then we obtain $\chi^\tau(\xi) = 0 = \chi^\tau(\xi')$ using that $\alpha \in \mathbb{E}$ since we assume $\Delta > 0$. Otherwise we use the i.h. for ν and obtain $\chi^\tau(\xi) = \chi^\tau(\nu) = \chi^\tau(\nu') = \chi^\tau(\xi')$.

Case 2: $\Xi^{*_1} \geq \Omega_1$.

Then $\xi' = \vartheta^\tau(\Xi' + \nu')$, and we apply the i.h. for Ξ and ν using the equivalence from above.

Case 3: Otherwise.

Then $\xi' = \vartheta^\tau(\Xi' + \alpha + \nu')$. In case of $\nu > 0$ we argue as in Case 2. Suppose finally that $\nu = 0$. We then have $\chi^\tau(\xi') = \chi^\tau(\vartheta^\tau(\Xi' + \alpha)) = \chi^\tau(\Xi')$ since $\Xi' < \Delta$ due to the fact that $\xi' \in (\alpha, \alpha^+)$, and the claim follows by an application of the i.h. for Ξ.

Part (b) is shown by induction on the buildup of $\xi \in \mathrm{T}^\tau_{\restriction\alpha}$. The interesting case is where $\xi = \vartheta_{k+1}(\Xi + \nu)$ with $\nu < \Omega_{k+2} \mid \Xi$. Since the claim is clear if $\Xi + \nu = 0$ we may assume that $\Xi + \nu > 0$. Setting $\xi' := \iota_{\tau,\alpha}(\xi)$, $\Xi' := \iota_{\tau,\alpha}(\Xi)$, and $\nu' := \iota_{\tau,\alpha}(\nu)$ we again use the equivalence

$$\nu = \sup_{\sigma<\nu} \vartheta_{k+1}(\Xi + \sigma) \vee \mathrm{logend}(\nu) = 0 \Leftrightarrow \nu' = \sup_{\sigma<\nu'} \vartheta_{k+1}(\Xi' + \sigma) \vee \mathrm{logend}(\nu') = 0.$$

Now, if $\nu = \sup_{\sigma<\nu} \vartheta_{k+1}(\Xi + \sigma) \vee \mathrm{logend}(\nu) = 0$ then $\chi^\alpha(\xi') = \chi^\alpha(\Xi') = 1$ and hence by the i.h. $\chi^\tau(\xi) = \chi^\tau(\Xi) = 0$. Otherwise $\chi^\alpha(\xi') = \chi^\alpha(\nu') = 1$ and hence by the i.h. $\chi^\tau(\xi) = \chi^\tau(\nu) = 0$. □

We now define two additional operators μ^τ_α and ϱ^τ_α that will play an essential role in the definition of tracking sequences and chains. As will be shown later, μ^τ_α yields the index of the largest of those $\leq_2$-connectivity components newly arising in the α-th component (where α is an epsilon number greater than τ) within a context locally indexed by τ. The set of contexts, called *reference points*, is defined in 3.18 and shown in Corollary 4.11 to match the set of appropriate relativization points for enumeration functions of connectivity components. Referring again to the context locally indexed by τ from above, the α-th component becomes a new context (within the outer context indicated by τ) giving itself rise to new, larger $\leq_2$-components. In fact, reference points are exactly those ordinals whose tracking sequence consists of increasing epsilon numbers (cf. 3.18).

Staying with the same τ and α as in the above explanation of $\mu^\tau_\alpha, \varrho^\alpha_\xi$ (see Definition 3.9) where $\xi \leq \mu^\tau_\alpha$ will be shown later to yield the index of that ν_ξ-relativized $\leq_1$-connectivity component which contains the largest $\leq_2$-successor of ν_ξ where ν_ξ is the ξ-th such newly arising $\leq_2$-component in the α-th component.

Definition 3.4. Let $\tau \in \mathbb{E}_1$ and $\alpha \in (\tau, \tau^\infty) \cap \mathbb{E}$, say $\alpha = \vartheta^\tau(\Delta + \eta)$ where $\Delta = \Omega_1 \cdot (\lambda + k)$ such that $\lambda \in \{0\} \cup \mathrm{Lim}$ and $k < \omega$. We define

$$\mu^\tau_\alpha := \omega^{\iota_{\tau,\alpha}(\lambda) + \chi^\alpha(\iota_{\tau,\alpha}(\lambda)) + k}.$$

The next lemma will justify inductive proofs along ht_τ which was introduced in [10]. The more refined estimation will also be used, especially when dealing with localizations. The subsequent algebraic lemmas concerning the notions of translation and base transformation introduced in [10] will later be used without explicit mention.

Lemma 3.5. $\mathrm{ht}_\alpha(\mu^\tau_\alpha) \leq \mathrm{ht}_\alpha(\lambda^\tau_\alpha) < \mathrm{ht}_\tau(\alpha)$ *and* $\mu^\tau_\alpha, (\mu^\tau_\alpha)^+ < \alpha^+$.

Proof. Immediate from the respective definitions, cf. Corollaries 7.3 and 7.6 of [10]. Note that we regard μ^τ_α and $(\mu^\tau_\alpha)^+$ elements of T^α, whereas $\alpha^+ = \vartheta^\tau(\Delta + \eta + 1) = \vartheta^\alpha(\Delta)$ as shown in [10]. □

Lemma 3.6. *Let $\alpha = \vartheta^\tau(\Delta + \eta) \in \mathbb{E}^{>\tau}$. Then*

$$\mu^\tau_\beta = \mu^\alpha_{\beta \mathfrak{t}^\tau_\alpha} = \mu^\alpha_\beta$$

for every $\beta = \vartheta^\tau(\Gamma + \rho) \in (\alpha, \alpha^+) \cap \mathbb{E}$.

Proof. Immediate from the respective definitions, cf. Lemma 7.7 of [10]. □

Lemma 3.7. *Let $\sigma, \tau \in \mathbb{E}$, $\sigma < \tau$, and $\alpha = \vartheta^\tau(\Delta + \eta) \in \mathrm{T}^{\tau}[\sigma] \cap (\tau, \tau^\infty) \cap \mathbb{E}$. Then $\mu^\tau_\alpha \in \mathrm{T}^\tau[\sigma]$ and $\pi_{\sigma,\tau}(\mu^\tau_\alpha) = \mu^\sigma_{\pi_{\sigma,\tau}(\alpha)}$, i.e. the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{T}^\tau[\sigma] \cap (\tau, \tau^\infty) \cap \mathbb{E} & \xrightarrow{\mu^\tau} & \mathrm{T}^\tau[\sigma] \\ \downarrow{\scriptstyle \pi_{\sigma,\tau}} & & \downarrow{\scriptstyle \pi_{\sigma,\tau}} \\ \mathrm{T}^\sigma \cap (\sigma, \sigma^\infty) \cap \mathbb{E} & \xrightarrow[\mu^\sigma]{} & \mathrm{T}^\sigma \end{array}$$

Proof. By Lemma 5.10 of [4] we have commutativity of $\omega^{\cdot}$ with $\pi_{\sigma,\tau}$. It is then easy to see that $\pi_{\sigma,\tau}(\Delta) = \Omega_1 \cdot (\pi_{\sigma,\tau}(\lambda) + k)$ and

$$\chi^{\pi_{\sigma,\tau}(\alpha)}\left(\iota_{\sigma,\pi_{\sigma,\tau}(\alpha)}(\pi_{\sigma,\tau}(\lambda))\right) = \chi^\alpha\left(\iota_{\tau,\alpha}(\lambda)\right)$$

where λ, k are as in the definition of μ^τ_α. An application of Lemma 7.9 of [10] now yields the claim. See also 7.10 of [10] for the corresponding lemma for λ^τ_α. □

For the next lemma recall the $\bar{\cdot}$-operator which was introduced in Section 8 of [10] and extended in Section 5 of [4].

Lemma 3.8. *Let $\tau \in \mathbb{E}_1$ and $\alpha \in \mathbb{E} \cap (\tau, \tau^\infty)$ and $\gamma \in \mathbb{E} \cap (\bar{\alpha}, \alpha)$. Then we have*

$$\pi^{-1}_{\gamma,\alpha}(\mu^\tau_\gamma) \le \mu^\tau_\alpha.$$

Proof. Notice that the claim is similar to the consequence of Lemma 8.2 of [10] which shows that $\pi^{-1}_{\gamma,\alpha}(\lambda^\tau_\gamma) < \lambda^\tau_\alpha$. Suppose $\gamma = \vartheta^\tau(\Gamma + \nu)$ and $\alpha = \vartheta^\tau(\Delta + \eta)$. Since $\gamma \in (\bar{\alpha}, \alpha)$ we have $\Gamma \le \Delta$. Let $\mu, \lambda \in \mathrm{Lim} \cup \{0\}$ and $k, l < \omega$ be such that $\Gamma = \Omega_1 \cdot (\mu + l)$ and $\Delta = \Omega_1 \cdot (\lambda + k)$. In the case $\mu = \lambda$ we have $l \le k$, and the claim follows since $\pi^{-1}_{\gamma,\alpha}(\iota_{\tau,\gamma}(\mu)) = \iota_{\tau,\alpha}(\lambda)$ using Lemma 7.8 of [10] which thanks to Lemma 3.2 also yields $\chi^\gamma(\iota_{\tau,\gamma}(\mu)) = \chi^\alpha(\iota_{\tau,\alpha}(\lambda))$. In the case $\mu < \lambda$ we obtain $\pi^{-1}_{\gamma,\alpha}(\iota_{\tau,\gamma}(\mu)) = \iota_{\tau,\alpha}(\mu) < \iota_{\tau,\alpha}(\lambda)$ by Lemma 7.2 of [10] which implies the claim since $\iota_{\tau,\alpha}(\lambda)$ is a limit ordinal greater than $\iota_{\tau,\alpha}(\mu)$. □

Definition 3.9. Let $\tau \in \mathbb{E}$ and $\alpha < \tau^\infty$ where $\mathrm{logend}(\alpha) = \lambda + k$ such that $\lambda \in \{0\} \cup \mathrm{Lim}$ and $k < \omega$. We define

$$\varrho^\tau_\alpha := \tau \cdot (\lambda + k \dot{-} \chi^\tau(\lambda)).$$

Lemma 3.10. $\varrho^\tau_\alpha \le \tau \cdot \mathrm{logend}(\alpha)$ *and* $\mathrm{ht}_\tau(\varrho^\tau_\alpha) \le \max\{1, \mathrm{ht}_\tau(\alpha)\}$.

Proof. Immediate by definition. □

Lemma 3.11. *Let $\sigma, \tau \in \mathbb{E}$, $\sigma < \tau$, and $\alpha \in \mathrm{T}^\tau[\sigma] \cap \tau^\infty$. Then we have $\varrho^\tau_\alpha \in \mathrm{T}^\tau[\sigma]$ and*

$$\pi_{\sigma,\tau}(\varrho^\tau_\alpha) = \varrho^\sigma_{\pi_{\sigma,\tau}(\alpha)},$$

i.e. the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{T}^\tau[\sigma] \cap \tau^\infty & \xrightarrow{\varrho^\tau} & \mathrm{T}^\tau[\sigma] \\ \downarrow{\scriptstyle \pi_{\sigma,\tau}} & & \downarrow{\scriptstyle \pi_{\sigma,\tau}} \\ \mathrm{T}^\sigma \cap \sigma^\infty & \xrightarrow[\varrho^\sigma]{} & \mathrm{T}^\sigma \end{array}$$

Proof. Straightforward. □

The lemma below shows the interrelations between the operators from [10] and the new ones.

Lemma 3.12. *Let $\tau \in \mathbb{E}_1$ and $\alpha = \vartheta^\tau(\Delta + \eta) \in (\tau, \tau^\infty) \cap \mathbb{E}$. Then we have*

(a) $\iota_{\tau,\alpha}(\Delta) = \varrho^\alpha_{\mu^\tau_\alpha}$ *and hence* $\lambda^\tau_\alpha = \varrho^\alpha_{\mu^\tau_\alpha} + \zeta^\tau_\alpha$.
(b) $\varrho^\alpha_\beta \le \lambda^\tau_\alpha$ *for every* $\beta \le \mu^\tau_\alpha$. *For* $\beta < \mu^\tau_\alpha$ *even* $\varrho^\alpha_\beta + \alpha \le \lambda^\tau_\alpha$.
(c) *If* $\mu^\tau_\alpha < \alpha$ *we have* $\mu^\tau_\alpha < \alpha \le \lambda^\tau_\alpha < \alpha^2$, *while otherwise*

$$\max\left((\mu^\tau_\alpha + 1) \cap \mathbb{E}\right) = \max\left((\lambda^\tau_\alpha + 1) \cap \mathbb{E}\right).$$

(d) *If* $\lambda^\tau_\alpha \in \mathbb{E}^{>\alpha}$ *we have* $\mu^\tau_\alpha = \lambda^\tau_\alpha \cdot \omega$ *in case of* $\chi^\alpha(\lambda^\tau_\alpha) = 1$ *and* $\mu^\tau_\alpha = \lambda^\tau_\alpha$ *otherwise.*

Proof. Immediate from the respective definitions. □

In order to see that the following central definition of *(relativized) tracking sequences* is sound one needs to verify the subsequent lemma along the way.

Definition 3.13. Let $\tau \in \mathbb{E}_1$ and $\alpha \in [\tau, \tau^\infty) \cap \mathbb{P}$. The **tracking sequence of** α **above** τ, $\mathrm{ts}^\tau(\alpha)$, is defined recursively in α as follows.

- If $\alpha \in \mathbb{M}^{>\tau}$ with τ-localization $\tau = \alpha_0, \ldots, \alpha_n = \alpha$ we set
$$i := \begin{cases} \max\{j \in \{1, \ldots, n-1\} \mid \mu^\tau_{\alpha_j} \geq \alpha\} & \text{if that exists} \\ n & \text{otherwise.} \end{cases}$$
  - If $i = n$ then $\mathrm{ts}^\tau(\alpha) := (\alpha)$.
  - If $i < n$ then $\mathrm{ts}^\tau(\alpha) := \mathrm{ts}^\tau(\alpha_i)^\frown(\alpha)$ is obtained from $\mathrm{ts}^\tau(\alpha_i)$ by appending α.
- $\alpha \notin \mathbb{M}^{>\tau}$ and $\alpha \leq \tau^\omega$: Then $\mathrm{ts}^\tau(\alpha) := (\alpha)$.
- Otherwise. Then $\bar{\alpha} \in [\tau, \alpha)$ and $\alpha =_{\mathrm{NF}} \bar{\alpha} \cdot \beta$ for some $\beta \in \mathbb{M}^{>1}$. Let $\mathrm{ts}^\tau(\bar{\alpha}) = (\alpha_1, \ldots, \alpha_n)$ and set $\alpha_0 := \tau$.[1]
  - If $\alpha_n \in \mathbb{E}^{>\alpha_{n-1}}$ and $\beta \leq \mu^\tau_{\alpha_n}$ then $\mathrm{ts}^\tau(\alpha) := (\alpha_1, \ldots, \alpha_n, \beta)$.
  - Otherwise. For $i \in \{1, \ldots, n\}$ let $(\beta^i_1, \ldots, \beta^i_{m_i})$ be $\mathrm{ts}^{\alpha_i}(\beta)$ provided $\beta > \alpha_i$, and set $m_i := 1$, $\beta^i_1 := \alpha_i \cdot \beta$ if $\beta \leq \alpha_i$. Define
$$i_0 := \max\left(\{1\} \cup \{j \in \{2, \ldots, n\} \mid \beta^j_1 \leq \mu^\tau_{\alpha_{j-1}}\}\right).$$
Then $\mathrm{ts}^\tau(\alpha) := (\alpha_1, \ldots, \alpha_{i_0-1}, \beta^{i_0}_1, \ldots, \beta^{i_0}_{m_{i_0}})$.

Instead of $\mathrm{ts}^1(\alpha)$ we also simply write $\mathrm{ts}(\alpha)$.

Remark. For $\alpha \in \mathbb{E} \cap (\tau, \tau^\infty)$ the tracking sequence of α above τ is exactly the τ-$\leq_1$-localization of α as in Definition 5.8 of [11], as follows from Corollary 5.9 of [11], Lemma 3.12, part (c), and Lemma 3.14, part (a), below. Notice that according to Lemma 4.9 of [11] there is convenient robustness of the above definition regarding the base to which the μ-operator refers.

Lemma 3.14. *Let $\tau \in \mathbb{E}_1$ and $\alpha \in [\tau, \tau^\infty) \cap \mathbb{P}$. Let further $(\alpha_1, \ldots, \alpha_n)$ be $\mathrm{ts}^\tau(\alpha)$, the tracking sequence of α above τ.*

(a) *If $\alpha \in \mathbb{M}$ then $\alpha_n = \alpha$ and $\mathrm{ts}^\tau(\alpha_i) = (\alpha_1, \ldots, \alpha_i)$ for $i = 1, \ldots, n$.*
(b) *If $\alpha =_{\mathrm{NF}} \eta \cdot \xi \notin \mathbb{M}$ then $\alpha_n \in \mathbb{P} \cap [\xi, \alpha]$ and $\alpha_n =_{\mathrm{NF}} \overline{\alpha_n} \cdot \xi$.*
(c) *$(\alpha_1, \ldots, \alpha_{n-1})$ is either empty or a strictly increasing sequence of epsilon numbers in the interval (τ, α).*
(d) *For $1 \leq i \leq n-1$ we have $\alpha_{i+1} \leq \mu^\tau_{\alpha_i}$, and if $\alpha_i < \alpha_{i+1}$ then $(\alpha_1, \ldots, \alpha_{i+1})$ is a subsequence of the τ-localization of α_{i+1}.*

Proof. Immediate by induction on α along the definition of $\mathrm{ts}^\tau(\alpha)$. □

The card provided by the following lemma, which establishes a $<$-$<_{\mathrm{lex}}$-order isomorphism of additive principal numbers and their tracking sequences relative to τ, will be played winningly in the next section when showing that tracking sequences click into place with the system of enumeration functions for (relativized) connectivity components in the core of $\mathcal{R}_2$.

Lemma 3.15. *Let $\tau \in \mathbb{E}_1$ and $\alpha, \gamma \in [\tau, \tau^\infty) \cap \mathbb{P}$, $\alpha < \gamma$. Then we have*
$$\mathrm{ts}^\tau(\alpha) <_{\mathrm{lex}} \mathrm{ts}^\tau(\gamma).$$

Proof. The proof is by induction on the natural sum $\alpha \# \gamma$ where τ may vary. We will make frequent use of Lemma 3.14.

Case 1. $\gamma \leq \tau^\omega$:
Then $\mathrm{ts}^\tau(\alpha) = (\alpha) <_{\mathrm{lex}} (\gamma) = \mathrm{ts}^\tau(\gamma)$.

Case 2. $\gamma \in \mathbb{M}^{>\tau}$:
Let $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$ be the τ-localization of γ and $\mathrm{ts}^\tau(\gamma) = (\gamma_{i_1}, \ldots, \gamma_{i_l})$. The claim is immediately verified if $l = 1$. Therefore suppose $l > 1$ and set $\gamma' := \gamma_{i_{l-1}}$. Note that $\gamma \leq \mu^\tau_{\gamma'}$. If $\alpha \leq \gamma'$ by the i.h. we obtain $\mathrm{ts}^\tau(\alpha) \leq_{\mathrm{lex}} \mathrm{ts}^\tau(\gamma') <_{\mathrm{lex}} \mathrm{ts}^\tau(\gamma)$. Otherwise we have $\gamma' < \alpha < \gamma$ whence by Lemma 5.3 of [4] the initial sequence $(\gamma_1, \ldots, \gamma')$ of $\boldsymbol{\gamma}$ is an initial sequence of the τ-localization of α. The subcase where $\alpha \leq \tau^\omega$ is trivial. Suppose next that $\alpha \in \mathbb{M}^{>\tau}$. Then $\mathrm{ts}^\tau(\gamma')$ is an initial sequence of $\mathrm{ts}^\tau(\alpha)$ which implies the claim. If $\alpha \notin \mathbb{M}^{>\tau} \cup (\tau^\omega + 1)$, say $\alpha =_{\mathrm{NF}} \bar{\alpha} \cdot \beta$, we have $\gamma' \leq \bar{\alpha}$ since $\gamma' \in \mathbb{M}$. By the i.h. we obtain $\mathrm{ts}^\tau(\gamma') \leq_{\mathrm{lex}} \mathrm{ts}^\tau(\bar{\alpha}) <_{\mathrm{lex}} \mathrm{ts}^\tau(\gamma)$ whence $\mathrm{ts}^\tau(\gamma')$ is an initial sequence of $\mathrm{ts}^\tau(\bar{\alpha})$. Let $\mathrm{ts}^\tau(\bar{\alpha}) = (\xi_1, \ldots, \xi_p)$. If $\xi_p \in \mathbb{E}^{>\xi_{p-1}}$ and $\beta \leq \mu^\tau_{\xi_p}$ we are done. This is particularly the case if $l - 1 = p$. Otherwise we have $l \leq p$, i.e. $\mathrm{ts}^\tau(\gamma')$ is a proper initial sequence of $\mathrm{ts}^\tau(\bar{\alpha})$. Let $(\beta^l_1, \ldots, \beta^l_{m_l})$ be according to the definition of $\mathrm{ts}^\tau(\alpha)$. Since $\beta^l_1 \leq \xi_l \cdot \beta \leq \alpha < \mu^\tau_{\gamma'}$ we have $i_0 \geq l$ for i_0 according to the definition of $\mathrm{ts}^\tau(\alpha)$. Thus the claim follows.

Case 3. $\gamma \notin \mathbb{M}^{>\tau} \cup (\tau^\omega + 1)$:
Let $\gamma =_{\mathrm{NF}} \bar{\gamma} \cdot \delta$. Let $(\zeta_1, \ldots, \zeta_q) := \mathrm{ts}^\tau(\bar{\gamma})$. The subcase $\alpha \leq \tau^\omega$ is again trivial. If $\alpha \in \mathbb{M}^{>\tau}$ then $\alpha \leq \bar{\gamma}$ whence by the i.h. $\mathrm{ts}^\tau(\alpha) \leq_{\mathrm{lex}} \mathrm{ts}^\tau(\bar{\gamma})$. We have $\mathrm{ts}^\tau(\bar{\gamma}) <_{\mathrm{lex}} \mathrm{ts}^\tau(\gamma)$ by definition. If finally $\alpha \notin \mathbb{M}^{>\tau} \cup (\tau^\omega + 1)$, say $\alpha =_{\mathrm{NF}} \bar{\alpha} \cdot \beta$, we are again done if

[1] As verified in part (b) of the lemma below we have $\beta \leq \alpha_n$.

$\alpha < \bar{\gamma}$. Suppose $\bar{\gamma} \le \alpha$ and let the decompositions of $\bar{\alpha}$ and $\bar{\gamma}$ into products of descending multiplicative principal numbers be given by $\alpha_1 \cdots \alpha_k$ and $\gamma_1 \cdots \gamma_l$, respectively. We then see that in presence of our current assumptions $(\gamma_1, \ldots, \gamma_l)$ must be an initial sequence of $(\alpha_1, \ldots, \alpha_k, \beta)$. The case $l = k+1$ is clear since $\mathrm{ts}^\tau(\bar{\gamma}) <_{\mathrm{lex}} \mathrm{ts}^\tau(\gamma)$. We henceforth assume $l \le k$. If $\zeta_q \in \mathbb{E}^{>\zeta_{q-1}}$ and $\delta \le \mu^\tau_{\zeta_q}$, it is easy to see that $\mathrm{ts}^\tau(\alpha) = (\zeta_1, \ldots, \zeta_q, \alpha_{l+1} \cdots \alpha_k \cdot \beta) <_{\mathrm{lex}} (\zeta_1, \ldots, \zeta_q, \delta) = \mathrm{ts}^\tau(\gamma)$. Otherwise let $j_0 \in \{1, \ldots, q\}$ be according to the definition of $\mathrm{ts}^\tau(\gamma)$ so that

$$\mathrm{ts}^\tau(\gamma) = (\zeta_1, \ldots, \zeta_{j_0-1}, \delta^{j_0}_1, \ldots, \delta^{j_0}_{n_{j_0}}).$$

We give a uniform treatment of the cases where $l \le k$ in this situation. Suppose that for some $\alpha^\star \in \mathbb{P} \cap [\bar{\gamma}, \gamma)$ (say, $\alpha^\star = \alpha_1 \cdots \alpha_i$ for some $i \in \{l, \ldots, k\}$) with $\mathrm{ts}^\tau(\alpha^\star) = (\eta_1, \ldots, \eta_r)$ we already know that $\eta_j = \zeta_j$ for $1 \le j \le j_0 - 1, r \ge j_0$, $\eta_{j_0} \in [\zeta_{j_0}, \delta^{j_0}_1]$, and $\mathrm{ts}^\tau(\alpha^\star) <_{\mathrm{lex}} \mathrm{ts}^\tau(\gamma)$. Let some $\varepsilon \in \mathbb{M} \cap (1, \delta)$ be given such that for $\alpha' := \alpha^\star \cdot \varepsilon$ we have $\alpha' =_{\mathrm{NF}} \alpha^\star \cdot \varepsilon$ (here we aim at $\varepsilon = \alpha_{i+1}$ if $i < k$ or $\varepsilon = \beta$ otherwise). We show that $\mathrm{ts}^\tau(\alpha')$ then again is of a form

$$\mathrm{ts}^\tau(\alpha') = (\eta_1, \ldots, \eta_{i_0-1}, \varepsilon^{i_0}_1, \ldots, \varepsilon^{i_0}_{m_{i_0}})$$

where $j_0 \le i_0$, $m_{i_0} \ge 1$, that its j_0-th component belongs to the interval $[\zeta_{j_0}, \delta^{j_0}_1]$, and that we have $\mathrm{ts}^\tau(\alpha') <_{\mathrm{lex}} \mathrm{ts}^\tau(\gamma)$. A $(k-l+1)$-fold iteration of this argument then yields $\mathrm{ts}^\tau(\alpha) <_{\mathrm{lex}} \mathrm{ts}^\tau(\gamma)$.

In case of $\eta_r \in \mathbb{E}^{>\eta_{r-1}}$ and $\varepsilon \le \mu^\tau_{\eta_r}$ we have $\mathrm{ts}^\tau(\alpha') = \mathrm{ts}^\tau(\alpha^\star)^\frown\varepsilon <_{\mathrm{lex}} \mathrm{ts}^\tau(\gamma)$ since $\varepsilon \le \eta_r$. Note that this case applies whenever $\mathrm{ts}^\tau(\alpha^\star)$ is a proper initial sequence of $\mathrm{ts}^\tau(\gamma)$.

Let us now suppose that $\mathrm{ts}^\tau(\alpha^\star)$ is not a proper initial sequence of $\mathrm{ts}^\tau(\gamma)$. Then there exists the least $s \in \{1, \ldots, r - j_0 + 1\}$ such that $\eta_j < \delta^{j_0}_s$, setting $j := j_0 - 1 + s$. Note that we then have $\delta^{j_0}_s \le \mu^\tau_{\eta_{j-1}}$.

If $\varepsilon \le \eta_j$ then $\varepsilon^j_1 = \eta_j \cdot \varepsilon < \delta^{j_0}_s \le \mu^\tau_{\eta_{j-1}}$ and hence $j_0 \le j \le i_0$ and $\mathrm{ts}^\tau(\alpha') <_{\mathrm{lex}} \mathrm{ts}^\tau(\gamma)$. Otherwise we first note that in the situation where $\varepsilon^j_1 < \delta^{j_0}_s$ or $\varepsilon = \delta^{j_0}_s$ we immediately obtain $j_0 \le j \le i_0$ and $\mathrm{ts}^\tau(\alpha') <_{\mathrm{lex}} \mathrm{ts}^\tau(\gamma)$. We now assume that $\varepsilon^j_1 \ge \delta^{j_0}_s$ and $\varepsilon > \delta^{j_0}_s$. Then $\delta > \delta^{j_0}_s$, i.e. $s < n_{j_0}$. We take a closer look at $\mathrm{ts}^{\eta_j}(\varepsilon)$ in comparison with $\mathrm{ts}^{\delta^{j_0}_s}(\delta) = (\delta^{j_0}_{s+1}, \ldots, \delta^{j_0}_{n_{j_0}})$. Note that $\varepsilon \in \left(\delta^{j_0}_s, (\delta^{j_0}_s)^+\right)$ whence by Lemma 6.5 of [10] the η_j-localization of ε is the concatenation of the η_j-localization of $\delta^{j_0}_s$ with the $\delta^{j_0}_s$-localization of ε. The i.h. applied to $\varepsilon < \delta$ yields $\mathrm{ts}^{\delta^{j_0}_s}(\varepsilon) <_{\mathrm{lex}} \mathrm{ts}^{\delta^{j_0}_s}(\delta)$. The assumption $\varepsilon^j_1 > \delta^{j_0}_s$ would imply $\varepsilon^j_1 \le \delta^{j_0}_{s+1} \le \mu^\tau_{\delta^{j_0}_s}$ and therefore contradict the fact that ε^j_1 is the first element of $\mathrm{ts}^{\eta_j}(\varepsilon)$. Thus $\varepsilon^j_1 = \delta^{j_0}_s \le \mu^\tau_{\eta_{j-1}}$ and hence $j_0 \le j \le i_0$ as well as $\mathrm{ts}^{\eta_j}(\varepsilon) = (\varepsilon^j_1)^\frown\mathrm{ts}^{\delta^{j_0}_s}(\varepsilon) <_{\mathrm{lex}} (\delta^{j_0}_s, \ldots, \delta^{j_0}_{n_{j_0}})$ which then allows for the desired conclusion. $\square$

The following definition will play an important role in the next section. It will turn out to provide refined upper bounds for (relativized) connectivity components.

Definition 3.16. Let $\tau \in \mathbb{E}_1$ and $\alpha \in (\tau, \tau^\infty) \cap \mathbb{E}$. We define

$$\widehat{\alpha} := \min\{\gamma \in \mathbb{M}^{>\alpha} \mid \mathrm{ts}^\alpha(\gamma) = (\gamma) \;\&\; \mu^\tau_\alpha < \gamma\}.$$

Remark. Note that in the above context we have $\widehat{\alpha} \le \alpha^+$. As is the case with α^+ we suppress the base τ in the notation $\widehat{\alpha}$ assuming that it will always be well understood from the respective context.

Lemma 3.17. *Let τ, α be as in the above definition. Then*

$$\widehat{\beta} \le \widehat{\alpha} \quad \textit{for any} \quad \beta \in \mathrm{T}^\alpha \cap \mathbb{E} \cap (\alpha, \mu^\tau_\alpha].$$

We further have $\lambda^\tau_\alpha < \widehat{\alpha}$.

Proof. We show that $\widehat{\alpha}$ satisfies the conditions on $\widehat{\beta}$ apart from minimality. We clearly have $\widehat{\alpha} \in \mathbb{M}^{>\beta}$. The assumption $\widehat{\alpha} \le \mu^\alpha_\beta$ would imply $\beta < \widehat{\alpha} < \beta^+$ and therefore contradict the property $\mathrm{ts}^\alpha(\widehat{\alpha}) = (\widehat{\alpha})$, which also entails $\mathrm{ts}^\beta(\widehat{\alpha}) = (\widehat{\alpha})$.

In order to show that $\lambda^\tau_\alpha < \widehat{\alpha}$ let $\alpha = \vartheta^\tau(\Delta + \eta)$. If $\mu^\tau_\alpha < \alpha$ we have $\lambda^\tau_\alpha < \alpha^2 < \alpha^\omega \le \widehat{\alpha}$ by Lemma 3.12. Now assume $\mu^\tau_\alpha \ge \alpha$. Since $\widehat{\alpha} \ge (\mu^\tau_\alpha)^\omega \ge \alpha^\omega$ we may assume $\lambda^\tau_\alpha \ge \alpha^\omega$, whence $\Delta \ge \Omega^\omega_1$. This shows that $(\mu^\tau_\alpha)^\omega > \omega^{\iota_{\tau,\alpha}(\Delta)} + \zeta^\tau_\alpha \ge \lambda^\tau_\alpha$. $\square$

We are now able to describe the set of ordinals that comprises the essential starting points of relativized connectivity components of $\mathrm{Core}(\mathcal{R}_2)$ for both relations $\le_1$ and $\le_2$. The purely arithmetical relevance of these ordinals will become clear during the following sections (cf. 4.11) while its semantical meaning concerning $\mathcal{R}_2$, namely of being the origins of infinite $\le_1$-chains along which new $\le_2$-components arise, will be proved at the end of this article.

Definition 3.18. The set RP of **reference points below** 1^∞ is defined by

$$\mathrm{RP} := \{0\} \cup \{\alpha \in \mathbb{P}^{<1^\infty} \mid \mathrm{ts}(\alpha) = (\alpha_1, \ldots, \alpha_n) \text{ where } \alpha_1 < \cdots < \alpha_n \in \mathbb{E}\}.$$

4. A facet structure of ordinals

In this section we introduce functions that will turn out to characterize the enumeration functions of (relativized) connectivity components of the core of $\mathcal{R}_2$. This characterization already takes crucial uniformity properties, particularly of relativized $\le_1$-connectivity components, into account and restricts the enumeration of components up to their "critical index" (cf. the remark preceding Definition 3.4), which in turn is necessary because of the nested occurrence of $\le_1$- and $\le_2$-components. Our simultaneously recursive Definition 4.4 will make use of the ordering $(\mathrm{ISeq}, <_{\mathrm{lex}})$ given below.

Lemma 4.1. *Let* ISeq *comprise all nonempty finite sequences* $(\alpha_1, \ldots, \alpha_n)$ *of ordinals below* 1^∞ *that satisfy* $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{E}$ *and* $\mathrm{ht}_{\alpha_i}(\alpha_{i+1}) < \mathrm{ht}_{\alpha_{i-1}}(\alpha_i)$ *for* $1 \le i \le n-1$ *where* $\alpha_0 := 1$. *Then the lexicographic ordering*

$$(\mathrm{ISeq}, <_{\mathrm{lex}})$$

is a well-ordering.

Proof. Suppose $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_n) \in \mathrm{ISeq}$. We set $l := \mathrm{ht}_1(\alpha_1)$ whence $n \le l + 1$. Any $\boldsymbol{\beta} := (\beta_1, \ldots, \beta_m) \in \mathrm{ISeq}$ such that $\boldsymbol{\beta} <_{\mathrm{lex}} \boldsymbol{\alpha}$ satisfies $\mathrm{ht}_1(\beta_1) \le \mathrm{ht}_1(\alpha_1)$, since $\beta_1 \le \alpha_1$, and therefore underlies the restriction $m \le l + 1$. □

We now define a set of ordinal sequences that will turn out to characterize the set of tracking sequences defined in Section 3 via an evaluation procedure defined in 4.8. The evaluations of *reference sequences* defined subsequently comprises RP as will be shown at the end of this section.

Definition 4.2. Let $\tau \in \mathbb{E}_1$. A nonempty sequence $(\alpha_1, \ldots, \alpha_n)$ of ordinals in the interval $[\tau, \tau^\infty)$ is called a τ**-tracking sequence** if

1. $(\alpha_1, \ldots, \alpha_{n-1})$ is either empty or a strictly increasing sequence of epsilon numbers greater than τ.
2. $\alpha_n \in \mathbb{P}, \alpha_n > 1$ if $n > 1$.
3. $\alpha_{i+1} \le \mu^\tau_{\alpha_i}$ for every $i \in \{1, \ldots, n-1\}$.

By TS^τ we denote the set of all τ-tracking sequences. Instead of TS^1 we also write TS.

Remark. Note that $\{\mathrm{ts}^\tau(\alpha) \mid \alpha \in \mathbb{P} \cap [\tau, \tau^\infty)\} \subseteq \mathrm{TS}^\tau$, that TS^τ is closed under nonempty initial sequences, and that $\mathrm{TS} \subseteq \mathrm{ISeq}$. It will be shown that in fact $\mathrm{TS} = \{\mathrm{ts}(\alpha) \mid \alpha \in \mathbb{P} \cap 1^\infty\}$. For $i \in \{1, \ldots, n-2\}$ the initial sequence $(\alpha_1, \ldots, \alpha_{i+1})$ is a subsequence of the τ-localization of α_{i+1}. This also holds for $i = n-1$ provided that $\alpha_{n-1} < \alpha_n$.

Definition 4.3. Let $\tau \in \mathbb{E}_1$. A sequence $\boldsymbol{\alpha}$ of ordinals below τ^∞ is a τ**-reference sequence** if

- $\boldsymbol{\alpha} = ()$ or
- $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathrm{TS}^\tau$ such that $\alpha_n \in \mathbb{E}^{>\alpha_{n-1}}$ (where $\alpha_0 := \tau$).

We denote the set of τ-reference sequences by RS^τ. In case of $\tau = 1$ we simply write RS and call its elements reference sequences.

Remark. Note that RS^τ is closed under initial sequences and that it contains the sequence $(\alpha_1, \ldots, \alpha_{n-1})$ for any $\alpha \in \mathbb{P} \cap \tau^\infty$ such that $\mathrm{ts}^\tau(\alpha) = (\alpha_1, \ldots, \alpha_n)$.

We are now going to define those functions which arithmetically characterize the enumerations of suitably relativized $\le_1$- and $\le_2$-components below the least α such that $\alpha <_1 \infty$ which in turn will eventually be proved to be equal to 1^∞ and comprise $\mathrm{Core}(\mathcal{R}_2)$. The soundness of this essential definition will be shown in the subsequent Lemma 4.5. Its semantical correctness can only be proved at the end of this article. We will then see that

- $\kappa^{()}$ enumerates those $\le_1$-minimal ordinals and $\mathrm{lh}(\kappa^{()}_\alpha) = \kappa^{()}_\alpha + \mathrm{dp}_{()}(\alpha)$, that
- $\nu^{\boldsymbol{\alpha}}$ (where the nonempty index sequence $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathrm{RS}$ codes the relativization point, say, α) enumerates the α-$\le_2$-minimal ordinals $\nu^{\boldsymbol{\alpha}}_\xi > \alpha$ $(1 \le \xi \le \mu^{\alpha_{n-1}}_{\alpha_n}$, setting $\alpha_0 := 1)$ up to the origin of the largest new[2] $\le_2$-connectivity component which satisfy $\mathrm{lh}(\nu^{\boldsymbol{\alpha}}_\xi) = \mathrm{lh}(\alpha \cdot 2) \le \mathrm{lh}(\alpha)$, while $\nu^{\boldsymbol{\alpha}}_0 = \alpha$ satisfies $\mathrm{lh}(\alpha) = \mathrm{lh}(\alpha \cdot 2)$ if and only if $n = 1$ or $n > 1$ and either $\chi^{\alpha_{n-1}}(\alpha_n) = 1$ or $\alpha_n = \mu^{\alpha_{n-2}}_{\alpha_{n-1}} = \lambda^{\alpha_{n-2}}_{\alpha_{n-1}}$, and that, referring to the same $\boldsymbol{\alpha}$ as above,
- $\nu^{\boldsymbol{\alpha}}_\xi + \kappa^{\boldsymbol{\alpha}}_\beta$ is the β-th $\nu^{\boldsymbol{\alpha}}_\xi$-$\le_1$-minimal ordinal for $\xi \le \mu^{\alpha_{n-1}}_{\alpha_n}$ and

$$\beta < \begin{cases} \varrho^{\alpha_n}_\xi + \alpha_n & \text{if } \xi < \mu^{\alpha_{n-1}}_{\alpha_n} \text{ and } \chi^{\alpha_n}(\xi) = 0, \\ \varrho^{\alpha_n}_\xi + 1 & \text{if } \xi < \mu^{\alpha_{n-1}}_{\alpha_n} \text{ and } \chi^{\alpha_n}(\xi) = 1, \\ \lambda^{\alpha_{n-1}}_{\alpha_n} + 1 & \text{if } \xi = \mu^{\alpha_{n-1}}_{\alpha_n}. \end{cases}$$

[2] By "new" we mean a connectivity component which cannot be obtained by translation of an isomorphic copy from below.

Notice that the case in the middle addresses the situation where the $\le_2$-component starting from $\nu^{\boldsymbol{\alpha}}_\xi$ non-trivially $\le_1$-connects back to the *main line starting from* α, that is: the image of $\nu^{\boldsymbol{\alpha}}$ together with all ordinals γ such that $\nu^{\boldsymbol{\alpha}}_\xi <_1 \gamma <_1 \nu^{\boldsymbol{\alpha}}_{\xi+1}$ for some ξ in the domain of $\nu^{\boldsymbol{\alpha}}$. "Non-trivial" means that $\varrho^{\alpha_n}_\xi > 0$, whence $\nu^{\boldsymbol{\alpha}}_\xi$ has its greatest $<_2$-successor in the $\varrho^{\alpha_n}_\xi$-th $\nu^{\boldsymbol{\alpha}}_\xi$-relativized $\le_1$-component, and "$\le_1$-connects back to the main line" means that $\nu^{\boldsymbol{\alpha}}_\xi + \kappa^{\boldsymbol{\alpha}}_{\varrho^{\alpha_n}_\xi} <_1 \nu^{\boldsymbol{\alpha}}_{\xi+1}$.[3]

It is crucial that the cofinality operator λ falls into place in the third case. This shows another parallel to the situation in Core$(\mathcal{R}_1^+)$; see [11].

$\mathrm{dp}_{\boldsymbol{\alpha}}(\beta)$, where we now assume that $\beta > 0$, satisfies the equation $\mathrm{lh}(\nu^{\boldsymbol{\alpha}}_\xi + \kappa^{\boldsymbol{\alpha}}_\beta) = \nu^{\boldsymbol{\alpha}}_\xi + \kappa^{\boldsymbol{\alpha}}_\beta + \mathrm{dp}_{\boldsymbol{\alpha}}(\beta)$ only if the $\nu^{\boldsymbol{\alpha}}_\xi$-relativized $\le_1$-component starting at $\nu^{\boldsymbol{\alpha}}_\xi + \kappa^{\boldsymbol{\alpha}}_\beta$ does not $\le_1$-connect back to any previously arising main line. If it does, let $\nu^{\boldsymbol{\alpha}'}_\eta$ be the largest element of the largest such main line such that $\nu^{\boldsymbol{\alpha}'}_\eta < \nu^{\boldsymbol{\alpha}}_\xi + \kappa^{\boldsymbol{\alpha}}_\beta$, whence we have $\nu^{\boldsymbol{\alpha}}_\xi + \kappa^{\boldsymbol{\alpha}}_\beta + \mathrm{dp}_{\boldsymbol{\alpha}}(\beta) = \nu^{\boldsymbol{\alpha}'}_{\eta+1}$. Here, the case $\nu^{\boldsymbol{\alpha}'}_\eta < \nu^{\boldsymbol{\alpha}}_\xi$ implies $\xi = \mu^{\alpha_{n-1}}_{\alpha_n}$ as will be shown later.

Definition 4.4. For any $\boldsymbol{\alpha} \in \mathrm{RS}$, say $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ where $n = 0$ in case of $\boldsymbol{\alpha} = ()$, we set $\alpha_0 := 1$ and define the functions

$$\kappa^{\boldsymbol{\alpha}} : \mathrm{dom}(\kappa^{\boldsymbol{\alpha}}) \to 1^\infty \text{ where } \mathrm{dom}(\kappa^{\boldsymbol{\alpha}}) := \begin{cases} 1^\infty & \text{if } n = 0 \\ \lambda^{\alpha_{n-1}}_{\alpha_n} + 1 & \text{otherwise,} \end{cases}$$

$$\mathrm{dp}_{\boldsymbol{\alpha}} : \mathrm{dom}(\kappa^{\boldsymbol{\alpha}}) \to 1^\infty,$$

and for $n > 0$ only

$$\nu^{\boldsymbol{\alpha}} : \mathrm{dom}(\nu^{\boldsymbol{\alpha}}) \to 1^\infty, \quad \text{where } \mathrm{dom}(\nu^{\boldsymbol{\alpha}}) := \mu^{\alpha_{n-1}}_{\alpha_n} + 1.$$

Along the way we use the following abbreviations

$$\boldsymbol{\alpha}_{|0} := (), \qquad \boldsymbol{\alpha}_{|i} := (\alpha_1, \ldots, \alpha_i) \quad (i = 1, \ldots, n),$$

and set $\beta' := (1/\bar{\beta}) \cdot \beta$ whenever $\beta \in \mathbb{P}$.
The clauses defining $\kappa^{\boldsymbol{\alpha}}$ are as follows.

- $\kappa^{\boldsymbol{\alpha}}_0 := 0$, $\kappa^{\boldsymbol{\alpha}}_1 := 1$,
- $\kappa^{\boldsymbol{\alpha}}_\beta := \kappa^{\boldsymbol{\alpha}}_\gamma + \mathrm{dp}_{\boldsymbol{\alpha}}(\gamma) + \kappa^{\boldsymbol{\alpha}}_\delta$ for $\beta =_{\mathrm{NF}} \gamma + \delta$,
- $\kappa^{\boldsymbol{\alpha}}_\beta := \kappa^{\boldsymbol{\alpha}_{|n-1}}_\beta$ if $n > 0$ and $\beta \in \mathbb{P} \cap (1, \alpha_n]$,
- $\kappa^{\boldsymbol{\alpha}}_\beta := \kappa^{\boldsymbol{\alpha}}_{\bar{\beta}+1} \cdot \beta'$ for $\beta \in \mathbb{P}^{>\alpha_n}$.

$\mathrm{dp}_{\boldsymbol{\alpha}}$ is defined by:

- $\mathrm{dp}_{\boldsymbol{\alpha}}(0) := 0$, $\mathrm{dp}_{\boldsymbol{\alpha}}(1) := 0$, and $\mathrm{dp}_{\boldsymbol{\alpha}}(\alpha_n) := 0$ in case of $n > 0$,
- $\mathrm{dp}_{\boldsymbol{\alpha}}(\beta) := \mathrm{dp}_{\boldsymbol{\alpha}}(\delta)$ if $\beta =_{\mathrm{NF}} \gamma + \delta$,
- $\mathrm{dp}_{\boldsymbol{\alpha}}(\beta) := \mathrm{dp}_{\boldsymbol{\alpha}_{|n-1}}(\beta)$ if $n > 0$ for $\beta \in \mathbb{P} \cap (1, \alpha_n)$,
- for $\beta \in \mathbb{P}^{>\alpha_n} - \mathbb{E}$ let $\gamma := (1/\alpha_n) \cdot \beta$ and $\log(\gamma) =_{\mathrm{ANF}} \gamma_1 + \cdots + \gamma_m$ and set [4]

  $$\mathrm{dp}_{\boldsymbol{\alpha}}(\beta) := \kappa^{\boldsymbol{\alpha}}_{\gamma_1} + \mathrm{dp}_{\boldsymbol{\alpha}}(\gamma_1) + \cdots + \kappa^{\boldsymbol{\alpha}}_{\gamma_m} + \mathrm{dp}_{\boldsymbol{\alpha}}(\gamma_m),$$

- for $\beta \in \mathbb{E}^{>\alpha_n}$ let $\boldsymbol{\gamma} := (\alpha_1, \ldots, \alpha_n, \beta)$, and set

  $$\mathrm{dp}_{\boldsymbol{\alpha}}(\beta) := \nu^{\boldsymbol{\gamma}}_{\mu^{\alpha_n}_\beta} + \kappa^{\boldsymbol{\gamma}}_{\lambda^{\alpha_n}_\beta} + \mathrm{dp}_{\boldsymbol{\gamma}}(\lambda^{\alpha_n}_\beta).$$

For $n > 0$ setting $\alpha := \kappa^{\boldsymbol{\alpha}_{|n-1}}_{\alpha_n}$ we define $\nu^{\boldsymbol{\alpha}}$ by

- $\nu^{\boldsymbol{\alpha}}_0 := \alpha$,
- $\nu^{\boldsymbol{\alpha}}_\beta := \nu^{\boldsymbol{\alpha}}_\gamma + \kappa^{\boldsymbol{\alpha}}_{\varrho^{\alpha_n}_\gamma} + \mathrm{dp}_{\boldsymbol{\alpha}}(\varrho^{\alpha_n}_\gamma) + \check{\chi}^{\alpha_n}(\gamma) \cdot \alpha$ if $\beta = \gamma + 1$,
- $\nu^{\boldsymbol{\alpha}}_\beta := \nu^{\boldsymbol{\alpha}}_\gamma + \kappa^{\boldsymbol{\alpha}}_{\varrho^{\alpha_n}_\gamma} + \mathrm{dp}_{\boldsymbol{\alpha}}(\varrho^{\alpha_n}_\gamma) + \nu^{\boldsymbol{\alpha}}_\delta$ if $\beta =_{\mathrm{NF}} \gamma + \delta \in \mathrm{Lim}$,
- $\nu^{\boldsymbol{\alpha}}_\beta := \alpha \cdot \beta$ for $\beta \in \mathbb{P} \cap (1, \alpha_n]$,
- $\nu^{\boldsymbol{\alpha}}_\beta := \nu^{\boldsymbol{\alpha}}_{\bar{\beta}+1} \cdot \beta'$ for $\beta \in \mathbb{P}^{>\alpha_n} - \mathbb{E}$,
- $\nu^{\boldsymbol{\alpha}}_\beta := \kappa^{\boldsymbol{\alpha}}_\beta$ for $\beta \in \mathbb{E}^{>\alpha_n}$.

Remark. Notice that TS comprises the sequences of a form $\boldsymbol{\alpha}^\frown\beta$ where $\boldsymbol{\alpha} \in \mathrm{RS}$ and $\beta \in \mathbb{P} \cap 1^\infty$ such that $\beta \in \mathbb{P} \cap \mathrm{dom}(\kappa^{()})$ if $\boldsymbol{\alpha} = ()$ and $\beta \in \mathbb{P}^{>1} \cap \mathrm{dom}(\nu^{\boldsymbol{\alpha}})$ otherwise.

3 We will use the term "main line" only in informal formulations, no proof will rely on this nevertheless intuitive and crucial notion.

4 This integrates the treatment of $\mathcal{R}_1$ from [1] into the setting of $\mathcal{R}_2$ as will be shown later.

We now verify the well-definedness of the functions introduced above and prove some basic properties and useful estimations.

Lemma 4.5. *Let $\boldsymbol{\alpha} \in \mathrm{RS}$, say $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ where $n = 0$ in case of $\boldsymbol{\alpha} = ()$, and set $\alpha_0 := 1$. The functions $\kappa^{\boldsymbol{\alpha}}$, $\mathrm{dp}_{\boldsymbol{\alpha}}$, and (in case of $n > 0$) $\nu^{\boldsymbol{\alpha}}$ are well-defined and satisfy the following properties.*

(a) *$\kappa^{\boldsymbol{\alpha}}$ is continuous, strictly increasing, and maps additive principal numbers to additive principal numbers.*
(b) *In case of $n > 0$ also $\nu^{\boldsymbol{\alpha}}$ is continuous, strictly increasing, and maps infinite additive principal numbers to additive principal numbers greater than $\alpha := \kappa^{\boldsymbol{\alpha}|n-1}_{\alpha_n}$.*
(c) *If $n > 0$ we have*

$$\kappa^{\boldsymbol{\alpha}}_{\beta} = \kappa^{\boldsymbol{\alpha}|n-1}_{\beta} \quad \textit{whenever } \beta \le \alpha_n$$

and

$$\mathrm{dp}_{\boldsymbol{\alpha}}(\beta) = \mathrm{dp}_{\boldsymbol{\alpha}|n-1}(\beta) \quad \textit{whenever } \beta < \alpha_n.$$

(d) *For $\beta \in \mathrm{dom}(\kappa^{\boldsymbol{\alpha}})$ we have the following estimations on $\mathrm{dp}_{\boldsymbol{\alpha}}(\beta)$:*
   1. *$\mathrm{dp}_{\boldsymbol{\alpha}}(\beta) < \kappa^{\boldsymbol{\alpha}}_{\beta}$ if $\beta \notin \mathbb{E}$, $\beta > 0$.*
   2. *$\kappa^{\boldsymbol{\alpha}}_{\beta} \cdot \omega \le \mathrm{dp}_{\boldsymbol{\alpha}}(\beta) < \kappa^{\boldsymbol{\alpha}}_{\beta} \cdot \widehat{\beta}$ if $\beta \in \mathbb{E}^{>\alpha_n}$.*
   3. *$\mathrm{dp}_{\boldsymbol{\alpha}}(\beta) < \kappa^{\boldsymbol{\alpha}}_{\beta} \cdot \mu^{\alpha_n}_{\beta} \cdot \omega$ if $\beta \in \mathbb{E}^{>\alpha_n}$ and $\mu^{\alpha_n}_{\beta} < \beta$.*

(e) *If $n > 0$ and $\beta \in \mathrm{dom}(\nu^{\boldsymbol{\alpha}}) - \mathbb{E}^{>\alpha_n}$ we have $\varrho^{\alpha_n}_{\beta} \in \mathrm{dom}(\kappa^{\boldsymbol{\alpha}})$ and*

$$\mathrm{mc}\left(\kappa^{\boldsymbol{\alpha}}_{\varrho^{\alpha_n}_{\beta}} + \mathrm{dp}_{\boldsymbol{\alpha}}(\varrho^{\alpha_n}_{\beta})\right) \le \nu^{\boldsymbol{\alpha}}_{\beta}.$$

(f) *If $n > 0$ we have*

$$\mathrm{Im}(\kappa^{\boldsymbol{\alpha}}), \mathrm{Im}(\nu^{\boldsymbol{\alpha}}) \subseteq \alpha \cdot \widehat{\alpha_n}.$$

Proof. The proof proceeds by induction along $<_{\mathrm{lex}}$ on ISeq in the following way:

Claim 4.6. *Given $\boldsymbol{\alpha}$ as above and β such that $\boldsymbol{\alpha}^\frown\beta \in \mathrm{ISeq}$ we prove:*

1. *If β is of a form $\gamma + 1$ where $\gamma \in \mathrm{dom}(\kappa^{\boldsymbol{\alpha}})$ then $\mathrm{dp}_{\boldsymbol{\alpha}}(\gamma)$ is well-defined and the claims concerning $\mathrm{dp}_{\boldsymbol{\alpha}}$ stated in the Lemma, including part (d), but excluding part (e), hold up to (and including) γ.*
2. *If $\beta \in \mathrm{dom}(\kappa^{\boldsymbol{\alpha}})$ then $\kappa^{\boldsymbol{\alpha}}_{\beta}$ is well-defined and the claims concerning $\kappa^{\boldsymbol{\alpha}}$ stated in the Lemma, excluding parts (d) and (e), hold up to (and including) β.*
3. *If $n > 0$ and β is of a form $\alpha_n \cdot (\gamma + 1)$ such that $\gamma \in \mathrm{dom}(\nu^{\boldsymbol{\alpha}}) - \mathbb{E}^{>\alpha_n}$ then part (e) holds up to and including γ.*
4. *If $n > 0$ and β is of a form $\alpha_n \cdot \gamma$ such that $\gamma \in \mathrm{dom}(\nu^{\boldsymbol{\alpha}})$ then $\nu^{\boldsymbol{\alpha}}_{\gamma}$ is well-defined and the claims concerning $\nu^{\boldsymbol{\alpha}}$ stated in the Lemma, excluding part (e), hold up to (and including) γ.*

Ad 1. The well-definedness of $\mathrm{dp}_{\boldsymbol{\alpha}}(\gamma)$ follows using the i.h. Notice the crucial point that in the case $\gamma \in \mathbb{E}^{>\alpha_n}$ we have

$$(\alpha_1, \ldots, \alpha_n, \gamma, \ldots) <_{\mathrm{lex}} (\alpha_1, \ldots, \alpha_n, \beta).$$

Part (c) concerning $\mathrm{dp}_{\boldsymbol{\alpha}}$ follows by i.h. and definition of $\mathrm{dp}_{\boldsymbol{\alpha}}$.

For part (d) we use that by the i.h. $\kappa^{\boldsymbol{\alpha}}$ has already been shown to be strictly increasing up to (and including) γ, and that by definition $\kappa^{\boldsymbol{\alpha}}_{\delta+1} = \kappa^{\boldsymbol{\alpha}}_{\delta} + \mathrm{dp}_{\boldsymbol{\alpha}}(\delta) + 1$. The case $\gamma \notin \mathbb{E}$ is then easily verified using the i.h. , showing part 1 of (d).

In order to see part 2 of (d) suppose $\gamma \in \mathbb{E}^{>\alpha_n}$ and set $\boldsymbol{\gamma} := \boldsymbol{\alpha}^\frown\gamma$. It is easy to see that $\mathrm{dp}_{\boldsymbol{\alpha}}(\gamma) \ge \nu^{\boldsymbol{\gamma}}_{\omega} = \kappa^{\boldsymbol{\alpha}}_{\gamma} \cdot \omega$. The estimation $\mathrm{dp}_{\boldsymbol{\alpha}}(\gamma) < \kappa^{\boldsymbol{\alpha}}_{\gamma} \cdot \widehat{\gamma}$ is verified as follows. By i.h. we have $\mathrm{Im}(\kappa^{\boldsymbol{\gamma}}), \mathrm{Im}(\nu^{\boldsymbol{\gamma}}) \subseteq \kappa^{\boldsymbol{\alpha}}_{\gamma} \cdot \widehat{\gamma}$, and $\mathrm{dp}_{\boldsymbol{\gamma}}(\lambda^{\alpha_n}_{\gamma}) < \kappa^{\boldsymbol{\gamma}}_{\lambda^{\alpha_n}_{\gamma}}$ if $\lambda^{\alpha_n}_{\gamma} \notin \mathbb{E}$. We have $\mathrm{dp}_{\boldsymbol{\gamma}}(\lambda^{\alpha_n}_{\gamma}) = 0$ if $\lambda^{\alpha_n}_{\gamma} = \gamma$, and if $\lambda^{\alpha_n}_{\gamma} \in \mathbb{E} \cap \gamma$ we obtain $\mathrm{dp}_{\boldsymbol{\gamma}}(\lambda^{\alpha_n}_{\gamma}) = \mathrm{dp}_{\boldsymbol{\alpha}}(\lambda^{\alpha_n}_{\gamma}) < \kappa^{\boldsymbol{\alpha}}_{\lambda^{\alpha_n}_{\gamma}+1} \le \kappa^{\boldsymbol{\alpha}}_{\gamma}$ using the i.h. If finally $\lambda^{\alpha_n}_{\gamma} \in \mathbb{E}^{>\gamma}$ we have $\mathrm{dp}_{\boldsymbol{\gamma}}(\lambda^{\alpha_n}_{\gamma}) < \kappa^{\boldsymbol{\gamma}}_{\lambda^{\alpha_n}_{\gamma}} \cdot \widehat{\lambda^{\alpha_n}_{\gamma}}$ by the i.h. Since $\widehat{\lambda^{\alpha_n}_{\gamma}} \le \widehat{\gamma}$ according to Lemmata 3.12 and 3.17, the latter expression is less than or equal to $\kappa^{\boldsymbol{\alpha}}_{\gamma} \cdot \widehat{\gamma}$. Thus $\mathrm{dp}_{\boldsymbol{\alpha}}(\gamma) < \kappa^{\boldsymbol{\alpha}}_{\gamma} \cdot \widehat{\gamma}$ follows.

For part 3 of (d) assume that $\gamma \in \mathbb{E}^{>\alpha_n}$ and $\mu^{\alpha_n}_{\gamma} < \gamma$. By Lemma 3.12 we have $\mu^{\alpha_n}_{\gamma} < \gamma \le \lambda^{\alpha_n}_{\gamma} < \gamma^2$, moreover $\delta := \mathrm{mc}(\lambda^{\alpha_n}_{\gamma}) \le \gamma \cdot \mu^{\alpha_n}_{\gamma}$, which using the i.h. and inspecting the definition of $\kappa^{\boldsymbol{\gamma}}$ yields $\kappa^{\boldsymbol{\gamma}}_{\delta} \le \kappa^{\boldsymbol{\alpha}}_{\gamma} \cdot \mu^{\alpha_n}_{\gamma}$ whence $\kappa^{\boldsymbol{\gamma}}_{\lambda^{\alpha_n}_{\gamma}} < \kappa^{\boldsymbol{\alpha}}_{\gamma} \cdot \mu^{\alpha_n}_{\gamma} \cdot \omega$. By definition we have $\nu^{\boldsymbol{\gamma}}_{\mu^{\alpha_n}_{\gamma}} = \kappa^{\boldsymbol{\alpha}}_{\gamma} \cdot \mu^{\alpha_n}_{\gamma}$, and using the i.h. we have $\mathrm{dp}_{\boldsymbol{\gamma}}(\lambda^{\alpha_n}_{\gamma}) < \kappa^{\boldsymbol{\gamma}}_{\lambda^{\alpha_n}_{\gamma}}$. Thus $\mathrm{dp}_{\boldsymbol{\alpha}}(\gamma) < \kappa^{\boldsymbol{\alpha}}_{\gamma} \cdot \mu^{\alpha_n}_{\gamma} \cdot \omega$.

Ad 2. Suppose $\beta \in \mathrm{dom}(\kappa^{\boldsymbol{\alpha}})$. Using the i.h. we immediately see that $\kappa^{\boldsymbol{\alpha}}_{\beta}$ is well-defined, that it is an additive principal number if β is, and that $\kappa^{\boldsymbol{\alpha}}_{\beta} = \kappa^{\boldsymbol{\alpha}|n-1}_{\beta} \le \alpha$ if $n > 0$ and $\beta \le \alpha_n$. We therefore immediately obtain the claim if $\beta \le \alpha_n$ and may from now on assume $\beta > \alpha_n$. We employ a side induction on $\gamma < \beta$ to verify $\kappa^{\boldsymbol{\alpha}}_{\gamma} < \kappa^{\boldsymbol{\alpha}}_{\beta}$ among the other remaining claims of 2.

Case 1: $\beta \notin \mathbb{P}$.

Then the i.h. immediately implies that $\kappa^{\boldsymbol{\alpha}}$ is strictly increasing and continuous up to and including β. If $n > 0$ and $\beta =_{\mathrm{NF}} \gamma + \delta$ we obtain $\kappa^{\boldsymbol{\alpha}}_{\beta} < \alpha \cdot \widehat{\alpha_n}$ using part d) (for $\boldsymbol{\alpha}^\frown(\gamma + 1)$) which gives $\mathrm{dp}_{\boldsymbol{\alpha}}(\gamma) < \kappa^{\boldsymbol{\alpha}}_{\gamma}$ if $\gamma \notin \mathbb{E}$, and $\mathrm{dp}_{\boldsymbol{\alpha}}(\gamma) < \kappa^{\boldsymbol{\alpha}}_{\gamma} \cdot \widehat{\gamma}$ if $\gamma \in \mathbb{E}$. In this latter case Lemma 3.17 yields $\widehat{\gamma} \le \widehat{\alpha_n}$, so that altogether we obtain $\kappa^{\boldsymbol{\alpha}}_{\beta} < \alpha \cdot \widehat{\alpha_n}$.

Case 2: $\beta \in \mathbb{P}$.

Setting $\beta' := (1/\bar{\beta}) \cdot \beta$ we have $\kappa^{\boldsymbol{\alpha}}_{\beta} = \kappa^{\boldsymbol{\alpha}}_{\bar{\beta}+1} \cdot \beta'$ according to the definition. In case of $n > 0$ we have $\kappa^{\boldsymbol{\alpha}}_{\bar{\beta}+1} < \alpha \cdot \widehat{\alpha_n}$ by the i.h. and $\beta' \le \beta \le \lambda^{\alpha_{n-1}}_{\alpha_n} < \widehat{\alpha_n}$ using Lemma 3.17. This shows $\kappa^{\boldsymbol{\alpha}}_{\beta} < \alpha \cdot \widehat{\alpha_n}$ if $n > 0$. It remains to show that $\kappa^{\boldsymbol{\alpha}}_{\beta}$ is the proper supremum of $\mathrm{Im}(\kappa^{\boldsymbol{\alpha}}_{\restriction\beta})$.

Subcase 2.1: $\beta = \bar{\beta} \cdot \omega$ (i.e. $\beta \notin \mathbb{L}$).

We then have $\kappa^{\boldsymbol{\alpha}}_{\beta} = (\kappa^{\boldsymbol{\alpha}}_{\bar{\beta}} + \mathrm{dp}_{\boldsymbol{\alpha}}(\bar{\beta})) \cdot \omega = \sup\{\kappa^{\boldsymbol{\alpha}}_{\bar{\beta}\cdot k} \mid k < \omega\}$.

Subcase 2.2: $\beta \in \mathbb{L}$.

From now on we may show $\kappa^{\boldsymbol{\alpha}}_{\gamma} < \kappa^{\boldsymbol{\alpha}}_{\beta}$ just for ordinals $\gamma \in (\bar{\beta}, \beta) \cap \mathbb{P} =: G$ since β is the proper supremum of G. $\kappa^{\boldsymbol{\alpha}}_{\beta}$ will then be shown to be the proper supremum of the $\kappa^{\boldsymbol{\alpha}}_{\gamma}$ for $\gamma \in G$. Since the case $\bar{\gamma} = \bar{\beta}$ is trivial in showing $\kappa^{\boldsymbol{\alpha}}_{\gamma} < \kappa^{\boldsymbol{\alpha}}_{\beta}$, in such verifications we may assume that $\bar{\gamma} \in (\bar{\beta}, \beta)$, whence also $\bar{\gamma}^+ < \beta$. Let $\gamma' := (1/\bar{\gamma}) \cdot \gamma$. We have $\kappa^{\boldsymbol{\alpha}}_{\gamma} = (\kappa^{\boldsymbol{\alpha}}_{\bar{\gamma}} + \mathrm{dp}_{\boldsymbol{\alpha}}(\bar{\gamma})) \cdot \gamma'$, and $\kappa^{\boldsymbol{\alpha}}_{\bar{\gamma}} < \kappa^{\boldsymbol{\alpha}}_{\beta}$ by the i.h.

Subcase 2.2.1: $\beta \notin \mathbb{M}$.

Then neither γ nor $\bar{\gamma}$ can be multiplicative principal numbers. We then have $\beta =_{\mathrm{NF}} \bar{\beta} \cdot \beta'$ as well as $\gamma =_{\mathrm{NF}} \bar{\gamma} \cdot \gamma'$, and it follows that $\gamma' < \beta'$. We have the already established $\mathrm{dp}_{\boldsymbol{\alpha}}(\bar{\gamma}) < \kappa^{\boldsymbol{\alpha}}_{\bar{\gamma}}$, and so $\kappa^{\boldsymbol{\alpha}}_{\gamma} = \kappa^{\boldsymbol{\alpha}}_{\bar{\gamma}} \cdot \gamma' < \kappa^{\boldsymbol{\alpha}}_{\bar{\beta}+1} \cdot \beta' = \kappa^{\boldsymbol{\alpha}}_{\beta}$. In order to see that $\kappa^{\boldsymbol{\alpha}}_{\beta} = \sup\{\kappa^{\boldsymbol{\alpha}}_{\gamma} \mid \gamma \in G\}$ notice that the proper supremum of the ordinals γ' corresponding with $\gamma \in G$ is β' and that $\kappa^{\boldsymbol{\alpha}}_{\bar{\beta}+1} \cdot \gamma' \le \kappa^{\boldsymbol{\alpha}}_{\bar{\gamma}+1} \cdot \gamma' = \kappa^{\boldsymbol{\alpha}}_{\gamma} < \kappa^{\boldsymbol{\alpha}}_{\bar{\beta}+1} \cdot \beta' = \kappa^{\boldsymbol{\alpha}}_{\beta}$.

Subcase 2.2.2: $\beta \in \mathbb{M}$.

Then we have $\beta' = \beta$, and in the more interesting case of $\bar{\gamma} \in \mathbb{E}$ we use the already established $\mathrm{dp}_{\boldsymbol{\alpha}}(\bar{\gamma}) < \kappa^{\boldsymbol{\alpha}}_{\bar{\gamma}} \cdot \widehat{\bar{\gamma}}$ together with $\widehat{\bar{\gamma}} \le \bar{\gamma}^+ < \beta$ to conclude $\kappa^{\boldsymbol{\alpha}}_{\gamma} < \kappa^{\boldsymbol{\alpha}}_{\bar{\gamma}} \cdot \widehat{\bar{\gamma}} \cdot \gamma' < \kappa^{\boldsymbol{\alpha}}_{\bar{\beta}+1} \cdot \beta = \kappa^{\boldsymbol{\alpha}}_{\beta}$. If $\beta \in \mathrm{Lim}(\mathbb{M})$ we now immediately see that $\kappa^{\boldsymbol{\alpha}}_{\beta} = \sup\{\kappa^{\boldsymbol{\alpha}}_{\gamma} \mid \gamma \in G\}$. Otherwise β has a form $\omega^{\omega^{\delta+1}}$ and by Lemma 5.10 of [4] we have $\bar{\beta} \in \mathbb{E}_1$ (being the predecessor of β in its τ-localization), and setting $\gamma_k := \omega^{\omega^{\delta}\cdot(k+1)}$ for $k < \omega$ we have $\beta = \sup\{\gamma_k \mid k < \omega\}$ and by definition obtain either $\kappa^{\boldsymbol{\alpha}}_{\gamma_k} = \kappa^{\boldsymbol{\alpha}}_{\bar{\beta}+1} \cdot (\omega^{\omega^{\delta}})^{(k+1)}$, namely when $\delta \notin \mathbb{E}$, or $\kappa^{\boldsymbol{\alpha}}_{\gamma_{k+1}} = \kappa^{\boldsymbol{\alpha}}_{\bar{\beta}+1} \cdot (\omega^{\omega^{\delta}})^{(k+1)}$ otherwise. This explicitly shows $\kappa^{\boldsymbol{\alpha}}_{\beta} = \sup\{\kappa^{\boldsymbol{\alpha}}_{\gamma} \mid \gamma \in G\}$.

Ad 3. Suppose that $n > 0$ and β is of a form $\alpha_n \cdot (\gamma + 1)$ such that $\gamma \in \mathrm{dom}(\nu^{\boldsymbol{\alpha}}) - \mathbb{E}^{>\alpha_n}$. Note that $\varrho^{\alpha_n}_{\gamma} \le \alpha_n \cdot \mathrm{logend}(\gamma) \le \alpha_n \cdot \gamma < \beta$, and that by part (b) of Lemma 3.12 we have $\varrho^{\alpha_n}_{\gamma} \le \lambda^{\alpha_{n-1}}_{\alpha_n}$. Inspecting the definition and using the i.h. we can read off the desired estimation.

Case 1: $\gamma \le \alpha_n$.

Subcase 1.1: $\gamma \in \mathbb{E}^{\le\alpha_n}$.

$$\varrho^{\alpha_n}_{\gamma} = \alpha_n \cdot \gamma, \qquad \nu^{\boldsymbol{\alpha}}_{\gamma} = \alpha \cdot \gamma = \kappa^{\boldsymbol{\alpha}}_{\alpha_n\cdot\gamma}, \quad \text{and} \quad \mathrm{dp}_{\boldsymbol{\alpha}}(\alpha_n \cdot \gamma) = \kappa^{\boldsymbol{\alpha}}_{\gamma} + \mathrm{dp}_{\boldsymbol{\alpha}}(\gamma) \le \alpha.$$

Subcase 1.2: $\gamma < \alpha_n$, $\gamma \notin \mathbb{E}$.

Let $\mathrm{logend}(\gamma) =: \delta =_{\mathrm{ANF}} \delta_1 + \cdots + \delta_l$. Then we have $\varrho^{\alpha_n}_{\gamma} = \alpha_n \cdot \delta$, $\nu^{\boldsymbol{\alpha}}_{\gamma} \ge \alpha \cdot \gamma$, $\kappa^{\boldsymbol{\alpha}}_{\alpha_n\cdot\delta} = \kappa^{\boldsymbol{\alpha}}_{\alpha_n\cdot\delta_1} + \mathrm{dp}_{\boldsymbol{\alpha}}(\alpha_n \cdot \delta_1) + \cdots + \kappa^{\boldsymbol{\alpha}}_{\alpha_n\cdot\delta_{l-1}} + \mathrm{dp}_{\boldsymbol{\alpha}}(\alpha_n \cdot \delta_{l-1}) + \kappa^{\boldsymbol{\alpha}}_{\alpha_n\cdot\delta_l} = \alpha \cdot \delta$, and $\mathrm{dp}_{\boldsymbol{\alpha}}(\alpha_n \cdot \delta) = \mathrm{dp}_{\boldsymbol{\alpha}}(\alpha_n \cdot \delta_l) < \alpha$.

Case 2: $\gamma =_{\mathrm{NF}} \delta + \xi > \alpha_n$.

Notice that $\varrho^{\alpha_n}_{\gamma} = \varrho^{\alpha_n}_{\xi}$. The i.h. directly applies to ξ, if $\xi \notin \mathbb{E}^{>\alpha_n}$. By the i.h. (part 4 of the Claim) we have $\nu^{\boldsymbol{\alpha}}_{\gamma} > \nu^{\boldsymbol{\alpha}}_{\xi}$. If on the other hand $\xi \in \mathbb{E}^{>\alpha_n}$, then $\varrho^{\alpha_n}_{\gamma} = \xi$ whence $\nu^{\boldsymbol{\alpha}}_{\gamma} \ge \nu^{\boldsymbol{\alpha}}_{\xi+1} = \kappa^{\boldsymbol{\alpha}}_{\xi} \cdot 2 + \mathrm{dp}_{\boldsymbol{\alpha}}(\xi) + \kappa^{\boldsymbol{\alpha}}_{\xi}$.

Case 3: $\gamma \in \mathbb{P}^{>\alpha_n} - \mathbb{E}$.

By definition $\nu^{\boldsymbol{\alpha}}_{\gamma} = \nu^{\boldsymbol{\alpha}}_{\bar{\gamma}+1} \cdot \gamma'$ where $\gamma' = (1/\bar{\gamma}) \cdot \gamma$. Let $\gamma =_{\mathrm{CNF}} \omega^{\gamma_1+\cdots+\gamma_k}$. Then $\varrho^{\alpha_n}_{\gamma} \le \alpha_n \cdot (\gamma_1 + \cdots + \gamma_k)$ and $\mathrm{mc}(\kappa^{\boldsymbol{\alpha}}_{\varrho^{\alpha_n}_{\gamma}} + \mathrm{dp}_{\boldsymbol{\alpha}}(\varrho^{\alpha_n}_{\gamma})) = \mathrm{mc}(\kappa^{\boldsymbol{\alpha}}_{\alpha_n\cdot\gamma_1} + \mathrm{dp}_{\boldsymbol{\alpha}}(\alpha_n \cdot \gamma_1))$.

Subcase 3.1: $\gamma \notin \mathbb{M}$, i.e. $k > 1$.

Then $\bar{\gamma} = \omega^{\gamma_1+\cdots+\gamma_{k-1}}$ and $\gamma =_{\mathrm{NF}} \bar{\gamma} \cdot \gamma'$. Hence $\mathrm{mc}(\kappa^{\boldsymbol{\alpha}}_{\alpha_n\cdot\gamma_1} + \mathrm{dp}_{\boldsymbol{\alpha}}(\alpha_n \cdot \gamma_1)) = \mathrm{mc}(\kappa^{\boldsymbol{\alpha}}_{\varrho^{\alpha_n}_{\bar{\gamma}}} + \mathrm{dp}_{\boldsymbol{\alpha}}(\varrho^{\alpha_n}_{\bar{\gamma}})) \le \nu^{\boldsymbol{\alpha}}_{\bar{\gamma}+1} < \nu^{\boldsymbol{\alpha}}_{\gamma}$.

Subcase 3.2: $\gamma \in \mathbb{M}$, i.e. $k = 1$.

Then $\gamma_1 \in (\alpha_n, \gamma) \cap \mathbb{P} - \mathbb{E}$, $\gamma' = \gamma$, and $\bar{\gamma} \in \mathbb{E}^{\ge\alpha_n}$. By i.h. we have the estimation $\mathrm{dp}_{\boldsymbol{\alpha}}(\alpha_n \cdot \gamma_1) < \kappa^{\boldsymbol{\alpha}}_{\alpha_n\cdot\gamma_1}$. Let $\gamma_1 =_{\mathrm{CNF}} \omega^{\delta_1+\cdots+\delta_l}$. If $\delta_1 = \alpha_n$ we have $\bar{\gamma} = \alpha_n$ and $\kappa^{\boldsymbol{\alpha}}_{\alpha_n\cdot\gamma_1} = \alpha \cdot \gamma_1 < \alpha \cdot \alpha_n \cdot \gamma = \nu^{\boldsymbol{\alpha}}_{\gamma}$. If otherwise $\delta_1 > \alpha_n$ we have $\alpha_n \cdot \gamma_1 = \gamma_1$ and obtain $\kappa^{\boldsymbol{\alpha}}_{\gamma_1} \le \kappa^{\boldsymbol{\alpha}}_{\bar{\gamma}+1} \cdot \gamma_1 \le \nu^{\boldsymbol{\alpha}}_{\bar{\gamma}+1} \cdot \gamma_1 < \nu^{\boldsymbol{\alpha}}_{\gamma}$.

Ad 4. Suppose that $n > 0$ and β is of a form $\alpha_n \cdot \gamma$ such that $\gamma \in \mathrm{dom}(\nu^{\boldsymbol{\alpha}})$. That $\nu^{\boldsymbol{\alpha}}_{\gamma} \in \mathbb{P}^{>\alpha}$ in case of $\gamma \in \mathbb{P}^{>1}$ will follow immediately once the well-definedness of $\nu^{\boldsymbol{\alpha}}_{\gamma}$ is shown. We consider the following cases.

Case 1: $\gamma =_{\mathrm{NF}} \delta + \xi$.

As in part 3 we note that $\varrho^{\alpha_n}_{\delta} \le \alpha_n \cdot \mathrm{logend}(\delta) \le \alpha_n \cdot \delta < \beta$ and that by part (b) of Lemma 3.12 we have $\varrho^{\alpha_n}_{\delta} \le \lambda^{\alpha_{n-1}}_{\alpha_n}$. By the i.h. and the already established parts 1 and 2 of Claim 4.6 we know that $\kappa^{\boldsymbol{\alpha}}$ and $\mathrm{dp}_{\boldsymbol{\alpha}}$ are well-defined and satisfy the claimed properties up to and including $\varrho_\delta^{\alpha_n}$. Thus $\nu_\gamma^{\boldsymbol\alpha}$ is well-defined. If $\xi = 1$ and $\chi^{\alpha_n}(\delta) = 1$ we easily verify that $\kappa^{\boldsymbol\alpha}_{\varrho_\delta^{\alpha_n}} + \mathrm{dp}_{\boldsymbol\alpha}(\varrho_\delta^{\alpha_n})$ is greater than 0 since $\varrho_\delta^{\alpha_n} > 0$ in this case. In all other cases, using the i.h. we immediately see that $\nu_\gamma^{\boldsymbol\alpha} > \nu_{\gamma'}^{\boldsymbol\alpha}$ for any $\gamma' < \gamma$, and that $\nu_\gamma^{\boldsymbol\alpha}$ is the supremum of $\mathrm{Im}(\nu^{\boldsymbol\alpha}_{\restriction\gamma})$ in the case $\xi > 1$. In order to see that $\nu_\gamma^{\boldsymbol\alpha} < \alpha\cdot\widehat{\alpha_n}$ we need to verify $\mathrm{dp}_{\boldsymbol\alpha}(\varrho_\delta^{\alpha_n}) < \alpha\cdot\widehat{\alpha_n}$. If $\varrho_\delta^{\alpha_n} \notin \mathbb{E}$ or $\varrho_\delta^{\alpha_n} = \alpha_n$ we have $\mathrm{dp}_{\boldsymbol\alpha}(\varrho_\delta^{\alpha_n}) < \kappa^{\boldsymbol\alpha}_{\varrho_\delta^{\alpha_n}}$, otherwise, i.e. $\varrho_\delta^{\alpha_n} \in \mathbb{E}^{>\alpha_n}$, we have $\varrho_\delta^{\alpha_n} \le \mu_{\alpha_n}^{\alpha_{n-1}}$ by Lemma 3.12 and therefore obtain $\widehat{\varrho_\delta^{\alpha_n}} \le \widehat{\alpha_n}$ by Lemma 3.17, which together with $\mathrm{dp}_{\boldsymbol\alpha}(\varrho_\delta^{\alpha_n}) < \kappa^{\boldsymbol\alpha}_{\varrho_\delta^{\alpha_n}} \cdot \widehat{\varrho_\delta^{\alpha_n}}$ yields $\mathrm{dp}_{\boldsymbol\alpha}(\varrho_\delta^{\alpha_n}) < \alpha\cdot\widehat{\alpha_n}$.

Case 2: $\gamma \in \mathbb{P} \cap (1, \alpha_n]$.

Then we have $\nu_\gamma^{\boldsymbol\alpha} = \alpha\cdot\gamma$. All claims follow immediately from the i.h. if $\gamma \in \mathbb{L}$. If $\gamma =_{\mathrm{NF}} \delta\cdot\omega$ we have $\nu_\gamma^{\boldsymbol\alpha} = \alpha\cdot\delta\cdot\omega = \sup\{\nu^{\boldsymbol\alpha}_{\delta\cdot k} \mid k < \omega\}$ since $\varrho^{\alpha_n}_{\delta\cdot(k+1)} = \varrho_\delta^{\alpha_n} \le \alpha_n\cdot\mathrm{logend}(\delta) \le \alpha_n\cdot\delta < {\alpha_n}^2$ and $\mathrm{dp}_{\boldsymbol\alpha}(\varrho_\delta^{\alpha_n}) < \alpha$, $\kappa^{\boldsymbol\alpha}_{\varrho_\delta^{\alpha_n}} = \alpha\cdot\mathrm{logend}(\delta)$, which can be seen inspecting the definition and using the i.h.

Case 3: $\alpha_n < \gamma = \bar\gamma\cdot\omega$.

Then, again inspecting the definition and using the i.h., $\nu_\gamma^{\boldsymbol\alpha} = \nu^{\boldsymbol\alpha}_{\bar\gamma+1}\cdot\omega = \sup\{\nu^{\boldsymbol\alpha}_{\bar\gamma\cdot k} \mid k < \omega\}$, and the claims concerning $\nu_\gamma^{\boldsymbol\alpha}$ follow by the i.h.

Case 4: $\gamma \in \mathbb{L}^{>\alpha_n} - \mathbb{E}$.

Then $\nu_\gamma^{\boldsymbol\alpha} = \nu^{\boldsymbol\alpha}_{\bar\gamma+1}\cdot\gamma'$ where $\gamma' = (1/\bar\gamma)\cdot\gamma$. By side induction on $\delta \in (\bar\gamma, \gamma) \cap \mathbb{P} =: D$ we show $\nu_\delta^{\boldsymbol\alpha} < \nu_\gamma^{\boldsymbol\alpha}$. Notice that $(\bar\gamma, \gamma) \cap \mathbb{E} = \emptyset$. Let $\delta \in D$ whence $\nu_\delta^{\boldsymbol\alpha} = \nu^{\boldsymbol\alpha}_{\bar\delta+1}\cdot\delta'$ where $\delta' = (1/\bar\delta)\cdot\delta < \gamma'$. The case $\bar\delta = \bar\gamma$ is trivial, so assume $\bar\delta \in D$. Since $\mathrm{mc}(\kappa^{\boldsymbol\alpha}_{\varrho_{\bar\delta}^{\alpha_n}} + \mathrm{dp}_{\boldsymbol\alpha}(\varrho_{\bar\delta}^{\alpha_n})) \le \nu^{\boldsymbol\alpha}_{\bar\delta}$ by the i.h., we also have $\nu^{\boldsymbol\alpha}_{\bar\delta+1} < \nu_\gamma^{\boldsymbol\alpha}$ by the i.h. This implies $\nu_\delta^{\boldsymbol\alpha} < \nu_\gamma^{\boldsymbol\alpha}$. In order to see continuity in γ notice that $\nu^{\boldsymbol\alpha}_{\bar\gamma+1}\cdot\delta' \le \nu^{\boldsymbol\alpha}_{\bar\delta+1}\cdot\delta' = \nu_\delta^{\boldsymbol\alpha}$ and $\sup\{\delta'+1 \mid \delta \in D\} = \gamma'$.

Case 5: $\gamma \in \mathbb{E}^{>\alpha_n}$.

We then have $\nu_\gamma^{\boldsymbol\alpha} = \kappa_\gamma^{\boldsymbol\alpha} = \kappa^{\boldsymbol\alpha}_{\bar\gamma+1}\cdot\gamma$ and set $D := (\bar\gamma, \gamma) \cap \mathbb{M}$. For any $\delta \in D \cap \mathbb{E}$ we have $\nu_\delta^{\boldsymbol\alpha} = \kappa_\delta^{\boldsymbol\alpha}$ and thus obtain $\nu_\delta^{\boldsymbol\alpha} < \nu_\gamma^{\boldsymbol\alpha}$ from the already established part 2 of Claim 4.6. Let $\delta \in D - \mathbb{E}$ whence $\nu_\delta^{\boldsymbol\alpha} = \nu^{\boldsymbol\alpha}_{\bar\delta+1}\cdot\delta$. If $\bar\delta = \bar\gamma$ or $\bar\delta \notin \mathbb{E}^{>\alpha_n}$ we proceed as in the previous case. Otherwise we have $\varrho_{\bar\delta}^{\alpha_n} = \bar\delta$ and $\kappa_\delta^{\boldsymbol\alpha} + \mathrm{dp}_{\boldsymbol\alpha}(\bar\delta) < \kappa_\gamma^{\boldsymbol\alpha}$ by part 2 of the Claim, hence $\nu_\delta^{\boldsymbol\alpha} < \nu_\gamma^{\boldsymbol\alpha}$. Continuity in γ follows by part 2 of the Claim if $\gamma \in \mathrm{Lim}(\mathbb{E})$. Otherwise we again have $D \cap \mathbb{E} = \emptyset$ and $\nu_\delta^{\boldsymbol\alpha} = \nu^{\boldsymbol\alpha}_{\bar\delta+1}\cdot\delta \ge \kappa^{\boldsymbol\alpha}_{\bar\gamma+1}\cdot\delta$ since $\nu^{\boldsymbol\alpha}_{\bar\gamma+1} \ge \kappa^{\boldsymbol\alpha}_{\bar\gamma+1}$. Therefore $\sup\{\nu_\delta^{\boldsymbol\alpha} \mid \delta \in D\} = \nu_\gamma^{\boldsymbol\alpha}$. This concludes the proof of Claim 4.6 and thus establishes Lemma 4.5. □

The following lemma reveals equations for κ- and ν-values referring to tracking sequences, thus providing a more intuitive redefinition of the κ- and ν-functions. Notice that using this alternative definition involving tracking sequences directly in 4.4 would have complicated the proof of Lemma 4.5 considerably.

Lemma 4.7. *Let $\boldsymbol\alpha \in \mathrm{RS}$, say $\boldsymbol\alpha = (\alpha_1, \ldots, \alpha_n)$ where $n = 0$ in case of $\boldsymbol\alpha = ()$, and set $\alpha_0 := 1$. Let $\beta \in \mathbb{M}^{>\alpha_n}$ and γ be the immediate predecessor of β in $\mathrm{ts}^{\alpha_n}(\beta)$ if that exists and α_n otherwise. If $\beta \in \mathrm{dom}(\kappa^{\boldsymbol\alpha})$ then*

$$\kappa_\beta^{\boldsymbol\alpha} = \kappa^{\boldsymbol\alpha}_{\gamma+1}\cdot\beta.$$

If $n > 0$ and $\beta \in \mathrm{dom}(\nu^{\boldsymbol\alpha})$ then

$$\nu_\beta^{\boldsymbol\alpha} = \nu^{\boldsymbol\alpha}_{\gamma+1}\cdot\beta.$$

For products $\delta =_{\mathrm{MNF}} \delta_1\cdots\delta_m > \alpha_n$ where $m > 1$ let γ be the immediate predecessor of δ_1 in $\mathrm{ts}^{\alpha_n}(\delta_1)$ if that exists and α_n otherwise. If $\delta \in \mathrm{dom}(\kappa^{\boldsymbol\alpha})$ we have

$$\kappa_\delta^{\boldsymbol\alpha} = \begin{cases} \kappa^{\boldsymbol\alpha}_{\gamma+1}\cdot\delta & \text{if } \delta_1 \in \mathbb{M}^{>\alpha_n} - \mathbb{E} \\ \kappa^{\boldsymbol\alpha}_{\delta_1+1}\cdot\delta_2\cdots\delta_m & \text{if } \delta_1 \in \mathbb{E}^{\ge\alpha_n}. \end{cases}$$

If $n > 0$ and $\delta \in \mathrm{dom}(\nu^{\boldsymbol\alpha})$ we similarly have

$$\nu_\delta^{\boldsymbol\alpha} = \begin{cases} \nu^{\boldsymbol\alpha}_{\gamma+1}\cdot\delta & \text{if } \delta_1 \in \mathbb{M}^{>\alpha_n} - \mathbb{E} \\ \nu^{\boldsymbol\alpha}_{\delta_1+1}\cdot\delta_2\cdots\delta_m & \text{if } \delta_1 \in \mathbb{E}^{\ge\alpha_n}. \end{cases}$$

Proof. The proof is by inspection of Definition 4.4 using Lemma 4.5, in particular part d) 2. Let $\alpha_n = \beta_0, \ldots, \beta_m = \beta$ be the τ-fine-localization of β. Since $\beta \in \mathbb{M}^{>\alpha_n}$ we have $\bar\beta = \beta_{m-1} \in \mathbb{E}^{\ge\alpha_n}$. Let $k \in \{0, \ldots, m-1\}$ be such that $\beta_k = \gamma$. One easily sees that $\widehat{\beta_{k+i}} \le \beta$ for $i = 1, \ldots, m-k-1$ (referring to base α_n). Thus $\kappa_\beta^{\boldsymbol\alpha} = \kappa^{\boldsymbol\alpha}_{\beta_{k+i}}\cdot\beta$ successively for $i = m-k-1, \ldots, i = 1$, and $\kappa_\beta^{\boldsymbol\alpha} = \kappa^{\boldsymbol\alpha}_{\beta_k+1}\cdot\beta$.

The claim concerning $\nu_\beta^{\boldsymbol\alpha}$ (where $n > 0$ is assumed) is derived easily from the result for $\kappa^{\boldsymbol\alpha}$ by inspecting the definition of $\nu_\beta^{\boldsymbol\alpha}$.

The remaining claims are now straightforward, using parts (d) and (e) of Lemma 4.5. □

Definition 4.8. $\mathrm{o} : \mathrm{TS} \to \mathbb{P} \cap 1^\infty$, the **evaluation function for tracking sequences**, is defined by setting for $\boldsymbol{\alpha}^\frown\beta \in \mathrm{TS}$

$$\mathrm{o}\,(\boldsymbol{\alpha}^\frown\beta) := \begin{cases} \kappa^{()}_\beta & \text{if } \boldsymbol{\alpha} = () \\ \nu^{\boldsymbol{\alpha}}_\beta & \text{otherwise.} \end{cases}$$

We additionally define $\mathrm{o}\,(()) := 0$ so that o is defined on all of RS.

The definition given next is crucial in the proof of the subsequent lemma. It will help us detect multiplicative normal forms of ordinals given as multiples of (relativized) connectivity components. These multiplicative normal forms are in turn necessary to find the tracking sequences of ordinals given as evaluations. We will thus be able to show that tracking sequences perfectly reverse enumerations of those relativized components in the core of $\mathcal{R}_2$ which are additively indecomposable.

Definition 4.9. For $\beta \in \mathbb{M} \cap 1^\infty$ and $\gamma \in \mathbb{E} \cap 1^\infty$ let $\mathrm{sk}_\beta(\gamma)$ be the maximal sequence $\delta_1, \ldots, \delta_l$ such that (setting $\delta_0 := 1$)

- $\delta_1 = \gamma$ and
- if $i \in \{1, \ldots, l-1\}$ & $\delta_i \in \mathbb{E}^{>\delta_{i-1}}$ & $\beta \le \mu_{\delta_i}$, then $\delta_{i+1} = \overline{\mu_{\delta_i} \cdot \beta}$.

Remark. Lemma 3.5 guarantees that the above definition terminates. We have $(\delta_1, \ldots, \delta_{l-1}) \in \mathrm{RS}$ and $(\delta_1, \ldots, \delta_l) \in \mathrm{TS}$. Notice that $\beta \le \delta_i$ for $i = 2, \ldots, l$.

Lemma 4.10. *For any* $\boldsymbol{\gamma} \in \mathrm{TS}$ *we have*

$$\mathrm{ts}(\mathrm{o}(\boldsymbol{\gamma})) = \boldsymbol{\gamma}.$$

For any $\alpha \in \mathbb{P} \cap 1^\infty$ *we have*

$$\mathrm{o}(\mathrm{ts}(\alpha)) = \alpha.$$

In other words, the mapping ts *is a* $<$-$<_{\mathrm{lex}}$*-order-isomorphism of* $\mathbb{P} \cap 1^\infty$ *onto* TS *with inverse* o.

Proof. By Lemma 3.15 the second and third claim follow from the first. The first claim is shown by induction along the ordering $<_{\mathrm{lex}}$ on TS. Let $\boldsymbol{\gamma} = \boldsymbol{\alpha}^\frown\beta \in \mathrm{TS}$, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathrm{RS}$, and $\alpha_0 := 1$. We set $\beta' := (1/\bar{\beta}) \cdot \beta$, so that $\beta' \in \mathbb{M}^{\le\beta}$, and $\alpha := \kappa^{\boldsymbol{\alpha}|n-1}_{\alpha_n}$. By Lemma 3.14 we have $\mathrm{lf}(\alpha) = \mathrm{lf}(\alpha_n) = \alpha_n$.

Case 1: $\beta \le \alpha_n$.
This is trivial if $n = 0$. If $n > 0$ we have $\mathrm{o}(\gamma) =_{\mathrm{NF}} \alpha \cdot \beta$, and the claim follows immediately from $\alpha_n \in \mathbb{E}^{>\alpha_{n-1}}$ and $\beta \le \mu^{\alpha_{n-1}}_{\alpha_n}$.

Case 2: $\beta > \alpha_n$.
We define

$$\tilde{\beta} := \begin{cases} \beta_{k-1} & \text{if } \beta \in \mathbb{M} \text{ where } \mathrm{ts}^{\alpha_n}(\beta) = (\beta_1, \ldots, \beta_k) \text{ and } \beta_0 := \alpha_n \\ \bar{\beta} & \text{otherwise.} \end{cases}$$

Case 2.1: $\tilde{\beta} = \alpha_n$.
If $n = 0$, using Lemma 4.7 we obtain $\mathrm{o}(\boldsymbol{\gamma}) = \beta$, and the claim follows immediately. Now suppose $n > 0$. If $\beta \notin \mathbb{M}$ we have $\mathrm{o}(\boldsymbol{\gamma}) =_{\mathrm{NF}} \nu^{\boldsymbol{\alpha}}_{\alpha_n} \cdot \beta'$, $\nu^{\boldsymbol{\alpha}}_{\alpha_n} = \alpha \cdot \alpha_n$, and by the i.h. we have $\mathrm{ts}(\nu^{\boldsymbol{\alpha}}_{\alpha_n}) = \boldsymbol{\alpha}^\frown\alpha_n$. The claim then follows, given that $\beta' \le \alpha_n$ and $\beta = \alpha_n \cdot \beta' \le \mu^{\alpha_{n-1}}_{\alpha_n}$. Now assume $\beta \in \mathbb{M}$, whence $\beta' = \beta$ and $\mathrm{o}(\boldsymbol{\gamma}) = \alpha \cdot \beta$. The case $\alpha < \beta$ is trivial, so let us assume that $\beta \le \alpha$. Then $\alpha \notin \mathbb{M}$, implying that $\alpha =_{\mathrm{NF}} \bar{\alpha} \cdot \alpha_n$. Let $\delta := \overline{\alpha \cdot \beta}$, whence $\beta \le \delta < \alpha < \delta \cdot \beta$. By Lemma 3.15 we have $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_l) := \mathrm{ts}(\delta) <_{\mathrm{lex}} \mathrm{ts}(\alpha) \overset{\text{i.h.}}{=} \boldsymbol{\alpha}$, hence $\mathrm{o}(\mathrm{ts}(\delta)) = \delta$ by the i.h. Now, $\boldsymbol{\delta}$ cannot be a proper initial sequence of $\boldsymbol{\alpha}$ since on the one hand $\beta \le \mathrm{lf}(\delta) = \mathrm{lf}(\delta_l) \le \delta_l$ according to Lemma 3.14 and on the other hand $\beta > \alpha_l$ in case of $l < n$. Hence there exists the minimal $l_0 \in \{1, \ldots, \min\{l-1, n\}\}$ such that $\delta_{l_0} < \alpha_{l_0}$. The sequence $(\delta_1, \ldots, \delta_{l_0})$ cannot be an initial sequence of $\mathrm{ts}(\delta \cdot \beta)$ because $\mathrm{ts}(\alpha) <_{\mathrm{lex}} \mathrm{ts}(\delta \cdot \beta)$ according to Lemma 3.15. Since $\delta_{l_0} \in (\alpha_{l_0-1}, \alpha_{l_0})$ and $\beta \in (\delta_{l_0}, \delta^+_{l_0})$ we have $\mathrm{ts}^{\delta_{l_0}}(\beta) = (\alpha_{l_0}, \ldots, \alpha_n, \beta)$ using Lemma 6.5 of [10] and Lemma 3.6, referring to base $\alpha_{l_0-1} = \delta_{l_0-1}$ where $\delta_0 := 1$ in case of $l_0 = 1$. Hence $\mathrm{ts}(\delta \cdot \beta) = \boldsymbol{\gamma}$.

Case 2.2: $\tilde{\beta} > \alpha_n$ & $\tilde{\beta} \notin \mathbb{E}$.
By Lemma 4.5 we have $\mathrm{o}(\boldsymbol{\gamma}) =_{\mathrm{NF}} \mathrm{o}(\boldsymbol{\alpha}^\frown\bar{\beta}) \cdot \beta'$. The claim is easily verified using that $\mathrm{ts}(\mathrm{o}(\boldsymbol{\alpha}^\frown\bar{\beta})) = \boldsymbol{\alpha}^\frown\bar{\beta}$ by the i.h.

Case 2.3: $\tilde{\beta} \in \mathbb{E}^{>\alpha_n}$.
Let $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_l) := \mathrm{sk}_{\beta'}(\tilde{\beta})$, $\boldsymbol{\delta}' := (\delta_1, \ldots, \delta_{l-1})$, $\delta_0 := 1$, and $\delta := \mathrm{o}(\boldsymbol{\alpha}^\frown\boldsymbol{\delta})$. Since $\boldsymbol{\alpha}^\frown\boldsymbol{\delta} <_{\mathrm{lex}} \boldsymbol{\gamma}$ we have $\mathrm{ts}(\delta) = \boldsymbol{\alpha}^\frown\boldsymbol{\delta}$ by i.h. By Lemma 3.14 we have $\mathrm{lf}(\delta) = \mathrm{lf}(\delta_l) \ge \beta'$, noticing that in the case $\beta = \beta' > \tilde{\beta}$ we have $\beta \in \mathbb{M}^{\le\mu_{\tilde{\beta}}}$ and therefore $l > 1$. The desired $\mathrm{ts}(\mathrm{o}(\boldsymbol{\gamma})) = \boldsymbol{\gamma}$ is directly verified once we prove that

$$\mathrm{o}(\boldsymbol{\gamma}) =_{\mathrm{NF}} \delta \cdot \beta'.$$

The above equality is verified by inspection of Definition 4.4, making use of Lemma 3.12, part (a), Lemma 4.5, parts (d) and (e), and Lemma 4.7. In case of $\delta_l \in \mathbb{E}^{>\delta_{l-1}}$ and $\mu_{\delta_l} < \beta'$ we have $\mu_{\delta_l} < \delta_l$ and therefore (by part (d) of 4.5)

$$\delta < \mathrm{dp}_{\boldsymbol{\alpha}^\frown\delta'}(\delta_l) < \delta\cdot\mu_{\delta_l}\cdot\omega < \delta\cdot\beta',$$

which is then also used in showing the last equality of

$$\begin{aligned}\mathrm{o}(\boldsymbol{\gamma}) &= \mathrm{dp}_{\boldsymbol{\alpha}}(\delta_1)\cdot\beta'\\ &= \mathrm{dp}_{\boldsymbol{\alpha}^\frown\delta_1}(\delta_2)\cdot\beta'\\ &= \dots\\ &= \mathrm{dp}_{\boldsymbol{\alpha}^\frown(\delta_1,\dots,\delta_{l-2})}(\delta_{l-1})\cdot\beta'\\ &= \delta\cdot\beta'.\end{aligned}$$

This concludes our proof. □

Corollary 4.11. *We obtain the following correspondence between* RP *and* RS*:*

$$\mathrm{RP} = \mathrm{Im}(\mathrm{o}_{\restriction\mathrm{RS}}) \quad \textit{and} \quad \mathrm{RS} = \mathrm{Im}(\mathrm{ts}_{\restriction\mathrm{RP}}),$$

defining $\mathrm{ts}(0) := ()$ *for convenience.* □

Using the set RP and its characterization in the above corollary we are now able to simplify the settings of relativization in the definition of $\kappa^{\boldsymbol{\alpha}}$ and $\nu^{\boldsymbol{\alpha}}$.

Definition 4.12. For $\alpha \in \mathrm{RP}$ we define $\kappa^\alpha := \kappa^{\mathrm{ts}(\alpha)}$, $\mathrm{dp}_\alpha := \mathrm{dp}_{\mathrm{ts}(\alpha)}$, and in case of $\alpha > 0$ we set $\nu^\alpha := \nu^{\mathrm{ts}(\alpha)}$.

In order to formulate the assignment of tracking chains to ordinals in Section 6 we need to introduce a suitable notion of tracking sequence relative to a given context. We first introduce an evaluation function for relativized tracking sequences.

Definition 4.13. Let $\alpha \in \mathrm{RP} - \{0\}$ with $\mathrm{ts}(\alpha) = (\alpha_1,\dots,\alpha_n) =: \boldsymbol{\alpha}$. We define

$$\mathrm{TS}^{\boldsymbol{\alpha}} := \{\boldsymbol{\gamma} \in \mathrm{TS}^{\alpha_n} \mid \gamma_1 \le \lambda_{\alpha_n}^{\alpha_{n-1}}\}$$

and for $\boldsymbol{\gamma}^\frown\beta \in \mathrm{TS}^{\boldsymbol{\alpha}}$

$$\mathrm{o}^\alpha(\boldsymbol{\gamma}^\frown\beta) := \begin{cases}\kappa^{\boldsymbol{\alpha}}_\beta & \text{if } \boldsymbol{\gamma} = ()\\ \nu^{\boldsymbol{\alpha}^\frown\boldsymbol{\gamma}}_\beta & \text{otherwise.}\end{cases}$$

For convenience we identify o^0 with o.

Remark. Note that this is well-defined thanks to part (c) of Lemma 3.12. Notice also that $\mathrm{TS}^{\boldsymbol{\alpha}}$ is a $<_{\mathrm{lex}}$-initial segment of TS^{α_n}. We have the following

Lemma 4.14. *Let α and $\boldsymbol{\alpha}$ be as in the above definition. Let $\lambda_1 := \mathrm{mc}(\lambda_{\alpha_n})$, and whenever λ_i is defined and $\lambda_i \in \mathbb{E}^{>\lambda_{i-1}}$ (setting $\lambda_0 := \alpha_n$), let $\lambda_{i+1} := \mu_{\lambda_i}$. If we denote the resulting vector by $(\lambda_1,\dots,\lambda_k) =: \boldsymbol{\lambda}$ then $\mathrm{TS}^{\boldsymbol{\alpha}}$ is the initial segment of TS^{α_n} with $<_{\mathrm{lex}}$-maximum $\boldsymbol{\lambda}$. We have*

$$\mathrm{o}^\alpha(\boldsymbol{\lambda}) = \mathrm{mc}(\kappa^{\boldsymbol{\alpha}}_{\lambda_{\alpha_n}} + \mathrm{dp}_{\boldsymbol{\alpha}}(\lambda_{\alpha_n})).$$

Proof. The proof is by evaluation of $\mathrm{mc}(\kappa^{\boldsymbol{\alpha}}_{\lambda_{\alpha_n}} + \mathrm{dp}_{\boldsymbol{\alpha}}(\lambda_{\alpha_n}))$ using parts (d) and (e) of Lemma 4.5. □

The analogue to Lemma 4.10 is as follows. Notice that we have to be careful regarding multiples of indices versus their evaluations.

Lemma 4.15. *Let α, $\boldsymbol{\alpha}$, and $\boldsymbol{\gamma}^\frown\beta \in \mathrm{TS}^{\boldsymbol{\alpha}}$ be as in the above definition. Then we have*

$$\mathrm{ts}^{\alpha_n}(\alpha_n\cdot((1/\alpha)\cdot\mathrm{o}^\alpha(\boldsymbol{\gamma}^\frown\beta))) = \boldsymbol{\gamma}^\frown\beta.$$

For $\delta \in \mathbb{P}\cap[\alpha_n, \alpha_n^\infty)$ such that $\mathrm{ts}^{\alpha_n}(\delta) \in \mathrm{TS}^{\boldsymbol{\alpha}}$ we have

$$\mathrm{o}^\alpha(\mathrm{ts}^{\alpha_n}(\delta)) = \alpha\cdot((1/\alpha_n)\cdot\delta).$$

Setting $\lambda := \alpha_n\cdot((1/\alpha)\cdot\mathrm{mc}(\kappa^{\boldsymbol{\alpha}}_{\lambda_{\alpha_n}} + \mathrm{dp}_{\boldsymbol{\alpha}}(\lambda_{\alpha_n})))$ we have

$$\mathrm{ts}^{\alpha_n}(\lambda) = \boldsymbol{\lambda} \in \mathrm{TS}^{\boldsymbol{\alpha}}$$

for $\boldsymbol{\lambda}$ as defined in Lemma 4.14, and the mapping ts^{α_n} is a $<$-$<_{\mathrm{lex}}$-order isomorphism of

$$\{\delta \in \mathbb{P}\cap[\alpha_n, \alpha_n^\infty) \mid \mathrm{ts}^{\alpha_n}(\delta) \in \mathrm{TS}^{\boldsymbol{\alpha}}\} = [\alpha_n, \lambda]\cap\mathbb{P}$$

with $\mathrm{TS}^{\boldsymbol{\alpha}}$.

Proof. Once the first claim of the lemma is shown by induction along $<_{\mathrm{lex}}$ on $\mathrm{TS}^{\boldsymbol{\alpha}}$, the remaining claims follow using Lemmas 3.15 and 4.14. In proving the first claim for $\boldsymbol{\gamma}^\frown\beta \in \mathrm{TS}^{\boldsymbol{\alpha}}$, say $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_m)$ where $m \geq 0$, we proceed in analogy with the course of proof of Lemma 4.10, replacing $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}^\frown\boldsymbol{\gamma}$, α_n with γ_m (setting $\gamma_0 := \alpha_n$), and α with $\gamma := \alpha_n \cdot ((1/\alpha) \cdot \mathrm{o}(\boldsymbol{\gamma}))$ in the case $m > 0$, where by the i.h. we have $\mathrm{ts}^{\alpha_n}(\gamma) = \boldsymbol{\gamma}$. □

Definition 4.16. Let $\alpha \in \mathrm{RP} - \{0\}$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) := \mathrm{ts}(\alpha)$. Set $\alpha_0 := 1$ and let $\beta \in \mathbb{P} \cap 1^\infty$. The **tracking sequence of** β **relative to** α**,** $\mathrm{ts}[\alpha](\beta)$, is defined as follows. Let $k \in \{0, \ldots, n\}$ be maximal such that $\tilde{\alpha}_k := \mathrm{o}(\boldsymbol{\alpha}_{\restriction k}) \leq \beta$.

$$\mathrm{ts}[\alpha](\beta) := \begin{cases} \mathrm{ts}(\beta) & \text{if } k = 0 \\ \mathrm{ts}^{\alpha_k}(\alpha_k \cdot ((1/\tilde{\alpha}_k) \cdot \beta)) & \text{if } k > 0. \end{cases}$$

For technical reasons define $\mathrm{ts}[0](\beta) := \mathrm{ts}(\beta)$.

Remark. $\mathrm{ts}[\alpha]$ aims at a tracking sequence with starting point $\tilde{\alpha}_k$ instead of 0. In the above situation for $\mathrm{ts}[\alpha](\beta)$ to make sense, i.e. to be related to α, we should have $\beta_1 \leq \lambda^{\alpha_{k-1}}_{\alpha_k}$ in case of $k > 0$ where β_1 is the first element of $\mathrm{ts}[\alpha](\beta)$. It is easy to see (using 3.12, 3.15, and 4.15) that this holds if $k \in (0, n)$. However, in case of $k = n$ this holds if and only if $\beta \leq \kappa^{\boldsymbol{\alpha}}_{\lambda_{\alpha_n}} + \mathrm{dp}_{\boldsymbol{\alpha}}(\lambda_{\alpha_n})$ as shown in 4.14 and 4.15.

Lemma 4.17. *Let α be as in the above definition and $\beta, \gamma \leq \kappa^{\boldsymbol{\alpha}}_{\lambda_{\alpha_n}} + \mathrm{dp}_{\boldsymbol{\alpha}}(\lambda_{\alpha_n})$ be additive principal numbers.*

(a) *With k as in the above definition and setting $\alpha_{n+1} := \lambda_{\alpha_n} + 1$ we have*

$$\alpha_k \leq \beta_1 < \alpha_{k+1}$$

where β_1 is the first element of $\mathrm{ts}[\alpha](\beta)$.

(b) *If $\beta < \gamma$ then*

$$\mathrm{ts}[\alpha](\beta) <_{\mathrm{lex}} \mathrm{ts}[\alpha](\gamma).$$

Proof. The lemma is proved by straightforward application of Lemmas 3.15 and 4.15, using part (a) to show part (b). □

5. Tracking chains

The preparations made in the previous sections have set the grounds to introduce the concept of tracking chains. Tracking chains will provide us with a grid on the segment of ordinals below 1^∞ whose resolution is sufficiently high to allow for a characterization of the relations $\leq_1$ and $\leq_2$ within the core of $\mathcal{R}_2$. Here we will first explicitly define tracking chains and then assign tracking chains to the ordinals below 1^∞. This assignment will be shown to exhaust the set of tracking chains in a one-one manner.

A tracking chain is a vector of index sequences whose first element always denotes a κ-index (possibly relativized from the second vector component on) and whose possible other elements denote ν-indices. Conditions on the indices that occur in a tracking chain will guarantee a unique and semantically correct representation of ordinals in the core by successively approaching them through more and more refined (relativized) $\leq_1$-components (the indices given by the first element in each sequence) and (relativized) $\leq_2$-components nested along the sequences, as was the case for tracking sequences which indeed characterize single component tracking chains. Thus, moving along the index sequences from left to right and from the upper sequence (vector component 1) downward to the lower sequences of the vector we obtain a unique "address" for any ordinal in the core. A few normal form conditions are necessary, mainly that non-zero indices on main lines always have priority over simple κ-indices, while the index 0 may only occur in one case, namely representing 0.

It might be very instructive for the reader to consider the restriction of tracking chains to vectors of single $\kappa^{()}$-indices below ε_0: These characterize the elements of the core of $\mathcal{R}_1$ (see [1]) that is, the initial segment of the core of $\mathcal{R}_2$ below ε_0. Within such a vector, the upper vector components down to the i-th represent the greatest $<_1$-predecessor of the ordinal represented by the upper vector components down to the $i+1$-st.

The approach of tracking chains can be generalized to a treatment of $\mathcal{R}_n$ (or even $\mathcal{R}_\omega$), rearranging the indices of relativized $\leq_i$-components so that $\leq_i$-connected components are listed downward along the i-th column while refinements to $\leq_j$-components (where $j < i$) start with a new line with an entry in the j-th column, thus leaving many entries in the resulting $<\omega \times n$-matrix possibly empty. Further considerations into this direction, however, would exceed the topic of the present paper.

The following two central definitions of this article, tracking chains and their maximal extension, should formally be considered one simultaneous definition.

Definition 5.1. We define a **tracking chain** to be a vector $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n)$ where $n \geq 1$, consisting of sequences $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \ldots, \alpha_{i,m_i})$ of ordinals below 1^∞ with $m_i \geq 1$ for $1 \leq i \leq n$, that satisfies certain conditions. We proceed by induction along the lexicographic ordering of the index pairs (n, m_n).

The **initial chains** $\boldsymbol{\alpha}_{\restriction(i,j)}$ **of** $\boldsymbol{\alpha}$ where $1 \leq i \leq n$ and $1 \leq j \leq m_i$ are

$$\boldsymbol{\alpha}_{\restriction(i,j)} := ((\alpha_{1,1}, \ldots, \alpha_{1,m_1}), \ldots, (\alpha_{i-1,1}, \ldots, \alpha_{i-1,m_{i-1}}), (\alpha_{i,1}, \ldots, \alpha_{i,j})).$$

By $\boldsymbol{\alpha}_{\restriction i}$ we abbreviate $\boldsymbol{\alpha}_{\restriction(i,m_i)}$. For technical convenience we set $\boldsymbol{\alpha}_{\restriction(1,0)} := ()$ and $\boldsymbol{\alpha}_{\restriction(i+1,0)} := \boldsymbol{\alpha}_{\restriction(i,m_i-1)}$ for $1 \leq i < n$.

A necessary condition on $\boldsymbol{\alpha}$ to be a tracking chain is that all its proper initial chains $\boldsymbol{\alpha}_{\restriction(i,j)}$ be tracking chains. We therefore suppose from now on that all proper initial chains of $\boldsymbol{\alpha}$ are tracking chains. Before we list further conditions on tracking chains we need to introduce some important terminology.

The vector $\boldsymbol{\tau} = (\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n)$ defined by $\tau_{i,j} := \mathrm{end}(\alpha_{i,j})$ for $1 \le i \le n$ and $1 \le j \le m_i$ is called the **chain associated with $\boldsymbol{\alpha}$**.

For $1 \le i \le n$ the ***i*-th reference index pair** $\mathrm{ref}(i)$ **of $\boldsymbol{\alpha}$** is $\mathrm{ref}(i) := (k, m_k - 1)$ if the maximal $k \in \{1, \ldots, i\}$ such that $m_k > 1$ exists, and $\mathrm{ref}(i) := (1, 0)$ otherwise. For technical convenience we set $\tau_{1,0} := 1$ and $\tau_{i+1,0} := \tau_{\mathrm{ref}(i)}$ for $1 \le i < n$, and $\alpha_{i,0} := \tau_{i,0}$ for $1 \le i \le n$.

The ***i*-th unit** $\tau_i^\star$ **of $\boldsymbol{\alpha}$** and its **index pair** $i^\star$ for $1 \le i \le n$ is defined as follows. Let $i^\star := (l, j)$ if there exist l, j where $l \in \{1, \ldots, i-1\}$ is maximal such that there exists a maximal $j \in \{1, \ldots, m_l - 1\}$ with $\tau_{l,j} \le \tau_{i,1}$, and $i^\star := (1, 0)$ otherwise.[5] We then set $\tau_i^\star := \tau_{i^\star}$

For $1 \le i \le n$ and $1 \le j \le m_i$ we define the **base** $\tau'_{i,j}$ **of** $\tau_{i,j}$ **in $\boldsymbol{\alpha}$** and its **index pair** $(i,j)'$ by $\tau'_{i,j} := \tau_{(i,j)'}$ where

$$(i,j)' := \begin{cases} i^\star & \text{if } j = 1 \\ (i, j-1) & \text{otherwise.} \end{cases}$$

For technical convenience we extend this notation to index pairs $(i, 0)$, $1 \le i \le n$, by

$$(i,0)' := \begin{cases} (1,0) & \text{if } i = 1 \\ (\mathrm{ref}(i-1))' & \text{otherwise.} \end{cases}$$

For $1 \le i \le n$ we define the ***i*-th maximal base** τ'_i **of $\boldsymbol{\alpha}$** by

$$\tau'_i := \tau'_{i,m_i}.$$

We set $(i,j)'' := ((i,j)')'$. By τ''_i we denote $\tau_{(i,m_i)''}$. In order to increase readability we write $\mu_{\tau_{i,j}}, \mu_{\tau'_i}$ for $\mu^{\tau'_{i,j}}_{\tau_{i,j}}, \mu^{\tau''_i}_{\tau'_i}$, respectively, and $\lambda_{\tau_{i,j}}, \lambda_{\tau'_i}$ for $\lambda^{\tau'_{i,j}}_{\tau_{i,j}}, \lambda^{\tau''_i}_{\tau'_i}$, respectively, provided those terms are defined.

We define the ***i*-th critical index of $\boldsymbol{\alpha}$** by

$$\rho_i := \begin{cases} \log\left((1/\tau_i^\star) \cdot \tau_{i,1}\right) + 1 & \text{if } m_i = 1 \\ \varrho^{\tau'_i}_{\tau_{i,m_i}} + \tau'_i & \text{if } m_i > 1 \ \&\ \tau_{i,m_i} < \mu_{\tau'_i} \ \&\ \chi^{\tau'_i}(\tau_{i,m_i}) = 0 \\ \varrho^{\tau'_i}_{\tau_{i,m_i}} + 1 & \text{if } m_i > 1 \ \&\ \tau_{i,m_i} < \mu_{\tau'_i} \ \&\ \chi^{\tau'_i}(\tau_{i,m_i}) = 1 \\ \lambda_{\tau'_i} + 1 & \text{otherwise} \end{cases}$$

whenever the terms that apply are defined. In order to clarify the tracking chain to which the ρ-notation refers we will sometimes write $\rho_i(\boldsymbol{\alpha})$ instead of ρ_i which is used when no ambiguity is likely.

The $<_{\mathrm{lex}}$-greatest index pair (i, j) of $\boldsymbol{\alpha}$ after which the elements of $\boldsymbol{\alpha}$ fall onto the main line starting at $\alpha_{i,j}$ is called the **critical main line index pair of $\boldsymbol{\alpha}$**. The formal definition is as follows:

If there exists a maximal $i \in \{1, \ldots, n\}$ such that there is a maximal $j \in \{1, \ldots, m_i - 1\}$ with $\alpha_{i,j+1} < \mu_{\tau_{i,j}}$ and if (i, j) satisfies the following conditions:

- $\chi^{\tau_{i,j}}(\tau_{i,j+1}) = 1$ and
- $\boldsymbol{\alpha}$ is reached by maximal 1-step extensions starting from $\boldsymbol{\alpha}_{\restriction i,j+1}$, according to Definition 5.2[6]

then (i, j) is called the critical main line index pair of $\boldsymbol{\alpha}$, written as $\mathrm{cml}(\boldsymbol{\alpha})$. Otherwise $\boldsymbol{\alpha}$ does not possess a critical main line index pair.

In order for $\boldsymbol{\alpha}$ to be a tracking chain the following explicitly enumerated conditions must hold:

1. $\alpha_{i,j} > 0$ for any $i \in \{1, \ldots, n\}$ and any $j \in \{1, \ldots, m_i\}$, unless $n = 1$ and $m_n = 1$ in which unique case $\alpha_{1,1} = 0$ is allowed.

5 Notice that since we are operating on additive principal numbers this is a divisibility condition.

6 Here Definition 5.2 is applied only to tracking chains that have already been defined.

2. For any $i \in \{1, \ldots, n\}$ such that $m_i > 1$ we have
$$\tau_{i,1}, \ldots, \tau_{i,m_i-1} \in \mathbb{E} \quad \text{and} \quad \tau_i^\star < \tau_{i,1} < \cdots < \tau_{i,m_i-1}.$$
3. $\alpha_{i,j+1} \le \mu_{\tau_{i,j}}$ for all i, j such that $1 \le i \le n$ and $1 \le j < m_i$.
4. $\alpha_{i+1,1} < \rho_i$ for any $i \in \{1, \ldots, n-1\}$.
5. $\alpha_{i+1,1} \ne \tau_{i,m_i}$ whenever $i \in \{1, \ldots, n-1\}$ and $\tau_i' < \tau_{i,m_i} \in \mathbb{E}$.
6. If $m_n = 1$ and if $\boldsymbol{\alpha}$ possesses a critical main line index pair $\mathrm{cml}(\boldsymbol{\alpha}) = (i, j)$ then $\tau_{n,1} \ne \tau_{i,j}$.

By TC we denote the set of all tracking chains. For $\boldsymbol{\alpha} \in \mathrm{TC}$ we define $\mathrm{dom}(\boldsymbol{\alpha})$ to be the set of all index pairs of $\boldsymbol{\alpha}$, that is
$$\mathrm{dom}(\boldsymbol{\alpha}) := \{(i, j) \mid 1 \le i \le n \;\&\; 1 \le j \le m_i\}.$$
By $(i, j)^+$ we denote the immediate $<_{\mathrm{lex}}$-successor of (i, j) in $\mathrm{dom}(\boldsymbol{\alpha})$ if that exists and $(n+1, 1)$ otherwise. For convenience we set $(i, 0)^+ := (i, 1)$.

An **extension** of a tracking chain $\boldsymbol{\alpha}$ is a tracking chain of which $\boldsymbol{\alpha}$ is an initial chain. A **1-step extension** is an extension by exactly one additional ordinal.

Due to frequent future occurrences we introduce the following notation for the modification of a tracking chain's last ordinal.
$$\boldsymbol{\alpha}[\xi] := \begin{cases} \boldsymbol{\alpha}_{\restriction n-1}{}^\frown(\alpha_{n,1}, \ldots, \alpha_{n,m_n-1}, \xi) & \text{if } \xi > 0 \vee (n, m_n) = (1, 1) \\ \boldsymbol{\alpha}_{\restriction n-1}{}^\frown(\alpha_{n,1}, \ldots, \alpha_{n,m_n-1}) & \text{if } \xi = 0 \wedge m_n > 1 \\ \boldsymbol{\alpha}_{\restriction n-1} & \text{if } \xi = 0 \wedge n > 1 \wedge m_n = 1. \end{cases}$$

Remark. Note that $\boldsymbol{\alpha}[\xi]$ might not be a tracking chain. This has to be verified when this notation is used. We have included cases where $\xi = 0$ for convenience, especially in the formulation of Theorem 7.9. It follows from the definition that for any $\xi \in (0, \alpha_{n,m_n})$ the following easy criterion holds:
$$\boldsymbol{\alpha}[\xi] \notin \mathrm{TC} \;\Leftrightarrow\; n > 1 \;\&\; m_n = 1 \;\&\; \tau_{n-1}' < \tau_{n-1,m_{n-1}} = \xi \in \mathbb{E}.$$
In this case we do have $\boldsymbol{\alpha}[\zeta] \in \mathrm{TC}$ for every $\zeta \in (0, \rho_{n-1}) - \{\tau_{n-1,m_{n-1}}\} \subseteq \tau_{n-1,m_{n-1}} + \tau_{n-1}'$.

The following definition describes a procedure to extend a given tracking chain stepwise in a maximal manner. It will be shown that this procedure terminates after finitely many steps.

Definition 5.2. Let $\boldsymbol{\alpha} \in \mathrm{TC}$ with components $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \ldots, \alpha_{i,m_i})$ for $1 \le i \le n$. The **extension candidate for $\boldsymbol{\alpha}$** is defined via the following case differentiation, setting $\tau := \tau_{n,m_n}$ and $\tau' := \tau_n'$:

1. $m_n = 1$: We consider three subcases:
   1.1. $\tau' = \tau$: Then $\boldsymbol{\alpha}$ is already maximal. An extension candidate for $\boldsymbol{\alpha}$ does not exist.
   1.2. $\tau' < \tau \in \mathbb{E}$: Then $\boldsymbol{\alpha}$ is extended by $\alpha_{n,2} := \mu_\tau$.
   1.3. Otherwise: Then $\boldsymbol{\alpha}$ is extended by $\alpha_{n+1,1} := \log\big((1/\tau') \cdot \tau\big)$.
2. $m_n > 1$: We consider three subcases.
   2.1. $\tau = 1$: Then $\boldsymbol{\alpha}$ is already maximal. An extension candidate for $\boldsymbol{\alpha}$ does not exist.
   2.2. $\tau' < \tau \in \mathbb{E}$: Here we consider another two subcases.
      2.2.1. $\tau = \mu_{\tau'} < \lambda_{\tau'}$: Then we extend $\boldsymbol{\alpha}$ by $\alpha_{n+1,1} := \lambda_{\tau'}$.
      2.2.2. Otherwise: Then $\boldsymbol{\alpha}$ is extended by $\alpha_{n,m_n+1} := \mu_\tau$.
   2.3. Otherwise: We consider again two subcases.
      2.3.1. $\tau < \mu_{\tau'}$: Then $\boldsymbol{\alpha}$ is extended by $\alpha_{n+1,1} := \varrho_\tau^{\tau'}$.
      2.3.2. Otherwise: Then $\boldsymbol{\alpha}$ is extended by $\alpha_{n+1,1} := \lambda_{\tau'}$.

If the extension candidate for $\boldsymbol{\alpha}$ exists we denote it by $\mathrm{ec}(\boldsymbol{\alpha})$, and if it is a tracking chain then we call it the **maximal 1-step extension of $\boldsymbol{\alpha}$**.

The iterated extension of $\boldsymbol{\alpha}$ starts with $t_0 := \boldsymbol{\alpha}$. Suppose t_n has already been defined. If t_n is maximal or is not a tracking chain, then the extension process ends with t_n. Otherwise we continue the extension process with the extension candidate t_{n+1} for t_n.

If after finitely many steps some t_{n_0} is reached which is a tracking chain that cannot be extended any further or whose extension candidate is not a tracking chain then we call t_{n_0} the **maximal extension of $\boldsymbol{\alpha}$,** $\mathrm{me}(\boldsymbol{\alpha})$. We define
$$\mathrm{me}^+(\boldsymbol{\alpha}) := \begin{cases} \mathrm{ec}(\mathrm{me}(\boldsymbol{\alpha})) & \text{if that exists} \\ \mathrm{me}(\boldsymbol{\alpha}) & \text{otherwise.} \end{cases}$$

Remark. Notice that any extension candidate which itself is not again a tracking chain cannot be extended any further. Note also that for any $\boldsymbol{\alpha} \in \mathrm{TC}$ for which $\mathrm{cml}(\boldsymbol{\alpha})$ exists, this same critical main line index pair is maintained along the process of stepwise maximal extension starting from $\boldsymbol{\alpha}$.

The following definition of *characteristic sequence* for a tracking chain is a characterization of the reversal of the sequence obtained when, starting with τ_{n,m_n}, $'$ is applied successively before reaching 1.

Definition 5.3. Let $\boldsymbol{\alpha} \in \mathrm{TC}$ with components $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \dots, \alpha_{i,m_i})$ for $1 \le i \le n$. The **characteristic sequence** $\mathrm{cs}(\boldsymbol{\alpha})$ **of** $\boldsymbol{\alpha}$ is defined by

$$\mathrm{cs}(\boldsymbol{\alpha}) := \mathrm{cs}(\boldsymbol{\alpha}_{\restriction n^\star})^\frown(\tau_{n,1}, \dots, \tau_{n,m_n})$$

where $\mathrm{cs}\,(()) := ()$. We define $\mathrm{cs}'(\boldsymbol{\alpha}) \in \mathrm{ISeq}$ as a sequence associated with $\boldsymbol{\alpha}$ crucial for inductive proofs along $<_{\mathrm{lex}}$. If $\mathrm{cs}(\boldsymbol{\alpha}) = \boldsymbol{\gamma}^\frown\beta$, setting

$$\delta = \begin{cases} (1/\tau_n^\star)\cdot\beta + 1 & \text{if } m_n = 1 \\ \beta + 2 & \text{if } m_n > 1 \end{cases}$$

we define $\mathrm{cs}'(\boldsymbol{\alpha})$ to be $\boldsymbol{\gamma}^\frown\delta$.

Lemma 5.4. *Let* $\boldsymbol{\alpha} \in \mathrm{TC}$.

(a) *If* $\boldsymbol{\alpha}^+$ *is a* 1*-step extension of* $\boldsymbol{\alpha}$ *then* $\mathrm{cs}'(\boldsymbol{\alpha}^+) <_{\mathrm{lex}} \mathrm{cs}'(\boldsymbol{\alpha})$.
(b) *The procedure of stepwise extension according to Definition 5.2 terminates after finitely many steps, that is,* $\mathrm{me}(\boldsymbol{\alpha})$ *exists.*
(c) *If* $\mathrm{ec}(\mathrm{me}(\boldsymbol{\alpha}))$ *exists then this candidate satisfies all conditions on tracking chains except for the 6th.*

Proof. Part (a) is shown by induction on $\mathrm{cs}'(\boldsymbol{\alpha}) \in \mathrm{ISeq}$ along $<_{\mathrm{lex}}$. In the case where $\boldsymbol{\alpha}^+$ has the form $\boldsymbol{\alpha}_{\restriction n-1}{}^\frown(\alpha_{n,1}, \dots, \alpha_{n,m_n}, \alpha_{n,m_n+1})$ we have $\tau_{n,1} \in \mathbb{E}^{>\tau_n^\star}$, and the claim follows directly. If on the other hand $\boldsymbol{\alpha}^+$ has the form $\boldsymbol{\alpha}^\frown(\alpha_{n+1,1})$, then we have

$$\mathrm{cs}'(\boldsymbol{\alpha}^+) = \mathrm{cs}(\boldsymbol{\alpha}_{\restriction(n+1)^\star})^\frown\left(\left(1/\tau_{(n+1)^\star}\right)\cdot\tau_{n+1,1} + 1\right).$$

Case 1: $m_n = 1$.
Then we have $\tau_{n+1,1} \le \alpha_{n+1,1} \le \log\left(\left(1/\tau_n^\star\right)\cdot\tau_{n,1}\right) \le \tau_{n,1}$, hence $(n+1)^\star \le_{\mathrm{lex}} n^\star$ and $\mathrm{cs}(\boldsymbol{\alpha}_{\restriction(n+1)^\star})$ is an initial sequence of $\mathrm{cs}(\boldsymbol{\alpha}_{\restriction n^\star})$. Moreover, $\tau_{n+1,1} < \tau_{n,1}$ since the assumption $\tau_{n+1,1} = \tau_{n,1}$ would imply $\tau_{n,1} \in \mathbb{E}^{>\tau_n^\star}$ and $\alpha_{n+1,1} = \tau_{n,1}$ which is not possible due to Condition 5 on tracking chains. This shows the claim in case of $\mathrm{cs}(\boldsymbol{\alpha}_{\restriction(n+1)^\star}) = \mathrm{cs}(\boldsymbol{\alpha}_{\restriction n^\star})$. The claim is seen easily if $\mathrm{cs}(\boldsymbol{\alpha}_{\restriction(n+1)^\star})$ is a proper initial sequence of $\mathrm{cs}(\boldsymbol{\alpha}_{\restriction n^\star})$.

Case 2: $m_n > 1$.
Let $\tau := \tau_{n,m_n}$ and $\tau' := \tau_{n,m_n-1} = \tau_n'$. If in this situation $\tau_{n+1,1} < \tau'$ then $\mathrm{cs}(\boldsymbol{\alpha}_{\restriction(n+1)^\star})$ is a proper initial sequence of $\mathrm{cs}(\tilde{\tau}')$ and the claim follows easily. Suppose finally that $\tau_{n+1,1} \ge \tau'$, that is, $(n+1)^\star = (n, m_n - 1)$. In order to then verify that $\tau + 2 > (1/\tau')\cdot\tau_{n+1,1} + 1$, notice that the contrary assumption $\tau'\cdot\tau < \tau_{n+1,1}$ in conjunction with the relation $\alpha_{n+1,1} < \rho_n$ implies, using part (a) of Lemma 3.12 in the case $\tau = \mu_{\tau'}$, $\tau_{n+1,1} \le \varrho_\tau^{\tau'} \le \tau'\cdot\tau < \tau_{n+1,1}$: Contradiction.
Part (b) is a consequence of part (a).
In order to see part (c) consider the situation where $\boldsymbol{\alpha}$ has an extension candidate $\boldsymbol{\alpha}' \notin \mathrm{TC}$. It is easy to verify that $\boldsymbol{\alpha}'$ satisfies Conditions 1 to 5 for tracking chains. Therefore $\boldsymbol{\alpha}'$ satisfies all conditions on tracking chains except for Condition 6. □

Lemma 5.5. *Let* $\boldsymbol{\alpha} \in \mathrm{TC}$ *be such that* $\mathrm{cml}(\boldsymbol{\alpha}) =: (i,j)$ *exists.*

(a) *If* $(i, j+1) = (n, m_n)$ *then* $\mathrm{ec}(\boldsymbol{\alpha})$ *exists and is a tracking chain.*
(b) *If* $(i, j+1) <_{\mathrm{lex}} (n, m_n)$ *then the following invariance properties hold (using terminology as in Definition 5.2)*
  - $\tau_{i,j} \le \tau'$,
  - *in case of* $m_n = 1$: $\tau > \tau_{i,j}$ *and* $\chi^{\tau_{i,j}}(\tau) = 1$,
  - *in case of* $m_n > 1$: $\tau = \mu_{\tau'}$ *and* $\chi^{\tau_{i,j}}(\tau') = 1$,

  and $\mathrm{ec}(\boldsymbol{\alpha})$ *exists. If* $\mathrm{ec}(\boldsymbol{\alpha}) \notin \mathrm{TC}$ *then* $\mathrm{ec}(\boldsymbol{\alpha})$ *is an extension of* $\boldsymbol{\alpha}$ *by some* $\alpha_{n+1,1}$*, and the last unit of* $\mathrm{ec}(\boldsymbol{\alpha})$ *has the index pair* $(n+1)^\star = (i,j)$.

Proof. Note that by assumption $\boldsymbol{\alpha}$ is an initial chain of $\mathrm{me}(\boldsymbol{\alpha}_{\restriction(i,j+1)})$. In order to show part (a) we use Lemma 5.4. If $\tau_{i,j} < \tau_{i,j+1} \in \mathbb{E}$ then case 2.2.2 of Definition 5.2 applies and the extension is clearly a tracking chain, otherwise case 2.3.1 applies. In this latter case $\boldsymbol{\alpha}$ is extended by $\alpha_{i+1,1} = \varrho_{\tau_{i,j+1}}^{\tau_{i,j}} = \tau_{i,j}\cdot\lambda$ where $\lambda := \log(\tau_{i,j+1})$ is a limit ordinal since $\tau_{i,j+1} \in \mathbb{L}^{\ge\tau_{i,j}}$ due to the assumption $\chi^{\tau_{i,j}}(\tau_{i,j+1}) = 1$. Hence $\tau_{i+1,1} > \tau_{i,j} = \tau_{i+1,1}'$, implying that $\mathrm{ec}(\boldsymbol{\alpha}) \in \mathrm{TC}$. Using Lemma 3.3 we have $\chi^{\tau_{i,j}}(\tau_{i,j+1}) = \chi^{\tau_{i,j}}(\lambda) = \chi^{\tau_{i,j}}(\mathrm{end}(\lambda)) = 1$ and hence also $\chi^{\tau_{i,j}}(\tau_{i+1,1}) = 1$.
By inspection of Definition 5.2 we can now clarify how $\mathrm{me}^+(\boldsymbol{\alpha})$ looks like. Consider index pairs (k,l) such that $(i, j+2) \le_{\mathrm{lex}} (k,l)$ in case of $\tau_{i,j} < \tau_{i,j+1} \in \mathbb{E}$ and $(i+1, 1) <_{\mathrm{lex}} (k,l)$ otherwise. We have the following cases.

- If $l > 1$ then $\alpha_{k,l} = \mu_{\tau_{k,l-1}}$ which in the case $l < m_k$ is an epsilon number and equal to $\lambda_{\tau_{k,l-1}}$.
- If $l = 1$ and $m_{k-1} > 1$ then $\alpha_{k,l} = \lambda_{\tau_{k-1,m_{k-1}-1}}$.
- Otherwise we have $\alpha_{k,l} = \log\left(\left(1/\tau_{(k-1)^\star}\right)\cdot\tau_{k-1,1}\right)$.

Part (b) is now shown by induction on the index pairs (n, m_n) ordered lexicographically. For the initial step $\boldsymbol{\alpha} = \mathrm{ec}(\boldsymbol{\alpha}_{\restriction(i,j+1)})$ part (a) yields the invariance properties. Suppose that the invariance properties hold for $\boldsymbol{\alpha}$. We verify that $\mathrm{ec}(\boldsymbol{\alpha})$ exists and show that in case of $\mathrm{ec}(\boldsymbol{\alpha}) \in \mathrm{TC}$ the invariance properties also hold for $\mathrm{ec}(\boldsymbol{\alpha})$. In the case $\mathrm{ec}(\boldsymbol{\alpha}) \notin \mathrm{TC}$ the respective claim follows from Lemma 5.4 and the inductive hypothesis which shows that there is no $\tau_{k,l} \le \tau_{i,j}$ with $(i, j) <_{\mathrm{lex}} (k, l)$.

Case 1: $m_n = 1$.
We need to show why $\tau = \tau'$ is not possible. In order to derive a contradiction let us assume this were the case. Then $\tau' \in \mathbb{E}^{>\tau_{i,j}}$, hence $\tau' = \tau_{k,l}$ for some (k, l) such that $(i, j) <_{\mathrm{lex}} (k, l) <_{\mathrm{lex}} (n, m_n)$ with $l < m_k$. Now we either have $\lambda_{\tau'} = \mu_{\tau'} = \alpha_{k,l+1}$ or $\lambda_{\tau'} = \alpha_{k+1,1}$, and using Lemma 3.3 we see that since $\chi^{\tau'}(\tau) = 1$ we have $\chi^{\tau'}(\lambda_{\tau'}) = 1$, which in turn implies $\chi^{\tau_{i,j}}(\tau') = 0$ using Lemma 3.3. Contradiction. Thus $\mathrm{ec}(\boldsymbol{\alpha})$ exists, and we proceed to verify the invariance properties for $\mathrm{ec}(\boldsymbol{\alpha})$ assuming that $\mathrm{ec}(\boldsymbol{\alpha}) \in \mathrm{TC}$. If case 1.2 of Definition 5.2 applies, that is, $\tau' < \tau \in \mathbb{E}$, and $\mathrm{ec}(\boldsymbol{\alpha})$ is obtained by appending $\alpha_{n,m_n+1} = \mu_\tau$, we have $\tau_{i,j} \le \tau' < \tau = \tau'_{n,m_n+1}$ and $\chi^{\tau_{i,j}}(\tau) = 1$. If on the other hand case 1.3 applies then $\alpha_{n+1,1} = \log\left((1/\tau') \cdot \tau\right)$ is appended to $\boldsymbol{\alpha}$. Here $\chi^{\tau_{i,j}}(\tau) = 1$ implies $\chi^{\tau_{i,j}}(\tau_{n+1,1}) = 1$, hence $\tau_{n+1,1} \ge \tau_{i,j}$ and thus also $\tau'_{n+1,1} \ge \tau_{i,j}$ using the i.h. Now, by the assumption $\mathrm{ec}(\boldsymbol{\alpha}) \in \mathrm{TC}$ the equality of $\tau_{n+1,1}$ and $\tau_{i,j}$ is excluded.

Case 2: $m_n > 1$.
Then the existence of $\mathrm{ec}(\boldsymbol{\alpha})$ is clear since $\tau = \mu_{\tau'} > 1$. We verify the invariance properties for $\mathrm{ec}(\boldsymbol{\alpha})$ assuming that it is a tracking chain. If case 2.2 of Definition 5.2 applies, that is, $\tau' < \tau \in \mathbb{E}$, and in particular case 2.2.1, that is $\tau = \mu_{\tau'} < \lambda_{\tau'}$, then $\mathrm{ec}(\boldsymbol{\alpha})$ is obtained by appending $\alpha_{n+1,1} = \lambda_{\tau'}$, and we argue as above where case 1.3 applied. If otherwise case 2.2.2 applies, that is $\tau = \mu_{\tau'} = \lambda_{\tau'} \in \mathbb{E}$, then $\alpha_{n,m_n+1} = \mu_\tau$ is appended to $\boldsymbol{\alpha}$ and we have $\tau'_{n,m_n+1} = \tau > \tau' \ge \tau_{i,j}$ as well as $\chi^{\tau_{i,j}}(\tau) = \chi^{\tau_{i,j}}(\lambda_{\tau'}) = \chi^{\tau_{i,j}}(\tau') = 1$. Finally, if case 2.3 and thus in particular case 2.3.2 applies, $\alpha_{n+1,1} = \lambda_{\tau'}$ is appended to $\boldsymbol{\alpha}$, and we argue as we did when cases 1.3 and 2.2.1 applied. □

Corollary 5.6. *Let $\boldsymbol{\alpha} \in \mathrm{TC}$ be maximal, i.e. $\boldsymbol{\alpha} = \mathrm{me}(\boldsymbol{\alpha})$, with maximal index pair (n, m_n). If $\mathrm{cml}(\boldsymbol{\alpha}) =: (i, j)$ exists then $\mathrm{ec}(\boldsymbol{\alpha})$ exists, and the extending index with index pair $(n + 1, 1)$ is a successor multiple of $\tau_{i,j}$ with $(n + 1)^\star = (i, j)$.* □

The following lemma clarifies, on a technical level, basic properties of tracking chains. It will be needed in proving Lemma 5.8 which in turn is needed for Lemma 5.10. These lemmas then reveal a crucial structural uniformity property of tracking chains (see parts (c) of 5.8 and 5.10).

Lemma 5.7. *Let $\boldsymbol{\alpha} \in \mathrm{TC}$ with components $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \dots, \alpha_{i,m_i})$ for $1 \le i \le n$.*

(a) *If $\mathrm{ref}(i) \neq (1, 0)$ then $\rho_i \le \lambda_{\tau_{\mathrm{ref}(i)}} + 1$.*
(b) *$\tau_i^\star \le \tau_{i,0}$. For $i \in \{1, \dots, n - 1\}$ we have $(i + 1)^\star \le_{\mathrm{lex}} \mathrm{ref}(i)$.*

Proof. Part (a) is shown inspecting the definition of ρ_i casewise from $i = 1$ up to $i = n$. Assume that $\mathrm{ref}(i) \neq (1, 0)$.

Case 1: $m_i = 1$. Then $i > 1$ and $\mathrm{ref}(i) = \mathrm{ref}(i - 1)$. We then have $\rho_i \le \tau_{i,1} + 1 \le \rho_{i-1} \le \lambda_{\tau_{\mathrm{ref}(i)}} + 1$ using the already shown instance $i - 1$ of the claim.

Case 2: $m_i > 1$ and $\tau_{i,m_i} < \mu_{\tau'_i}$. Then $\tau_{\mathrm{ref}(i)} = \tau'_i$. By Lemma 3.12 part (b) we obtain $\rho_i \le \lambda_{\tau_{\mathrm{ref}(i)}} + 1$.

Case 3: Otherwise. Then again $\tau_{\mathrm{ref}(i)} = \tau'_i$ and the claim follows.

Part (b) is obvious from the definition. □

Lemma 5.8. *Let $\boldsymbol{\alpha} \in \mathrm{TC}$ with components $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \dots, \alpha_{i,m_i})$ for $1 \le i \le n$.*

(a) *$\mathrm{cs}(\boldsymbol{\alpha}_{\restriction i^\star})$ is an initial sequence of $\mathrm{cs}(\boldsymbol{\alpha}_{\restriction(i,0)})$ which is proper if $\tau_i^\star < \tau_{i,0}$.*
(b) *If $\tau_i^\star > 1$ then $\tau_{i,1} \le \lambda_{\tau_{i^\star}}$.*
(c) *We have $\mathrm{cs}(\boldsymbol{\alpha}) \in \mathrm{ISeq}$, more precisely, if $\mathrm{cs}(\boldsymbol{\alpha}) = \boldsymbol{\gamma}^\frown\delta$ then*

$$\boldsymbol{\gamma} \in \mathrm{RS} \quad \textit{and} \quad \delta = \tau_{n,m_n} \in \begin{cases} \mathrm{dom}(\kappa^{\boldsymbol{\gamma}}) & \text{if } m_n = 1 \\ \mathrm{dom}(\nu^{\boldsymbol{\gamma}}) & \text{otherwise.} \end{cases}$$

For $i \in \{1, \dots, n\}$ and $0 \le j < m_i$ we even have $\mathrm{cs}(\boldsymbol{\alpha}_{\restriction(i,j)}) \in \mathrm{RS}$.

Proof. Part (a) is clear if $i^\star = (1, 0)$. Otherwise we apply part (b) of the previous lemma to see that $\tau_i^\star$ is an element of $\mathrm{cs}(\boldsymbol{\alpha}_{\restriction(i,0)})$, and the claim follows.

Part (b) is shown for $i = 1$ up to $i = n$ successively. For $i = 1$ we have $i^\star = (1, 0)$, so there is nothing to show. Now suppose $i \in (1, n)$ and $\tau_{i+1}^\star > 1$. By part (b) of the previous lemma this implies $\mathrm{ref}(i) \neq (1, 0)$, hence $\tau_{i+1,1} \le \lambda_{\tau_{\mathrm{ref}(i)}}$ by part (a) of the previous lemma. We are done if $\tau_{i+1}^\star = \tau_{\mathrm{ref}(i)}$, otherwise we apply part (a) and use part (c) of Lemma 3.12 as well as the already shown instances of part (b) to conclude the claim.

Part (c) holds due to Conditions 2, 3, 4, using part (c) of Lemma 3.12 and part (b). □

Definition 5.9. Let $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n)$ where $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \ldots, \alpha_{i,m_i})$ be a tracking chain with associated chain $\boldsymbol{\tau}$.

The **evaluations** $\tilde{\tau}_{i,j}$ **and** $\tilde{\alpha}_{i,j}$ for $1 \le i \le n$ and $0 \le j \le m_i$ are defined by[7]

$$\tilde{\tau}_{1,0} := \tilde{\alpha}_{1,0} := 0,$$
$$\tilde{\tau}_{i+1,0} := \tilde{\alpha}_{i+1,0} := \tilde{\tau}_{i,m_i-1} \quad \text{for } 1 \le i < n,$$

and for $1 \le i \le n$ and $1 \le j < m_i$

$$\tilde{\tau}_{i,1} := \kappa^{\tilde{\tau}_{i,0}}_{\tau_{i,1}}, \qquad \tilde{\tau}_{i,j+1} := \nu^{\tilde{\tau}_{i,j}}_{\tau_{i,j+1}}$$

and

$$\tilde{\alpha}_{i,1} := \kappa^{\tilde{\tau}_{i,0}}_{\alpha_{i,1}}, \qquad \tilde{\alpha}_{i,j+1} := \nu^{\tilde{\tau}_{i,j}}_{\alpha_{i,j+1}}.$$

The **initial values** $\{o_{i,j}(\boldsymbol{\alpha}) \mid 1 \le i \le n, 1 \le j \le m_i\}$ **of** $\boldsymbol{\alpha}$ are defined – setting for convenience $m_0 := 0$ and $o_{0,0}(\boldsymbol{\alpha}) := 0$ – for $i = 1, \ldots, n$ by

$$o_{i,1}(\boldsymbol{\alpha}) := o_{i-1,m_{i-1}}(\boldsymbol{\alpha}) + \tilde{\alpha}_{i,1}$$

and

$$o_{i,j+1}(\boldsymbol{\alpha}) := o_{i,j}(\boldsymbol{\alpha}) + (-\tilde{\tau}_{i,j} + \tilde{\alpha}_{i,j+1}) \quad \text{for } 1 \le j < m_i.$$

We define the **value of** $\boldsymbol{\alpha}$ by $o(\boldsymbol{\alpha}) := o_{n,m_n}(\boldsymbol{\alpha})$ which is the terminal initial value of $\boldsymbol{\alpha}$.

Remark. The correction $-\tilde{\tau}_{i,j}$ in the above definition avoids double summation: Consider the easy example of the chain $((\varepsilon_0, 1))$ which codes $\varepsilon_0 \cdot 2$. Notice that $-\tilde{\tau}_{i,j} + \tilde{\alpha}_{i,j+1}$ is always a non-zero multiple of $\tilde{\tau}_{i,j}$. We clearly have $o_{i,j}(\boldsymbol{\alpha}) = o(\boldsymbol{\alpha}_{\restriction(i,j)})$.

Notice that by definition the evaluation of a tracking chain whose single component is a tracking sequence is equal to the evaluation of that tracking sequence, which justifies the use of the same notation. Clearly, all evaluations defined above yield ordinals below 1^∞.

Lemma 5.10. (a) *The evaluations of the above definition are well-defined.*
(b) *In the situation of Definition 5.9 for all $i \in \{1, \ldots, n\}$ we have*

$$\tilde{\tau}_{i,1} = \kappa^{\tilde{\tau}_{i^\star}}_{\tau_{i,1}}.$$

(c) *For all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m_i\}$ such that $\tau_{i,1} \in \mathbb{E}^{>\tau_i^\star}$ and $\tau_{i,j} > 1$ we have*

$$\operatorname{ts}(\tilde{\tau}_{i,j}) = \left\{ \begin{array}{ll} (\tau_{i,1}, \ldots, \tau_{i,j}) & \text{if } \tau_i^\star = 1 \\ \operatorname{ts}(\tilde{\tau}_{i^\star})^\frown(\tau_{i,1}, \ldots, \tau_{i,j}) & \text{otherwise} \end{array} \right\} = \operatorname{cs}(\boldsymbol{\alpha}_{\restriction(i,j)}).$$

(d) *Suppose $(i,j) \le_{\text{lex}} (k,l)$ where $j < m_i$ and $l < m_k$ for index pairs from* $\operatorname{dom}(\boldsymbol{\alpha})$. *If $\operatorname{ts}(\tilde{\tau}_{i,j}) = \operatorname{ts}(\tilde{\tau}_{k,l})$ then $(i,j) = (k,l)$.*

Proof. The verification of all claims of the lemma proceeds simultaneously along $<_{\text{lex}}$ on the index pairs (i,j) where $1 \le i \le n$ and $0 \le j \le m_i$.

Part (a) uses already proved instances of the lemma, if necessary, and Lemma 5.8, part (c), in order to see that $\tilde{\tau}_{i,j} \in \mathrm{RP}$ when $j < m_i$. Condition 4 of Definition 5.1 together with Lemma 5.7, part (a), implies that $\alpha_{i,1} \in \operatorname{dom}(\kappa^{\tilde{\tau}_{i,0}})$ whereas Condition 3 guarantees that $\alpha_{i,j+1} \in \operatorname{dom}(\nu^{\tilde{\tau}_{i,j}})$ for $j = 1, \ldots, m_i - 1$.

For part (b) we use the already shown instance $(i, 0)$ of part (c) of the lemma together with Lemma 5.8, part (a), to verify that, in case of $\tau_i^\star > 1$, $\operatorname{ts}(\tilde{\tau}_{i^\star})$ is an initial sequence of $\operatorname{ts}(\tilde{\tau}_{i,0})$. If both sequences are equal we are done, otherwise the immediate successor of (the $<_{\text{lex}}$-maximal occurrence of) $\tau_i^\star$ in $\operatorname{ts}(\tilde{\tau}_{i,0})$ is an epsilon number greater than $\tau_i^\star$ (by definition of $i^\star$). Using Lemma 5.8, part (b), we obtain $\tau_{i,1} \le \lambda_{\tau_i^\star}$ if $\tau_i^\star > 1$. Hence $\tau_{i,1} \in \operatorname{dom}(\kappa^{\tau_i^\star})$.

In order to prove part (c) assume that $\tau_{i,j} > 1$ as well as $\tau_{i,1} \in \mathbb{E}^{>\tau_i^\star}$ (which is implicit in the case $m_i > 1$ where it holds by Condition 2 of Definition 5.1). The second equality of part (c) follows immediately using the already shown instance $i^\star$ of the lemma. If $\tau_i^\star = 1$ we are done due to the already shown well-definedness of $\tilde{\tau}_{i,j}$ and Lemma 4.10. Otherwise, by part (c) of Lemma 5.8 we have $\operatorname{ts}(\tilde{\tau}_{i^\star}) \in \mathrm{RS}$ and by Lemma 5.8, part (b), together with Lemma 3.12, part (c), we obtain $\tau_{i,1} \le \mu_{\tau_i^\star}$, whence by definition $\nu^{\tilde{\tau}_{i^\star}}_{\tau_{i,1}} = \kappa^{\tilde{\tau}_{i^\star}}_{\tau_{i,1}} = \tilde{\tau}_{i,1}$, using part (b). We now see that $\tilde{\tau}_{i,j} = o(\operatorname{cs}(\boldsymbol{\alpha}_{\restriction(i,j)}))$, and Lemma 4.10 yields part (c).

Part (d) follows from part (c) by comparing the index pairs involved starting from the first elements of $\operatorname{ts}(\tilde{\tau}_{i,j})$ and $\operatorname{ts}(\tilde{\tau}_{k,l})$. □

[7] The well-definedness of these evaluations follows from the conditions on tracking chains together with the next lemma. Notice that the notation $\tilde{\tau}_{i,j}$ depends on the underlying tracking chain which will always be understood from the context in which the ˜-notation is used.

We continue with estimations on the values of tracking chains that will allow us to establish an order isomorphism between tracking chains and their evaluations. It is convenient to use the following notion of depth for tracking chains that will turn out to characterize the $\le_1$-reach of the ordinal, say α, coded by the tracking chain unless $\alpha <_1$-connects to some $\eta + \nu_{\xi+1}$ where $\eta + \nu_\xi \le_1 \alpha$ is maximal such that ξ is not the maximum index of that ν-function, in which case this notion of depth characterizes the distance between α and $\eta + \nu_{\xi+1}$.

Definition 5.11. Let $\alpha < 1^\infty$ and $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n) \in \mathrm{TC}$ where $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \ldots, \alpha_{i,m_i})$. We call $\boldsymbol{\alpha}$ a **tracking chain for** α if $\alpha = \mathrm{o}(\boldsymbol{\alpha})$.

$\mathrm{dp}(\boldsymbol{\alpha})$ is defined as follows: Let $\tau := \tau_{n,m_n}$ and $\tau' := \tau'_n$.

$$\mathrm{dp}(\boldsymbol{\alpha}) := \begin{cases} \mathrm{dp}_{\tilde{\tau}'}(\tau) & \text{if } m_n = 1 \\ \kappa^{\tilde{\tau}'}_{\varrho_\tau} + \mathrm{dp}_{\tilde{\tau}'}(\varrho_\tau) + \check{\chi}^{\tau'}(\tau) \cdot \tilde{\tau}' & \text{if } m_n > 1 \;\&\; \tau < \mu_{\tau'} \\ \kappa^{\tilde{\tau}'}_{\lambda_{\tau'}} + \mathrm{dp}_{\tilde{\tau}'}(\lambda_{\tau'}) & \text{if } m_n > 1 \;\&\; \tau = \mu_{\tau'}. \end{cases}$$

Remark. In case of $m_n > 1$ and $\tau < \mu_{\tau'}$ we have

$$\mathrm{o}(\boldsymbol{\alpha}) + \mathrm{dp}(\boldsymbol{\alpha}) = \mathrm{o}(\boldsymbol{\alpha}[\alpha_{n,m_n} + 1])$$

as follows from the definition of the ν-functions.

We now proceed with a lemma that will allow us to establish an order isomorphism between tracking chains and their evaluations in 5.14. The explicit computations in parts (c), (d), and (e) of the following lemma will be used in proving Lemma 6.2.

Lemma 5.12. *Let $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n) \in \mathrm{TC}$ where $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \ldots, \alpha_{i,m_i})$ and set $\alpha := \mathrm{o}(\boldsymbol{\alpha})$. If there is no extension of $\boldsymbol{\alpha}$ then*

$$\mathrm{dp}(\boldsymbol{\alpha}) = 0.$$

Otherwise let $\boldsymbol{\alpha}^+$ be a 1-step extension of $\boldsymbol{\alpha}$, or $\mathrm{ec}(\boldsymbol{\alpha})$ if that exists. Then $\boldsymbol{\alpha}$ is of a form either

$$\boldsymbol{\alpha}^+ = \boldsymbol{\alpha}^\frown(\alpha_{n+1,1}) \quad \text{or} \quad \boldsymbol{\alpha}^+ = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{n-1}, \boldsymbol{\alpha}_n{}^\frown\alpha_{n,m_n+1}),$$

and we set $\alpha_{n,m_n+1} := 0$ if $\boldsymbol{\alpha}^+$ is of the former, and $\alpha_{n+1,1} := 0$ if $\boldsymbol{\alpha}^+$ is of the latter form.

If $\boldsymbol{\alpha}^+ \notin \mathrm{TC}$ then

$$\tau_{n+1,1} = \tau_{\mathrm{cml}(\boldsymbol{\alpha})} \quad \text{and} \quad \mathrm{end}(\mathrm{dp}(\boldsymbol{\alpha})) = \tilde{\tau}_{\mathrm{cml}(\boldsymbol{\alpha})}.$$

Otherwise let the vector $\boldsymbol{\alpha}'$ be obtained from $\boldsymbol{\alpha}^+$ by adding 1 to its last ordinal (that is either $\alpha_{n+1,1}$ or α_{n,m_n+1}). Setting $\tau := \tau_{n,m_n}$ and $\tau' := \tau'_n$ we obtain an estimation of $\mathrm{o}(\boldsymbol{\alpha}^+) + \mathrm{dp}(\boldsymbol{\alpha}^+)$ depending on the following cases:

1. $\boldsymbol{\alpha}^+ \neq \mathrm{ec}(\boldsymbol{\alpha})$:
   (a) $\boldsymbol{\alpha}' \notin \mathrm{TC}$: *In this case we have $m_n > 1$, $\tau = \mu_{\tau'} \in \mathbb{E} \cap (\tau', \lambda_{\tau'})$, $\alpha_{n,m_n+1} = \mu_\tau$, and*
   $$\mathrm{o}(\boldsymbol{\alpha}^+) + \mathrm{dp}(\boldsymbol{\alpha}^+) < \mathrm{o}(\boldsymbol{\alpha}^\frown((\tau + 1))) \le \alpha + \mathrm{dp}(\boldsymbol{\alpha}).$$
   (b) $\boldsymbol{\alpha}' \in \mathrm{TC}$: *Then we have*
   $$\mathrm{o}(\boldsymbol{\alpha}^+) + \mathrm{dp}(\boldsymbol{\alpha}^+) \le \mathrm{o}(\boldsymbol{\alpha}') \le \alpha + \mathrm{dp}(\boldsymbol{\alpha}) \quad \text{and} \quad \mathrm{o}(\boldsymbol{\alpha}^+) + \mathrm{dp}(\boldsymbol{\alpha}^+) < \alpha + \mathrm{dp}(\boldsymbol{\alpha}).$$
2. $\boldsymbol{\alpha}^+ = \mathrm{ec}(\boldsymbol{\alpha})$:
   (a) *$m_n > 1$ and $\tau < \mu_{\tau'}$:*
      i. $\chi^{\tau'}(\tau) = 0$: *Then*
      $$\mathrm{o}(\boldsymbol{\alpha}^+) + \mathrm{dp}(\boldsymbol{\alpha}^+) < \mathrm{o}(\boldsymbol{\alpha}^\frown((\varrho_\tau + 1))) < \alpha + \mathrm{dp}(\boldsymbol{\alpha}).$$
      ii. $\chi^{\tau'}(\tau) = 1$: *Then* $\mathrm{cml}(\mathrm{me}(\boldsymbol{\alpha})) = (n, m_n - 1)$, $\mathrm{me}^+(\boldsymbol{\alpha}) \notin \mathrm{TC}$, *and*
      $$\mathrm{o}(\boldsymbol{\alpha}^+) + \mathrm{dp}(\boldsymbol{\alpha}^+) = \alpha + \mathrm{dp}(\boldsymbol{\alpha}).$$
   (b) *Otherwise: Then again*
   $$\mathrm{o}(\boldsymbol{\alpha}^+) + \mathrm{dp}(\boldsymbol{\alpha}^+) = \alpha + \mathrm{dp}(\boldsymbol{\alpha}).$$

For any extension $\boldsymbol{\beta}$ of $\boldsymbol{\alpha}$ we have

(a) $\mathrm{o}(\boldsymbol{\beta}) + \mathrm{dp}(\boldsymbol{\beta}) \le \alpha + \mathrm{dp}(\boldsymbol{\alpha})$.
(b) $\mathrm{o}(\boldsymbol{\beta}) < \alpha + \mathrm{dp}(\boldsymbol{\alpha})$ *if $m_n > 1$ and $\tau < \mu_{\tau'}$.*
(c) *If $m_n > 1$, $\tau < \mu_{\tau'}$, and $\chi^{\tau'}(\tau) = 1$ we have*

$$\mathrm{o}(\mathrm{me}(\boldsymbol{\alpha})) + \mathrm{dp}(\mathrm{me}(\boldsymbol{\alpha})) = \alpha + \mathrm{dp}(\boldsymbol{\alpha}) = \mathrm{o}(\boldsymbol{\alpha}[\alpha_{n,m_n} + 1]).$$

(d) *If $\boldsymbol{\alpha}$ does not possess a critical main line index pair $\mathrm{cml}(\boldsymbol{\alpha})$ then $\mathrm{dp}(\mathrm{me}(\boldsymbol{\alpha})) = 0$ and*

$$\mathrm{o}(\mathrm{me}(\boldsymbol{\alpha})) = \begin{cases} \alpha + \mathrm{dp}_{\tilde{\tau}'}(\tau) & \text{if } m_n = 1 \\ \alpha + \kappa^{\tilde{\tau}'}_{\varrho^{\tau'}_\tau} + \mathrm{dp}_{\tilde{\tau}'}(\varrho^{\tau'}_\tau) & \text{if } m_n > 1 \;\&\; \tau < \mu_{\tau'} \\ \alpha + \kappa^{\tilde{\tau}'}_{\lambda_{\tau'}} + \mathrm{dp}_{\tilde{\tau}'}(\lambda_{\tau'}) & \text{otherwise} \end{cases}$$

which only deviates from $\alpha + \mathrm{dp}(\boldsymbol{\alpha})$ in the case $m_n > 1 \;\&\; \tau < \mu_{\tau'}$.

(e) *If* $\mathrm{cml}(\boldsymbol{\alpha}) =: (i, j)$ *exists then*

$$\begin{aligned}\mathrm{o}(\mathrm{me}(\boldsymbol{\alpha})) + \mathrm{dp}(\mathrm{me}(\boldsymbol{\alpha})) &= \alpha + \mathrm{dp}(\boldsymbol{\alpha})\\ &= \begin{cases}\alpha + \mathrm{dp}_{\tilde{\tau}'}(\tau) & \textit{if } m_n = 1\\ \alpha + \kappa^{\tilde{\tau}'}_{\varrho^{\tau'}_{\tau}} + \mathrm{dp}_{\tilde{\tau}'}(\varrho^{\tau'}_{\tau}) & \textit{if } (n, m_n) = (i, j+1)\\ \alpha + \kappa^{\tilde{\tau}'}_{\lambda_{\tau'}} + \mathrm{dp}_{\tilde{\tau}'}(\lambda_{\tau'}) & \textit{otherwise}\end{cases}\\ &= \mathrm{o}(\boldsymbol{\alpha}_{\restriction(i,j+1)}[\alpha_{i,j+1} + 1]),\end{aligned}$$

and

$$\mathrm{dp}(\mathrm{me}(\boldsymbol{\alpha})) = \kappa^{\tilde{\tau}'_r}_{\tau_{i,j}(\xi+1)}$$

where, say, (r, k_r) *is the* $<_{\mathrm{lex}}$*-greatest index pair of* $\mathrm{me}(\boldsymbol{\alpha})$ *and* $\tau_{i,j}(\xi + 1)$ *for suitable* $\xi \geq 0$ *is the extending index of* $\mathrm{ec}(\mathrm{me}(\boldsymbol{\alpha}))$ *according to Corollary 5.6.*

Proof. First of all we verify the two statements made at the beginning of the lemma. If there is no extension of $\boldsymbol{\alpha}$ it follows from the definitions of TC and dp that $\mathrm{dp}(\boldsymbol{\alpha}) = 0$. In case of $\boldsymbol{\alpha}^+ \notin \mathrm{TC}$ we have $\boldsymbol{\alpha}^+ = \mathrm{ec}(\boldsymbol{\alpha})$, and by Corollary 5.6 we have $\tau_{n+1,1} = \tau_{\mathrm{cml}(\boldsymbol{\alpha})} \leq \tau'$ and $\tau_{\mathrm{cml}(\boldsymbol{\alpha})} \in \mathrm{ts}(\tilde{\tau}')$ using part (c) of Lemma 5.10. In the case $m_n = 1$ we must have $\tau' < \tau \notin \mathbb{E}$ and $\alpha_{n+1,1} = \log((1/\tau') \cdot \tau)$. Hence

$$\mathrm{dp}(\boldsymbol{\alpha}) = \mathrm{dp}_{\tilde{\tau}'}(\tau) = \kappa^{\tilde{\tau}'}_{\alpha_{n+1,1}}$$

because $\mathrm{dp}_{\tilde{\tau}'}(\alpha_{n+1,1}) = \mathrm{dp}_{\tilde{\tau}'}(\tau_{n+1,1}) = 0$, and

$$\mathrm{end}(\kappa^{\tilde{\tau}'}_{\alpha_{n+1,1}}) = \kappa^{\tilde{\tau}'}_{\tau_{\mathrm{cml}(\boldsymbol{\alpha})}} = \tilde{\tau}_{\mathrm{cml}(\boldsymbol{\alpha})}.$$

If on the other hand $m_n > 1$, the assumption $\tau < \mu_{\tau'}$ would imply that $\mathrm{cml}(\boldsymbol{\alpha}) = (n, m_n - 1)$, which according to part (a) of Lemma 5.5 would entail $\boldsymbol{\alpha}^+ \in \mathrm{TC}$. We therefore have $\tau = \mu_{\tau'}$ and $\alpha_{n+1,1} = \lambda_{\tau'}$. Thus

$$\mathrm{dp}(\boldsymbol{\alpha}) = \kappa^{\tilde{\tau}'}_{\lambda_{\tau'}} + \mathrm{dp}_{\tilde{\tau}'}(\lambda_{\tau'}) = \kappa^{\tilde{\tau}'}_{\alpha_{n+1,1}} \quad \text{and} \quad \mathrm{end}(\kappa^{\tilde{\tau}'}_{\alpha_{n+1,1}}) = \tilde{\tau}_{\mathrm{cml}(\boldsymbol{\alpha})}$$

as above. From now on we assume that $\boldsymbol{\alpha}^+ \in \mathrm{TC}$ and proceed by showing the assertions concerning $\boldsymbol{\alpha}^+$.

Ad 1 (a). The assumptions imply $m_n > 1$, $\tau = \mu_{\tau'} \in \mathbb{E} \cap (\tau', \lambda_{\tau'})$, and $\alpha_{n,m_n+1} = \mu_\tau$ as stated. We easily compute $\mathrm{o}(\boldsymbol{\alpha}^+) = \alpha + \nu^{\tilde{\tau}}_{\mu_\tau}$, $\mathrm{dp}(\boldsymbol{\alpha}^+) = \kappa^{\tilde{\tau}}_{\lambda_\tau} + \mathrm{dp}_{\tilde{\tau}}(\lambda_\tau)$, $\mathrm{o}(\boldsymbol{\alpha}^\frown((\tau+1))) = \alpha + \mathrm{dp}_{\tilde{\tau}'}(\tau) + 1$, and $\mathrm{dp}(\boldsymbol{\alpha}) = \kappa^{\tilde{\tau}'}_{\lambda_{\tau'}} + \mathrm{dp}_{\tilde{\tau}'}(\lambda_{\tau'})$. Since $\tau < \lambda_{\tau'}$ Lemma 4.5 yields "$\leq$", and "$<$" follows by definition of $\mathrm{dp}_{\tilde{\tau}'}(\tau)$.

Ad 1 (b). In the case $\tau \in \mathbb{E}^{>\tau'}$ & $0 < \alpha_{n,m_n+1} < \mu_\tau$ we have $\mathrm{o}(\boldsymbol{\alpha}^+) + \mathrm{dp}(\boldsymbol{\alpha}^+) = \mathrm{o}(\boldsymbol{\alpha}') < \alpha + \mathrm{dp}(\boldsymbol{\alpha})$ where "$<$" again follows from Lemma 4.5, which also applies in the remaining cases where we have $\mathrm{o}(\boldsymbol{\alpha}^+) + \mathrm{dp}(\boldsymbol{\alpha}^+) < \mathrm{o}(\boldsymbol{\alpha}') \leq \alpha + \mathrm{dp}(\boldsymbol{\alpha})$.

Ad 2 (a) i. Here both cases concerning the form of $\boldsymbol{\alpha}^+$ are possible, that is, $\alpha_{n,m_n+1} = \mu_\tau$ and $\alpha_{n+1,1} = \varrho_\tau$. The claim follows again inspecting the definitions and using Lemma 4.5.

Ad 2 (a) ii. The assertions $\mathrm{cml}(\mathrm{me}(\boldsymbol{\alpha})) = (n, m_n - 1)$ and $\mathrm{me}^+(\boldsymbol{\alpha}) \notin \mathrm{TC}$ follow by definition of the maximal extension and Corollary 5.6. The stated equation is easy to verify.

Ad 2 (b). This again follows directly from the involved definitions.

We now show first part (a), then parts (b) and (c), by induction on $\mathrm{cs}'(\boldsymbol{\alpha})$ along $<_{\mathrm{lex}}$ since any proper extension $\boldsymbol{\beta}$ of $\boldsymbol{\alpha}$ can be broken up into the first 1-step extension $\boldsymbol{\alpha}^+$ of $\boldsymbol{\alpha}$ and the extension of $\boldsymbol{\alpha}^+$ to $\boldsymbol{\beta}$. Part (a) is then immediate. Part (b) is easily seen in the case $\chi^{\tau'}(\tau) = 0$. Now assume $\chi^{\tau'}(\tau) = 1$. We observe that only successive maximal 1-step extensions, calling $\boldsymbol{\delta}$ one such, can and do maintain the equality $\mathrm{o}(\boldsymbol{\delta}) + \mathrm{dp}(\boldsymbol{\delta}) = \alpha + \mathrm{dp}(\boldsymbol{\alpha})$, a procedure that according to Corollary 5.6 leads to the final extension candidate $\mathrm{me}^+(\boldsymbol{\alpha})$ which is not a tracking chain. We thus have $\mathrm{o}(\mathrm{me}(\boldsymbol{\alpha})) + \mathrm{dp}(\mathrm{me}(\boldsymbol{\alpha})) = \alpha + \mathrm{dp}(\boldsymbol{\alpha})$, which is equal to $\mathrm{o}(\boldsymbol{\alpha}[\alpha_{n,m_n} + 1])$, and for any proper extension, say, $\boldsymbol{\delta}$ of $\mathrm{me}(\boldsymbol{\alpha})$ we have $\mathrm{o}(\boldsymbol{\delta}) + \mathrm{dp}(\boldsymbol{\delta}) < \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha})) + \mathrm{dp}(\mathrm{me}(\boldsymbol{\alpha}))$. Thus parts (b) and (c) follow.

In order to see part (d) first observe that since $\mathrm{me}(\boldsymbol{\alpha})$ does not have an extension candidate $\mathrm{ec}(\mathrm{me}(\boldsymbol{\alpha}))$, it follows that $\mathrm{dp}(\mathrm{me}(\boldsymbol{\alpha})) = 0$. In all cases except for the situation where $m_n > 1$ and $\tau < \mu_{\tau'}$ we know from the already shown parts that $\mathrm{o}(\mathrm{me}(\boldsymbol{\alpha})) = \alpha + \mathrm{dp}(\boldsymbol{\alpha})$. Now, in the case $m_n > 1$ and $\tau < \mu_{\tau'}$ we have $\chi^{\tau'}(\tau) = 0$. If $\boldsymbol{\alpha}$ is already maximal, we are done. Otherwise we have $\mathrm{me}(\boldsymbol{\alpha}) = \mathrm{me}(\mathrm{ec}(\boldsymbol{\alpha}))$ and face two cases:

Case 1: $\tau' < \tau \in \mathbb{E}$. Then $\mathrm{ec}(\boldsymbol{\alpha})$ extends $\boldsymbol{\alpha}$ by an additional index $\alpha_{n,m_n+1} = \mu_\tau$, and using that $\varrho^{\tau'}_{\tau} = \tau$ we obtain

$$\begin{aligned}\mathrm{o}(\mathrm{me}(\boldsymbol{\alpha})) &= \mathrm{o}(\mathrm{me}(\mathrm{ec}(\boldsymbol{\alpha})))\\ &= \mathrm{o}(\mathrm{ec}(\boldsymbol{\alpha})) + \mathrm{dp}(\mathrm{ec}(\boldsymbol{\alpha}))\\ &= \alpha + \nu^{\tilde{\tau}}_{\mu_\tau} + \kappa^{\tilde{\tau}}_{\lambda_\tau} + \mathrm{dp}_{\tilde{\tau}}(\lambda_\tau)\\ &= \alpha + \kappa^{\tilde{\tau}'}_{\varrho^{\tau'}_{\tau}} + \mathrm{dp}_{\tilde{\tau}'}(\varrho^{\tau'}_{\tau}).\end{aligned}$$

Case 2: Otherwise. Then $\mathrm{ec}(\boldsymbol{\alpha})$ extends $\boldsymbol{\alpha}$ by an additional index $\alpha_{n+1,1} = \varrho_\tau^{\tau'}$, and directly obtain the claim.

For part (e) we know by the already shown parts that in the cases where $(n, m_n) \neq (i, j+1)$ we have $\alpha + \mathrm{dp}(\boldsymbol{\alpha}) = \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha})) + \mathrm{dp}(\mathrm{me}(\boldsymbol{\alpha}))$. Now, in the situation $(n, m_n) = (i, j+1)$ we know by part (a) of Lemma 5.5 that $\mathrm{ec}(\boldsymbol{\alpha})$ exists and is a tracking chain. We can then conclude $\alpha + \mathrm{dp}(\boldsymbol{\alpha}) = \mathrm{o}(\mathrm{ec}(\boldsymbol{\alpha})) + \mathrm{dp}(\mathrm{ec}(\boldsymbol{\alpha})) = \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha})) + \mathrm{dp}(\mathrm{me}(\boldsymbol{\alpha}))$. The claim regarding $\mathrm{dp}(\mathrm{me}(\boldsymbol{\alpha}))$ follows from the definitions using Corollary 5.6. □

We now define a well-ordering $<_{\mathrm{TC}}$ on TC such that the evaluation function o on TC becomes order preserving, as shown in the sequel.

Definition 5.13. We define a linear ordering $<_{\mathrm{TC}}$ on TC as follows. Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathrm{TC}$ be given, say, of the form

$$\boldsymbol{\alpha} = ((\alpha_{1,1}, \ldots, \alpha_{1,m_1}), \ldots, (\alpha_{n,1}, \ldots, \alpha_{n,m_n}))$$

and

$$\boldsymbol{\beta} = ((\beta_{1,1}, \ldots, \beta_{1,k_1}), \ldots, (\beta_{l,1}, \ldots, \beta_{l,k_l})).$$

Let (i, j) where $1 \leq i \leq \min\{n, l\}$ and $1 \leq j \leq \min\{m_i, k_i\}$ be $<_{\mathrm{lex}}$-maximal such that $\boldsymbol{\alpha}_{\restriction(i,j)} = \boldsymbol{\beta}_{\restriction(i,j)}$, if that exists, and $(i, j) := (1, 0)$ otherwise.

$$\begin{aligned}\boldsymbol{\alpha} <_{\mathrm{TC}} \boldsymbol{\beta} \;:\Leftrightarrow\; & (i, j) = (n, m_n) \neq (l, k_l)\\ & \vee (j < \min\{m_i, k_i\} \;\&\; \alpha_{i,j+1} < \beta_{i,j+1})\\ & \vee (j = m_i < k_i \;\&\; i < n \;\&\; \alpha_{i+1,1} < \tau_{i,j})\\ & \vee (j = k_i < m_i \;\&\; i < l \;\&\; \tau_{i,j} < \beta_{i+1,1})\\ & \vee (j = k_i = m_i \;\&\; i < \min\{n, l\} \;\&\; \alpha_{i+1,1} < \beta_{i+1,1})\end{aligned}$$

$$\boldsymbol{\alpha} \leq_{\mathrm{TC}} \boldsymbol{\beta} \;:\Leftrightarrow\; \boldsymbol{\alpha} <_{\mathrm{TC}} \boldsymbol{\beta} \vee \boldsymbol{\alpha} = \boldsymbol{\beta}.$$

Lemma 5.14. *For all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathrm{TC}$ we have*

$$\boldsymbol{\alpha} <_{\mathrm{TC}} \boldsymbol{\beta} \Leftrightarrow \mathrm{o}(\boldsymbol{\alpha}) < \mathrm{o}(\boldsymbol{\beta}).$$

Proof. Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathrm{TC}$ such that $\boldsymbol{\alpha} <_{\mathrm{TC}} \boldsymbol{\beta}$ be given. We show $\mathrm{o}(\boldsymbol{\alpha}) < \mathrm{o}(\boldsymbol{\beta})$. The lemma then follows since it is easy to check that $<_{\mathrm{TC}}$ is a linear ordering of TC. The evaluation of a tracking chain is strictly increasing along its initial values. The claim follows from the strict monotonicity of the κ- and ν-functions shown in Lemma 4.5 using Lemma 5.12. □

Corollary 5.15. *For any $\alpha < 1^\infty$ there exists at most one tracking chain for α.* □

In the next section we will establish that the evaluation o on tracking chains is a mapping onto 1^∞ and define its inverse, which will be called tc. We will thus obtain an order isomorphism between $(1^\infty, <)$ and $(\mathrm{TC}, <_{\mathrm{TC}})$.

6. Assignment of tracking chains to the ordinals below 1^∞

By the following definition we assign finite sequences of ordinal vectors to the ordinals below 1^∞ which meet all conditions for tracking chains stated in Definition 5.1. Moreover, it will be shown that TC from 5.1 is a characterization of the set $\{\mathrm{tc}(\alpha) \mid \alpha < 1^\infty\}$ as defined below.

Definition 6.1. For $\alpha < 1^\infty$ we define the **tracking chain assigned to** α, $\mathrm{tc}(\alpha)$, recursively as follows. We define $\mathrm{tc}(0) := ((0))$, and if $\alpha \in \mathbb{P}$ we set $\mathrm{tc}(\alpha) := (\mathrm{ts}(\alpha))$. Now suppose $\mathrm{tc}(\alpha) = \boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n)$ to be the tracking chain already assigned to some $\alpha > 0$, where $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \ldots, \alpha_{i,m_i})$ for $1 \leq i \leq n$, and let $\beta \in \mathbb{P}^{\leq \mathrm{end}(\alpha)}$. For technical reasons we set $\alpha_{n+1,1} := 0$ and $m_{n+1} := 1$. The definition of $\mathrm{tc}(\alpha + \beta)$, the tracking chain assigned to $\alpha + \beta$, requires the following preparations.

- For $1 \leq i \leq n$ and $0 \leq j < m_i$ let

$$(\beta_1^{i,j}, \ldots, \beta_{r_{i,j}}^{i,j}) := \mathrm{ts}[\tilde{\tau}_{i,j}](\beta),$$

  writing simply $(\beta_1, \ldots, \beta_r)$ in the case $(i, j) = (1, 0)$.
- Let (i_0, j_0), where $1 \leq i_0 \leq n$ and $1 \leq j_0 < m_{i_0}$, be $<_{\mathrm{lex}}$-maximal with

$$\alpha_{i_0,j_0+1} < \mu_{\tau_{i_0,j_0}}$$

  if that exists, otherwise set $(i_0, j_0) := (1, 0)$.
- Let (k_0, l_0) be either $(1, 0)$ or satisfy $1 \leq k_0 \leq n+1$ and $1 \leq l_0 \leq m_{k_0}$, so that (k_0, l_0) is $<_{\mathrm{lex}}$-minimal with $(i_0, j_0) \leq_{\mathrm{lex}} (k_0, l_0)$ and
  1. for all $k \in \{k_0, \ldots, n\}$ we have

$$\alpha_{k+1,1} + \beta_1^{k,m_k-1} \geq \rho_k$$

2. for all $k \in \{k_0, \dots, n\}$ and all $l \in \{1, \dots, m_k - 2\}$ such that $(k_0, l_0) <_{\text{lex}} (k, l)$ we have

$$\tau_{k,l+1} + \beta_1^{k,l} > \lambda_{\tau_{k,l}}.$$

Case 1: $(i_0, j_0) = (k_0, l_0)$. Then there are three subcases:

1.1: $\beta < \tilde{\tau}_{i_0,j_0}$. Then $\text{tc}(\alpha + \beta)$ is defined to be

$$\boldsymbol{\alpha}_{\restriction(i_0,j_0+1)}{}^\frown \left(\varrho^{\tau_{i_0,j_0}}_{\tau_{i_0,j_0+1}} + \beta_1^{i_0,j_0}, \beta_2^{i_0,j_0}, \dots, \beta^{i_0,j_0}_{r_{i_0,j_0}}\right).$$

1.2: $\beta = \tilde{\tau}_{i_0,j_0}$. Then $\text{tc}(\alpha + \beta)$ is defined by

$$\boldsymbol{\alpha}_{\restriction(i_0,j_0+1)}[\alpha_{i_0,j_0+1} + 1].$$

1.3: $\beta > \tilde{\tau}_{i_0,j_0}$. Then there is an $r_0 < r$ such that, setting $\beta_0 := 1$, $\beta_{r_0} = \tau_{i_0,j_0}$, and $\text{tc}(\alpha + \beta)$ is defined

$$\boldsymbol{\alpha}_{\restriction i_0-1}{}^\frown(\alpha_{i_0,1}, \dots, \alpha_{i_0,j_0}, \alpha_{i_0,j_0+1} + \beta_{r_0+1}, \beta_{r_0+2}, \dots, \beta_r).$$

Case 2: $(i_0, j_0) <_{\text{lex}} (k_0, l_0)$. Then there are the following subcases:

2.1: $k_0 = n + 1$ and $\beta_1^{n,m_n-1} = \tau_{n,m_n} \in \mathbb{E}^{>\tau'_n}$. Then $\beta = \tilde{\tau}_{n,m_n}$, and $\text{tc}(\alpha + \beta)$ is defined by

$$\boldsymbol{\alpha}_{\restriction n-1}{}^\frown(\alpha_{n,1}, \dots, \alpha_{n,m_n}, 1).$$

2.2: $k_0 \le n$, $l_0 \in \{1, \dots, m_{k_0} - 2\}$ and $\tau_{k_0,l_0+1} + \beta_1^{k_0,l_0} \le \lambda_{\tau_{k_0,l_0}}$, we define $\text{tc}(\alpha + \beta)$ by

$$\boldsymbol{\alpha}_{\restriction(k_0,l_0+1)}{}^\frown \left(\tau_{k_0,l_0+1} + \beta_1^{k_0,l_0}, \beta_2^{k_0,l_0}, \dots, \beta^{k_0,l_0}_{r_{k_0,l_0}}\right),$$

provided this vector satisfies Condition 6 of Definition 5.1, otherwise we have $\beta = \tilde{\tau}_{i_0,j_0}$, and $\text{tc}(\alpha + \beta)$ is defined as in case 1.2.

2.3: Otherwise. Then $k_0 > i_0$ and $\alpha_{k+1,1} + \beta_1^{k,m_k-1} < \rho_k$ for $k := k_0 - 1$, and $\text{tc}(\alpha + \beta)$ is defined by

$$\boldsymbol{\alpha}_{\restriction k}{}^\frown \left(\alpha_{k_0,1} + \beta_1^{k,m_k-1}, \beta_2^{k,m_k-1}, \dots, \beta^{k,m_k-1}_{r_{k,m_k-1}}\right),$$

provided this vector satisfies Condition 6 of Definition 5.1, otherwise we have $\beta = \tilde{\tau}_{i_0,j_0}$, and $\text{tc}(\alpha + \beta)$ is defined as in case 1.2.

Remark. Case 1.3 uniformly covers two quite different situations: The situation $(i_0, j_0) = (1, 0)$ will be shown to correspond to the scenario where adding β to α means to jump into a larger $\le_1$-connectivity component, whereas the situation $(i_0, j_0) \ne (1, 0)$ corresponds to jumping into a larger $\le_2$-connectivity component on the surrounding main line. Notice that we could have incorporated Case 1.2 into Case 1.3, say, by setting $\beta_{r+1} := 1$. Case 2.1 takes care of Condition 5 of Definition 5.1.

Lemma 6.2. *Let* $\alpha < 1^\infty$.

(a) $\text{tc}(\alpha) \in \text{TC}$, *i.e.* $\text{tc}(\alpha)$ *meets all conditions of Definition* 5.1.
(b) *There exists exactly one tracking chain for* α, *namely* $\text{tc}(\alpha)$ *satisfies* $\text{o}(\text{tc}(\alpha)) = \alpha$.

Proof. We prove both parts of the lemma simultaneously by induction on α. The case $\alpha = 0$ is trivial, and using Lemma 4.10 we see that the claims hold whenever $\alpha \in \mathbb{P}$. Now suppose the claims have been shown for some $\alpha > 0$ with assigned tracking chain $\text{tc}(\alpha) = \boldsymbol{\alpha}$ as in the definition and suppose $\beta \le \text{end}(\alpha)$ so that we have the inductive hypothesis available for any ordinal below $\alpha + \beta$. We adopt the terminology of the previous definition and commence proving the inductive step for $\alpha + \beta$ by showing the following claims.

Claim 6.3. *If* $\beta \le \tilde{\tau}_{i_0,j_0}$ *then* $\beta_1^{i_0,j_0} \le \tau_{i_0,j_0}$. *If additionally* $\chi^{\tau_{i_0,j_0}}(\tau_{i_0,j_0+1}) = 1$ *and* $(i_0, j_0) \le_{\text{lex}} (k, l)$ *for some index pair* (k, l) *of* $\text{me}(\boldsymbol{\alpha}_{\restriction(i_0,j_0+1)})$ *with* $l < m_k$ *then* $\beta^{k,l} \le \tau_{i_0,j_0}$. *In both assertions equality holds if and only if* $\beta = \tilde{\tau}_{i_0,j_0}$.

In order to show the claim let us assume that $\beta \le \tilde{\tau}_{i_0,j_0}$. This assumption implies that $(i_0, j_0) \ne (1, 0)$. In the case $\beta_1^{i_0,j_0} = \tau_{i_0,j_0}$ the assumption implies $r_{i_0,j_0} = 1$ and $\beta = \tilde{\tau}_{i_0,j_0}$. On the other hand, in case of $\beta = \tilde{\tau}_{i_0,j_0}$ we clearly have $\text{ts}[\tilde{\tau}_{i_0,j_0}](\beta) = (\tau_{i_0,j_0})$. The assertion concerning (k, l) can easily be recuced to the one concerning (i_0, j_0): By Lemma 5.5 $\text{ts}(\tilde{\tau}_{i_0,j_0})$ is an initial sequence of $\text{ts}(\tilde{\tau}_{k,l})$ and therefore

$$\text{ts}[\tilde{\tau}_{k,l}](\beta) = (\beta_1^{k,l}, \dots, \beta^{k,l}_{r_{k,l}}) = (\beta_1^{i_0,j_0}, \dots, \beta^{i_0,j_0}_{r_{i_0,j_0}}) = \text{ts}[\tilde{\tau}_{i_0,j_0}](\beta).$$

Now assume $\beta < \tilde{\tau}_{i_0,j_0}$. By Lemma 3.15 we have $\text{ts}(\beta) <_{\text{lex}} \text{ts}(\tilde{\tau}_{i_0,j_0}) =: (\gamma_1, \dots, \gamma_s)$. By part (a) of Lemma 4.17 for some $0 \le k < s$ we have, setting $\gamma_0 := 1$, $\gamma_k \le \beta_1^{i_0,j_0} < \gamma_{k+1} \le \tau_{i_0,j_0}$. This concludes the proof of Claim 6.3.

Claim 6.4. *If* $(i_0, j_0) \neq (1, 0)$ *and* $\chi^{\tau_{i_0,j_0}}(\tau_{i_0,j_0+1}) = 1$ *then* $\beta \leq \tilde{\tau}_{i_0,j_0}$ *implies* $(i_0, j_0) <_{\text{lex}} (k_0, l_0)$.

For the proof of this claim assume $\beta \leq \tilde{\tau}_{i_0,j_0}$ and let (i, j) be the $\leq_{\text{lex}}$-maximal index pair such that $\boldsymbol{\alpha}_{\restriction(i,j)}$ is a common initial chain of $\boldsymbol{\alpha}$ and $\text{me}(\boldsymbol{\alpha}_{\restriction i_0,j_0+1})$, hence $(i_0, j_0 + 1) \leq_{\text{lex}} (i, j)$. By Corollary 5.6 we know that $\text{ec}(\boldsymbol{\alpha}_{\restriction(i,j)})$ exists. In order to derive a contradiction we now assume that $(i_0, j_0) = (k_0, l_0)$ and discuss the possible cases in the definition of $\text{ec}(\boldsymbol{\alpha}_{\restriction(i,j)})$. For convenience of notation we set $\tau := \tau_{i,j}$ and $\tau' := \tau'_{i,j}$.

Case 1: $j = 1$. Then we have $m_i = 1$ by the maximality of (i, j), $i > i_0$ and thus $(i_0, j_0 + 1) <_{\text{lex}} (i, j)$.

Subcase 1.1: $\tau' < \tau \in \mathbb{E}$. By Lemma 5.5 and the assumption $k_0 = i_0 < i$ we then have

$$\tau_{i_0,j_0} \leq \tau = \log((1/\tau') \cdot \tau) < \rho_i \leq \alpha_{i+1,1} + \beta_1^{i,m_i-1}.$$

But according to Claim 6.3 we have $\beta_1^{i,m_i-1} \leq \tau_{i_0,j_0}$, and by Condition 5 for tracking chains we have $\alpha_{i+1,1} < \tau$, whence $\alpha_{i+1,1} + \beta_1^{i,m_i-1} < \rho_i$. Contradiction.

Subcase 1.2: Otherwise. Then $\alpha_{i+1,1}$ is strictly less than $\log((1/\tau') \cdot \tau)$ which is the extending index of $\text{ec}(\boldsymbol{\alpha}_{\restriction(i,j)})$ and according to Lemma 5.5 and Corollary 5.6 a proper multiple of τ_{i_0,j_0}. We run into the same contradiction as in Subcase 1.1.

Case 2: $j > 1$. Then $\tau' = \tau_{i,j-1}$.

Subcase 2.1: $\tau' < \tau \in \mathbb{E}$.

2.1.1: $\tau = \mu_{\tau'} < \lambda_{\tau'}$. Then $(i_0, j_0 + 1) <_{\text{lex}} (i, j)$ which implies $(k_0, l_0) <_{\text{lex}} (i, j - 1)$. The extending index of $\text{ec}(\boldsymbol{\alpha}_{\restriction(i,j)})$ is then $\lambda_{\tau'}$. If $j < m_i$ we obtain the contradiction $\tau + \beta_1^{i,j-1} < \lambda_{\tau'}$, otherwise we obtain the contradiction $\alpha_{i+1,1} + \beta_1^{i,j-1} < \rho_i = \lambda_{\tau'} + 1$ in a similar fashion as in Case 1.

2.1.2: Otherwise. The extending index of $\text{ec}(\boldsymbol{\alpha}_{\restriction(i,j)})$ is then μ_τ, and $m_i = j$. By Condition 5 for tracking chains $\alpha_{i+1,1} \neq \tau$. By the assumptions of this case and using Lemma 5.5 we have $\tau_{i_0,j_0} < \tau$. We first consider the case $(i, j) = (i_0, j_0 + 1)$. Then $\alpha_{i+1,1} < \tau$ and $\rho_i = \tau + 1$. We obtain the contradiction $\alpha_{i+1,1} + \beta^{i,j-1} < \rho_i$, again using the previous claim. Now assume $(i_0, j_0 + 1) <_{\text{lex}} (i, j)$. Again we have $\rho_i = \tau + 1$, $\alpha_{i+1,1} < \tau$, and we run into the same contradiction.

Subcase 2.2: Otherwise. Then again $m_i = j$.

2.2.1: $\tau < \mu_{\tau'}$. This can only occur if $(i, j) = (i_0, j_0 + 1)$, thus $\tau' = \tau_{i_0,j_0}$ and $\tau = \tau_{i_0,j_0+1}$. The extending index of $\text{ec}(\boldsymbol{\alpha}_{\restriction(i,j)})$ is $\varrho_\tau^{\tau'}$, and $\rho_i = \varrho_\tau^{\tau'} + 1$. Lemma 5.5 yields $\chi^{\tau'}(\varrho_\tau^{\tau'}) = 1$. We are then confronted with the contradiction $\alpha_{i+1,1} + \beta_1^{i,j} < \rho_i$.

2.2.2: Otherwise, that is, $\tau = \mu_{\tau'}$. This implies $(i_0, j_0 + 1) <_{\text{lex}} (i, j)$, and the extending index of $\text{ec}(\boldsymbol{\alpha}_{\restriction(i,j)})$ is $\lambda_{\tau'}$ which again is a proper multiple of τ_{i_0,j_0}. Thus $\alpha_{i+1,1} + \beta^{i,j-1} \leq \lambda_{\tau'} < \lambda_{\tau'} + 1 = \rho_i$. Contradiction. Our assumption $(i_0, j_0) = (k_0, l_0)$ therefore cannot hold true, which concludes the proof of Claim 6.4.

We are now prepared to verify the lemma for each of the six clauses of the assignment of $\text{tc}(\alpha + \beta)$ to $\alpha + \beta$. However, a uniform argument exploiting the careful choice of the index pair (k_0, l_0) will be given in the last part of this proof to complete the treatment of the single clauses.

Case 1: $(i_0, j_0) = (k_0, l_0)$.

Subcase 1.1: $\beta < \tilde{\tau}_{i_0,j_0}$. Then clearly $(i_0, j_0) \neq (1, 0)$, by Claim 6.3 $\beta_1^{i_0,j_0} < \tau_{i_0,j_0}$, and Claim 6.4 yields $\chi^{\tau_{i_0,j_0}}(\tau_{i_0,j_0+1}) = 0$. Using Lemma 4.17 we obtain $\text{tc}(\alpha + \beta) \in \text{TC}$, and since $\alpha_{i_0+1,1} + \beta_1^{i_0,j_0} \geq \rho_{i_0}$ we must have $m_{i_0} > j_0 + 1$ and hence $\varrho_{\tau_{i_0,j_0+1}}^{\tau_{i_0,j_0}} = \tau_{i_0,j_0+1}$. According to part (d) 2 of Lemma 4.5, Lemma 4.15, and part (d) of Lemma 5.12 we have

$$\begin{aligned} \text{o}(\text{tc}(\alpha + \beta)) &= \text{o}(\boldsymbol{\alpha}_{\restriction(i_0,j_0+1)}) + \text{dp}_{\tilde{\tau}_{i_0,j_0}}(\tau_{i_0,j_0+1}) + \beta \\ &= \text{o}(\text{me}(\boldsymbol{\alpha}_{\restriction(i_0,j_0+1)})) + \beta. \end{aligned}$$

It remains to be shown that this is equal to $\alpha + \beta$.

Subcase 1.2: $\beta = \tilde{\tau}_{i_0,j_0}$. Again, $(i_0, j_0) \neq (1, 0)$, by Claim 6.3 we have $\beta^{i_0,j_0} = (\tau_{i_0,j_0})$, and Claim 6.4 yields $\chi^{\tau_{i_0,j_0}}(\tau_{i_0,j_0+1}) = 0$. $\text{tc}(\alpha + \beta) \in \text{TC}$ is immediate. We compute similarly as above

$$\begin{aligned} \text{o}(\text{tc}(\alpha + \beta)) &= \text{o}(\boldsymbol{\alpha}_{\restriction(i_0,j_0+1)}) + \kappa^{\tilde{\tau}_{i_0,j_0}}_{\varrho_{\tau_{i_0,j_0+1}}} + \text{dp}_{\tilde{\tau}_{i_0,j_0}}(\varrho_{\tau_{i_0,j_0+1}}) + \beta \\ &= \text{o}(\text{me}(\boldsymbol{\alpha}_{\restriction(i_0,j_0+1)})) + \beta, \end{aligned}$$

and again it remains to be shown that this is equal to $\alpha + \beta$.

Subcase 1.3: $\beta > \tilde{\tau}_{i_0,j_0}$. Making use of Lemma 5.12 we observe that

$$\tilde{\tau}_{i_0,j_0} < \beta \leq \text{end}(\alpha) = \text{end}(\tilde{\alpha}_{n,m_n}) < \nu^{\tilde{\tau}_{i_0,j_0}}_{\mu_{\tau_{i_0,j_0}}},$$

which, realizing that due to Lemma 4.10 $\mathrm{ts}(\nu^{\tilde{\tau}_{i_0,j_0}}_{\mu_{\tau_{i_0,j_0}}}) = \mathrm{ts}(\tilde{\tau}_{i_0,j_0})^\frown\mu_{\tau_{i_0,j_0}}$, according to Lemma 3.15 implies

$$\mathrm{ts}(\tilde{\tau}_{i_0,j_0}) <_{\mathrm{lex}} \mathrm{ts}(\beta) <_{\mathrm{lex}} \mathrm{ts}(\tilde{\tau}_{i_0,j_0})^\frown\mu_{\tau_{i_0,j_0}},$$

whence $\mathrm{ts}(\tilde{\tau}_{i_0,j_0})$ is a proper initial segment of $\mathrm{ts}(\beta)$. Thus there is an $r_0 < r$ such that $\tau_{i_0,j_0} = \beta_{r_0}$, leaving the possibility $r_0 = 0$ for the case $(i_0, j_0) = (1, 0)$. We now see that $\mathrm{tc}(\alpha+\beta) \in \mathrm{TC}$. Using again Lemmas 4.10 and 5.12 we obtain

$$\mathrm{o}(\mathrm{tc}(\alpha+\beta)) = \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}_{\restriction(i_0,j_0+1)})) + \beta,$$

and leave showing that this is equal to $\alpha+\beta$ for later.

Case 2: $(i_0, j_0) <_{\mathrm{lex}} (k_0, l_0)$.

Subcase 2.1: $k_0 = n+1$ and $\beta_1^{n,m_n-1} = \tau_{n,m_n} \in \mathbb{E}^{>\tau'_n}$. Since $\beta \le \tilde{\tau}_{n,m_n}$ we then have $\beta = \tilde{\tau}_{n,m_n}$, and $\mathrm{tc}(\alpha+\beta) \in \mathrm{TC}$ is clear. Since $k_0 = n+1$ we have $\tau_{n,m_n} < \rho_n$, and realizing that $-\tilde{\tau}_{n,m_n} + \nu_1^{\tilde{\tau}_{n,m_n}} = \tilde{\tau}_{n,m_n}$ we obtain

$$\mathrm{o}(\mathrm{tc}(\alpha+\beta)) = \alpha+\beta.$$

Subcase 2.2: $k_0 \le n$, $l_0 \in \{1,\ldots,m_{k_0}-2\}$ and $\tau_{k_0,l_0+1} + \beta_1^{k_0,l_0} \le \lambda_{\tau_{k_0,l_0}}$.

2.2.1: $\boldsymbol{\alpha}_{\restriction(k_0,l_0+1)}{}^\frown\left(\tau_{k_0,l_0+1}+\beta_1^{k_0,l_0}, \beta_2^{k_0,l_0}, \ldots, \beta_{r_{k_0,l_0}}^{k_0,l_0}\right)$ satisfies Condition 6 for tracking chains. Then $\mathrm{tc}(\alpha+\beta)$ is defined by this vector which is easily seen to be a tracking chain. Note that since $\tau_{k_0,l_0+1} = \mu_{\tau_{k_0,l_0}} \in \mathbb{E}^{>\tau_{k_0,l_0}} \cap \lambda_{\tau_{k_0,l_0}}$ we have $\boldsymbol{\alpha}_{\restriction k_0,l_0+2} <_{\mathrm{TC}} \mathrm{ec}(\boldsymbol{\alpha}_{\restriction k_0,l_0+1})$, implying that $\boldsymbol{\alpha}_{\restriction k_0,l_0+2}$ does not possess a critical main line index pair. Part (d) of Lemma 5.12 therefore yields

$$\mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}_{\restriction k_0,l_0+2})) = \mathrm{o}(\boldsymbol{\alpha}_{\restriction k_0,l_0+2}) + \kappa^{\tilde{\tau}_{k_0,l_0+1}}_{\lambda_{\tau_{k_0,l_0+1}}} + \mathrm{dp}_{\tilde{\tau}_{k_0,l_0+1}}(\lambda_{\tau_{k_0,l_0+1}}).$$

We now compute using Lemma 4.15

$$\begin{aligned}\mathrm{o}(\mathrm{tc}(\alpha+\beta)) &= \mathrm{o}(\boldsymbol{\alpha}_{\restriction k_0,l_0+1}) + \mathrm{dp}_{\tilde{\tau}_{k_0,l_0}}(\tau_{k_0,l_0+1}) + \beta\\ &= \mathrm{o}(\boldsymbol{\alpha}_{\restriction k_0,l_0+1}) + \nu^{\tilde{\tau}_{k_0,l_0+1}}_{\tau_{k_0,l_0+2}} + \kappa^{\tilde{\tau}_{k_0,l_0+1}}_{\lambda_{\tau_{k_0,l_0+1}}} + \mathrm{dp}_{\tilde{\tau}_{k_0,l_0+1}}(\lambda_{\tau_{k_0,l_0+1}}) + \beta\\ &= \mathrm{o}(\boldsymbol{\alpha}_{\restriction k_0,l_0+2}) + \kappa^{\tilde{\tau}_{k_0,l_0+1}}_{\lambda_{\tau_{k_0,l_0+1}}} + \mathrm{dp}_{\tilde{\tau}_{k_0,l_0+1}}(\lambda_{\tau_{k_0,l_0+1}}) + \beta\\ &= \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}_{\restriction k_0,l_0+2})) + \beta\end{aligned}$$

and leave the task of showing this to be equal to $\alpha+\beta$ for later.

2.2.2: Otherwise. Then $\mathrm{tc}(\alpha+\beta) = \boldsymbol{\alpha}_{\restriction(i_0,j_0+1)}[\alpha_{i_0,j_0+1}+1] \in \mathrm{TC}$. The assumptions making up this case imply $r_{k_0,l_0} = 1$, $\beta_1^{k_0,l_0} = \tau_{i_0,j_0}$, $\tau_{k_0,l_0+1} + \beta_1^{k_0,l_0} = \lambda_{\tau_{k_0,l_0}}$, which is the extending index of $\mathrm{ec}(\boldsymbol{\alpha}_{\restriction k_0,l_0+1}) \notin \mathrm{TC}$, and thus $\mathrm{me}(\boldsymbol{\alpha}_{\restriction i_0,j_0+1}) = \boldsymbol{\alpha}_{\restriction k_0,l_0+1}$. Noticing that $\mathrm{dp}_{\tilde{\tau}_{k_0,l_0}}(\lambda_{\tau_{k_0,l_0}}) = 0$ part (e) of Lemma 5.12 now conveys the computation

$$\begin{aligned}\mathrm{o}(\mathrm{tc}(\alpha+\beta)) &= \mathrm{o}(\boldsymbol{\alpha}_{\restriction i_0,j_0+1}) + \kappa^{\tilde{\tau}_{i_0,j_0}}_{\varrho_{\tau_{i_0,j_0+1}}} + \mathrm{dp}_{\tilde{\tau}_{i_0,j_0}}(\varrho_{\tau_{i_0,j_0+1}})\\ &= \mathrm{o}(\boldsymbol{\alpha}_{\restriction k_0,l_0+1}) + \kappa^{\tilde{\tau}_{k_0,l_0}}_{\lambda_{\tau_{k_0,l_0}}}\\ &= \mathrm{o}(\boldsymbol{\alpha}_{\restriction k_0,l_0+1}) + \mathrm{dp}_{\tilde{\tau}_{k_0,l_0}}(\tau_{k_0,l_0+1}) + \beta\\ &= \mathrm{o}(\boldsymbol{\alpha}_{\restriction k_0-1}{}^\frown(\alpha_{k_0,1},\ldots,\alpha_{k_0,l_0+1},\mu_{\tau_{k_0,l_0+1}})) + \kappa^{\tilde{\tau}_{k_0,l_0+1}}_{\lambda_{\tau_{k_0,l_0+1}}} + \mathrm{dp}_{\tilde{\tau}_{k_0,l_0+1}}(\lambda_{\tau_{k_0,l_0+1}}) + \beta\\ &= \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}_{\restriction k_0,l_0+2})) + \beta\end{aligned}$$

where the last equality holds, since the tracking chain $\boldsymbol{\alpha}_{\restriction k_0-1}{}^\frown(\alpha_{k_0,1},\ldots,\alpha_{k_0,l_0+1},\mu_{\tau_{k_0,l_0+1}})$ does not possess a critical main line index pair, according to part (d) of Lemma 5.12. That this is equal to $\alpha+\beta$ will be shown later.

Subcase 2.3: Otherwise. Then $k_0 > i_0$, $l_0 = 1$, and $\alpha_{k+1,1} + \beta_1^{k,m_k-1} < \rho_k$ for $k := k_0 - 1$.

2.3.1: The vector $\boldsymbol{\alpha}_{\restriction k}{}^\frown\left(\alpha_{k_0,1}+\beta_1^{k,m_k-1}, \beta_2^{k,m_k-1}, \ldots, \beta_{r_{k,m_k-1}}^{k,m_k-1}\right)$ satisfies Condition 6 for tracking chains. Then $\mathrm{tc}(\alpha+\beta)$ is defined by this vector which using part a) of Lemma 5.7 is easily seen to be a tracking chain. Let us first assume that $k_0 = n+1$. Using Lemma 4.15 we then have

$$\mathrm{o}(\mathrm{tc}(\alpha+\beta)) = \alpha+\beta.$$

Now we suppose $k_0 \le n$. We observe that $\boldsymbol{\alpha}_{\restriction k_0,1}$ does not possess a critical main line index pair since $\alpha_{k_0,1} < \rho_k \dot{-} 1$ where $\tau_{k,m_k} < \mu_{\tau'_k}$ is possible only if $(k, m_k) = (i_0, k_0 + 1)$. Now Lemmas 4.15 and 5.12, part (d), yield

$$\begin{aligned} \mathrm{o}(\mathrm{tc}(\alpha+\beta)) &= \mathrm{o}(\boldsymbol{\alpha}_{\restriction k_0,1}) + \mathrm{dp}_{\tilde{\tau}_{k,m_k-1}}(\tau_{k_0,1}) + \beta \\ &= \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}_{\restriction k_0,l_0})) + \beta \end{aligned}$$

which will be shown to be equal to $\alpha + \beta$.

2.3.2: Otherwise. Then $\mathrm{tc}(\alpha+\beta) = \boldsymbol{\alpha}_{\restriction(i_0,j_0+1)}[\alpha_{i_0,j_0+1} + 1] \in \mathrm{TC}$. In this final case we have $r_{k,m_k-1} = 1$, $\beta^{k,m_k-1} = (\tau_{i_0,j_0})$, $\alpha_{k_0,1} + \tau_{i_0,j_0} = \rho_k \dot{-} 1$, and $\mathrm{me}(\boldsymbol{\alpha}_{\restriction i_0,j_0+1}) = \boldsymbol{\alpha}_{\restriction k}$. By part (a) of Lemma 5.5 we either have $\rho_k \dot{-} 1 = \log((1/\tau'_k) \cdot \tau_{k,1})$ in the case $m_k = 1$ where $k > i_0$, or we have $\rho_k \dot{-} 1 = \lambda_{\tau'_k}$ in the case $m_k > 1$ where $\tau_{k,m_k} = \mu_{\tau'_k}$. $\alpha_{k_0,1}$ must be a (possibly zero in the case $k_0 = n+1$) multiple of τ_{i_0,j_0} since if not, by part (a) of Lemma 5.12 we would have

$$\mathrm{o}(\boldsymbol{\alpha}_{\restriction k}) < \alpha < \mathrm{o}(\boldsymbol{\alpha}_{\restriction k}) + \kappa^{\tilde{\tau}'_{k_0}}_{\tau_{k_0,1}+1} < \mathrm{o}(\boldsymbol{\alpha}_{\restriction k}) + \beta$$

where we have used that our assumption would entail $\tau_{k_0,1} < \tau_{i_0,j_0}$ whence τ'_{k_0} would be an element of $\mathrm{ts}(\tilde{\tau}_{i_0,j_0})$ and therefore $\mathrm{dp}_{\tilde{\tau}'_{k_0}}(\tau_{k_0,1}) = \mathrm{dp}_{\tilde{\tau}_{i_0,j_0}}(\tau_{k_0,1}) < \kappa^{\tilde{\tau}_{i_0,j_0}}_{\tau_{i_0,j_0}} = \tilde{\tau}_{i_0,j_0} = \beta$. This would mean that $\mathrm{end}(\alpha) < \beta$ which is not the case. We are now prepared for another twofold application of Lemma 5.12, first part (e), then part (d). In the case $k_0 = n+1$ we are finished with the second equation while otherwise we continue the computation as shown.

$$\begin{aligned} \mathrm{o}(\mathrm{tc}(\alpha+\beta)) &= \mathrm{o}(\boldsymbol{\alpha}_{\restriction i_0,j_0+1}) + \kappa^{\tilde{\tau}_{i_0,j_0}}_{\varrho_{\tau_{i_0,j_0+1}}} + \mathrm{dp}_{\tilde{\tau}_{i_0,j_0}}(\varrho_{\tau_{i_0,j_0+1}}) \\ &= \mathrm{o}(\boldsymbol{\alpha}_{\restriction k}) + \kappa^{\tilde{\tau}_{k,m_k-1}}_{\alpha_{k_0,1}} + \mathrm{dp}_{\tilde{\tau}_{k,m_k-1}}(\alpha_{k_0,1}) + \beta \\ &= \mathrm{o}(\boldsymbol{\alpha}_{\restriction k_0,1}) + \mathrm{dp}_{\tilde{\tau}'_{k_0}}(\tau_{k_0,1}) + \beta \\ &= \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}_{\restriction k_0,l_0})) + \beta \end{aligned}$$

which in the case $k_0 \le n$ will be shown below to be equal to $\alpha + \beta$.

We are now going to show the equalities left open in the single cases. Notice that all cases where $k_0 = n+1$ are finished already. We therefore assume $k_0 \le n$ from now on, whence $\beta_1^{n,m_n-1} \ge \rho_n$. In the first step we show that

$$\mathrm{o}(\mathrm{me}(\boldsymbol{\alpha})) + \beta = \alpha + \beta. \tag{1}$$

This is clear if $\mathrm{me}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$. If $\boldsymbol{\alpha} <_{\mathrm{TC}} \mathrm{me}(\boldsymbol{\alpha})$ we have to consider three cases in each of which we use Lemma 5.12.

If $m_n = 1$ then $\alpha < \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha})) \le \alpha + \mathrm{dp}_{\tilde{\tau}'_n}(\tau_{n,1})$, and referring to Lemmas 5.8 and 5.10 we have

$$\mathrm{dp}_{\tilde{\tau}'_n}(\tau_{n,1}) = \kappa^{\tilde{\tau}_{n,0}}_{\log((1/\tau'_n)\cdot\tau_{n,1})} + \mathrm{dp}_{\tilde{\tau}_{n,0}}(\log((1/\tau'_n)\cdot\tau_{n,1})).$$

By part (b) of Lemma 4.17 the assumption $\beta \le \mathrm{dp}_{\tilde{\tau}'_n}(\tau_{n,1})$ would imply $\beta_1^{n,0} < \log((1/\tau'_n)\cdot\tau_{n,1}) + 1 = \rho_n$ which is not the case.

Now assume $m_n > 1$ and $\tau_{n,m_n} < \mu_{\tau'_n}$. This is only possible if $(n, m_n) = (i_0, j_0+1)$. We then have $\alpha < \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha})) \le \alpha + \kappa^{\tilde{\tau}_{i_0,j_0}}_{\varrho_{\tau_{i_0,j_0+1}}} + \mathrm{dp}_{\tilde{\tau}_{i_0,j_0}}(\varrho_{\tau_{i_0,j_0+1}})$. Here the assumption $\beta \le \kappa^{\tilde{\tau}_{i_0,j_0}}_{\varrho_{\tau_{i_0,j_0+1}}} + \mathrm{dp}_{\tilde{\tau}_{i_0,j_0}}(\varrho_{\tau_{i_0,j_0+1}})$ would entail the contradiction $\beta_1^{i_0,j_0} < \rho_n$.

Otherwise we have $m_n > 1$ and $\tau_{n,m_n} = \mu_{\tau'_n}$. Then we have $\alpha < \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha})) \le \alpha + \kappa^{\tilde{\tau}'_n}_{\lambda_{\tau'_n}} + \mathrm{dp}_{\tilde{\tau}'_n}(\lambda_{\tau'_n})$, and the assumption $\beta \le \kappa^{\tilde{\tau}'_n}_{\lambda_{\tau'_n}} + \mathrm{dp}_{\tilde{\tau}'_n}(\lambda_{\tau'_n})$ would lead to the contradiction $\beta_1^{n,m_n-1} < \lambda_{\tau'_n} + 1 = \rho_n$. Thus in all cases we have $\mathrm{o}(\mathrm{me}(\boldsymbol{\alpha})) + \beta = \alpha + \beta$ as claimed.

We now have to show that for index pairs $(i,j) \in \mathrm{dom}(\boldsymbol{\alpha}) - \{(n, m_n)\}$ which are lexicographically greater than or equal to the index pair occurring in the respective case above we have

$$\mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}_{\restriction(i,j)})) + \beta = \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}_{\restriction(i,j)^+})) + \beta. \tag{2}$$

This means that regarding the equations to be proven in Case 1 we assume $(i_0, j_0+1) \le_{\mathrm{lex}} (i,j)$, regarding those to be shown in Case 2.2 we assume $(k_0, l_0+2) \le_{\mathrm{lex}} (i,j)$, and regarding Case 2.3 we assume $(k_0, l_0) \le_{\mathrm{lex}} (i,j)$. Let such an index pair (i,j) be given. We may assume that $\mathrm{me}(\boldsymbol{\alpha}_{\restriction(i,j)^+}) <_{\mathrm{TC}} \mathrm{me}(\boldsymbol{\alpha}_{\restriction(i,j)})$ since in the case of equality there is nothing to show, while $\mathrm{me}(\boldsymbol{\alpha}_{\restriction(i,j)}) <_{\mathrm{TC}} \mathrm{me}(\boldsymbol{\alpha}_{\restriction(i,j)^+})$ is not possible, for if this were the case we would have $(i,j) = (i_0, j_0+1)$, $\chi^{\tau_{i_0,j_0}}(\tau_{i_0,j_0+1}) = 0$, $\rho_{i_0} = \varrho_{\tau_{i_0,j_0+1}} + \tau_{i_0,j_0}$, $(i,j)^+ = (i+1, 1)$, and $\alpha_{i+1,1} = \varrho_{i_0,j_0+1} + \xi$ for some $\xi \in (0, \tau_{i_0,j_0})$, which by Lemma 5.12 would imply that $\beta \le \mathrm{end}(\alpha) < \tilde{\tau}_{i_0,j_0}$ whence we would be in Case 1.1, running into the contradiction $\alpha_{i_0+1} + \beta_1^{i_0,j_0} < \rho_{i_0}$. We therefore have $\boldsymbol{\alpha}_{\restriction(i,j)^+} <_{\mathrm{TC}} \mathrm{ec}(\boldsymbol{\alpha}_{\restriction(i,j)})$ and consider the two possibilities for $(i,j)^+$:

- $(i,j)^+ = (i,j+1)$. Then we have $\alpha_{i,j+1} = \mu_{\tau_{i,j}}$, since $(i_0,j_0) <_{\mathrm{lex}} (i,j)$. Due to the fact that $\boldsymbol{\alpha}_{\restriction(i,j+1)}$ is not a maximal 1-step extension of $\boldsymbol{\alpha}_{\restriction(i,j)}$ we have $j > 1$, $\tau_{i,j} = \mu_{\tau_{i,j-1}} \in \mathbb{E} \cap (\tau_{i,j-1}, \lambda_{\tau_{i,j-1}})$, $\mathrm{ec}(\boldsymbol{\alpha}_{\restriction(i,j)}) = \boldsymbol{\alpha}_{\restriction(i,j)}{}^\frown(\lambda_{\tau_{i,j-1}})$, and $(i_0, j_0+1) <_{\mathrm{lex}} (i,j)$. In particular, $\boldsymbol{\alpha}_{\restriction(i,j+1)}$ does not possess a critical main line index pair. Part (d) of Lemma 5.12 yields

$$\begin{aligned}\mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}_{\restriction(i,j+1)})) &= \mathrm{o}(\boldsymbol{\alpha}_{\restriction(i,j+1)}) + \kappa^{\tilde{\tau}_{i,j}}_{\lambda_{\tau_{i,j}}} + \mathrm{dp}_{\tilde{\tau}_{i,j}}(\lambda_{\tau_{i,j}})\\ &= \mathrm{o}(\boldsymbol{\alpha}_{\restriction(i,j)}) + \nu^{\tilde{\tau}_{i,j}}_{\mu_{\tau_{i,j}}} + \kappa^{\tilde{\tau}_{i,j}}_{\lambda_{\tau_{i,j}}} + \mathrm{dp}_{\tilde{\tau}_{i,j}}(\lambda_{\tau_{i,j}})\\ &= \mathrm{o}(\boldsymbol{\alpha}_{\restriction(i,j)}) + \mathrm{dp}_{\tilde{\tau}_{i,j-1}}(\tau_{i,j}).\end{aligned}$$

  Another extensive application of Lemma 5.12 provides us with

$$\mathrm{o}(\boldsymbol{\alpha}_{\restriction(i,j)}) + \mathrm{dp}_{\tilde{\tau}_{i,j-1}}(\tau_{i,j}) < \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}_{\restriction(i,j)})) \le \mathrm{o}(\boldsymbol{\alpha}_{\restriction(i,j)}) + \kappa^{\tilde{\tau}_{i,j-1}}_{\lambda_{\tau_{i,j-1}}} + \mathrm{dp}_{\tilde{\tau}_{i,j-1}}(\lambda_{\tau_{i,j-1}}).$$

  Now setting $\delta := -\tau_{i,j} + \lambda_{\tau_{i,j-1}}$, the assumption $\beta \le \kappa^{\tilde{\tau}_{i,j-1}}_{\delta} + \mathrm{dp}_{\tilde{\tau}_{i,j-1}}(\delta)$ would imply, by Lemma 4.17, that $\beta_1^{i,j-1} \le \delta$ and hence $\tau_{i,j} + \beta_1^{i,j-1} \le \lambda_{\tau_{i,j-1}}$ which is not the case: In Cases 1 and 2.2 we always have $(k_0, l_0) <_{\mathrm{lex}} (i, j-1)$, while Case 2.3 presupposes that $\tau_{k_0,l_0+1} + \beta_1^{k_0,l_0} > \lambda_{\tau_{k_0,l_0}}$, which covers the only possibility where $(k_0,l_0) = (i,j-1)$.
- $(i,j)^+ = (i+1,1)$. We then have $j = m_i$ and consider three subcases.

  If $m_i = 1$ then $\alpha_{i+1,1} < \log((1/\tau_i')\cdot\tau_{i,1}) = \rho_i \dot{-} 1$, hence $\boldsymbol{\alpha}_{\restriction i+1,1}$ does not possess a critical main line index pair. By Lemma 5.12 we have

$$\begin{aligned}\mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}_{\restriction(i+1,1)})) &= \mathrm{o}(\boldsymbol{\alpha}_{\restriction(i+1,1)}) + \mathrm{dp}_{\tilde{\tau}'_{i+1}}(\tau_{i+1,1})\\ &= \mathrm{o}(\boldsymbol{\alpha}_{\restriction(i,1)}) + \kappa^{\tilde{\tau}_{i,0}}_{\alpha_{i+1,1}} + \mathrm{dp}_{\tilde{\tau}_{i,0}}(\tau_{i+1,1})\\ &< \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}_{\restriction(i,1)}))\\ &\le \mathrm{o}(\boldsymbol{\alpha}_{\restriction(i,1)}) + \mathrm{dp}_{\tilde{\tau}'_i}(\tau_{i,1})\\ &= \mathrm{o}(\boldsymbol{\alpha}_{\restriction(i,1)}) + \kappa^{\tilde{\tau}_{i,0}}_{\log((1/\tau_i')\cdot\tau_{i,1})} + \mathrm{dp}_{\tilde{\tau}_{i,0}}(\log((1/\tau_i')\cdot\tau_{i,1})).\end{aligned}$$

  By setting $\delta := -\alpha_{i+1,1} + \log((1/\tau_i')\cdot\tau_{i,1})$ and assuming $\beta \le \kappa^{\tilde{\tau}_{i,0}}_{\delta} + \mathrm{dp}_{\tilde{\tau}_{i,0}}(\delta)$ we would obtain $\alpha_{i+1,1} + \beta_1^{i,0} < \rho_i$ which because of $i \ge k_0$ is not the case. Thus Eq. (2) holds in the case $m_i = 1$.

  If $(i, m_i) = (i_0, j_0+1)$ then only Case 1 is possible, and it follows that $\alpha_{i+1,1} < \varrho_{\tau_{i_0,j_0+1}} < \rho_{i_0}$. Lemma 5.12 supplies us with

$$\begin{aligned}\mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}_{\restriction i+1,1})) &= \mathrm{o}(\boldsymbol{\alpha}_{\restriction(i_0,j_0+1)}) + \kappa^{\tilde{\tau}_{i_0,j_0}}_{\alpha_{i+1,1}} + \mathrm{dp}_{\tilde{\tau}_{i_0,j_0}}(\alpha_{i+1,1})\\ &< \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}_{\restriction(i_0,j_0+1)}))\\ &\le \mathrm{o}(\boldsymbol{\alpha}_{\restriction(i_0,j_0+1)}) + \kappa^{\tilde{\tau}_{i_0,j_0}}_{\varrho_{\tau_{i_0,j_0+1}}} + \mathrm{dp}_{\tilde{\tau}_{i_0,j_0}}(\varrho_{\tau_{i_0,j_0+1}}),\end{aligned}$$

  and setting $\delta := -\alpha_{i+1,1} + \varrho_{\tau_{i_0,j_0+1}}$ the assumption $\beta \le \kappa^{\tilde{\tau}_{i_0,j_0}}_{\delta} + \mathrm{dp}_{\tilde{\tau}_{i_0,j_0}}(\delta)$ would have the consequence $\alpha_{i+1,1} + \beta_1^{i_0,j_0} < \rho_{i_0}$ which is not the case. We therefore have (2) in this special case.

  Finally, if $m_i > 1$ and $(i_0, j_0+1) <_{\mathrm{lex}} (i, m_i)$ then $\alpha_{i+1,1} < \lambda_{\tau_{i,m_i-1}} = \rho_i \dot{-} 1$. Lemma 5.12 yields

$$\begin{aligned}\mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}_{\restriction(i+1,1)})) &= \mathrm{o}(\boldsymbol{\alpha}_{\restriction(i,m_i)}) + \kappa^{\tilde{\tau}_{i,m_i-1}}_{\alpha_{i+1,1}} + \mathrm{dp}_{\tilde{\tau}_{i,m_i-1}}(\alpha_{i+1,1})\\ &< \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}_{\restriction(i,m_i)}))\\ &\le \mathrm{o}(\boldsymbol{\alpha}_{\restriction(i,m_i)}) + \kappa^{\tilde{\tau}_{i,m_i-1}}_{\lambda_{\tau_{i,m_i-1}}} + \mathrm{dp}_{\tilde{\tau}_{i,m_i-1}}(\lambda_{\tau_{i,m_i-1}}),\end{aligned}$$

  and setting $\delta := -\alpha_{i+1,1} + \lambda_{\tau_{i,m_i-1}}$ the assumption $\beta \le \kappa^{\tilde{\tau}_{i,m_i-1}}_{\delta} + \mathrm{dp}_{\tilde{\tau}_{i,m_i-1}}(\delta)$ would imply the contradictory $\alpha_{i+1,1} + \beta_1^{i,m_i-1} < \rho_i$. Consequently, Eq. (2) follows also in this situation.

This concludes the proof of (2). From the Eqs. (1) and (2) all claimed equalities follow, completing the proof of Lemma 6.2. □

Corollary 6.5. tc *is a* $<$-$<_{\mathrm{TC}}$*-order isomorphism between* 1^∞ *and* TC *with inverse* o. *We thus have*

$$\mathrm{tc}(\mathrm{o}(\boldsymbol{\alpha})) = \boldsymbol{\alpha}$$

for any $\boldsymbol{\alpha} \in \mathrm{TC}$ *and*

$$\alpha < \beta \Leftrightarrow \mathrm{tc}(\alpha) <_{\mathrm{TC}} \mathrm{tc}(\beta)$$

for all $\alpha, \beta < 1^\infty$. □

Corollary 6.6. *Let $\alpha < 1^\infty$ and $\mathrm{tc}(\alpha) =: \boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n)$, where $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \ldots, \alpha_{i,m_i})$ for $1 \le i \le n$. Then we have*

$$\mathrm{tc}(\alpha + \mathrm{dp}(\boldsymbol{\alpha})) = \begin{cases} \boldsymbol{\alpha}[\alpha_{n,m_n} + 1] & \text{if } m_n > 1 \ \&\ \tau_{n,m_n} < \mu_{\tau_{n,m_n-1}} \\ \mathrm{me}(\boldsymbol{\alpha}) & \text{otherwise.} \end{cases}$$

Let $\beta < 1^\infty$. Then $\mathrm{tc}(\beta)$ is a proper extension of $\mathrm{tc}(\alpha)$ if and only if

$$\beta \in \begin{cases} (\alpha, \alpha + \mathrm{dp}(\boldsymbol{\alpha})) & \text{if } m_n > 1 \ \&\ \tau_{n,m_n} < \mu_{\tau_{n,m_n-1}} \\ (\alpha, \alpha + \mathrm{dp}(\boldsymbol{\alpha})] & \text{otherwise.} \end{cases}$$

Proof. Using the above corollary the first claim now follows by Definition 5.11 of dp (see the remark there) and Lemma 5.12. The left-to-right direction of the second claim follows from Lemma 5.12. For the right-to-left direction assume that β lies within the respective interval. By the above corollary we have $\mathrm{tc}(\alpha) <_{\mathrm{TC}} \mathrm{tc}(\beta)$.

Case 1: $m_n > 1 \ \&\ \tau_{n,m_n} < \mu_{\tau_{n,m_n-1}}$. Then by the above corollary

$$\mathrm{tc}(\beta) <_{\mathrm{TC}} \boldsymbol{\alpha}[\alpha_{n,m_n} + 1].$$

Case 2: Otherwise. Then by the above corollary $\mathrm{tc}(\beta) \le_{\mathrm{TC}} \mathrm{me}(\boldsymbol{\alpha})$. □

7. Arithmetical characterization of $\le_1$ and $\le_2$

It is only now that we are prepared to compute the relations $\le_1$ and $\le_2$ below the least $\alpha \in \mathrm{Ord}$ such that every pure pattern of order 2 has a covering below α, which will be shown to be equal to 1^∞, the proof-theoretic ordinal of $\mathrm{KP}\ell_0$.

First of all we provide criteria for elementary substructurehood that allow us to avoid dealing with formulas. As already applied in [1,9,11] we have the following folklore criterion for Σ_1-elementary substructure for finite relational languages.

Proposition 7.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be structures for a finite language without function symbols. $\mathcal{A}$ is a Σ_1-elementary substructure of $\mathcal{B}$ if and only if $\mathcal{A}$ is a substructure of $\mathcal{B}$ and whenever X is a finite subset of $|\mathcal{A}|$ and Y is a finite subset of $|\mathcal{B}| - |\mathcal{A}|$ then there exists a subset $\tilde{Y}$ of $|\mathcal{A}|$ such that*

$$X \cup Y \cong_X X \cup \tilde{Y}.$$

Proof. The proof is elementary and given in full detail in [8]. □

Lemma 7.2. 1. *In $\mathcal{R}_1$ we have (see [1])*

$$\alpha \le_1 \alpha + 1 \ \Leftrightarrow\ \alpha \in \mathrm{Lim}.$$

2. *In $\mathcal{R}_2$ we have*

$$\alpha \le_1 \alpha + 1 \ \Leftrightarrow\ \alpha \in \mathrm{Lim} \ \&\ \forall \beta(\beta <_2 \alpha \Rightarrow \alpha = \sup\{\gamma < \alpha \mid \beta \le_2 \gamma\}).$$

Proof. We show part 2. Recall Lemma 1.2. Let us first assume that $\alpha <_1 \alpha + 1$. It is easy to see that $\alpha \in \mathrm{Lim}$. If there were some β such that $\beta <_2 \alpha$ and $\delta := \sup\{\gamma < \alpha \mid \beta \le_2 \gamma\} < \alpha$ then we would have $\beta \le_2 \delta$, and $\alpha + 1 \models \exists x > \delta\ \beta <_2 x$ while $\alpha \not\models \exists x > \delta\ \beta <_2 x$. Hence the right hand side of the equivalence holds whenever $\alpha <_1 \alpha + 1$.

Now suppose the right hand side of the equivalence holds. We use 7.1 to show that $\alpha <_1 \alpha + 1$. Let $X \subseteq_{\mathrm{fin}} \alpha$ be given and let $Y := \{\alpha\}$. We set $X_i' := \{x \in X \mid x \not<_i \alpha\}$ for $i = 1, 2$. There is $\mu < \alpha$ such that

$$\forall x \in X_i'\ \forall \xi \in (\mu, \alpha)\ x \not<_i \xi,\ \ i = 1, 2.$$

In the case $X_2' = X$ we may choose any ordinal $\tilde{\alpha} \in (\mu, \alpha)$ and set $\tilde{Y} := \{\tilde{\alpha}\}$ while otherwise there exists $\beta := \max(X - X_2')$ so that we have $\gamma \le_2 \beta$ for all $\gamma \in X - X_2'$, then choose some $\delta \in (\mu, \alpha)$ such that $\beta <_2 \delta$ and set $\tilde{Y} := \{\delta\}$. It is now easy to see that we have $X \cup Y \cong X \cup \tilde{Y}$. □

For the readers' convenience we recall results shown in [1]. Let $\mathrm{lh}_1^{\mathcal{R}_i}(\alpha)$ be $\max\{\beta \mid \alpha \le_1^{\mathcal{R}_i} \beta\}$ if that exists and ∞ otherwise.

Theorem 7.3 ([1]). 1. $\mathcal{R}_1 \cong \mathcal{R}_1 \cap [\alpha + 1, \infty)$ *for all α.*
2. $\mathrm{lh}_1^{\mathcal{R}_1}(\varepsilon_0 \cdot (1 + \eta)) = \infty$ *for all η.*
3. *For $\alpha =_{\mathrm{CNF}} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ ($n > 0$) with $\alpha_n =_{\mathrm{ANF}} \rho_1 + \cdots + \rho_m < \alpha$ we have*

$$\mathrm{lh}_1^{\mathcal{R}_1}(\alpha) = \alpha + \mathrm{lh}_1^{\mathcal{R}_1}(\rho_1) + \cdots + \mathrm{lh}_1^{\mathcal{R}_1}(\rho_m).$$

The approach to obtain the above result is to consider the connectivity components of $\le_1$ in $\mathcal{R}_1$ and to compute the enumeration function $\alpha \mapsto \kappa_\alpha$ of the $\le_1$-minimal ordinals. In this computation the translation invariance of $\mathcal{R}_1$ (see the first claim of the theorem) plays an essential role.

A criterion for the relation $\le_2$ that will turn out to actually characterize $\le_2$ in $\mathcal{R}_2$ is

Proposition 7.4. *Suppose $\alpha < \beta$. If for all $X \subseteq_{\rm fin} \alpha$ and all $Y \subseteq_{\rm fin} [\alpha, \beta)$ there exists $\tilde{Y}$ such that:*

1. $X < \tilde{Y} < \alpha$ *and*
2. $\exists h : X \cup \tilde{Y} \xrightarrow{\cong} X \cup Y$ *such that for all finite $\tilde{Y}^+$ with $\tilde{Y} \subseteq \tilde{Y}^+ \subseteq \alpha$*

$$\exists Y^+ \subseteq \beta,\ h^+ \supseteq h \quad \text{s.t. } h^+ : X \cup \tilde{Y}^+ \xrightarrow{\cong} X \cup Y^+$$

then $\alpha <_2 \beta$.

Proof. Let ordinals α, β such that $\alpha < \beta$ be given and assume the criterion of the lemma holds. Using 7.1 we obtain $\alpha <_1 \beta$. Let $\varphi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ be a quantifier free formula of $\mathcal{L}(\mathcal{R}_2)$ with all free variables shown and let $\boldsymbol{\xi} \subseteq \alpha$ be a list of parameters matching the length of $\boldsymbol{x}$. Assume first that

$$\alpha \models \exists \boldsymbol{y}\ \forall \boldsymbol{z}\ \varphi(\boldsymbol{\xi}, \boldsymbol{y}, \boldsymbol{z}).$$

Let $\boldsymbol{\eta} \subseteq \alpha$ be a list of witnesses for $\boldsymbol{y}$ so that $\alpha \models \forall \boldsymbol{z}\ \varphi(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{z})$. Thanks to $\alpha <_1 \beta$ we have $\beta \models \forall \boldsymbol{z}\ \varphi(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{z})$, whence $\beta \models \exists \boldsymbol{y}\ \forall \boldsymbol{z}\ \varphi(\boldsymbol{\xi}, \boldsymbol{y}, \boldsymbol{z})$. Now let us assume that

$$\beta \models \exists \boldsymbol{y}\ \forall \boldsymbol{z}\ \varphi(\boldsymbol{\xi}, \boldsymbol{y}, \boldsymbol{z})$$

and let $\boldsymbol{\eta} \subseteq \beta$ be witnesses for $\boldsymbol{y}$. Without loss of generality we may assume that $\boldsymbol{\eta} \subseteq [\alpha, \beta)$ since we could regard witnesses below α as parameters in the list $\boldsymbol{\xi}$. Set $X := \{\xi \mid \xi \in \boldsymbol{\xi}\}$, $Y := \{\eta \mid \eta \in \boldsymbol{\eta}\}$. By the criterion there exists a set $\tilde{Y}$ such that conditions 1 and 2 of the criterion hold. Let $h : X \cup \tilde{Y} \xrightarrow{\cong} X \cup Y$ be according to condition 2 and let $\tilde{\boldsymbol{\eta}} := h^{-1}[\boldsymbol{\eta}]$. We claim

$$\alpha \models \forall \boldsymbol{z}\ \varphi(\boldsymbol{\xi}, \tilde{\boldsymbol{\eta}}, \boldsymbol{z}).$$

In order to show this claim let $\tilde{\boldsymbol{\zeta}} \subseteq \alpha$ matching $\boldsymbol{z}$ be given. Set $\tilde{Y}^+ := \tilde{Y} \cup \tilde{\boldsymbol{\zeta}}$ and let h^+ and Y^+ be according to the criterion so that for $\boldsymbol{\zeta} := h^+[\tilde{\boldsymbol{\zeta}}]$. Since $\beta \models \varphi(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta})$ and $X \cup \tilde{Y}^+ \cong X \cup Y^+$ we then have $\alpha \models \varphi(\boldsymbol{\xi}, \tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\zeta}})$ and thus

$$\alpha \models \exists \boldsymbol{y}\ \forall \boldsymbol{z}\ \varphi(\boldsymbol{\xi}, \boldsymbol{y}, \boldsymbol{z})$$

concluding the proof of the criterion. □

Remark. The above type of criterion can be generalized to the higher levels as well as to structures with underlying arithmetic. Notice that if the criterion holds for pairs of ordinals α, β and β, γ, showing that $\alpha <_2 \beta <_2 \gamma$ then the criterion also holds for α, γ. Also, if the criterion holds for pairs α, γ_ι, where $\iota \in I$ for some nonempty set I, showing that $\alpha <_2 \gamma_\iota$ for all $\iota \in I$, then the criterion also holds for $\alpha, \sup\{\gamma_\iota \mid \iota \in I\}$.

Example. In $\mathcal{R}_2$ we have $\varepsilon_0 \cdot \omega <_2 \varepsilon_0 \cdot (\omega + 1)$. This is the least such pair of ordinals in $\mathcal{R}_2$. The least $\le_1$-predecessor of $\varepsilon_0 \cdot \omega$ is ε_0 which is the least element of the $<_1$-chain of the multiples of ε_0 up to $\varepsilon_0 \cdot (\omega + 1)$. In general, as an elementary observation, any ordinal that has a proper $<_2$-successor is the supremum of an infinite $<_1$-chain:

Lemma 7.5. *If $\alpha <_2 \beta$ then α is the* sup *of an infinite $<_1$-chain.*

Proof. For any $\rho < \alpha$ we have $\beta \models \exists x\ \forall y > x\ (\rho < x <_1 y)$. Hence the same holds true in α. We obtain $\rho_1 <_1 \rho_2 <_1 \rho_3 <_1 \cdots <_1 \alpha$. □

Another useful elementary observation is the following

Lemma 7.6. *Suppose $\alpha <_2 \beta$, $X \subseteq_{\rm fin} \alpha$, and $\emptyset \neq Y \subseteq_{\rm fin} [\alpha, \beta)$.*

1. *There exist cofinally many $\tilde{Y} \subseteq \beta$ such that $X \cup \tilde{Y} \cong X \cup Y$. More generally, for any $Z \subseteq_{\rm fin} \alpha$ with $X < Z$, if $\alpha \models \forall x \exists \tilde{Z}\ (x < \tilde{Z} \wedge \text{“}X \cup Z \cong X \cup \tilde{Z}\text{”})$ then this also holds in β.*
2. *Cofinally in α, copies $\tilde{Y} \subseteq \alpha$ of Y can be chosen which besides $X < \tilde{Y}$ and $X \cup \tilde{Y} \cong X \cup Y$ also “maintain $\le_1$-connections to β”: For any $y \in Y$ such that $y <_1 \beta$ the corresponding $\tilde{y}$ satisfies $\tilde{y} <_1 \alpha$.*

Proof. By $\alpha <_1 \beta$ we have $\alpha \models \forall r\ \exists \tilde{Y}\ (r < \tilde{Y} \wedge \text{“}X \cup \tilde{Y} \cong X \cup Y\text{”})$ where “$X \cup \tilde{Y} \cong X \cup Y$” means that the diagram of $X \cup Y$ in the language of $\mathcal{R}_2$ holds accordingly for $X \cup \tilde{Y}$. By $\alpha <_2 \beta$ this also holds in β. This shows the first part of the lemma. The second part follows from $\alpha <_2 \beta$ by letting $\{y \in Y \mid y <_1 \beta\} = \{y_1, \ldots, y_k\}$ and noting that for any parameter $\xi < \alpha$

$$\beta \models \exists \tilde{Y} > \xi\ \forall r > \tilde{Y} \left(\text{“}X \cup \tilde{Y} \cong X \cup Y\text{”} \wedge \bigwedge_{i=1}^{k} \tilde{y}_i <_1 r \right)$$

which then also holds in α. Notice that we used $\tilde{y}_i$ to indicate the element of $\tilde{Y}$ corresponding to $y_i \in Y$ which can be done by using e.g. increasing enumerations of the elements in Y and $\tilde{Y}$. □

We now introduce some terminology which will be helpful in the statement of Theorem 7.9.

Definition 7.7. Let $\alpha \in \mathrm{Ord}$. We define $\mathrm{Pred}_i(\alpha)$ to be the set of all $<_i$-predecessors of α for $i = 1, 2$.

$$\mathrm{Pred}_i(\alpha) := \{\beta \mid \beta <_i \alpha\}.$$

We define terms for the greatest $<_i$-predecessor of α, $i = 1, 2$, if such exist.

$$\mathrm{pred}_i(\alpha) := \begin{cases} \max(\mathrm{Pred}_i(\alpha)) & \text{if that exists} \\ 0 & \text{otherwise.} \end{cases}$$

The class of all $\le_i$-successors of α is denoted by

$$\mathrm{Succ}_i(\alpha) := \{\beta \mid \alpha \le_i \beta\},$$

in analogy to $\mathrm{lh}_1^{\mathcal{R}_1}$ we define

$$\mathrm{lh}_i(\alpha) := \begin{cases} \max(\mathrm{Succ}_i(\alpha)) & \text{if that exists} \\ \infty & \text{otherwise,} \end{cases}$$

and we will make use of the abbreviation $\mathrm{lh} := \mathrm{lh}_1$.

Definition 7.8. Given substructures X and Y of $\mathcal{R}_2$, a mapping $h : X \hookrightarrow Y$ is a *covering of X into Y*, if

1. h is an injection of X into Y that preserves $\le$, and
2. h maintains $\le_i$-connections for $i = 1, 2$, i.e. $\forall \alpha, \beta \in X\ (\alpha \le_i \beta \Rightarrow h(\alpha) \le_i h(\beta))$.

We call h a *covering of X* if it is a covering from X into $\mathcal{R}_2$. We call Y a *cover* of X if there is a covering of X with image Y.

Recall the definition of $\mathrm{cml}(\boldsymbol{\alpha})$, the critical main line index pair of $\boldsymbol{\alpha} \in \mathrm{TC}$, in 5.1, as well as the notion of maximal extension $\mathrm{me}(\boldsymbol{\alpha})$ of a tracking chain $\boldsymbol{\alpha}$, defined in 5.2. Also recall the notations $\tilde{\tau}_{i,j}$ and $\mathrm{o}_{i,j}(\boldsymbol{\alpha})$ for the (i, j)-th initial value of $\boldsymbol{\alpha}$ from Definition 5.9 as well as the notations $i^\star$ for the index pair of the i-th unit $\tau_i^\star$ of $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}[\xi]$ for modification of the last entry of a tracking chain from Definition 5.1.

Theorem 7.9. *Let $\alpha < 1^\infty$ and $\mathrm{tc}(\alpha) = \boldsymbol{\alpha}$ where $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \ldots, \alpha_{i,m_i})$ for $1 \le i \le n$.*

(a) *We have*

$$\alpha \text{ is } \le_1\text{-minimal} \;\Leftrightarrow\; (n, m_n) = (1, 1)$$

and

$$\mathrm{pred}_1(\alpha) = \begin{cases} \mathrm{o}_{n-1,m_{n-1}}(\boldsymbol{\alpha}) & \text{if } m_n = 1 \text{ and } n > 1 \\ \mathrm{o}(\boldsymbol{\alpha}[\xi]) & \text{if } m_n > 1,\ \alpha_{n,m_n} = \xi + 1, \text{ and } \chi^\tau(\xi) = 0 \\ \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}[\xi])) & \text{if } m_n > 1,\ \alpha_{n,m_n} = \xi + 1, \text{ and } \chi^\tau(\xi) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

In the case where $m_n > 1$ and $\alpha_{n,m_n} \in \mathrm{Lim}$ we have

$$\mathrm{Pred}_1(\alpha) = \bigcup_{\xi < \alpha_{n,m_n}} \mathrm{Pred}_1(\mathrm{o}(\boldsymbol{\alpha}[\xi])).$$

(b) *We have*

$$\alpha \text{ is } \le_2\text{-minimal} \;\Leftrightarrow\; m_n \le 2 \text{ and } \tau_n^\star = 1,$$

and in terms of pred_2 *we have, setting $(i_0, j_0) := n^\star$,*

$$\mathrm{pred}_2(\alpha) = \begin{cases} \mathrm{o}_{n,m_n-1}(\boldsymbol{\alpha}) & \text{if } m_n > 2 \\ \mathrm{o}_{i_0,j_0+1}(\boldsymbol{\alpha}) & \text{if } m_n \le 2 \text{ and } \tau_n^\star > 1 \\ 0 & \text{otherwise.} \end{cases}$$

The criterion of Proposition 7.4 holds for any pair γ, α such that $\gamma <_2 \alpha$.

(c) *Along $<_{\mathrm{lex}}\restriction_{\mathrm{dom}(\boldsymbol{\alpha})}$*

$$\left(\mathrm{o}_{i,j}(\boldsymbol{\alpha})\right)_{(i,j)\in\mathrm{dom}(\boldsymbol{\alpha})}$$

is a strictly increasing $\le_1$-chain. For any $i \le n$ such that $m_i > 1$ the sequence

$$\left(\mathrm{o}_{i,2}(\boldsymbol{\alpha}), \ldots, \mathrm{o}_{i,m_i}(\boldsymbol{\alpha})\right)$$

is a strictly increasing $\le_2$-chain, and for any $j \in (1, m_i]$ and $\xi < \alpha_{i,j}$ we have

$$\mathrm{o}(\boldsymbol{\alpha}_{\restriction(i,j)}[\xi]) <_1 \mathrm{o}_{i,j}(\boldsymbol{\alpha}).$$

(d) *If* $m_n = 1, \alpha_{n,1} = \xi + 1$ *for some* ξ, *let* $\delta := \mathrm{o}_{n-1,m_{n-1}}(\boldsymbol{\alpha})$, β *be such that* $\alpha = \beta + 1$, *and* $X := \mathrm{Pred}_2(\delta) \cup \{\delta\}$. *There exists a finite set* $Z \subseteq (\delta, \alpha)$ *such that there is no cover* $X \cup \tilde{Z}$ *of* $X \cup Z$ *with* $X < \tilde{Z}$ *and* $X \cup \tilde{Z} \subseteq \beta$.

Proof. The proof is by induction on α, where part (c) is an easy consequence of parts (a) and (b). We will make frequent use of Corollary 6.5. In the case $\alpha = 0$, equivalently $\boldsymbol{\alpha} = ((0))$, there is nothing to show, so let us assume that $\alpha > 0$, whence $\alpha_{n,m_n} > 0$. We distinguish between cases concerning m_n and whether α_{n,m_n} is a limit or a successor ordinal.

Case 1: $m_n = 1$.

Subcase 1.1: $\alpha_{n,1}$ is a successor ordinal, say $\alpha_{n,1} = \xi + 1$.

We then have $\tau_{n,1} = 1$, $\tau_n^\star = 1$, and α is a successor ordinal, say $\alpha = \beta + 1$. By 7.6 α is clearly $\le_2$-minimal. Let $\delta := \mathrm{o}_{n-1,m_{n-1}}(\boldsymbol{\alpha})$ and notice that $\beta = \delta + \kappa_\xi^{\tilde{\tau}_{n,0}} + \mathrm{dp}_{\tilde{\tau}_{n,0}}(\xi)$. Note further that the tracking chain of any ordinal in the interval $[\delta, \beta]$ has the initial chain $\boldsymbol{\alpha}_{\restriction(n-1,m_{n-1})}$. In the case $n = 1$ we have to show that α is $\le_1$-minimal. This will be the special case $\delta = 0$. Generally, for $n \ge 1$ we now show that α is δ-$\le_1$-minimal, which follows from part (d). In order to prove part (d) let us first consider the case $\xi = 0$. Then $\delta = \beta$ and α is clearly δ-$\le_1$-minimal. We trivially choose $Z := \emptyset$.

Now let us assume that $\xi =_{\mathrm{ANF}} \xi_1 + \cdots + \xi_r > 0$. Since $\boldsymbol{\alpha} \in \mathrm{TC}$, we then have $\boldsymbol{\alpha}[\xi] \in \mathrm{TC}$ if and only if Condition 5 of Definition 5.1 holds, and accordingly set

$$\boldsymbol{\gamma} := \begin{cases} \boldsymbol{\alpha}_{\restriction n-2}{}^\frown(\alpha_{n-1,1}, \ldots, \alpha_{n-1,m_{n-1}}, \mu_{\tau_{n-1,m_{n-1}}}) & \text{if } n > 1 \text{ and } \xi = \tau_{n-1,m_{n-1}} \in \mathbb{E}^{>\tau'_{n-1}} \\ \boldsymbol{\alpha}[\xi] & \text{otherwise.} \end{cases}$$

Let $\mathrm{tc}(\beta) =: \boldsymbol{\beta}$, where $\boldsymbol{\beta}_i = (\beta_{i,1}, \ldots, \beta_{i,k_i})$ for $i = 1, \ldots, l$, which by Lemma 5.12, part (d), is equal to $\mathrm{me}(\boldsymbol{\gamma})$ since, again because of $\boldsymbol{\alpha} \in \mathrm{TC}$, we know that $\boldsymbol{\gamma}$ (and hence also $\boldsymbol{\beta}$) does not possess a critical main line index pair. Let $\boldsymbol{\sigma}$ be the chain associated with $\boldsymbol{\beta}$ and set $k_0 := 0$. The i.h. yields $\delta < \gamma := \mathrm{o}(\boldsymbol{\gamma}) \le_1 \beta$, $\delta <_1 \gamma$ if $\delta > 0$, and we clearly have $k_l = 1$ by the choice of $\boldsymbol{\gamma}$ and the definition of me. Hence there exists a $\le_{\mathrm{lex}}$-minimal index pair $(p, 1) \in \mathrm{dom}(\boldsymbol{\beta})$ such that both $p \ge n > 1$ and $\beta_{p,1} \notin \mathbb{E}^{>\sigma_p^\star}$. Let $\eta := \mathrm{o}_{p-1,k_{p-1}}(\boldsymbol{\beta})$. Notice that due to the minimality of p the case $k_{p-1} = 1$ can only occur when $p = n$, $m_{n-1} = 1$, and hence $\delta = \eta$. Setting $\boldsymbol{\beta}' := \boldsymbol{\beta}_{\restriction(p,1)}$ and $\beta' := \mathrm{o}(\boldsymbol{\beta}')$, in general have

$$\delta \le \eta, \quad \gamma \le \beta' = \eta + \kappa_{\beta_{p,1}}^{\tilde{\sigma}_{p,0}}, \quad \text{and} \quad \beta' + \mathrm{dp}_{\tilde{\sigma}_{p,0}}(\beta_{p,1}) = \beta$$

using part (b) of Lemma 5.10, which implies that $\mathrm{dp}_{\tilde{\sigma}_{p,0}}(\beta_{p,1}) = \mathrm{dp}_{\tilde{\sigma}_p^\star}(\sigma_{p,1})$, and again Lemma 5.12, part (d). Setting $X_\eta := \mathrm{Pred}_2(\eta) \cup \{\eta\}$, note that $X_\eta \cap \delta \subseteq X$. We now consider cases regarding $\beta_{p,1}$ in order to define in each case a finite set $Z_\eta \subseteq (\eta, \alpha)$ such that there does not exist any cover $X_\eta \cup \tilde{Z}_\eta$ of $X_\eta \cup Z_\eta$ with $X_\eta < \tilde{Z}_\eta$ and $X_\eta \cup \tilde{Z}_\eta \subseteq \beta$.

- $\sigma_{p,1} = 1$. Then $l = p$, and by the i.h. applied to $\beta' = \beta$, which is of the form $\beta = \beta'' + 1$, there is $Z' \subseteq_{\mathrm{fin}} (\eta, \beta)$, with the property that there is no cover $X_\eta \cup \tilde{Z}'$ of $X_\eta \cup Z'$ such that $X_\eta < \tilde{Z}'$ and $X_\eta \cup \tilde{Z}' \subseteq \beta''$. Let

  $$Z_\eta := Z' \cup \{\beta\}.$$

  Clearly, if there were a set $\tilde{Z}_\eta \subseteq (\eta, \beta)$ such that $X_\eta \cup \tilde{Z}_\eta$ is a cover of $X_\eta \cup Z_\eta$ then $X_\eta \cup (\tilde{Z}_\eta \cap \max(\tilde{Z}_\eta))$ would be a cover of $X_\eta \cup Z'$ which is contained in β''.
- $\beta_{p,1} = \sigma_p^\star \in \mathbb{E}$. Then $\boldsymbol{\beta}'$ is maximal, implying that $l = p$ and $\beta' = \beta$. Note that by Lemma 4.5 and in awareness of the remark following Definition 5.1

  $$\beta = \sup\{\mathrm{o}(\boldsymbol{\beta}'[\zeta]) \mid 0 < \zeta < \beta_{p,1}\}.$$

  By the i.h. and using Lemma 7.2 we see that β is a successor-$<_2$-successor of its greatest $<_2$-predecessor $\mathrm{o}_{i,j+1}(\boldsymbol{\beta})$ where $(i, j) := p^\star$. Clearly, $\mathrm{o}_{i,j+1}(\boldsymbol{\beta}) \in X_\eta$. Accordingly,

  $$Z_\eta := \{\beta\}$$

  has the requested property.
- $\beta_{p,1} =_{\mathrm{NF}} \zeta + \sigma_{p,1}$ where $\zeta, \sigma_{p,1} > 1$. Since $\zeta + 1, \sigma_{p,1} + 1 < \beta_{p,1}$ we can apply the i.h. to $\beta'' := \mathrm{o}(\boldsymbol{\beta}'[\zeta + 1])$ and $\beta''' := \mathrm{o}(\boldsymbol{\beta}'[\sigma_{p,1} + 1])$, obtaining sets Z' and Z'' according to the claim, respectively. We then set

  $$Z_\eta := Z' \cup \left(\beta'' + \left(-\eta + Z''\right)\right).$$

  Z_η has the desired property due to the fact that

  $$\beta''' \quad \cong \quad \eta + 1 \cup [\beta'', \alpha)$$

  which in turn follows from the i.h. Clearly, we exploit the i.h. regarding β'' in order to see that a hypothetical cover of $X_\eta \cup Z_\eta$ would imply the existence of a cover of $X_\eta \cup Z''$.

- Otherwise. Then $\beta_{p,1} = \sigma_{p,1} \notin \mathbb{E}_1$, and we have $k_p = 1$, $(p+1, 1) \in \mathrm{dom}(\boldsymbol{\beta})$, and

$$0 < \beta_{p+1,1} = \log((1/\sigma_p^\star) \cdot \sigma_{p,1}) < \sigma_{p,1}.$$

By the i.h. applied to $\boldsymbol{\beta}'[\beta_{p+1,1} + 1]$ we obtain a set $Z' \subseteq (\eta, \mathrm{o}(\boldsymbol{\beta}'[\beta_{p+1,1} + 1]))$ according to the claim. We set

$$Z_\eta := \{\beta'\} \cup \left(\beta' + (-\eta + Z')\right).$$

Arguing toward contradiction let us assume there were a set $\tilde{Z}_\eta \subseteq (\eta, \beta)$ with $X_\eta < \tilde{Z}_\eta$ such that $X_\eta \cup \tilde{Z}_\eta \subseteq \beta$ is a cover of $X_\eta \cup Z_\eta$. Since by the i.h. $\beta' \le_1 \beta$, thus $\beta' \le_1 Z_\eta$ and hence $\mu := \min(\tilde{Z}_\eta) \le_1 \tilde{Z}_\eta$, using criterion 7.1 we find cofinally many copies of $\tilde{Z}_\eta$ below β'. We may therefore assume that $\tilde{Z}_\eta \subseteq (\eta, \beta')$ and moreover for some $\nu \in (0, \beta_{p,1})$ such that $\nu \ge \sigma_p^\star$ and $\log((1/\sigma_p^\star) \cdot \nu) < \beta_{p+1,1}$ (clearly satisfying $\boldsymbol{\beta}'[\nu] \in \mathrm{TC}$)

$$\tilde{Z}_\eta^- := \tilde{Z}_\eta - \{\mu\} \subseteq \left(\beta''', \beta''\right)$$

where $\beta''' := \mathrm{o}(\boldsymbol{\beta}'[\nu])$ and $\beta'' := \mathrm{o}(\boldsymbol{\beta}'[\nu + 1])$. Setting

$$\tilde{Z}' := \eta + (-\beta''' + \tilde{Z}_\eta^-)$$

and using that due to the i.h. we have

$$\eta + 1 \cup \left(\beta''', \beta''\right) \cong \eta + (-\beta''' + \beta'')$$

we obtain a cover $X_\eta \cup \tilde{Z}'$ of $X_\eta \cup Z'$ with $X_\eta < \tilde{Z}'$ and $X_\eta \cup \tilde{Z}' \subseteq \mathrm{o}(\boldsymbol{\beta}'[\beta_{p+1,1}])$, which contradicts the i.h.

Now, in the case $\delta = \eta$ we are done, choosing $Z := Z_\eta$. Let us therefore assume that $\delta < \eta$. We claim that for every index pair $(i, j) \in \mathrm{dom}(\boldsymbol{\beta})$ with $(n-1, m_{n-1}) \le_{\mathrm{lex}} (i, j) <_{\mathrm{lex}} (p, 1)$, setting for convenience $\eta_{i,j} := \mathrm{o}_{i,j}(\boldsymbol{\beta})$, there is $Z_{i,j} \subseteq_{\mathrm{fin}} (\eta_{i,j}, \alpha)$ such that, setting $X_{i,j} := \mathrm{Pred}_2(\eta_{i,j}) \cup \{\eta_{i,j}\}$, there does not exist any cover $X_{i,j} \cup \tilde{Z}_{i,j}$ of $X_{i,j} \cup Z_{i,j}$ with $X_{i,j} < \tilde{Z}_{i,j}$ and $X_{i,j} \cup \tilde{Z}_{i,j} \subseteq \beta$. This is shown by induction on the finite number of 1-step extensions from $\boldsymbol{\beta}_{\restriction(i,j)}$ to $\boldsymbol{\beta}'$. The initial step where $(i, j) = (p-1, k_{p-1})$ and $\eta_{i,j} = \eta$ has been shown above. Now assume $(i, j) <_{\mathrm{lex}} (p-1, k_{p-1})$ and let $(s, t) := (i, j)^+$. Let $X_{s,t} := \mathrm{Pred}_2(\eta_{s,t}) \cup \{\eta_{s,t}\}$ and $Z_{s,t} \subseteq (\eta_{s,t}, \alpha)$ be according to the i.h. The i.h. provides us with knowledge of the $<_i$-predecessors of $\eta_{s,t}$ $(i = 1, 2)$, which in turn is in $\le_1$-relation with any element in $Z_{s,t}$. We consider cases regarding (s, t).

- If $(s, t) = (i, j + 1)$, letting $\sigma := \sigma_{i,j}$ and $\sigma' := \sigma'_{i,j}$ we have $\beta_{i,j+1} = \mu_\sigma$. The i.h. applied to[8] $\mathrm{o}(\boldsymbol{\beta}_{\restriction(i,j)}{}^\frown(\bar{\sigma} + 1))$ yields a set $Z_{\bar{\sigma}} \subseteq (\eta_{i,j}, \mathrm{o}(\boldsymbol{\beta}_{\restriction(i,j)}{}^\frown(\bar{\sigma} + 1)))$ according to the claim. We now define

$$Z_{i,j} := \{\eta_{s,t}\} \cup (\eta_{s,t} + (-\eta_{i,j} + Z_{\bar{\sigma}})) \cup \{\mathrm{o}(\boldsymbol{\beta}_{\restriction(s,t)}{}^\frown(\sigma))\} \cup Z_{s,t}$$

and assume that there were a cover $X_{i,j} \cup \tilde{Z}_{i,j}$ of $X_{i,j} \cup Z_{i,j}$ with $X_{i,j} < \tilde{Z}_{i,j}$ and $X_{i,j} \cup \tilde{Z}_{i,j} \subseteq \beta$. Notice that since $\eta_{s,t} <_2 \mathrm{o}(\boldsymbol{\beta}_{\restriction(s,t)}{}^\frown(\sigma))$ the image $\mu := \min(\tilde{Z}_{i,j})$ of $\eta_{s,t}$ must have a $<_2$-successor and therefore, by the i.h. and Lemma 7.5, a tracking chain ending with a limit ν-index. Using criterion 7.1 the assumption can be fortified to assuming (noticing that we have $\kappa_\sigma^{\tilde{\sigma}'} = \tilde{\sigma}$ and $\mathrm{tc}(\eta_{i,j} + \tilde{\sigma}) = \boldsymbol{\beta}_{\restriction i-1}{}^\frown(\beta_{i,1}, \ldots, \beta_{i,j}, 1)$)

$$\tilde{Z}_{i,j} \subseteq [\eta_{i,j} + \kappa_\zeta^{\tilde{\sigma}'}, \eta_{i,j} + \kappa_{\zeta+1}^{\tilde{\sigma}'}) =: I$$

for a least ζ, which using the i.h. can easily be seen to satisfy $\zeta \in \mathbb{E} \cap (\bar{\sigma}, \sigma)$ and

$$\mu <_1 \mathrm{o}(\mathrm{me}(\boldsymbol{\beta}_{\restriction(i,j)}{}^\frown(\zeta))) = \eta_{i,j} + \kappa_\zeta^{\tilde{\sigma}'} + \mathrm{dp}_{\tilde{\sigma}'}(\zeta) = \max(\tilde{Z}_{i,j}).$$

The minimality of ζ moreover allows us to assume that $\mathrm{o}(\boldsymbol{\beta}_{\restriction(i,j)}{}^\frown(\zeta, \nu)) \le_2 \mu$ for some index $\nu \le \mu_\zeta$ for the following reasons: In case of $\mu < \mathrm{o}(\boldsymbol{\beta}_{\restriction(i,j)}{}^\frown(\zeta, \mu_\zeta))$ there is a least $\nu > 0$ such that $\mu <_1 \mathrm{o}(\boldsymbol{\beta}_{\restriction(i,j)}{}^\frown(\zeta, \nu+1))$, and by the i.h. we have (making use of Lemma 5.5) $\mathrm{o}(\boldsymbol{\beta}_{\restriction(i,j)}{}^\frown(\zeta, \nu)) \le_2 \mathrm{pred}_1(\mathrm{o}(\boldsymbol{\beta}_{\restriction(i,j)}{}^\frown(\zeta, \nu+1)))$. If on the other hand $\mu \ge \mathrm{o}(\boldsymbol{\beta}_{\restriction(i,j)}{}^\frown(\zeta, \mu_\zeta))$ the assumption $\mathrm{o}(\boldsymbol{\beta}_{\restriction(i,j)}{}^\frown(\zeta, \mu_\zeta)) \not\le_2 \mu$ would imply, using the i.h. regarding $\le_2$-predecessors of μ, that there is a least $q > i$ such that $\mathrm{o}_{q,1}(\mathrm{me}(\boldsymbol{\beta}_{\restriction(i,j)}{}^\frown(\zeta))) <_1 \mu$ with a corresponding index ρ such that $\mathrm{end}(\rho) < \zeta$ – contradicting the minimality of ζ. We may furthermore strengthen the assumption $\mathrm{o}(\boldsymbol{\beta}_{\restriction(i,j)}{}^\frown(\zeta, \nu)) \le_2 \mu$ for some index $\nu \le \mu_\zeta$ to actual equality of μ and $\mathrm{o}(\boldsymbol{\beta}_{\restriction(i,j)}{}^\frown(\zeta, \nu))$ since it is easy to check that this still results in a cover of $X_{i,j} \cup Z_{i,j}$ with the assumed properties.

Since $\zeta \in (\bar{\sigma}, \sigma)$, setting $\varphi := \pi_{\zeta,\sigma}^{-1}$ we have $\varphi(\lambda_\zeta) < \lambda_\sigma$ (cf. Lemma 8.2 of [10]) and $\varphi(\mu_\zeta) \le \mu_\sigma$ by Lemma 3.8. The vectors in the $<_{\mathrm{TC}}$-segment $\mathrm{tc}[I]$ of TC have a form

$$\boldsymbol{\iota} = \boldsymbol{\beta}_{\restriction(i,j)}{}^\frown(\boldsymbol{\zeta}, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_g)$$

[8] Here we use the $\bar{\cdot}$ operator in the context $\mathrm{T}^{\sigma'}$, hence $\bar{\sigma} \ge \sigma'$.

where $\boldsymbol{\zeta} = (\zeta, \zeta_1, \ldots, \zeta_h)$ with $g, h \geq 0$. Let

$$\boldsymbol{\zeta}' := \begin{cases} (\beta_{i,1}, \ldots, \beta_{i,j}, 1) & \text{if } h = 0 \\ (\beta_{i,1}, \ldots, \beta_{i,j}, 1 + \varphi(\zeta_1), \varphi(\zeta_2), \ldots, \varphi(\zeta_h)) & \text{otherwise.} \end{cases}$$

Let $g_0 \in \{1, \ldots, g\}$ be minimal such that $\xi_{g_0,1} < \zeta$ if that exists, and $g_0 = g + 1$ otherwise. We can now define the base transformation of $\boldsymbol{\iota}$ by

$$t(\boldsymbol{\iota}) := \boldsymbol{\beta}_{\restriction i-1}{}^\frown \left(\boldsymbol{\zeta}', \varphi(\boldsymbol{\xi}_1), \ldots, \varphi(\boldsymbol{\xi}_{g_0-1}), \boldsymbol{\xi}_{g_0}, \ldots, \boldsymbol{\xi}_g\right).$$

In order to clarify the definition note that $t(\boldsymbol{\iota}) = \boldsymbol{\beta}_{\restriction i-1}{}^\frown \left(\boldsymbol{\zeta}'\right)$ in case of $g = 0$. The part $\left(\boldsymbol{\xi}_{g_0}, \ldots, \boldsymbol{\xi}_g\right)$, which is empty in case of $g_0 = g + 1$, refers to the addition of a parameter below $\mathrm{o}(\boldsymbol{\beta}_{\restriction(i,j)}{}^\frown(\zeta))$ which is the reason why the relevant indices are not subject to base transformation. It is easy to see that $t(\boldsymbol{\iota}) \in \mathrm{TC}$ and therefore

$$t : \mathrm{tc}[I] \to \mathrm{TC}, \quad \text{with } \mathrm{o}[\mathrm{Im}(t)] \subseteq [\eta_{i,j} + \tilde{\sigma}, \beta).$$

Using t and applying the i.h. in combination with the commutativity of φ with all operators acting on the indices, as shown in 7.10 of [10] and Section 3, we obtain

$$\eta_{i,j} + 1 \cup I \;\cong\; \eta_{i,j} + 1 \cup \mathrm{o}[\mathrm{Im}(t)]$$

since thanks to $\sigma' < \zeta < \sigma$ it is easy to see that $\eta_{i,j} + \kappa^{\tilde{\sigma}'}_{\zeta}$ and $\eta_{i,j} + \tilde{\sigma}$ have the same greatest $<_2$-predecessor (which then is less than or equal to $\eta_{i,j}$) unless both are $\leq_2$-minimal. The set $\tilde{\tilde{Z}}_{i,j} := \mathrm{o} \circ t \circ \mathrm{tc}[\tilde{Z}_{i,j}]$ therefore gives rise to another cover of $X_{i,j} \cup Z_{i,j}$ with the assumed properties. We have $\min(\tilde{\tilde{Z}}_{i,j}) = \mathrm{o}(\boldsymbol{\beta}_{\restriction(s,t)}[\varphi(\nu)])$, corresponding to $\mu = \mathrm{o}(\boldsymbol{\beta}_{\restriction(i,j)}{}^\frown(\zeta, \nu))$. In the case $\varphi(\nu) < \mu_\sigma = \beta_{s,t}$, thanks to criterion 7.1 we firstly may assume that $\tilde{\tilde{Z}}_{i,j}$ is contained in the interval

$$[\mathrm{o}(\boldsymbol{\beta}_{\restriction(s,t)}[\varphi(\nu)]), \mathrm{o}(\boldsymbol{\beta}_{\restriction(s,t)}[\varphi(\nu) + 1])) =: J,$$

and finally we may as well assume that $\tilde{\tilde{Z}}_{i,j} \subseteq [\eta_{s,t}, \beta)$ since otherwise, as seen directly from the i.h., we exploit the isomorphism

$$\eta_{i,j} + 1 \cup J \;\cong\; \eta_{i,j} + 1 \cup (\eta_{s,t} + (-\mathrm{o}(\boldsymbol{\beta}_{\restriction(s,t)}[\varphi(\nu)]) + J))$$

which shifts J into the interval $[\eta_{s,t}, \beta)$. We have now transformed the originally assumed cover $X_{i,j} \cup \tilde{Z}_{i,j}$ to a cover $X_{i,j} \cup \tilde{\tilde{Z}}_{i,j}$ of $X_{i,j} \cup Z_{i,j}$ which fixes $\eta_{s,t} = \min(\tilde{\tilde{Z}}_{i,j})$ and still has the assumed property $X_{i,j} \cup \tilde{\tilde{Z}}_{i,j} \subseteq \beta$.

Now, defining $\tilde{Z}_{s,t}$ to be the subset corresponding to $Z_{s,t}$ in $\tilde{\tilde{Z}}_{i,j}$ we obtain a cover $X_{s,t} \cup \tilde{Z}_{s,t}$ of $X_{s,t} \cup Z_{s,t}$ that satisfies $X_{s,t} < \tilde{Z}_{s,t}$ and $X_{s,t} \cup \tilde{Z}_{s,t} \subseteq \beta$. Contradiction.

- If otherwise $(s, t) = (i + 1, 1)$ then we have $\beta_{s,t} = \sigma_{s,t} \in \mathbb{E}^{>\sigma^\star_s}$ (by the minimality of p) and $(s, t)^+ = (i + 1, 2)$ with $\beta_{i+1,2} = \mu_{\sigma_{s,t}}$. We define

  $$Z_{i,j} := Z_{s,t}$$

  and assume there were a cover $X_{i,j} \cup \tilde{Z}_{i,j}$ of $X_{i,j} \cup Z_{i,j}$ with $X_{i,j} < \tilde{Z}_{i,j}$ and $X_{i,j} \cup \tilde{Z}_{i,j} \subseteq \beta$. By the i.h. and, if necessary, an application of criterion 7.1, we may assume that $\tilde{Z}_{i,j} \subseteq (\eta_{i,j}, \eta_{s,t})$. The i.h. shows that we have the following isomorphism

  $$\eta_{s,t} \;\cong\; \eta_{i,j} + 1 \cup (\eta_{s,t}, \mathrm{o}(\boldsymbol{\beta}_{\restriction i}{}^\frown(\beta_{s,t}, 1))),$$

  which shows that defining $\tilde{\tilde{Z}}_{i,j} := \eta_{s,t} + (-\eta_{i,j} + \tilde{Z}_{i,j})$ we obtain another cover $X_{i,j} \cup \tilde{\tilde{Z}}_{i,j}$ of $X_{i,j} \cup Z_{i,j}$ with the assumed properties. We now claim that $X_{s,t} \cup \tilde{\tilde{Z}}_{i,j}$ is a cover of $X_{s,t} \cup Z_{s,t}$ with $X_{s,t} < \tilde{\tilde{Z}}_{i,j}$ and $X_{s,t} \cup \tilde{\tilde{Z}}_{i,j} \subseteq \beta$, contradicting the i.h. Indeed, we have $\eta_{s,t} <_1 Z_{i,j}, \tilde{\tilde{Z}}_{i,j}$ and $\eta_{s,t} \not\leq_2 \nu$ for any $\nu \in Z_{i,j} \cup \tilde{\tilde{Z}}_{i,j}$, and for any ν such that $\nu <_2 \eta_{s,t}$ we have $\nu \leq \eta_{i,j}$ and either $\nu = \eta_{i,j}$, which belongs to $X_{i,j}$, or $\nu < \eta_{i,j}$, implying that $\nu <_2 \eta_{i,j}$ and hence also $\nu \in X_{i,j}$.

This finishes the proof of part (d). Assuming $n > 1$ from now on we show using criterion 7.1 that $\delta <_1 \alpha$ as claimed in part (a). Let finite sets $X \subseteq \delta$ and $Y \subseteq [\delta, \alpha)$ be given. Without loss of generality we may assume that $\delta \in Y$. We are going to define a set $\tilde{Y}$ such that $X < \tilde{Y} < \delta$ and $X \cup \tilde{Y} \cong X \cup Y$, distinguishing between two cases, the second of which will require base transformation.

Subcase 1.1.1: $m_{n-1} = 1$. Since $\alpha_{n,1} < \varrho_{n-1} = \log((1/\tau^\star_{n-1}) \cdot \tau_{n-1,1}) + 1$ we see that $\alpha_{n-1,1}$ is a limit of ordinals $\eta < \alpha_{n-1,1}$ such that $\log((1/\tau^\star_{n-1}) \cdot \mathrm{end}(\eta)) \geq \xi$. Now choose such an index η large enough so that $\eta > \alpha_{n-1,1} \dot{-} \mathrm{end}(\alpha_{n-1,1})$, $\tau^\star_{n-1} \leq \mathrm{end}(\eta) < \tau_{n-1,1}$, and $X < \mathrm{o}(\boldsymbol{\alpha}_{\restriction n-1,1}[\eta]) =: \gamma$. Notice using the i.h. that γ and δ have the same $<_i$-predecessors ($i = 1, 2$). We will define a translation mapping t in terms of tracking chains resulting in an isomorphic copy of the interval

$[\delta,\beta]$ starting from γ. The tracking chain of an ordinal $\zeta\in[\delta,\beta]$ has a form $\boldsymbol{\iota}:=\boldsymbol{\alpha}_{\restriction(n-1,1)}{}^\frown\boldsymbol{\zeta}$ where $\boldsymbol{\zeta}=(\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_g)$, $g\ge 0$, and $\boldsymbol{\zeta}_i=(\zeta_{i,1},\ldots,\zeta_{i,w_i})$ for $1\le i\le g$. Let

$$\boldsymbol{\zeta}':=\begin{cases}((\eta,1),\boldsymbol{\zeta}_2,\ldots,\boldsymbol{\zeta}_g) & \text{if } g>0\ \&\ \zeta_{1,1}=\mathrm{end}(\eta)\in\mathbb{E}^{>\tau^\star_{n-1}}\ \&\ w_1=1\\ ((\eta,1+\zeta_{1,2},\zeta_{1,3},\ldots,\zeta_{1,w_1}),\boldsymbol{\zeta}_2,\ldots,\boldsymbol{\zeta}_g) & \text{if } g>0\ \&\ \zeta_{1,1}=\mathrm{end}(\eta)\in\mathbb{E}^{>\tau^\star_{n-1}}\ \&\ w_1>1\\ ((\eta),\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_g) & \text{otherwise}\end{cases}$$

where the first two cases take care of Condition 5 of Definition 5.1 since the situation $\mathrm{end}(\eta)\in\mathbb{E}^{>\tau^\star_{n-1}}$ cannot be avoided, and define

$$t(\boldsymbol{\iota}):=\boldsymbol{\alpha}_{\restriction(n-2,m_{n-2})}{}^\frown\boldsymbol{\zeta}'.$$

The mapping t gives rise to the translation mapping $\mathrm{o}\circ t\circ\mathrm{tc}:[\delta,\beta]\to[\gamma,\gamma+\kappa^{\bar\tau_{n,0}}_\xi+\mathrm{dp}_{\bar\tau_{n,0}}(\xi)]$, and by the i.h. we have

$$[0,\gamma+\kappa^{\bar\tau_{n,0}}_\xi+\mathrm{dp}_{\bar\tau_{n,0}}(\xi)]\ \cong\ [0,\gamma)\cup[\delta,\beta].$$

This shows that in order to obtain $X\cup\tilde Y\cong X\cup Y$ we may choose

$$\tilde Y:=\gamma+(-\delta+Y).$$

Subcase 1.1.2: $m_{n-1}>1$. Let $\sigma:=\tau_{n-1,m_{n-1}-1}$ and $\sigma':=\tau'_{n-1,m_{n-1}-1}$. If $\alpha_{n-1,m_{n-1}}\in\mathrm{Lim}$ let α' be a successor ordinal below it, large enough to satisfy $\mathrm{o}(\boldsymbol{\alpha}_{\restriction n-1}[\alpha'])>X$, otherwise let $\alpha':=\alpha_{n-1,m_{n-1}}\dot-1$. Notice that we have $\rho_{n-1}\ge\sigma$ and $\xi<\lambda_\sigma$. We consider the following subcases:

- $\xi<\sigma$. Here we can argue comfortably as in the treatment of Subcase 1.1.1, however, in the special case where $\chi^\sigma(\alpha')=1$ consider $\boldsymbol{\gamma}:=\mathrm{me}(\boldsymbol{\alpha}_{\restriction n-1}[\alpha'])$. Using Corollary 5.6 and part (e) of Lemma 5.12 we know that $\mathrm{ec}(\boldsymbol{\gamma})$ exists and is of a form $\sigma\cdot(\zeta+1)$ for some ζ as well as that the maximal extension of $\boldsymbol{\alpha}_{\restriction n-1}[\alpha']$ to $\boldsymbol{\gamma}$ does not add epsilon bases (in the sense of Definition 5.1) between σ' and σ. In the cases where $\chi^\sigma(\alpha')=0$ we set $\boldsymbol{\gamma}:=\boldsymbol{\alpha}_{\restriction n-1}[\alpha']$. Clearly, σ is a limit of ordinals η such that $\log((1/\sigma')\cdot\mathrm{end}(\eta))=\xi+1$, which guarantees that $\mathrm{end}(\eta)>\sigma'$, and η can be chosen large enough so that setting

$$\nu:=\begin{cases}\sigma\cdot\zeta+\eta & \text{if } \chi^\sigma(\alpha')=1\\ \varrho^\sigma_{\alpha'}+\eta & \text{if } \alpha'\in\mathrm{Lim}\ \&\ \chi^\sigma(\alpha')=0\\ \eta & \text{otherwise}\end{cases}$$

  we obtain $X<\mathrm{o}(\boldsymbol{\gamma}^\frown(\nu))=:\tilde\delta$. Observe that by the i.h. $\tilde\delta$ and δ then have the same $<_2$-predecessors and the same $<_1$-predecessors below $\tilde\delta$. The i.h. shows that

$$\tilde\delta+\kappa^{\tilde\sigma}_\xi+\mathrm{dp}_{\tilde\sigma}(\xi)+1\ \cong\ \tilde\delta\cup[\delta,\beta]$$

  whence choosing

$$\tilde Y:=\tilde\delta+(-\delta+Y)$$

  satisfies our needs.
- $\xi\ge\sigma$. Then we consequently have $\alpha_{n-1,m_{n-1}}\in\mathrm{Lim}$, $\sigma\in\mathrm{Lim}(\mathbb{E})$, and according to Lemma 8.1 of [10] σ is a limit of $\rho\in\mathbb{E}$ with $\varphi(\lambda^{\sigma'}_\rho)\ge\xi$ where $\varphi:=\pi^{-1}_{\rho,\sigma}$. Note that for any $y\in Y$ the tracking chain $\mathrm{tc}(y)$ is an extension of $\mathrm{tc}(\delta)$, and is of a form

$$\mathrm{tc}(y)=\boldsymbol{\alpha}_{\restriction n-2}{}^\frown(\alpha_{n-1,1},\ldots,\alpha_{n-1,m_{n-1}},\zeta^y_{0,1},\ldots,\zeta^y_{0,k_0(y)})^\frown\boldsymbol{\zeta}^y$$

  where $k_0(y)\ge 0$, $\boldsymbol{\zeta}^y=(\boldsymbol{\zeta}^y_1,\ldots,\boldsymbol{\zeta}^y_{r(y)})$, $r(y)\ge 0$, and $\boldsymbol{\zeta}^y_i=(\zeta^y_{i,1},\ldots,\zeta^y_{i,k_i(y)})$ with $k_i(y)\ge 1$ for $i=1,\ldots,r(y)$. Notice that $k_0(y)>0$ implies that $\tau_{n-1,m_{n-1}}\in\mathbb{E}^{>\sigma}$ and $\xi\ge\tau_{n-1,m_{n-1}}$. We now define $r_0(y)\in\{1,\ldots,r(y)\}$ to be minimal such that $\zeta^y_{r_0(y),1}<\sigma$ if that exists, and $r_0(y):=r(y)+1$ otherwise. For convenience let $\zeta^y_{r(y)+1,1}:=0$. Using Lemma 8.1 of [10] we may choose an epsilon number $\rho\in(\sigma',\sigma)$ satisfying $\xi\in\mathrm{T}^\sigma[\rho]$ and $\lambda_\rho\ge\pi(\xi)$, where $\pi:=\pi_{\rho,\sigma}$, large enough so that

$$\zeta^y_{r_0(y),1},\ \zeta^y_{i,j}\in\mathrm{T}^\sigma[\rho]$$

  for every $y\in Y$, every $i\in[0,r_0(y))$, and every $j\in\{1,\ldots,k_i(y)\}$. We may now map δ to $\tilde\delta:=\mathrm{o}(\boldsymbol{\alpha}_{\restriction n-1}[\alpha']^\frown(\rho,\mu_\rho))$, easily verifying using the i.h. that δ and $\tilde\delta$ have the same $<_2$-predecessors in Ord and the same $<_1$-predecessors in X.

The additional requirement $\rho > \bar{\sigma}$ yields the bounds $\varphi(\lambda_\rho) < \lambda_\sigma$ (cf. Lemma 8.2 of [10]) and $\varphi(\mu_\rho) \le \mu_\sigma$ by Lemma 3.8. Let $\tilde{\rho} := \kappa_\rho^{\tilde{\sigma}'}$ and

$$I := [\tilde{\delta}, \tilde{\delta} + \kappa^{\tilde{\rho}}_{\pi(\xi)} + \mathrm{dp}_{\tilde{\rho}}(\pi(\xi))].$$

As in the proof of part (d) we are now going to define

$$t : \mathrm{tc}[I] \to \mathrm{TC}, \quad \text{with } \mathrm{o}[\mathrm{Im}(t)] \subseteq [\delta, \alpha)$$

as follows. Any tracking chain $\boldsymbol{\iota} \in \mathrm{tc}[I]$ has a form

$$\boldsymbol{\alpha}_{\restriction n-1}[\alpha']^\frown\left((\rho, \mu_\rho, \zeta_{0,1}, \dots, \zeta_{0,k_0}), \boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_r\right)$$

where $k_0 \ge 0$, $r \ge 0$, and $\boldsymbol{\zeta}_i = (\zeta_{i,1}, \dots, \zeta_{i,k_i})$ with $k_i \ge 1$ for $1 \le i \le r$. Let $r_0 \in \{1, \dots, r\}$ be minimal such that $\zeta_{r_0,1} < \rho$ if that exists and $r_0 := r + 1$ otherwise. We distinguish between two groups of cases, as follows:

1. $k_0 = 0$: If $k_1 > 0$ and $\varphi(\zeta_{1,1}) = \tau_{n-1,m_{n-1}} \in \mathbb{E}^{>\sigma}$ we define the auxiliary vector

$$\boldsymbol{\zeta}'_1 := \begin{cases} (1) & \text{if } k_1 = 1 \\ (1 + \varphi(\zeta_{1,2}), \varphi(\zeta_{1,3}), \dots, \varphi(\zeta_{1,k_1})) & \text{otherwise} \end{cases}$$

and then define

$$t(\boldsymbol{\iota}) := \boldsymbol{\alpha}_{\restriction n-2}{}^\frown\left(\boldsymbol{\alpha}_{n-1}{}^\frown\boldsymbol{\zeta}'_1, \varphi(\boldsymbol{\zeta}_2), \dots, \varphi(\boldsymbol{\zeta}_{r_0-1}), \boldsymbol{\zeta}_{r_0}, \dots, \boldsymbol{\zeta}_r\right),$$

whereas otherwise we smoothly set

$$t(\boldsymbol{\iota}) := \boldsymbol{\alpha}_{\restriction n-1}{}^\frown\left(\varphi(\boldsymbol{\zeta}_1), \dots, \varphi(\boldsymbol{\zeta}_{r_0-1}), \boldsymbol{\zeta}_{r_0}, \dots, \boldsymbol{\zeta}_r\right).$$

2. $k_0 > 0$: Clearly, this can occur only if $\mu_\rho \in \mathbb{E}^{>\rho}$. If $\varphi(\mu_\rho) = \tau_{n-1,m_{n-1}} \in \mathbb{E}^{>\sigma}$ we smoothly define

$$t(\boldsymbol{\iota}) := \boldsymbol{\alpha}_{\restriction n-2}{}^\frown\left(\boldsymbol{\alpha}_{n-1}{}^\frown\varphi(\boldsymbol{\zeta}_0), \varphi(\boldsymbol{\zeta}_1), \dots, \varphi(\boldsymbol{\zeta}_{r_0-1}), \boldsymbol{\zeta}_{r_0}, \dots, \boldsymbol{\zeta}_r\right),$$

whereas otherwise

$$t(\boldsymbol{\iota}) := \boldsymbol{\alpha}_{\restriction n-1}{}^\frown\left(\boldsymbol{\zeta}'_0, \varphi(\boldsymbol{\zeta}_1), \dots, \varphi(\boldsymbol{\zeta}_{r_0-1}), \boldsymbol{\zeta}_{r_0}, \dots, \boldsymbol{\zeta}_r\right)$$

where

$$\boldsymbol{\zeta}'_0 := \begin{cases} (\varphi(\mu_\rho)) & \text{if } \zeta_{0,1} = 1 \\ (\varphi(\mu_\rho), -1 + \varphi(\zeta_{0,1}), \varphi(\zeta_{0,2}), \dots, \varphi(\zeta_{0,k_0})) & \text{otherwise.} \end{cases}$$

By our choice of ρ we now have $Y \subseteq \mathrm{o}[\mathrm{Im}(t)]$, and defining

$$\tilde{Y} := \mathrm{o} \circ t^{-1} \circ \mathrm{tc}[Y]$$

we obtain the desired copy of Y, since using the i.h. it is easy to check that, setting $A := \mathrm{o}(\boldsymbol{\alpha}_{\restriction n-1}[\alpha']) + 1$,

$$A \cup I \;\cong\; A \cup \mathrm{o}[\mathrm{Im}(t)].$$

Subcase 1.2: $\alpha_{n,1} \in \mathrm{Lim}$.

In the case $n = 1$ using Lemma 4.5 we have

$$\alpha = \sup\{\mathrm{o}(((\xi))) \mid \xi < \alpha_{1,1}\}$$

which by the i.h. is a proper supremum of $<_1$-minimal ordinals. Hence α is $\le_i$-minimal for $i = 1, 2$ and we are done.

Let us now assume that $n > 1$, and let $\delta := \mathrm{o}_{n-1,m_{n-1}}(\boldsymbol{\alpha})$. By Lemma 4.5 we have

$$\alpha = \sup\{\mathrm{o}(\boldsymbol{\alpha}[\xi]) \mid 0 < \xi < \alpha_{n,1} \;\&\; \boldsymbol{\alpha}[\xi] \in \mathrm{TC}\},$$

and part (a) of the claim for α follows from the i.h. applied to the $\mathrm{o}(\boldsymbol{\alpha}[\xi])$ for $\xi \in (0, \alpha_{n,1})$ such that $\boldsymbol{\alpha}[\xi] \in \mathrm{TC}$. δ is therefore the greatest $<_1$-predecessor of α.

We now turn to the proof of part (b). In the case $\tau^\star_n = 1$ we have to show that α is $\le_2$-minimal. Arguing toward contradiction let us assume that there exists γ such that $\gamma <_2 \alpha$. Then clearly $\gamma \le_2 \delta$ and hence $\mathrm{tc}(\gamma)$ is seen to be a proper initial chain of $\boldsymbol{\alpha}$, say $\gamma = \mathrm{o}_{i,j+1}(\boldsymbol{\alpha})$ for some i, j such that $(i, j+1) \in \mathrm{dom}(\boldsymbol{\alpha})$ and $i < n$. Due to the i.h. and Lemma 7.5 we know that $m_{n-1} > 1$ in case of $\gamma = \delta$. In case of $\tau_{n,1} < \alpha_{n,1}$ let η be such that $\alpha_{n,1} =_{\mathrm{NF}} \eta + \tau_{n,1}$, otherwise let $\eta := 0$. Let $\beta := \delta + \kappa_\eta^{\tilde{\tau}_{n,0}} + \mathrm{dp}_{\tilde{\tau}_{n,0}}(\eta)$ so that $\beta + \tilde{\tau}_{n,1} = \alpha$. Notice that according to our assumptions $\tau_{i,j} > \tau_{n,1} > 1$. Let $\zeta := \mathrm{o}_{i,j}(\boldsymbol{\alpha})$. Applying part (d) of the i.h. to $X := \mathrm{Pred}_2(\zeta) \cup \{\zeta\}$ there exists a finite set $Z \subseteq (\zeta, \zeta + \kappa^{\tilde{\tau}_{i,j}}_{\tau_{n,1}+1})$ such that there is no cover $X \cup \tilde{Z}$ of $X \cup Z$ with $X < \tilde{Z}$ and $X \cup \tilde{Z} \subseteq \zeta + \tilde{\tau}_{n,1}$ (note that $\kappa^{\tilde{\tau}_{i,j}}_{\tau_{n,1}} = \tilde{\tau}_{n,1}$ and $\mathrm{dp}_{\tilde{\tau}_{i,j}}(\tau_{n,1}) = 0$). By the i.h. we know that

$$\zeta + 1 + \tilde{\tau}_{i,j} \;\cong\; \zeta + 1 \cup (\mathrm{o}(\boldsymbol{\alpha}_{\restriction(i,j+1)}[\xi]), \mathrm{o}(\boldsymbol{\alpha}_{\restriction(i,j+1)}[\xi]) + \tilde{\tau}_{i,j})$$

for every $\xi \in (0, \alpha_{i,j+1})$. In the case $\alpha_{i,j+1} \in \mathrm{Lim}$ we directly see that below γ there are cofinally many copies $\tilde{Z}_\gamma$ such that $X \cup Z \cong X \cup \tilde{Z}_\gamma$. In the case $\alpha_{i,j+1} \notin \mathrm{Lim}$ we have by the i.h. (cf. Subcase 1.1.2, $\xi < \sigma$, above) $\gamma = \mathrm{pred}_1(\gamma) + \kappa^{\tilde{\tau}_{i,j}}_{\tau_{i,j}\cdot(\nu+1)}$ for some ν and, setting $\gamma' := \mathrm{pred}_1(\gamma) + \kappa^{\tilde{\tau}_{i,j}}_{\tau_{i,j}\cdot\nu+1}$,

$$\zeta + 1 + \tilde{\tau}_{i,j} \ \cong\ \zeta + 1 \cup [\gamma', \gamma),$$

and within the interval $[\gamma', \gamma)$ we find cofinally many intervals of the form

$$(\gamma' + \kappa^{\tilde{\tau}_{i,j}}_{\lambda}, \gamma' + \kappa^{\tilde{\tau}_{i,j}}_{\lambda+\tau_{n,1}+1}]$$

where $\lambda < \tau_{i,j}$ such that $\lambda + \tau_{n,1}$ is in normal form, isomorphic to $(\zeta, \zeta + \tilde{\tau}_{n,1} + 1]$ over $\zeta + 1$. Hence in any case there are copies $\tilde{Z}_\gamma$ of Z cofinally in γ such that $X \cup Z \cong X \cup \tilde{Z}_\gamma$. By Lemma 7.6 and our assumption $\gamma <_2 \alpha$ we now obtain copies $\tilde{Z}_\alpha$ of Z cofinally below α (and hence above β) such that $X \cup Z \cong X \cup \tilde{Z}_\alpha$. The i.h. reassures us of the isomorphism

$$\zeta + 1 + \tilde{\tau}_{n,1} \ \cong\ \zeta + 1 \cup (\beta, \alpha),$$

noting that the ordinals of the interval (β, α) cannot have any $<_2$-predecessors in $(\zeta, \beta]$ and that the tracking chains of the ordinals in $(\zeta, \zeta + \tilde{\tau}_{n,1}) \cup (\beta, \alpha)$ have the proper initial chain $\boldsymbol{\alpha}_{\restriction(i,j)}$. This provides us, however, with a copy $\tilde{Z} \subseteq (\zeta, \zeta + \tilde{\tau}_{n,1})$ of Z such that $X \cup Z \cong X \cup \tilde{Z}$, contradicting our choice of X and Z, whence $\gamma <_2 \alpha$ is impossible. In the case $\tau^\star_n = 1$ the ordinal α is therefore $\le_2$-minimal.

From now on let us assume that $\tau^\star_n > 1$, and let $(i, j) \in \mathrm{dom}(\boldsymbol{\alpha})$ be such that $n^\star = (i, j)$. We have to show that $\mathrm{pred}_2(\alpha) = \mathrm{o}_{i,j+1}(\boldsymbol{\alpha}) =: \gamma$. The above argument showing the $\le_2$-minimality of α in the case $\tau^\star_n = 1$ relativizes straightforwardly to showing that α is γ-$\le_2$-minimal. The next step is to verify that $\gamma <_2 \alpha$. In the situation $\tau^\star_n < \tau_{n,1}$ the ordinal α is a limit of $<_2$-successors of γ (whose greatest $<_2$-predecessor is γ). This follows from the i.h. noticing that $\alpha_{n,1}$ is a limit of indices which are successor multiples of $\tau^\star_n$. We are left to consider the situation $\tau^\star_n = \tau_{n,1}$. Here we show $\gamma <_2 \alpha$ using criterion 7.4. To this end let $X \subseteq_{\mathrm{fin}} \gamma$ and $Y \subseteq_{\mathrm{fin}} [\gamma, \alpha)$ be given. Without loss of generality we may assume that $\gamma \in Y$. Set $\tau := \tau_{i,j}$ and $\tilde{\tau} := \tilde{\tau}_{i,j}$. Let (k, l) be the $<_{\mathrm{lex}}$-maximum index pair in $\mathrm{dom}(\boldsymbol{\alpha})$ such that $(i, j+1) <_{\mathrm{lex}} (k, l) <_{\mathrm{lex}} (n, 1)$ and $\alpha_{k+1,1} < \rho_k \dot{-} 1$ in case of $(k, l)^+ = (k + 1, 1)$ and $\tau_{k,l} < \rho_k(\boldsymbol{\alpha}_{\restriction(k,l)}) \dot{-} 1$ in case of $(k, l)^+ = (k, l + 1)$, if that exists, and $(k, l) := (i, j + 1)$ otherwise. We then have $\alpha_{s,t+1} = \mu_{\tau_{s,t}}$ whenever $(k, l) <_{\mathrm{lex}} (s, t + 1) \in \mathrm{dom}(\boldsymbol{\alpha})$ due to Corollary 5.6 since $\tau_{n,1} = \tau$ and $\boldsymbol{\alpha}$ is maximal. Moreover, we have $\boldsymbol{\alpha} = \mathrm{me}(\boldsymbol{\alpha}_{\restriction(k,l)^+})$. In case of $\tau_{k,l} < \alpha_{k,l}$ let η be such that $\alpha_{k,l} =_{\mathrm{NF}} \eta + \tau_{k,l}$, otherwise set $\eta := 0$. Let $\beta := \mathrm{o}_{k,l}(\boldsymbol{\alpha})$. For the reader's convenience we are going to discuss the following cases in full detail. Subcase 1.2.1.2 below will treat the situation where a genuinely larger $\le_2$-connectivity component arises.

Subcase 1.2.1: $(k, l) = (i, j + 1)$. Let $\varrho := \alpha_{i+1,1}$ if $(i, j + 1)^+ = (i + 1, 1)$ and $\varrho := \tau_{i,j+1}$ (which then is an epsilon number greater than τ) otherwise. Lemma 3.3 allows us to conclude $\chi^\tau(\varrho) = 1$, and by Lemma 5.12 we have

$$\alpha = \gamma + \kappa^{\tilde{\tau}}_{\varrho} + \mathrm{dp}_{\tilde{\tau}}(\varrho).$$

Let $\lambda \in \mathrm{Lim} \cup \{0\}$ and $p < \omega$ be such that $\mathrm{logend}(\alpha_{i,j+1}) = \lambda + p$. Then we have $\varrho^\tau_{\alpha_{i,j+1}} = \tau \cdot (\lambda + p \dot{-} \chi^\tau(\lambda))$. It follows from $\mathrm{end}(\alpha) = \tilde{\tau}$ that ϱ must have the form $\varrho = \tau \cdot \xi$ for some $\xi \in (0, \lambda + p \dot{-} \chi^\tau(\lambda)]$.

1.2.1.1: $\varrho < \varrho^\tau_{\alpha_{i,j+1}}$. In this case it is easy to check that $\alpha_{i,j+1}$ is a supremum of indices $\eta + \nu$ such that $\varrho \le \varrho^\tau_{\eta+\nu}$ and $\chi^\tau(\nu) = 0$: If $\chi^\tau(\lambda) = 1$ we distinguish between $p \le 1$, where we have $\varrho^\tau_{\alpha_{i,j+1}} = \tau \cdot \lambda$ and $\xi < \lambda$, and $p > 1$, where $\alpha_{i,j+1} = \sup\{\eta + \omega^{\lambda+p-1} \cdot r \mid r \in (0, \omega)\}$, $\varrho^\tau_{\alpha_{i,j+1}} = \tau \cdot (\lambda + p - 1)$, and $\varrho^\tau_{\eta+\omega^{\lambda+p-1}\cdot r} = \tau \cdot (\lambda + p - 2)$. If on the other hand $\chi^\tau(\lambda) = 0$ we have $\varrho < \tau \cdot \lambda$ in case of $p = 0$, while for $p > 0$ we again obtain $\alpha_{i,j+1} = \sup\{\eta + \omega^{\lambda+p-1} \cdot r \mid r \in (0, \omega)\}$, however with $\varrho \le \varrho^\tau_{\eta+\omega^{\lambda+p-1}\cdot r} = \tau \cdot (\lambda + p - 1)$. By the i.h. we have

$$\gamma_\nu := \mathrm{o}(\boldsymbol{\alpha}_{\restriction(i,j+1)}[\eta + \nu]) <_2 \gamma_\nu + \kappa^{\tilde{\tau}}_{\varrho} + \mathrm{dp}_{\tilde{\tau}}(\varrho)$$

and

$$\alpha_\nu := \gamma_\nu + \kappa^{\tilde{\tau}}_{\varrho} + \mathrm{dp}_{\tilde{\tau}}(\varrho) \ \cong\ \gamma_\nu \cup [\gamma, \alpha) \tag{3}$$

for the ν specified above. Choose ν as specified above large enough so that $X \subseteq \gamma_\nu$ and let Y_ν be the isomorphic copy of Y according to (3). By the i.h. we obtain a copy $\tilde{Y} \subseteq \gamma_\nu$ according to criterion 7.4. Let $\tilde{Y}^+$ with $\tilde{Y} \subseteq \tilde{Y}^+ \subseteq \gamma$ be given, and set $U := X \cup \tilde{Y}^+ \cap \gamma_\nu$, $V := \tilde{Y}^+ - \gamma_\nu$. Since by the i.h. clearly $\gamma_\nu <_1 \gamma$ we obtain a copy $\tilde{V}$ such that $U < \tilde{V} \subseteq \gamma_\nu$ and $U \cup \tilde{V} \cong U \cup V$. Setting $\tilde{Y}^+_\nu := (\tilde{Y}^+ \cap \gamma_\nu) \cup \tilde{V}$, hence $\tilde{Y} \subseteq \tilde{Y}^+_\nu \subseteq \gamma_\nu$, the criterion yields an appropriate extension $Y^+_\nu \subseteq \alpha_\nu$ such that $X \cup \tilde{Y}^+_\nu \cong X \cup Y^+_\nu$ extends $X \cup \tilde{Y} \cong X \cup Y_\nu$. Now let Y^+ be the isomorphic copy of Y^+_ν according to (3). This provides us with the extension of Y according to $\tilde{Y}^+$ as required by criterion 7.4.

1.2.1.2: $\varrho = \varrho^\tau_{\alpha_{i,j+1}}$. Recalling that we have $\chi^\tau(\varrho) = 1$ this implies $(i, j + 1)^+ = (i + 1, 1)$ according to Corollary 5.6 which also shows that here the case $p = 0$ does not occur. We now have $\alpha_{i,j+1} = \sup\{\eta + \omega^{\lambda+p-1} \cdot r \mid r \in (0, \omega)\}$, and in the case $\chi^\tau(\lambda) = 1 \,\&\, p = 1$ we have $\varrho = \varrho^\tau_{\eta+\omega^\lambda\cdot r}$, while in the remaining cases $\varrho = \varrho^\tau_{\eta+\omega^{\lambda+p-1}\cdot r} + \tau$. In case of $\chi^\tau(\lambda) = 0$ the ordinal

ϱ is equal to $\tau \cdot (\lambda + p)$ whereas it is equal to $\tau \cdot (\lambda + p - 1)$ if $\chi^\tau(\lambda) = 1$. Let $r \in (0, \omega)$ be large enough so that $X \subseteq \gamma_\nu$ where $\nu := \omega^{\lambda+p-1} \cdot r$ and $\gamma_\nu := \mathrm{o}(\boldsymbol{\alpha}_{\restriction(i,j+1)}[\eta + \nu])$. Setting $\alpha_\nu := \mathrm{o}(\boldsymbol{\alpha}_{\restriction(i,j+1)}[\eta + \nu + 1])$ we obtain $\alpha_\nu = \gamma_\nu + \kappa^{\tilde{\tau}}_{\varrho} + \mathrm{dp}_{\tilde{\tau}}(\varrho)$ by Lemma 5.12 in the case $\chi^\tau(\lambda) = 1$ & $p = 1$, while otherwise $\alpha_\nu = \gamma_\nu + \kappa^{\tilde{\tau}}_{\varrho^\tau_{\eta+\nu}} + \mathrm{dp}_{\tilde{\tau}}(\varrho^\tau_{\eta+\nu}) + \tilde{\tau} = \gamma_\nu + \kappa^{\tilde{\tau}}_{\varrho}$. Now the i.h. yields

$$\alpha_\nu \;\cong\; \gamma_\nu \cup [\gamma, \alpha), \tag{4}$$

and we choose $\tilde{Y}$ to be the isomorphic copy of Y under this isomorphism. Let $\tilde{Y}^+$ with $\tilde{Y} \subseteq \tilde{Y}^+ \subseteq \gamma$ be given. Let $U := X \cup \tilde{Y}^+ \cap \alpha_\nu$ and $V := \tilde{Y}^+ - \alpha_\nu$. Since by the i.h. we have $\alpha_\nu <_1 \gamma$ there exists $\tilde{V}$ with $U < \tilde{V} < \alpha_\nu$ and $U \cup \tilde{V} \cong U \cup V$. Now let Y^+ be the copy of $(\tilde{Y}^+ \cap \alpha_\nu) \cup \tilde{V}$ under (4). This choice satisfies the requirements of criterion 7.4.

Subcase 1.2.2: $(i, j + 1) <_{\mathrm{lex}} (k, l)$. We argue similarly as in Subcases 1.1.1 and 1.1.2 above.

1.2.2.1: $l = 1$. This subcase corresponds to Subcase 1.1.1. Here we have $(k, l)^+ = (k + 1, 1)$ and $\alpha_{k+1,1} < \rho_k \dot{-} 1 = \log((1/\tau^\star_k) \cdot \tau_{k,1})$. We see that $\alpha_{k,1}$ is a limit of ordinals $\eta + \nu < \alpha_{k,1}$ such that $\tau^\star_k < \mathrm{end}(\nu) < \tau_{k,1}$ and $\log((1/\tau^\star_k) \cdot \mathrm{end}(\nu)) \geq \alpha_{k+1,1}$, and choosing ν large enough we may assume that $Y \cap \beta \subseteq \mathrm{o}(\boldsymbol{\alpha}_{\restriction(k,1)}[\eta + \nu]) =: \beta_\nu$. Using the i.h. and setting $\alpha_\nu := \beta_\nu + \kappa^{\tilde{\tau}_{k,0}}_{\alpha_{k+1,1}} + \mathrm{dp}_{\tilde{\tau}_{k,0}}(\alpha_{k+1,1})$ we now obtain the isomorphism

$$\alpha_\nu \;\cong\; \beta_\nu \cup [\beta, \alpha) \tag{5}$$

via a mapping of the corresponding tracking chains defined similarly as in Subcase 1.1.1. In fact, since $\gamma <_2 \alpha_\nu$ by the i.h., proving that $\gamma <_2 \alpha$ shows that this isomorphism extends to the suprema, that is, mapping α_ν to α. Exploiting (5) and using that the criterion holds for γ, α_ν we can now straightforwardly show that the criterion holds for γ, α.

1.2.2.2: $l > 1$. Here we proceed in parallel with Subcase 1.1.2. Let $\xi := \alpha_{k+1,1}$ in case of $(k, l)^+ = (k + 1, 1)$ and $\xi := \tau_{k,l}$ otherwise, whence

$$\alpha = \mathrm{o}_{k,l}(\boldsymbol{\alpha}) + \kappa^{\tilde{\sigma}}_{\xi} + \mathrm{dp}_{\tilde{\sigma}}(\xi).$$

Let further $\sigma := \tau_{k,l-1}$ and $\sigma' := \tau'_{k,l-1}$. In the case $\alpha_{k,l} \in \mathrm{Lim}$ let $\alpha' \in (\eta, \alpha_{k,l})$ be a successor ordinal large enough so that $Y \cap [\mathrm{o}(\boldsymbol{\alpha}_{\restriction(k,l)}[\alpha']), \beta) = \emptyset$, otherwise let $\alpha' := \alpha_{k,l} \dot{-} 1$. Notice that we have $\rho_k(\boldsymbol{\alpha}_{\restriction(k,l)}) \geq \sigma$ and $\xi < \lambda_\sigma$.

- $\xi < \sigma$. In the special case where $\chi^\sigma(\alpha') = 1$ consider $\boldsymbol{\alpha}' := \mathrm{me}(\boldsymbol{\alpha}_{\restriction(k,l)}[\alpha'])$. Using Corollary 5.6 and part (e) of Lemma 5.12 we know that $\mathrm{ec}(\boldsymbol{\alpha}')$ exists and is of a form $\sigma \cdot (\zeta + 1)$ for some ζ as well as that the maximal extension of $\boldsymbol{\alpha}_{\restriction(k,l)}[\alpha']$ to $\boldsymbol{\alpha}'$ does not add epsilon bases between σ' and σ. In the cases where $\chi^\sigma(\alpha') = 0$ we set $\boldsymbol{\alpha}' := \boldsymbol{\alpha}_{\restriction(k,l)}[\alpha']$. Clearly, σ is a limit of ordinals ρ such that $\log((1/\sigma') \cdot \mathrm{end}(\rho)) = \xi + 1$, which guarantees that $\mathrm{end}(\rho) > \sigma'$, and ρ can be chosen large enough so that setting

  $$\nu := \begin{cases} \sigma \cdot \zeta + \rho & \text{if } \chi^\sigma(\alpha') = 1 \\ \varrho^\sigma_{\alpha'} + \rho & \text{if } \alpha' \in \mathrm{Lim} \;\&\; \chi^\sigma(\alpha') = 0 \\ \rho & \text{otherwise} \end{cases}$$

  we obtain, setting $\beta_\nu := \mathrm{o}(\boldsymbol{\alpha}'^\frown(\nu))$, $Y \cap [\beta_\nu, \beta) = \emptyset$. Observe that by the i.h. β_ν and β then have the same $<_2$-predecessors and the same $<_1$-predecessors below β_ν. The i.h. shows that

  $$\alpha_\nu := \beta_\nu + \kappa^{\tilde{\sigma}}_{\xi} + \mathrm{dp}_{\tilde{\sigma}}(\xi) \;\cong\; \beta_\nu \cup [\beta, \alpha) \quad \text{and} \quad \gamma <_2 \alpha_\nu$$

  which we can exploit to show that criterion 7.4 holds for γ, α from its validity for γ, α_ν, implying that the above isomorphism extends to mapping α_ν to α.
- $\xi \geq \sigma$. Then we consequently have $\alpha_{k,l} \in \mathrm{Lim}$, $\sigma \in \mathrm{Lim}(\mathbb{E})$, and according to Lemma 8.1 of [10] σ is a limit of $\rho \in \mathbb{E}$ with $\varphi(\lambda^{\sigma'}_{\rho}) \geq \xi$ where $\varphi := \pi^{-1}_{\rho,\sigma}$. Note that for any $y \in Y - \beta$ the tracking chain $\mathrm{tc}(y)$ is an extension of $\mathrm{tc}(\beta)$, and is of a form

  $$\mathrm{tc}(y) = \boldsymbol{\alpha}_{\restriction k-1}{}^\frown(\alpha_{k,1}, \ldots, \alpha_{k,l}, \zeta^y_{0,1}, \ldots, \zeta^y_{0,k_0(y)})^\frown \boldsymbol{\zeta}^y$$

  where $k_0(y) \geq 0$, $\boldsymbol{\zeta}^y = (\boldsymbol{\zeta}^y_1, \ldots, \boldsymbol{\zeta}^y_{r(y)})$, $r(y) \geq 0$, and $\boldsymbol{\zeta}^y_i = (\zeta^y_{i,1}, \ldots, \zeta^y_{i,k_i(y)})$ with $k_i(y) \geq 1$ for $i = 1, \ldots, r(y)$. Notice that $k_0(y) > 0$ implies that $\tau_{k,l} \in \mathbb{E}^{>\sigma}$ and $\xi \geq \tau_{k,l}$. We now define $r_0(y) \in \{1, \ldots, r(y)\}$ to be minimal such that $\zeta^y_{r_0(y),1} < \sigma$ if that exists, and $r_0(y) := r(y) + 1$ otherwise. For convenience let $\zeta^y_{r(y)+1,1} := 0$. Using Lemma 8.1 of [10] we may choose an epsilon number $\rho \in (\sigma', \sigma)$ satisfying $\xi \in \mathrm{T}^{\sigma}[\rho]$ and $\lambda_\rho \geq \pi(\xi)$, where $\pi := \pi_{\rho,\sigma}$, large enough so that

  $$\zeta^y_{r_0(y),1}, \zeta^y_{i,j} \in \mathrm{T}^{\sigma}[\rho]$$

  for every $y \in Y$, every $i \in [0, r_0(y))$, and every $j \in \{1, \ldots, k_i(y)\}$. We may now map β to $\beta_\rho := \mathrm{o}(\boldsymbol{\alpha}_{\restriction(k,l)}[\alpha']^\frown(\rho, \mu_\rho))$, easily verifying using the i.h. that β and β_ρ have the same $<_2$-predecessors in Ord and the same $<_1$-predecessors in

$X\cup(Y\cap\beta_\rho)$. The additional requirement $\rho>\bar\sigma$ yields the bounds $\varphi(\lambda_\rho)<\lambda_\sigma$ (cf. Lemma 8.2 of [10]) and $\varphi(\mu_\rho)\le\mu_\sigma$ by Lemma 3.8. Let $\tilde\rho:=\kappa^{\tilde\sigma'}_\rho$, $\alpha_\rho:=\beta_\rho+\kappa^{\tilde\rho}_{\pi(\xi)}+\mathrm{dp}_{\tilde\rho}(\pi(\xi))$, and

$$I:=[\beta_\rho,\alpha_\rho).$$

In the same way as in Subcase 1.1.2 we can now define

$$t:\mathrm{tc}[I]\to\mathrm{TC},\quad \text{with } \mathrm{o}[\mathrm{Im}(t)]\subseteq[\beta,\alpha)$$

so that, setting $A:=\mathrm{o}(\boldsymbol{\alpha}_{\restriction(k,l)}[\alpha'])+1$, by the i.h.

$$A\cup I\ \cong\ A\cup\mathrm{o}[\mathrm{Im}(t)]\quad\text{and}\quad \gamma<_2\alpha_\rho.$$

By our choice of α' we have $X\cup(Y\cap\beta)\subseteq\mathrm{o}(\boldsymbol{\alpha}_{\restriction(k,l)}[\alpha'])$, and by our choice of ρ we have $Y-\beta\subseteq\mathrm{o}[\mathrm{Im}(t)]$, hence exploiting the mapping $\mathrm{o}\circ t^{-1}\circ\mathrm{tc}$ we can now derive the validity of criterion 7.4 for γ,α from its validity for γ,α_ρ, again implying that the above isomorphism extends to mapping α_ρ to α.

Case 2: $m_n>1$.

Subcase 2.1: α_{n,m_n} is a successor ordinal, say $\alpha_{n,m_n}=\xi+1$.
Let $\tau:=\tau_{n,m_n-1}$ and $\alpha':=\mathrm{o}(\boldsymbol{\alpha}[\xi])$. We consider cases for $\chi^\tau(\xi)$:

Subcase 2.1.1: $\chi^\tau(\xi)=0$. In order to verify part (a) we have to show that $\mathrm{pred}_1(\alpha)=\alpha'$. By Lemma 4.5 we have

$$\alpha=\sup\{\mathrm{o}(\boldsymbol{\alpha}[\xi]^\frown(\varrho^\tau_\xi+\eta))\mid\eta\in(0,\tau)\},$$

which by the i.h. is a proper supremum over ordinals whose greatest $<_1$-predecessor is α'.

We now proceed to prove part (b). We first consider the special case $\xi=0$. By part (a) $\alpha'=\mathrm{o}_{n,m_n-1}(\boldsymbol{\alpha})$ is the greatest $<_1$-predecessor of α. Then if $m_n=2$ by the i.h. α' is either $\le_1$-minimal or has a greatest $<_1$-predecessor, and thus $\alpha'\not<_2\alpha$ by Lemma 7.5, as claimed. Clearly, any $<_2$-predecessor of α then must be a $<_2$-predecessor of α' as well. If $\mathrm{pred}_2(\alpha')>0$ then using the i.h. α is seen to be the supremum of $<_2$-successors of $\mathrm{pred}_2(\alpha')$ like α' itself, hence $\mathrm{pred}_2(\alpha)=\mathrm{pred}_2(\alpha')$, as claimed. If on the other hand $m_n>2$ then $\alpha'<_2\alpha$ as according to the i.h. α then is the supremum of $<_2$-successors of α', hence $\mathrm{pred}_2(\alpha)=\alpha'$, as claimed.

From now on let us assume that $\xi>0$. If ξ is a successor ordinal then by the i.h. α' has a greatest $<_1$-predecessor, is itself the greatest $<_1$-predecessor of α, and Lemma 7.5 therefore yields $\alpha'\not<_2\alpha$. In the case $m_n=2\ \&\ \tau^\star_n=1$ the $\le_2$-minimality follows then from the $\le_2$-minimality of α', while in the remaining cases α is easily seen to be the supremum of ordinals with the same greatest $<_2$-predecessor as claimed for α.

We are left with the case that $\xi\in\mathrm{Lim}$. Then α' is the greatest $<_1$-predecessor, hence α is α'-$\le_2$-minimal, and showing that $\alpha'\not<_2\alpha$ will imply the claim as above. Arguing toward contradiction let us assume that $\alpha'<_2\alpha$. Let $X:=\mathrm{Pred}_2(\alpha')\cup\{\alpha'\}$ and $Z\subseteq(\alpha',\alpha'+\kappa^{\tilde\tau}_{\varrho^\tau_\xi+1})$ be sets according to part (d) of the i.h. for which there does not exist any cover $X\cup\tilde Z$ such that $X<\tilde Z$ and $X\cup\tilde Z\subseteq\alpha'+\kappa^{\tilde\tau}_{\varrho^\tau_\xi}+\mathrm{dp}_{\tilde\tau}(\varrho^\tau_\xi)$. We set $X':=X-\{\alpha'\}$ and $Z':=\{\alpha'\}\cup Z$. By Lemma 7.6 we obtain cofinally many copies $\tilde Z'$ below α' such that $X'<\tilde Z'$ and $X'\cup\tilde Z'\cong X'\cup Z'$ with the property that $\tilde\alpha':=\min\tilde Z'<_1\alpha'$. Let $\nu\in(0,\xi)$ be such that $\mathrm{o}(\boldsymbol{\alpha}[\nu])\le\tilde\alpha'<\mathrm{o}(\boldsymbol{\alpha}[\nu+1])$. Choosing $\tilde Z'$ large enough we may assume that $X'<\mathrm{o}(\boldsymbol{\alpha}[\nu])$ and $\mathrm{logend}(\nu)<\mathrm{logend}(\xi)$, hence $\varrho^\tau_\nu\le\varrho^\tau_\xi$. Notice that if $\mathrm{o}(\boldsymbol{\alpha}[\nu])<\tilde\alpha'$ the i.h. yields $\chi^\tau(\nu)=1$ and $\mathrm{pred}_1(\mathrm{o}(\boldsymbol{\alpha}[\nu+1]))=\mathrm{me}(\mathrm{o}(\boldsymbol{\alpha}[\nu]))\ge\alpha'$ and thus $\mathrm{o}(\boldsymbol{\alpha}[\nu])\le_2\tilde\alpha'$. We may therefore assume that $\tilde\alpha'=\mathrm{o}(\boldsymbol{\alpha}[\nu])$ since exchanging these ordinals would still result in a cover of $X'\cup Z'$. Because $\mathrm{o}(\boldsymbol{\alpha}[\nu+1])<_1\alpha'$ by the i.h. we may further assume that $\tilde Z'\subseteq\mathrm{o}(\boldsymbol{\alpha}[\nu+1])$. Noticing that in the case $\varrho^\tau_\nu=\varrho^\tau_\xi$ we must have $\chi^\tau(\nu)=1$ and by the i.h. $\mathrm{o}(\boldsymbol{\alpha}[\nu]^\frown(\varrho^\tau_\nu))<_1\alpha'$ we finally may assume that $X'\cup\tilde Z'\subseteq\mathrm{o}(\boldsymbol{\alpha}[\nu]^\frown(\zeta))$ for some $\zeta<\varrho^\tau_\xi$ with $\min\tilde Z'=\mathrm{o}(\boldsymbol{\alpha}[\nu])$ so that $X'\cup\tilde Z'$ is a cover of $X'\cup Z'$. Since by i.h.

$$\mathrm{o}(\boldsymbol{\alpha}[\nu]^\frown(\zeta))\ \cong\ \mathrm{o}(\boldsymbol{\alpha}[\nu])\cup[\alpha',\mathrm{o}(\boldsymbol{\alpha}[\xi]^\frown(\zeta))),$$

setting

$$\tilde Z:=(\alpha'+(-\mathrm{o}(\boldsymbol{\alpha}[\nu])+\tilde Z'))-\{\alpha'\}$$

results in a cover $X\cup\tilde Z$ of $X\cup Z$ with $X<\tilde Z$ and $X\cup\tilde Z\subseteq\alpha'+\kappa^{\tilde\tau}_{\varrho^\tau_\xi}+\mathrm{dp}_{\tilde\tau}(\varrho^\tau_\xi)$. Contradiction.

Subcase 2.1.2: $\chi^\tau(\xi)=1$. Part (a) claims that, setting $\boldsymbol{\delta}:=\mathrm{me}(\boldsymbol{\alpha}[\xi])$, $\mathrm{pred}_1(\alpha)=\mathrm{o}(\boldsymbol{\delta})=:\delta$. By part (e) of Lemma 5.12 the extending index of $\mathrm{ec}(\boldsymbol{\delta})$ is of a form $\tau\cdot(\eta+1)$ for some η. Notice that $\mathrm{cml}(\boldsymbol{\delta})=(n,m_n-1)$. By Lemma 4.5 we then have

$$\alpha=\sup\{\mathrm{o}(\boldsymbol{\delta}^\frown(\tau\cdot\eta+\zeta))\mid\zeta\in(0,\tau)\},$$

which by the i.h. is a proper supremum over ordinals whose greatest $<_1$-predecessor is δ.

As to part (b) we first show that α is α'-$\le_2$-minimal, arguing similarly as in the proof of (relativized) $\le_2$-minimality in Subcase 1.2., but providing the argument explicitly again for the reader's convenience. We will then prove $\alpha'\not<_2\alpha$

which as above implies the claim. Recall that we have $\mathrm{pred}_1(\alpha) = \mathrm{o}(\boldsymbol{\delta}) = \delta$ according to part (a). Any $<_2$-predecessor γ of α then satisfies $\gamma \le_2 \delta$, so that by the i.h. $\boldsymbol{\gamma} := \mathrm{tc}(\gamma)$ is an initial chain of $\boldsymbol{\delta}$ extending $\boldsymbol{\alpha}[\xi]$. Let $\boldsymbol{\delta} = (\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_r)$ where $\boldsymbol{\delta}_i = (\delta_{i,1}, \ldots, \delta_{i,k_i})$ for $1 \le i \le r$ with associated chain $\boldsymbol{\sigma}$. Then $r \ge n$, $k_n \ge m_n$, $\delta_{n,m_n} = \xi$, and $\delta_{i,j} = \alpha_{i,j}$ for all $(i,j) \in \mathrm{dom}(\boldsymbol{\delta})$ such that $(i,j) <_{\mathrm{lex}} (n, m_n)$. According to Lemma 5.5 we have $(n, m_n) <_{\mathrm{lex}} (r, k_r)$, and by part (e) of Lemma 5.12 we have

$$\alpha = \delta + \kappa^{\tilde{\sigma}'_r}_{\tau\cdot(\eta+1)}.$$

By Lemma 5.5 and the i.h. we know that $\alpha' <_2 \delta$. Arguing toward contradiction let us assume that $\gamma > \alpha'$, thus $\gamma = \mathrm{o}_{i,j+1}(\boldsymbol{\delta})$ for some $(i, j+1) \in \mathrm{dom}(\boldsymbol{\delta})$ with $(n, m_n) <_{\mathrm{lex}} (i, j+1)$. We then have $\sigma_{i,j} > \tau$, and set $\zeta := \mathrm{o}_{i,j}(\boldsymbol{\delta})$ as well as $\beta := \delta + \kappa^{\tilde{\sigma}'_r}_{\tau\cdot\eta} + \mathrm{dp}_{\tilde{\sigma}'_r}(\tau\cdot\eta)$, so that $\alpha = \beta + \tilde{\tau}$. Applying part (d) of the i.h. to $X := \mathrm{Pred}_2(\zeta) \cup \{\zeta\}$ there exists a finite set $Z \subseteq (\zeta, \zeta + \tilde{\tau} + 1)$ such that there is no cover $X \cup \tilde{Z}$ of $X \cup Z$ with $X < \tilde{Z}$ and $X \cup \tilde{Z} \subseteq \zeta + \tilde{\tau}$. By the i.h. we know that

$$\zeta + 1 + \tilde{\sigma}_{i,j} \;\cong\; \zeta + 1 \cup (\mathrm{o}(\boldsymbol{\delta}_{\restriction(i,j+1)}[\nu]), \mathrm{o}(\boldsymbol{\delta}_{\restriction(i,j+1)}[\nu]) + \tilde{\sigma}_{i,j})$$

for every $\nu \in (0, \delta_{i,j+1})$. Since $\delta_{i,j+1} = \mu_{\sigma_{i,j}} \in \mathbb{P}$ we directly see that below γ there are cofinally many copies $\tilde{Z}_\gamma$ such that $X \cup Z \cong X \cup \tilde{Z}_\gamma$. By Lemma 7.6 and our assumption $\gamma <_2 \alpha$ we now obtain copies $\tilde{Z}_\alpha$ of Z cofinally below α (and hence above β) such that $X \cup Z \cong X \cup \tilde{Z}_\alpha$. The i.h. reassures us of the isomorphism

$$\zeta + 1 + \tilde{\tau} \;\cong\; \zeta + 1 \cup (\beta, \alpha),$$

noting that the ordinals of the interval (β, α) cannot have any $<_2$-predecessors in $(\zeta, \beta]$ and that the tracking chains of the ordinals in $(\zeta, \zeta + \tilde{\tau}) \cup (\beta, \alpha)$ have the proper initial chain $\boldsymbol{\delta}_{\restriction(i,j)}$. This provides us, however, with a copy $\tilde{Z} \subseteq (\zeta, \zeta + \tilde{\tau})$ of Z such that $X \cup Z \cong X \cup \tilde{Z}$, contradicting our choice of X and Z, whence $\gamma <_2 \alpha$ is impossible. Therefore α is α'-$\le_2$-minimal.

We now show that $\alpha' \not<_2 \alpha$. In order to reach a contradiction let us assume to the contrary that $\alpha' <_2 \alpha$. Under this assumption we can prove the following variant of part (d):

Claim 7.10. *Suppose $\alpha' <_2 \alpha$ and let $X := \mathrm{Pred}_2(\alpha') \cup \{\alpha'\}$. There exists a finite set $Z \subseteq (\alpha', \alpha]$ such that there is no cover $X \cup \tilde{Z}$ of $X \cup Z$ with $X < \tilde{Z}$ and $X \cup \tilde{Z} \subseteq \alpha$.*

The proof of the above claim both builds upon part (d) and is similar to its proof, but for the reader's convenience we give it in detail. We are going to show that for every index pair $(i,j) \in \mathrm{dom}(\boldsymbol{\delta})$ such that $(n, m_n) \le_{\mathrm{lex}} (i,j) \le_{\mathrm{lex}} (r, k_r)$, setting $\eta_{i,j} := \mathrm{o}_{i,j}(\boldsymbol{\delta})$ and $X_{i,j} := \mathrm{Pred}_2(\eta_{i,j}) \cup \{\eta_{i,j}\}$, there exists a finite set $Z_{i,j} \subseteq (\eta_{i,j}, \alpha]$ such that there is no cover $X_{i,j} \cup \tilde{Z}_{i,j}$ of $X_{i,j} \cup Z_{i,j}$ with $X_{i,j} < \tilde{Z}_{i,j}$ and $X_{i,j} \cup \tilde{Z}_{i,j} \subseteq \alpha$. We proceed by induction on the finite number of 1-step extensions from $\boldsymbol{\delta}_{\restriction(i,j)}$ to $\boldsymbol{\delta}$: The initial step is $(i,j) = (r, k_r)$, hence $\eta_{i,j} = \delta$. Recalling that $\alpha = \delta + \kappa^{\tilde{\sigma}'_r}_{\tau\cdot(\eta+1)}$, we can apply part (d) of the i.h. to $\delta + \kappa^{\tilde{\sigma}'_r}_{\tau\cdot\eta+1}$ to obtain a set $Z' \subseteq (\delta, \delta + \kappa^{\tilde{\sigma}'_r}_{\tau\cdot\eta+1})$ such that there does not exist any cover $X_{i,j} \cup \tilde{Z}'$ of $X_{i,j} \cup Z'$ with $X_{i,j} < \tilde{Z}'$ and $X_{i,j} \cup \tilde{Z}' \subseteq \delta + \kappa^{\tilde{\sigma}'_r}_{\tau\cdot\eta} + \mathrm{dp}_{\tilde{\sigma}'_r}(\tau\cdot\eta)$. Defining

$$Z_{i,j} := Z' \cup \{\alpha\}$$

and noticing that by our assumption we have $\alpha' <_2 \alpha$ and that by the i.h. there are no $<_2$-successors of α' in the interval (δ, α), it is easy to check that $Z_{i,j}$ has the required property. Let us now assume that $(i,j) <_{\mathrm{lex}} (r, k_r)$ and set $(s,t) := (i,j)^+$.

- $(s,t) = (i+1, 1)$. By Lemma 5.5 we have $\sigma_{s,t} > \tau \in \mathbb{E}$ and hence $\sigma^\star_s \ge \tau$. Notice that the case $\sigma_{s,t} = \sigma^\star_s$ cannot occur since then $\mathrm{ec}(\boldsymbol{\delta}_{\restriction(s,t)})$ would not exist. We discuss the remaining possibilities for $\delta_{s,t}$:

  1. $\delta_{s,t} \in \mathbb{E}^{>\sigma^\star_s}$. We then argue as in the corresponding case in the proof of part (d). We therefore define

     $$Z_{i,j} := Z_{s,t}.$$

     That this choice is adequate is shown as in the proof of part (d).
  2. Otherwise. Let $\sigma := \sigma_{i+1,0}$ and $\tilde{\sigma} := \tilde{\sigma}_{i+1,0}$. In case of $\delta_{s,t} > \sigma_{s,t}$ let ζ be such that $\delta_{s,t} =_{\mathrm{NF}} \zeta + \sigma_{s,t}$, otherwise set $\zeta := 0$. If $\zeta > 0$ let $Z_\zeta \subseteq (\eta_{i,j}, \eta_{i,j} + \kappa^{\tilde{\sigma}}_{\zeta+1})$ be the set according to part (d) of the i.h. so that there does not exist any cover $X_{i,j} \cup \tilde{Z}_\zeta$ of $X_{i,j} \cup Z_\zeta$ with $X_{i,j} < \tilde{Z}_\zeta$ and $X_{i,j} \cup \tilde{Z}_\zeta \subseteq \eta_{i,j} + \kappa^{\tilde{\sigma}}_\zeta + \mathrm{dp}_{\tilde{\sigma}}(\zeta)$, otherwise set $Z_\zeta := \emptyset$. We now define

     $$Z_{i,j} := Z_\zeta \cup \{\eta_{s,t}\} \cup Z_{s,t}.$$

     In order to show that this choice of $Z_{i,j}$ satisfies the claim let us assume to the contrary the existence of a set $\tilde{Z}_{i,j}$ such that $X_{i,j} \cup \tilde{Z}_{i,j}$ is a cover of $X_{i,j} \cup Z_{i,j}$ with $X_{i,j} < \tilde{Z}_{i,j}$ and $X_{i,j} \cup \tilde{Z}_{i,j} \subseteq \alpha$. Let $Z' := \{\eta_{s,t}\} \cup Z_{s,t}$ and $\tilde{Z}'$ be the subset of $\tilde{Z}_{i,j}$ corresponding to Z'. Due to the property of Z_ζ in the case $\zeta > 0$ we have

     $$\tilde{Z}' \subseteq [\eta_{i,j} + \kappa^{\tilde{\sigma}}_{\zeta+1}, \alpha),$$

     and by an application of Proposition 7.1 to $\eta_{s,t} <_1 \alpha$ we obtain that – keeping the same $<_2$-predecessors – there are cofinally many copies

     $$\tilde{Z}' \subseteq [\eta_{i,j} + \kappa^{\tilde{\sigma}}_{\zeta+1}, \eta_{s,t})$$

below $\eta_{s,t}$. The ordinal $\mu := \min \tilde{Z}'$ corresponds to $\eta_{s,t}$ in $Z_{i,j}$, and since $\mu \leq_1 \tilde{Z}'$ we see that there exists $\nu \in (\zeta, \delta_{s,t})$ such that

$$\tilde{Z}' - \{\mu\} \subseteq (\eta', \eta'')$$

where $\eta' := \eta_{i,j} + \kappa^{\tilde{\sigma}}_{\nu}$ and $\eta'' := \eta_{i,j} + \kappa^{\tilde{\sigma}}_{\nu+1}$, which again we may assume to satisfy $\nu \geq \sigma^{\star}_s$ and $\log((1/\sigma^{\star}_s) \cdot \nu) < \log((1/\sigma^{\star}_s) \cdot \sigma_{s,t})$. By the i.h. we have

$$\eta'' \cong \eta' \cup [\eta_{s,t}, \eta_{s,t} + (-\eta' + \eta''))$$

since $\eta_{s,t}$ and η' have the same $<_2$-predecessors. Exploiting this isomorphism and noticing that $X_{s,t} - \{\eta_{s,t}\} \subseteq X_{i,j}$ we obtain a copy $\tilde{Z}_{s,t}$ of $\tilde{Z}' - \{\mu\}$ such that $X_{s,t} \cup \tilde{Z}_{s,t}$ is a cover of $X_{s,t} \cup Z_{s,t}$ with $X_{s,t} < \tilde{Z}_{s,t}$ and $X_{s,t} \cup \tilde{Z}_{s,t} \subseteq \alpha$. Contradiction.

- $(s, t) = (i, j+1)$. Setting $\sigma := \sigma_{i,j}$ and $\sigma' := \sigma'_{i,j}$ we then have $\delta_{i,j+1} = \mu_{\sigma}$ and proceed as in the corresponding case in the proof of part (d). Applying part (d) of the i.h. to $\mathrm{o}(\boldsymbol{\delta}_{\restriction(i,j)}{}^\frown(\bar{\sigma}+1))$ yields a set $Z_{\bar{\sigma}} \subseteq (\eta_{i,j}, \eta_{i,j} + \kappa^{\bar{\sigma}'}_{\bar{\sigma}+1})$ such that there does not exist a cover $X_{i,j} \cup \tilde{Z}_{\bar{\sigma}}$ of $X_{i,j} \cup Z_{\bar{\sigma}}$ with $X_{i,j} < \tilde{Z}_{\bar{\sigma}}$ and $X_{i,j} \cup \tilde{Z}_{\bar{\sigma}} \subseteq \eta_{i,j} + \kappa^{\bar{\sigma}'}_{\bar{\sigma}} + \mathrm{dp}_{\bar{\sigma}'}(\bar{\sigma})$. We now define

$$Z_{i,j} := \{\eta_{s,t}\} \cup (\eta_{s,t} + (-\eta_{i,j} + Z_{\bar{\sigma}})) \cup \{\mathrm{o}(\boldsymbol{\delta}_{\restriction(s,t)}{}^\frown(\sigma))\} \cup Z_{s,t}.$$

  In order to show that $Z_{i,j}$ has the desired property we assume that there were a cover $X_{i,j} \cup \tilde{Z}_{i,j}$ of $X_{i,j} \cup Z_{i,j}$ with $X_{i,j} < \tilde{Z}_{i,j}$ and $X_{i,j} \cup \tilde{Z}_{i,j} \subseteq \alpha$ and then argue as in the corresponding case in the proof of part (d) in order to drive the assumption into a contradiction.

The final instance $(i, j) = (n, m_n)$ establishes Claim 7.10.

We can now derive a contradiction similarly as in the previous subcase. Let X, Z be as in the above claim. Without loss of generality me may assume that $\mathrm{pred}_1(\alpha) = \delta \in Z$. We set $X' := X - \{\alpha'\}$ and $Z' := \{\alpha'\} \cup Z - \{\alpha\}$. By Lemma 7.6 we obtain cofinally many copies $\tilde{Z}'$ below α' such that $X' < \tilde{Z}'$ and $X' \cup \tilde{Z}' \cong X' \cup Z'$ with the property that all $\leq_1$-connections to α are maintained. Let $\tilde{\alpha}' := \min \tilde{Z}'$ and notice that $\tilde{\alpha}' \leq_2 \gamma$ for all $\gamma \in \tilde{Z}'$ such that $\gamma <_1 \alpha'$. Let $\nu \in (0, \xi)$ be such that $\mathrm{o}(\boldsymbol{\alpha}[\nu]) \leq \tilde{\alpha}' < \mathrm{o}(\boldsymbol{\alpha}[\nu+1])$. Choosing $\tilde{Z}'$ large enough we may assume that $X' < \mathrm{o}(\boldsymbol{\alpha}[\nu])$ and $\mathrm{logend}(\nu) < \mathrm{logend}(\xi)$, hence $\varrho^{\tau}_{\nu} \leq \varrho^{\tau}_{\xi}$. Notice that the i.h. yields $\chi^{\tau}(\nu) = 1$ and $\mathrm{pred}_1(\mathrm{o}(\boldsymbol{\alpha}[\nu+1])) = \mathrm{me}(\mathrm{o}(\boldsymbol{\alpha}[\nu])) \geq \alpha'$ and thus $\mathrm{o}(\boldsymbol{\alpha}[\nu]) \leq_2 \tilde{\alpha}'$. We may therefore assume that $\tilde{\alpha}' = \mathrm{o}(\boldsymbol{\alpha}[\nu])$ since exchanging these ordinals would still result in a cover of $X' \cup Z'$. Because $\mathrm{o}(\boldsymbol{\alpha}[\nu+1]) <_1 \alpha'$ by the i.h. we may further assume that $\tilde{Z}' \subseteq \mathrm{o}(\boldsymbol{\alpha}[\nu+1])$. Noticing that since $\chi^{\tau}(\nu) = \chi^{\tau}(\xi) = 1$ we have $\nu \cdot \omega < \xi$, and setting

$$\tilde{\tilde{\alpha}} := \mathrm{o}(\boldsymbol{\alpha}[\nu \cdot \omega]) + \kappa^{\tilde{\tau}}_{\varrho^{\tau}_{\nu\cdot\omega}} + \mathrm{dp}_{\tilde{\tau}}(\varrho^{\tau}_{\nu\cdot\omega})$$

we can use the isomorphism

$$\mathrm{o}(\boldsymbol{\alpha}[\nu+1]) \cong \mathrm{o}(\boldsymbol{\alpha}[\nu]) \cup [\mathrm{o}(\boldsymbol{\alpha}[\nu \cdot \omega]), \tilde{\tilde{\alpha}}),$$

which is established by the i.h., in order to shift $\tilde{Z}'$ by the translation $\tilde{\tilde{Z}}' := \mathrm{o}(\boldsymbol{\alpha}[\nu \cdot \omega]) + (-\mathrm{o}(\boldsymbol{\alpha}[\nu]) + \tilde{Z}')$. This results in the cover $X' \cup \tilde{\tilde{Z}}'$ of $X' \cup Z'$. By the i.h. we know that

$$\mathrm{o}(\boldsymbol{\alpha}[\nu \cdot \omega]) <_2 \tilde{\tilde{\alpha}} = \mathrm{o}(\boldsymbol{\alpha}[\nu \cdot \omega]) + (-\mathrm{o}(\boldsymbol{\alpha}[\nu]) + \mathrm{o}(\boldsymbol{\alpha}[\nu+1]))$$

and that for all $\gamma \in \tilde{Z}'$ such that $\gamma <_1 \alpha'$ the corresponding element in $\tilde{\tilde{Z}}'$ satisfies

$$\mathrm{o}(\boldsymbol{\alpha}[\nu \cdot \omega]) + (-\mathrm{o}(\boldsymbol{\alpha}[\nu]) + \gamma) <_1 \tilde{\tilde{\alpha}}.$$

Since $\varrho^{\tau}_{\nu\cdot\omega} < \varrho^{\tau}_{\xi}$, setting $\tilde{\alpha} := \alpha' + \kappa^{\tilde{\tau}}_{\varrho^{\tau}_{\nu\cdot\omega}} + \mathrm{dp}_{\tilde{\tau}}(\varrho^{\tau}_{\nu\cdot\omega})$, we may finally exploit the isomorphism

$$\tilde{\tilde{\alpha}} + 1 \cong \mathrm{o}(\boldsymbol{\alpha}[\nu \cdot \omega]) \cup [\alpha', \tilde{\alpha}]$$

so that setting

$$\tilde{Z} := (\alpha' + (-\mathrm{o}(\boldsymbol{\alpha}[\nu \cdot \omega]) + (\tilde{\tilde{Z}}' \cup \{\tilde{\tilde{\alpha}}\}))) - \{\alpha'\}$$

we obtain the cover $X \cup \tilde{Z}$ of $X \cup Z$ which satisfies $X < \tilde{Z}$ and $X \cup \tilde{Z} \subseteq \alpha$. Contradiction.

Subcase 2.2: $\alpha_{n,m_n} \in \mathrm{Lim}$.

Part (a) follows from the i.h. by part (b) of Lemma 4.5 which shows that

$$\alpha = \sup\{\mathrm{o}(\boldsymbol{\alpha}[\xi]) \mid \xi \in (0, \alpha_{n,m_n})\}.$$

In order to see part (b) we simply observe that according to part (a) and the i.h. $(\mathrm{o}(\boldsymbol{\alpha}[\xi]))_{\xi<\alpha_{n,m_n}}$ is a $<_1$-chain of ordinals either $\leq_2$-minimal as claimed for α or with the same greatest $<_2$-predecessor as claimed for α. □

Corollary 7.11. 1^{∞} *is* $\leq_1$*-minimal.*

Proof. That 1^{∞} is $\leq_1$-minimal immediately follows from part (a) of Theorem 7.9 since clearly

$$\sup\{\kappa^0_{\xi} \mid \xi < 1^{\infty}\} = 1^{\infty}$$

is a non-attained supremum of $\leq_1$-minimal ordinals. □

Definition 7.12. Let $\boldsymbol{\alpha} \in \mathrm{TC}$ where $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \ldots, \alpha_{i,m_i})$ for $1 \le i \le n$ and set

$$\boldsymbol{\alpha}^\star := \begin{cases} \boldsymbol{\alpha} & \text{if } m_n = 1 \\ \boldsymbol{\alpha}[\mu_{\tau_{n,m_n-1}}] & \text{otherwise.} \end{cases}$$

We define the **greatest branch-off index pair of $\boldsymbol{\alpha}$**, $\mathrm{gbo}(\boldsymbol{\alpha})$, by

$$\mathrm{gbo}(\boldsymbol{\alpha}) := \begin{cases} \mathrm{gbo}(\boldsymbol{\alpha}_{\restriction(i,j+1)}) & \text{if } (i,j) := \mathrm{cml}(\boldsymbol{\alpha}^\star) \text{ exists} \\ (n, m_n) & \text{otherwise.} \end{cases}$$

Corollary 7.13. *Let $\alpha < 1^\infty$ with $\mathrm{tc}(\alpha) = \boldsymbol{\alpha}$ where $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \ldots, \alpha_{i,m_i})$ for $1 \le i \le n$.*

(a) *If $m_n = 1$ then*

$$\mathrm{Succ}_2(\alpha) = \{\alpha\} \quad \textit{and} \quad \mathrm{lh}_2(\alpha) = \alpha.$$

In the case $m_n > 1$ let ν, ξ be such that $\kappa^{\tilde{\tau}}_{\varrho} + \mathrm{dp}_{\tilde{\tau}}(\varrho) = \tilde{\tau} \cdot \nu + \xi$ and $\xi < \tilde{\tau}$ where $\tau := \tau_{n,m_n-1}$, $\tilde{\tau} := \tilde{\tau}_{n,m_n-1}$, and $\varrho := \varrho^{\tau}_{\tau_{n,m_n}}$. Then setting

$$\eta_{\max} := \nu \dot{-} \chi^{\tau}(\tau_{n,m_n})$$

we have

$$\mathrm{Succ}_2(\alpha) = \{\alpha + \tilde{\tau} \cdot \eta \mid \eta \le \eta_{\max}\} \quad \textit{and} \quad \mathrm{lh}_2(\alpha) = \alpha + \tilde{\tau} \cdot \eta_{\max}.$$

(b) *Setting $(n_0, m_0) := \mathrm{gbo}(\boldsymbol{\alpha})$ and $m := m_0 \dot{-} 2 + 1$ we have*

$$\mathrm{lh}(\alpha) = \mathrm{o}_{n_0,m}(\boldsymbol{\alpha}) + \mathrm{dp}_{\tilde{\tau}_{n_0,m-1}}(\tau_{n_0,m}).$$

Proof. For the proof of part (a) we first observe that by Theorem 7.9 and Corollary 7.11, for any β, if $\alpha <_1 \beta$ then $\beta < 1^\infty$. Next we notice that the case $m_n = 1$ is trivial, since according to part (b) of Theorem 7.9 α cannot be the $<_2$-predecessor of any ordinal. Let us now assume that $m_n > 1$. By Theorem 7.9 we know that for any β such that $\alpha <_2 \beta$ $\mathrm{tc}(\alpha)$ is a proper initial chain of $\mathrm{tc}(\beta)$ and β lies within the interval specified in Corollary 6.6. We argue by induction on β, where according to Corollary 6.6

$$\beta \in \begin{cases} (\alpha, \alpha + \mathrm{dp}(\boldsymbol{\alpha})) & \text{if } m_n > 1 \;\&\; \tau_{n,m_n} < \mu_{\tau_{n,m_n-1}} \\ (\alpha, \alpha + \mathrm{dp}(\boldsymbol{\alpha})] & \text{otherwise,} \end{cases}$$

and show that $\alpha <_2 \beta$ if and only if $\beta \in \{\alpha + \tilde{\tau} \cdot \eta \,|\, 0 < \eta \le \eta_{\max}\}$. It is easy to check that $\alpha + \tilde{\tau} \cdot \eta_{\max}$ is the greatest multiple of $\tilde{\tau}$ in the interval provided by Corollary 6.6 which also yields that $\mathrm{tc}(\beta)$ is a proper extension of $\mathrm{tc}(\alpha)$ for any β in the interval. Let (r, k_r) be the $<_{\mathrm{lex}}$-maximal index pair of the proper extension $\boldsymbol{\beta} := \mathrm{tc}(\beta)$ of $\boldsymbol{\alpha}$. Notice that $(n, m_n) <_{\mathrm{lex}} (r, k_r)$ and $\mathrm{end}(\beta) = \mathrm{end}(\tilde{\tau}_{r,k_r})$. Clearly, we then have $n \le r$, $k_i = m_i$ for $i < n$, and $m_n \le k_n$ (which is strict in case of $r = n$). In the case $k_r = 2 \;\&\; \tau_{r,2} = 1$ we obtain the claimed equivalence from the i.h. using Theorem 7.9. We may therefore exclude this special case in the following argumentation.

Case 1: $k_r > 2$. Then according to part (b) of Theorem 7.9 the greatest $<_2$-predecessor of β is $\mathrm{o}_{r,k_r-1}(\boldsymbol{\beta})$ which is greater than or equal to α. It follows that $\tilde{\tau}_{r,k_r}$ is a multiple of $\tilde{\tau}_{r,k_r-1}$. In order to derive a contradiction we assume that

$$\tilde{\tau}_{r,k_r-1} < \tilde{\tau} \le \tilde{\tau}_{r,k_r}.$$

This implies $\tau_{r,k_r} > 1$ and $\mathrm{ts}(\tilde{\tau}_{r,k_r-1}) <_{\mathrm{lex}} \mathrm{ts}(\tilde{\tau}) \le_{\mathrm{lex}} \mathrm{ts}(\tilde{\tau}_{r,k_r})$ by 3.15, and using part (c) of Lemma 5.10 we obtain that $\mathrm{ts}(\tilde{\tau}_{r,k_r-1})$ is a proper initial sequence of $\mathrm{ts}(\tilde{\tau})$ which contradicts part (d) of Lemma 5.10 since $(n, m_n - 1) <_{\mathrm{lex}} (r, k_r - 1)$. We therefore either have $\tilde{\tau} \le \tilde{\tau}_{r,k_r-1}$, which implies that $\alpha \le_2 \mathrm{o}_{r,k_r-1}(\boldsymbol{\beta}) <_2 \beta$ using the i.h. if needed, or we have $\tilde{\tau}_{r,k_r-1}, \tilde{\tau}_{r,k_r} < \tilde{\tau}$, whence by the i.h. $\alpha \not\le_2 \mathrm{o}_{r,k_r-1}(\boldsymbol{\beta})$, implying that also $\alpha \not\le_2 \beta$; see Lemma 1.2.

Case 2: $k_r \le 2$ and $\tau^\star_r > 1$. This implies $n < r$. Let $(i, j) := r^\star$, $\sigma := \tau_{i,j}$, and $\tilde{\sigma} := \tilde{\tau}_{i,j}$. Then by part (b) of Theorem 7.9 the greatest $<_2$-predecessor of β is $\mathrm{o}_{i,j+1}(\boldsymbol{\beta})$. We have $\sigma = \tau^\star_r \le \tau_{r,1}$, $\tilde{\sigma} \le \tilde{\tau}_{r,1}$ (by part (b) of Lemma 5.10), and if $k_r = 2$ then $\tilde{\tau}_{r,2}$ is a multiple of $\tilde{\tau}_{r,1}$.

Subcase 2.1: $(i, j) = (n, m_n - 1)$. Then $\tilde{\sigma} = \tilde{\tau} \le \tilde{\tau}_{r,1} \le \mathrm{end}(\beta)$ and $\mathrm{o}_{i,j+1}(\boldsymbol{\beta}) = \alpha$.

Subcase 2.2: $(i, j) <_{\mathrm{lex}} (n, m_n - 1)$. Then we have $\sigma \le \tau_{r,1} < \tau$, $\tilde{\sigma} \le \tilde{\tau}_{r,1} = \kappa^{\tilde{\sigma}}_{\tau_{r,1}} < \tilde{\tau}$ by parts (b) and (c) of Lemma 5.10, and $\mathrm{o}_{i,j+1}(\boldsymbol{\beta}) < \alpha$. We have to show that $\mathrm{end}(\tilde{\tau}_{r,k_r}) < \tilde{\tau}$. In the case $k_r = 1$ we are done, otherwise we have $\sigma < \tau_{r,1}$, and $\tau_{r,2} > 1$ according to our argumentation above, hence $\mathrm{ts}(\tilde{\tau}_{r,2}) = \mathrm{ts}(\tilde{\sigma})^\frown(\tau_{r,1}, \tau_{r,2})$ using part (c) of Lemma 5.10. In order to derive a contradiction let us assume that $\tilde{\tau} \le \tilde{\tau}_{r,2}$. Then we have $\mathrm{ts}(\tilde{\tau}_{r,1}) <_{\mathrm{lex}} \mathrm{ts}(\tilde{\tau}) \le_{\mathrm{lex}} \mathrm{ts}(\tilde{\tau}_{r,2})$ by Lemma 3.15. As in Case 1 we obtain that $\mathrm{ts}(\tilde{\tau}_{r,1})$ is a proper initial sequence of $\mathrm{ts}(\tilde{\tau})$. Since $(n, m_n - 1) <_{\mathrm{lex}} (r, 1)$ this contradicts part (d) of Lemma 5.10.

Subcase 2.3: $(n, m_n - 1) <_{\mathrm{lex}} (i, j)$. We then have $\alpha < \mathrm{o}_{i,j+1}(\boldsymbol{\beta}) <_2 \beta$. In the case $\tilde{\tau} \le \tilde{\tau}_{i,j+1}$ by the i.h. we have $\alpha <_2 \mathrm{o}_{i,j+1}(\boldsymbol{\beta})$ and have to verify that $\tilde{\tau} \le \tilde{\tau}_{r,k_r}$. Assuming $\tilde{\sigma} < \tilde{\tau}$ we obtain that $\mathrm{ts}(\tilde{\sigma})$ is a proper initial sequence of $\mathrm{ts}(\tilde{\tau})$, again due to Lemma 3.15 and part (c) of Lemma 5.10, which since $(n, m_n - 1) <_{\text{lex}} (i, j)$ contradicts part (d) of Lemma 5.10. Thus $\tilde{\tau} \le \tilde{\sigma} \le \tilde{\tau}_{r,1} \le \tilde{\tau}_{r,k_r}$.

Otherwise we have $\tilde{\tau} > \tilde{\tau}_{i,j+1} > \tilde{\sigma}$, and by the i.h. we have $\alpha \not\le_2 \mathrm{o}_{i,j+1}(\boldsymbol{\beta})$ and hence $\alpha \not\le_2 \beta$. We have to show that $\tilde{\tau}_{r,k_r} < \tilde{\tau}$. Let us assume to the contrary that $\tilde{\tau} \le \tilde{\tau}_{r,k_r}$. Notice that in the special case $k_r = 1$ & $\tau_{r,1} \notin \mathbb{E}^{>\sigma}$ we have

$$\tilde{\tau}_{r,1} \le \nu^{\tilde{\sigma}}_{\mu_\sigma},$$

which using Lemma 3.12 is seen as follows: $\tau_{r,1}$ is a multiple of σ, hence $\tau_{r,1} \le \varrho^{\sigma}_{\mu_\sigma}$. If $\mu_\sigma \notin \mathbb{E}^{>\sigma}$ then by part (e) of Lemma 4.5

$$\tilde{\tau}_{r,1} = \kappa^{\tilde{\sigma}}_{\tau_{r,1}} \le \kappa^{\tilde{\sigma}}_{\rho_{\mu_\sigma}} \le \nu^{\tilde{\sigma}}_{\mu_\sigma},$$

while otherwise directly $\tau_{r,1} \le \mu_\sigma = \rho_{\mu_\sigma}$ and

$$\tilde{\tau}_{r,1} \le \kappa^{\tilde{\sigma}}_{\mu_\sigma} = \nu^{\tilde{\sigma}}_{\mu_\sigma}.$$

Since $\mathrm{ts}(\nu^{\tilde{\sigma}}_{\mu_\sigma}) = \mathrm{ts}(\tilde{\sigma})^\frown\mu_\sigma$ by Lemma 4.10 we then obtain from our assumptions that

$$\mathrm{ts}(\tilde{\sigma}) <_{\text{lex}} \mathrm{ts}(\tilde{\tau}) \le_{\text{lex}} \mathrm{ts}(\tilde{\sigma})^\frown\mu_\sigma,$$

whence $\mathrm{ts}(\tilde{\sigma})$ is a proper initial sequence of $\mathrm{ts}(\tilde{\tau})$, contradicting part (d) of Lemma 5.10 since $(n, m_n - 1) <_{\text{lex}} (i, j)$. In the remaining cases we again obtain that $\mathrm{ts}(\tilde{\sigma})$ is a proper initial sequence of $\mathrm{ts}(\tilde{\tau})$ contradicting part (d) of Lemma 5.10.

Case 3: Otherwise. Then we have $k_r \le 2$, $\tau^\star_r = 1$, and again $n < r$. According to part (b) of Theorem 7.9 β does not have any $<_2$-predecessor. We have to show that $\tilde{\tau}_{r,k_r} < \tilde{\tau}$. Since $\tau^\star_r = 1$ we have $\tau_{r,1} < \tau$ and hence $\tilde{\tau}_{r,1} = \kappa^0_{\tau_{r,1}} < \kappa^0_{\tau_{k,l}} \le \tilde{\tau}$ where (k, l) is the index pair of the first element of $\mathrm{ts}(\tilde{\tau})$ according to part (c) of Lemma 5.10, which implies that $\tau_{r,1} < \tau_{k,l}$. In the case $k_r = 1$ we are done, otherwise we have $\tau_{r,2} > 1$ according to our earlier assumption. By part (c) of Lemma 5.10 we have $\mathrm{ts}(\tilde{\tau}_{r,2}) = (\tau_{r,1}, \tau_{r,2})$. The assumption $\tilde{\tau} \le \tilde{\tau}_{r,2}$ then implies that $\mathrm{ts}(\tilde{\tau}_{r,1})$ is a proper initial sequence of $\mathrm{ts}(\tilde{\tau})$ which because of $(n, m_n - 1) <_{\text{lex}} (r, 1)$ contradicts part (d) of Lemma 5.10.

In order to show part (b) let $\boldsymbol{\alpha}' := (\boldsymbol{\alpha}_{\restriction(n_0,m_0)})^\star$ using the $\star$-notation from Definition 7.12, according to which the vector $\boldsymbol{\alpha}'$ does not possess a critical main line index pair. Using Lemma 5.12 part (d) we obtain

$$\mathrm{o}_{n_0,m}(\boldsymbol{\alpha}) + \mathrm{dp}_{\tilde{\tau}_{n_0,m-1}}(\tau_{n_0,m}) = \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}')).$$

We first show that

$$\alpha \le_1 \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}')). \tag{6}$$

Since in the case $m_0 = 1$ we have $(n_0, m_0) = (n, m_n)$ whence there is nothing to show, we may assume that $m_0 > 1$. By part (c) of Theorem 7.9 we have $\alpha \le_1 \mathrm{o}(\boldsymbol{\alpha}^\star) \le_1 \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}^\star))$. If $\mathrm{cml}(\boldsymbol{\alpha}^\star)$ does not exist, that is $\boldsymbol{\alpha}' = \boldsymbol{\alpha}^\star$, we are done with showing (6). Otherwise let $\mathrm{cml}(\boldsymbol{\alpha}^\star) =: (i_1, j_1)$ and let l_0 be maximal so that for all $l \in (0, l_0)$ $\mathrm{cml}((\boldsymbol{\alpha}_{\restriction(i_l,j_l+1)})^\star) =: (i_{l+1}, j_{l+1})$ exists. Clearly, the sequence of index pairs we obtain in this way is $<_{\text{lex}}$-decreasing and by Definition 7.12 $(i_{l_0}, j_{l_0} + 1) = (n_0, m_0)$. Using parts (a) and (c) of Theorem 7.9 we now obtain the chain of inequations

$$\alpha \le_1 \mathrm{o}(\boldsymbol{\alpha}^\star) \le_1 \mathrm{o}((\boldsymbol{\alpha}_{\restriction(i_1,j_1+1)})^\star) \le_1 \ldots \le_1 \mathrm{o}((\boldsymbol{\alpha}_{\restriction(i_{l_0},j_{l_0}+1)})^\star) = \mathrm{o}(\boldsymbol{\alpha}') \le_1 \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}')).$$

We claim that

$$\mathrm{pred}_1(\mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}')) + 1) < \mathrm{o}(\boldsymbol{\alpha}'). \tag{7}$$

To this end note that $\mathrm{tc}(\mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}')) + 1)$ must be of a form $\boldsymbol{\alpha}_{\restriction i}{}^\frown(\alpha_{i+1,1} + 1)$ where $i \le n_0$, and by part (a) of Theorem 7.9 $\mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}')) + 1$ either does not have any $<_1$-predecessor or the greatest $<_1$-predecessor is $\mathrm{o}(\alpha_{i-1,m_{i-1}})$. Hence (7) follows, which implies that $\alpha \not\le_1 \mathrm{o}(\mathrm{me}(\boldsymbol{\alpha}')) + 1$. We thus have $\mathrm{lh}(\alpha) = \mathrm{o}_{n_0,m}(\boldsymbol{\alpha}) + \mathrm{dp}_{\tilde{\tau}_{n_0,m-1}}(\tau_{n_0,m})$. □

Corollary 7.14. *Any pure pattern of order 2 has a covering below 1^∞, the least such ordinal.*

Proof. Consider the maximal extensions of the tracking chains $\mathrm{tc}(|\,\mathrm{ID}_n\,|) = ((|\,\mathrm{ID}_n\,|))$ of the proof-theoretic ordinals of the theories ID_n where $n \in (0, \omega)$. Omitting the first initial value $|\,\mathrm{ID}_n\,|$ we obtain a $<_2$-chain connecting $n + 2$ ordinals when considering $\mathrm{me}(\mathrm{tc}(|\,\mathrm{ID}_n\,|))$. This shows that any pure pattern of order 2 has a covering below 1^∞. By part (d) of Theorem 7.9 we obtain pure patterns of order 2 contained in the ordinals

$$\kappa^0_{|\,\mathrm{ID}_n\,|+1} = \mathrm{o}(\mathrm{me}(\mathrm{tc}(|\,\mathrm{ID}_n\,|))) + 1 = |\,\mathrm{ID}_n\,| + \mathrm{dp}_0(|\,\mathrm{ID}_n\,|) + 1$$

for which there does not exist a covering contained in $|\,\mathrm{ID}_n\,| + \mathrm{dp}_0(|\,\mathrm{ID}_n\,|)$. □

8. Conclusion

As mentioned in the introduction this article provides the basis for both a full arithmetical characterization of $\mathcal{R}_2$, thereby showing that the sequence $(\tau_\xi)_{\xi\in\mathrm{Ord}}$ as defined in [10], that is, $\tau_0 = 1^\infty$, $\tau_{\xi+1} = \tau_\xi^\infty$, and $\tau_\lambda = \lim\{\tau_\xi \mid \xi < \lambda\}$, starts with

$\tau_0 <_1 \tau_1$ and then connects $\tau_\xi <_2 \tau_\eta$ for any ξ, η such that $0 < \xi < \eta$ where the greatest $<_2$-predecessor of any $\tau_{\xi+1}$ is τ_ξ for $\xi > 0$, while τ_1 is $\le_2$-minimal. The method to be applied in order to generalize the results of this article is base transformation in the sense of [10].

The characterization of isominimal substructures and the core of $\mathcal{R}_2$ will be supplied by effective assignments between pure patterns of order 2 and finite sets of ordinals in hull notations, following the style of [12,4]. The key to such assignments will be the concept of tracking chains that was introduced in the present article.

Future work will extend the methods introduced here to variants of $\mathcal{R}_2$ such as $\mathcal{R}_2^+$ and higher orders. A considerable gain in strength beyond the proof-theoretical ordinal of KPI is claimed. The starting point for a more powerful ordinal arithmetic is given in [7].

Index

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