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Universal torsors of Del Pezzo surfaces and homogeneous spaces

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Abstract

Let $\mathrm{Cox}(S_r)$ be the homogeneous coordinate ring of the blow-up S_r of $\mathbb{P}^2$ in r general points, i.e., a smooth Del Pezzo surface of degree $9-r$. We prove that for $r \in \{6, 7\}$, $\mathrm{Proj}(\mathrm{Cox}(S_r))$ can be embedded into G_r/P_r, where G_r is an algebraic group with root system given by the primitive Picard lattice of S_r and $P_r \subset G_r$ is a certain maximal parabolic subgroup.

Keywords: Cox ring; Del Pezzo surface; Homogeneous space

1. Introduction

In this note we continue our investigations of universal torsors over Del Pezzo surfaces over an algebraically closed field $\mathbb{K}$ of characteristic 0. The blow-up S_r of $\mathbb{P}^2$ in $r \leqslant 8$ points in *general position* (i.e., no three on a line, no six on a conic, no eight and one of them singular on a cubic curve) is a smooth Del Pezzo surface of degree $9-r$; we will assume that $r \in \{3, \dots, 7\}$. A smooth Del Pezzo surface of degree 3 (respectively degree 2) is a smooth cubic surface in $\mathbb{P}^3$ (respectively a double cover of $\mathbb{P}^2$ ramified in a smooth curve of degree 4). The Picard group $\mathrm{Pic}(S_r)$ is a lattice with a non-degenerate symmetric linear form $(\cdot,\cdot)$, the *intersection form*. It is well known that $\mathrm{Pic}(S_r)$ contains a canonical root system R_r, which carries the action of the associated Weyl group W_r, see Table 1 and [15].

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Table 1
The root systems associated to Del Pezzo surfaces

r	3	4	5	6	7
R_r	$\mathbf{A}_2+\mathbf{A}_1$	$\mathbf{A}_4$	$\mathbf{D}_5$	$\mathbf{E}_6$	$\mathbf{E}_7$
N_r	6	10	16	27	56

It was a general expectation that the Weyl group symmetry on $\mathrm{Pic}(S_r)$ should be a reflection of a geometric link between Del Pezzo surfaces and algebraic groups. Here we show that universal torsors of smooth Del Pezzo surfaces of degree 2 and 3 admit an embedding into a certain flag variety for the corresponding algebraic group. The degree 5 case goes back to Salberger (talk at the Borel seminar Bern, June 1993) following Mumford [16], and independently Skorobogatov [20]. The degree 4 case was treated in the thesis of Popov [17, Chapter 6]. The existence of such an embedding in general was conjectured by Batyrev in his lecture at the conference *Diophantine geometry* (Universität Göttingen, June 2004). Skorobogatov announced related work in progress (joint with Serganova) at the conference *Cohomological approaches to rational points* (MSRI, March 2006).

As in [2, Section 2], the simple roots of R_{r-1} (with $R_2=\mathbf{A}_2$) can be identified with a subset I_r of the simple roots of R_r such that the edges in the Dynkin diagrams are respected. The complement of I_r in R_r consists of exactly one simple root α_r, with associated fundamental weight ϖ_r. Let G_r be a simply connected linear algebraic group associated to R_r, and fix a Borel group containing a maximal torus. The Weyl group W_r acts on the weight lattice of G_r. The fundamental representation ϱ_r of G_r with highest weight ϖ_r has dimension N_r as listed in Table 1; the weights of ϱ_r can be identified with classes of curves $E\subset S_r$ with self-intersection number $(E,E)=-1$, so-called (-1)-curves.

Let P_r be the maximal parabolic subgroup corresponding to I_r. By [19], we can regard the G_r-orbit H_r of the weight space to ϖ_r as the affine cone over G_r/P_r, and H_r is given by quadratic equations in affine space $\mathbb{A}^{N_r}$. For $r=6$, the equations are all partial derivatives of a certain cubic form on the 27-dimensional representation ϱ_6 of G_6. For $r=7$, equations can be found in [11]. In both cases, the equations were already known to E. Cartan in the 19th century. See Section 3 for more details on G_r, ϱ_r, and H_r for $r\in\{6,7\}$.

The *universal torsor* $\mathcal{T}_r$ over S_r is defined as follows: Let $\mathcal{L}_0,\dots,\mathcal{L}_r$ be a basis of $\mathrm{Pic}(S_r)\cong\mathbb{Z}^{r+1}$, and let $\mathcal{L}_i^\circ:=\mathcal{L}_i\setminus\{\text{zero-section}\}$. Then

$$\mathcal{T}_r:=\mathcal{L}_0^\circ\times_{S_r}\cdots\times_{S_r}\mathcal{L}_r^\circ.$$

It is a $T_{\mathrm{NS}}(S_r)$-bundle over S_r, where $T_{\mathrm{NS}}(S_r)$ is the Néron–Severi torus of S_r.

The *total coordinate ring*, or *Cox ring* of S_r is defined as

$$\mathrm{Cox}(S_r):=\bigoplus_{(\nu_0,\dots,\nu_r)\in\mathbb{Z}^{r+1}}H^0\big(S_r,\mathcal{L}_0^{\otimes\nu_0}\otimes\cdots\otimes\mathcal{L}_r^{\otimes\nu_r}\big)$$

as a vectorspace, and the multiplication is induced by the multiplication of sections (see [2,13]). It is naturally graded by $\mathrm{Pic}(S_r)$, and it is generated by N_r sections corresponding to the (-1)-curves on S_r. The ideal of relations in $\mathrm{Cox}(S_r)$ is generated by certain quadratic relations which are homogeneous with respect to the $\mathrm{Pic}(S_r)$-grading (see [2] for more details). Let

$$\mathbb{A}(S_r):=\mathrm{Spec}\big(\mathrm{Cox}(S_r)\big)\subset\mathbb{A}^{N_r}$$

be the corresponding affine variety. The universal torsor $\mathcal{T}_r$ is an open subset of $\mathbb{A}(S_r)$ (cf. [13]). See [8,12] for the calculation of universal torsors and Cox rings for *singular* Del Pezzo surfaces and [7, Chapter 3] for the smooth cases.

Universal torsors can be applied to *Manin's conjecture* (see [1,10]) on the number of rational points of bounded height on Del Pezzo surfaces. For the moment, let the Del Pezzo surface S_r be defined over a number field k, and let $H : S_r(k) \to \mathbb{R}$ be the anticanonical height function. Let U be the complement of the (-1)-curves on S_r. Then Manin's conjecture predicts that

$$N_{U,H}(B) := \#\{\mathbf{x} \in U(k) \mid H(\mathbf{x}) \leqslant B\}$$

behaves asymptotically as

$$N_{U,H}(B) \sim c \cdot B \cdot (\log B)^{n-1}$$

for some positive constant c, where n is the rank of the Picard group (over k) of S_r.

In results concerning Manin's conjecture for various Del Pezzo surfaces (see [3] and [9, Section 1] for an overview), often the first step is a translation of the counting problem for rational points on S_r to the counting of integral points in certain ranges on a universal torsor $\mathcal{T}_r$. The number of these points on $\mathcal{T}_r$ can then be estimated using techniques from analytic number theory.

Salberger [18] gave a proof of Manin's conjecture for toric varieties, which include smooth Del Pezzo surfaces of degree $\geqslant 6$, using universal torsors. De la Bretèche [4] used Salberger's and Skorobogatov's description of the universal torsor as a homogeneous space in his proof of the asymptotic formula for a Del Pezzo surface of degree 5. In lower degrees, Manin's conjecture has been proved only for examples of singular Del Pezzo surfaces (see [5] for a singular cubic surface, using the universal torsor), but it is a general expectation that universal torsors should lead to a proof of Manin's conjecture also in the remaining smooth cases.

We have seen that both $\mathbb{A}(S_r)$ and H_r can be viewed as embedded into $\mathbb{A}^{N_r}$, with a natural identification of the coordinates. For the embedding of $\mathbb{A}(S_r)$, we have some freedom: As the generators of $\mathrm{Cox}(S_r)$ are canonical only up to a non-zero constant, we can choose a rescaling factor for each of the N_r coordinates, giving a N_r-parameter family of embeddings of $\mathbb{A}(S_r)$ into affine space. The task is to find a rescaling such that $\mathbb{A}(S_r)$ is embedded into H_r.

More precisely, we start with an arbitrary embedding

$$\mathbb{A}(S_r) \subset \mathbb{A}_r := \mathrm{Spec}\big(\mathbb{K}\big[\xi(E) \mid E \text{ is a } (-1)\text{-curve on } S_r\big]\big) \cong \mathbb{A}^{N_r},$$

and view

$$H_r \subset \mathbb{A}'_r := \mathrm{Spec}\big(\mathbb{K}\big[\xi'(E) \mid E \text{ is a } (-1)\text{-curve on } S_r\big]\big) \cong \mathbb{A}^{N_r}$$

as embedded into a different affine space. An isomorphism $\phi_r : \mathbb{A}_r \to \mathbb{A}'_r$ such that

$$\phi_r^*\big(\xi'(E)\big) = \xi''(E) \cdot \xi(E)$$

for each of the N_r coordinates, with $\xi''(E) \in \mathbb{K}^* := \mathbb{K} \setminus \{0\}$, is called a *rescaling*, and the factors $\xi''(E)$ are (a system of) *rescaling factors*. A rescaling ϕ_r which embeds $\mathbb{A}(S_r)$ into H_r is called a *good rescaling*.

Our main result is:

Theorem 1. *Let S_r be a smooth Del Pezzo surface of degree $9-r$ and $\mathbb{A}(S_r)$ the affine variety described above. Let H_r be the affine cone over the flag variety G_r/P_r associated to the root system R_r as in Table* 1.

For $r \in \{6,7\}$, there exists a $(N_{r-2}+2)$-parameter family of good rescalings ϕ_r which embed $\mathbb{A}(S_r)$ into H_r.

Remark 2. The number $N_{r-2}+2$ of parameters is 12 for $r=6$, respectively 18 for $r=7$.

The rescaling factors are naturally graded by $\operatorname{Pic}(S_r) \cong \mathbb{Z}^{r+1}$, and we will see in Section 4 that the conditions for good rescaling are homogeneous with respect to this grading. Therefore, for each good rescaling ϕ_r, there is a $(r+1)$-parameter family of good rescalings which differ from ϕ_r only by the action of $T_{\mathrm{NS}}(S_r) \cong \mathbb{G}_{\mathrm{m}}^{r+1}$. Similarly, $T_{\mathrm{NS}}(S_r)$ acts on $\mathbb{A}(S_r)$, and it is easy to see that the image of $\mathbb{A}(S_r)$ in H_r is the same for all good rescalings in the same $(r+1)$-parameter family. Therefore, the $(N_{r-2}+2)$-parameter family of good rescalings gives rise to a $(N_{r-2}-r+1)$-parameter family (where $N_{r-2}-r+1$ equals 5 for $r=6$, respectively 10 for $r=7$) of images of $\mathbb{A}(S_r)$ in H_r.

For $r=5$, we have $N_{r-2}-r+1=2$, and by [17, Section 6.3], there is a two-parameter family of images of $\mathbb{A}(S_5)$ under good rescalings in H_5.

In Section 2, we summarize results of [2,7] on Cox rings of Del Pezzo surfaces of degree 3 and 2. In Section 3, we recall the classical equations for the homogeneous spaces G_r/P_r and give a simplified description on a certain Zariski open subset; this will help to find good rescalings. In Section 4, we derive conditions on the rescaling factors in terms of the description of $\operatorname{Cox}(S_r)$ and G_r/P_r. In Sections 5 and 6, we determine good rescalings in degree 3 and 2, finishing the proof of Theorem 1.

2. Cox rings of Del Pezzo surfaces

In this section, we describe the Cox ring of Del Pezzo surfaces of degree 3 and 2 in order to fix some notation. The results can be found in [2,7].

Let $r \in \{6,7\}$. Without loss of generality, we may assume that four of the r blown-up points in $\mathbb{P}^2$ giving S_r are in the positions

$$p_1=(1:0:0), \qquad p_2=(0:1:0), \qquad p_3=(0:0:1), \qquad p_4=(1:1:1).$$

By [2, Theorem 3.2], the generators of $\operatorname{Cox}(S_r)$ are sections $\xi(E)$ vanishing in a (-1)-curve E on S_r. Their number is N_r as in Table 1. We use the same symbols $\xi(E)$ for the coordinates in $\mathbb{A}_r$. Let $-K_r$ be the anticanonical divisor class of S_r.

A (k)-ruling $D \in \operatorname{Pic}(S_r)$ is the sum of two (-1)-curves whose intersection number is k, cf. [2, Definition 4.6] and [7, Definition 3.1]. The relations come in groups of $r-3$ for each (1)-ruling D. We will denote them by

$$F_{D,1}, \ldots, F_{D,r-3}.$$

For $r=7$, we have further relations corresponding to the (2)-ruling $-K_7$. The ideal J_r generated by these relations defines $\mathbb{A}(S_r)$ (see [2, Theorem 4.9] for $r \leqslant 6$ and [7, Theorem 3.2] for $r=7$).

For $r = 6$, we assume

$$p_5 = (1 : a : b), \qquad p_6 = (1 : c : d).$$

The $N_6 = 27$ (-1)-curves E are denoted by E_i, $m_{i,j}$, and Q_i corresponding to the six blown-up points, the 15 transforms of the lines through two of the six points, and the six transforms of the conics through five points as described in [7, Section 3.3]. Let $\eta_i := \xi(E_i)$, $\mu_{i,j} := \xi(m_{i,j})$, and $\lambda_i := \xi(Q_i)$. We order them in the following way:

$$\eta_1, \dots, \eta_6, \quad \mu_{1,2}, \dots, \mu_{1,6}, \mu_{2,3}, \dots, \mu_{2,6}, \mu_{3,4}, \dots, \mu_{5,6}, \quad \lambda_1, \dots, \lambda_6.$$

The (1)-rulings are $-K_6 - E$, where E runs through the (-1)-curves. The 81 equations $F_{-K_6-E,i}$ are listed in [7, Section 3.3].

For $r = 7$, let

$$p_5 = (1 : a_1 : b_1), \qquad p_6 = (1 : a_2 : b_2), \qquad p_7 = (1 : a_3 : b_3).$$

The $N_7 = 56$ (-1)-curves $E_i, m_{i,j}, Q_{i,j}, C_i$ and the corresponding generators $\xi(E)$ are described in [7, Section 3.4]. They are ordered as

$$\eta_1, \dots, \eta_7, \quad \mu_{1,2}, \dots, \mu_{1,7}, \mu_{2,3}, \dots, \mu_{6,7}, \quad \nu_{1,2}, \dots, \nu_{1,7}, \nu_{2,3}, \dots, \nu_{6,7}, \quad \lambda_1, \dots, \lambda_7.$$

By [7, Theorem 3.2], there are 529 relations $F_{D,i}$ (four for each of the 126 (1)-rulings, and 25 for the (2)-ruling $-K_7$) between them. We do not want to list them here, as they can be determined by the method of [7, Lemma 3.3] in a straightforward manner.

3. Homogeneous spaces

In this section, we examine the equations defining the affine cone $H_r \subset \mathbb{A}'_r$ over G_r/P_r for $r \in \{6, 7\}$. For the N_r coordinates $\xi'(E)$ of $\mathbb{A}'_r$, we also use the names η'_i, $\mu'_{i,j}$, λ'_i, and furthermore $\nu'_{i,j}$ in the case $r = 7$, with the obvious correspondence to the coordinates of $\mathbb{A}_r$ as in the previous section.

In particular, we show that H_r is a complete intersection on the open subset U_r of $\mathbb{A}'_r$ where the coordinates $\eta'_1, \dots, \eta'_r$ are non-zero.

We will see that H_r is defined by quadratic relations which are homogeneous with respect to the $\mathrm{Pic}(S_r)$-grading. For each (1)-ruling D, we have exactly one relation p_D of degree D, and furthermore in the case $r = 7$, we have eight relations $p^{(1)}_{-K_7}, \dots, p^{(8)}_{-K_7}$ where we use the convention $p_{-K_7} := p^{(1)}_{-K_7}$. For any possibility to write D as the sum of two (-1)-curves E, E', the relation p_D has a term $\xi'(E)\xi'(E')$ with a non-zero coefficient.

Definition 3. For a (-1)-curve E, let U_E be the open subset of $\mathbb{A}'_r$ where $\xi'(E)$ is non-zero. Let $\mathcal{N}(E)_k$ be the set of (-1)-curves E' with $(E, E') = k$, and let $\Xi'(E)_k$ be the set of the corresponding $\xi'(E')$. Let $\mathcal{N}(E)_{>k}$ and $\Xi'(E)_{>k}$ be defined similarly, but with the condition $(E, E') > k$.

Let $U_r \subset \mathbb{A}'_r$ be the intersection of $U_{E_1}, \dots, U_{E_r}$.

Note that $\mathcal{N}(E)_0$ has exactly N_{r-1} elements because we can identify its elements with the (-1)-curves on S_{r-1}. Since the only (-1)-curve intersecting E negatively is E itself, the number of elements of $\mathcal{N}(E)_{>0}$ is $N_r - N_{r-1} - 1$.

Proposition 4. *Let*

$$\Phi_r : H_r \cap U_r \to U_{r-1} \times \left(\mathbb{A}^1 \setminus \{0\}\right)$$

be the projection to the coordinates $\xi'(E) \in \Xi'(E_1)_0$ and η'_1. The map Φ_r is an isomorphism. The dimension of H_r is $N_{r-1} + 1$.

Proof. If $D = E_1 + E$ is a (1)-ruling, then $(E_1, D) = 0$, and all variables occurring in p_D besides η'_1 and $\xi'(E)$ are elements of $\Xi'(E_1)_0$. For $\eta'_1 \neq 0$, the relation p_D expresses $\xi'(E)$ in terms of η'_1 and $\Xi'(E_1)_0$.

For a (2)-ruling $D = E_1 + E$, we have $(E_1, D) = 1$, so the relation p_D expresses $\xi'(E)$ in terms of η'_1 and monomials $\xi'(E'_i)\xi'(E''_i)$ where $\xi(E'_i) \in \Xi'(E_1)_0$ and $\xi'(E''_i) \in \Xi'(E_1)_1$. Using the expressions for the elements of $\Xi'(E_1)_1$ of the first step, this shows that we can express the coordinates $\Xi'(E_1)_{>0}$ in terms of η'_1 and $\Xi'(E_1)_0$ by using the $N_r - N_{r-1} - 1$ relations g_{E_1+E} for $E \in \mathcal{N}(E)_{>0}$. This allows us to construct a map

$$\Psi_r : U_{r-1} \times \left(\mathbb{A}^1 \setminus \{0\}\right) \to \mathbb{A}'_r.$$

It remains to show that the image of Ψ_r is in H_r, i.e., that the resulting point also satisfies the remaining equations which define H_r. This is done in Lemmas 6 and 7 below.

Remark 5. Proposition 4 is also true if we enlarge U_r to U_{E_1} and U_{r-1} to $\mathbb{A}'_{r-1}$. However, the proofs of Lemmas 6 and 7 are slightly simplified by restricting to U_r.

First, let $r = 6$. Consider the cubic form in $N_6 = 27$ variables

$$F(M_1, M_2, M_3) := \det M_1 + \det M_2 + \det M_3 - \operatorname{tr}(M_1 M_2 M_3),$$

where

$$M_1 := \begin{pmatrix} \eta'_1 & \lambda'_1 & \mu'_{2,3} \\ \eta'_2 & \lambda'_2 & \mu'_{1,3} \\ \eta'_3 & \lambda'_3 & \mu'_{1,2} \end{pmatrix}, \qquad M_2 := \begin{pmatrix} \lambda'_4 & \lambda'_5 & \lambda'_6 \\ \eta'_4 & \eta'_5 & \eta'_6 \\ \mu'_{5,6} & \mu'_{4,6} & \mu'_{4,5} \end{pmatrix},$$

and

$$M_3 := \begin{pmatrix} \mu'_{1,4} & \mu'_{2,4} & \mu'_{3,4} \\ \mu'_{1,5} & \mu'_{2,5} & \mu'_{3,5} \\ \mu'_{1,6} & \mu'_{2,6} & \mu'_{3,6} \end{pmatrix}.$$

By [21, Proposition 1.6], the group of invertible $N_6 \times N_6$-matrices which leave invariant F is a simply connected linear algebraic group G_6 of type $\mathbf{E}_6$.

Note that the terms of $\operatorname{tr}(M_1 M_2 M_3)$ are $M_1^{(i,j)} M_2^{(j,k)} M_3^{(k,i)}$ for $i, j, k \in \{1, 2, 3\}$ (where $M_a^{(b,c)}$ is the entry (b, c) of the matrix M_a), so the number of terms of F is $3 \cdot 6 + 3^3 = 45$. Each is a product of three variables $\xi'(E), \xi'(E'), \xi'(E'')$ such that the corresponding (-1)-curves E, E', E'' on S_6 form a triangle, and their divisor classes add up to $-K_6$. The coefficient is $+1$ in the nine cases

$$\begin{array}{ccc} \eta'_1\mu'_{1,2}\lambda'_2, & \eta'_2\mu'_{2,3}\lambda'_3, & \eta'_3\mu'_{1,3}\lambda'_1, \\ \eta'_4\mu'_{4,6}\lambda'_6, & \eta'_5\mu'_{4,5}\lambda'_4, & \eta'_6\mu'_{5,6}\lambda'_5, \\ \mu'_{1,4}\mu'_{2,5}\mu'_{3,6}, & \mu'_{1,5}\mu'_{2,6}\mu'_{3,4}, & \mu'_{1,6}\mu'_{2,4}\mu'_{3,5} \end{array}$$

and -1 in the remaining 36 cases. (Of course, there is some choice here, for example by permuting the indices $1, \dots, 6$, but it is not as simple as choosing any 9 of the 45 terms to have the coefficient $+1$. See [14, Section 5] for more details.)

Let α_6 be the simple root at the end of one of the "long legs" in the Dynkin diagram $\mathbf{E}_6$. Let ϖ_6 be the associated fundamental weight. The action of G_6 on $\mathbb{K}^{N_6}$ is a N_6-dimensional irreducible representation of G_6 whose highest weight is ϖ_6 (cf. [6, Section 20.2]). The orbit H_6 of the weight space of ϖ_6 is described by the vanishing of the N_6 partial derivatives of the cubic form F (see [22, Section III.2.5]).

The derivative with respect to $\xi'(E)$ contains five terms $\pm\xi'(E')\xi'(E'')$ corresponding to the five ways to write the (1)-ruling $D := -K_6 - E$ as the sum of two intersecting (-1)-curves E', E''. We will denote it by $p_D = p_{-K_6-E}$.

Lemma 6. *For $\eta'_1 \neq 0$ and any values of*

$$\Xi'(E_1)_0 = \{\eta'_2, \dots, \eta'_6, \mu'_{2,3}, \dots, \mu'_{5,6}, \lambda'_1\}$$

with non-zero $\eta'_2, \dots, \eta'_6$, the equations p_{E_1+E} for

$$E \in \mathcal{N}(E_1)_1 = \{m_{1,2}, \dots, m_{1,6}, Q_2, \dots, Q_6\}$$

define a point of H_6.

Proof. As $T_{\mathrm{NS}}(S_6)$ acts on H_6 and $\{E_1, \dots, E_6\}$ is a subset of a basis of $\mathrm{Pic}(S_6)$, we may assume that $\eta'_1 = \dots = \eta'_6 = 1$. Then for $i \in \{2, \dots, 6\}$, the equation $p_{E_1+m_{1,i}}$ allows us to express $\mu'_{1,i}$ in terms of the remaining $\mu'_{i,j}$:

$$\begin{aligned} \mu'_{1,2} &= \mu'_{2,3} + \mu'_{2,4} + \mu'_{2,5} + \mu'_{2,6}, & \mu'_{1,3} &= \mu'_{2,3} - \mu'_{3,4} - \mu'_{3,5} - \mu'_{3,6}, \\ \mu'_{1,4} &= -\mu'_{2,4} - \mu'_{3,4} + \mu'_{4,5} - \mu'_{4,6}, & \mu'_{1,5} &= -\mu'_{2,5} - \mu'_{3,5} - \mu'_{4,5} + \mu'_{5,6}, \\ \mu'_{1,6} &= -\mu'_{2,6} - \mu'_{3,6} + \mu'_{4,6} - \mu'_{5,6}. \end{aligned}$$

Furthermore, for $i \in \{2, \dots, 6\}$, we can use $p_{E_1+Q_i}$ in order to express λ'_i in terms of λ'_1 and $\mu'_{j,k}$:

$$\begin{aligned} \lambda'_2 &= \mu'_{3,4}\mu'_{5,6} + \mu'_{3,5}\mu'_{4,6} + \mu'_{3,6}\mu'_{4,5} + \lambda'_1, \\ \lambda'_3 &= -\mu'_{2,4}\mu'_{5,6} - \mu'_{2,5}\mu'_{4,6} - \mu'_{2,6}\mu'_{4,5} + \lambda'_1, \\ \lambda'_4 &= -\mu'_{2,3}\mu'_{5,6} + \mu'_{2,5}\mu'_{3,6} - \mu'_{2,6}\mu'_{3,5} - \lambda'_1, \\ \lambda'_5 &= -\mu'_{2,3}\mu'_{4,6} - \mu'_{2,4}\mu'_{3,6} + \mu'_{2,6}\mu'_{3,4} - \lambda'_1, \\ \lambda'_6 &= -\mu'_{2,3}\mu'_{4,5} + \mu'_{2,4}\mu'_{3,5} - \mu'_{2,5}\mu'_{3,4} - \lambda'_1. \end{aligned}$$

Table 2
Rulings and relations defining G_7/P_7

Symbol	(1)-Ruling $D = D_I^{(n)}$	Relation p_D
$D_i^{(1)}$	$H - E_i$	v_i^8
$D_{i,j,k}^{(2)}$	$2H - (E_1 + \cdots + E_7) + E_i + E_j + E_k$	u^{ijk8}
$D_{i,j}^{(3)}$	$3H - (E_1 + \cdots + E_7) + E_i - E_j$	v_j^i
$D_{i,j,k,l}^{(4)}$	$4H - 2(E_1 + \cdots + E_7) + E_i + E_j + E_k + E_l$	u^{ijkl}
$D_i^{(5)}$	$5H - 2(E_1 + \cdots + E_7) + E_i$	v_8^i

By substituting and expanding, we check that the remaining 17 relations are fulfilled. Therefore, the resulting point lies in H_6. □

Next, we obtain similar results in the case $r = 7$ with $N_7 = 56$. By [21, Corollary 2.6], a simply connected linear algebraic group G_7 of type $\mathbf{E}_7$ is obtained as the identity component of the group of invertible $N_7 \times N_7$-matrices which leave invariant a certain quartic form defined on a vectorspace of dimension N_7 as in [21, Section 2.1]. The action of G_7 on this vectorspace is an irreducible representation whose highest weight ϖ_7 is the fundamental weight corresponding to the simple root α_7 at the end of the "longest leg" of the Dynkin diagram $\mathbf{E}_7$ (cf. [6, Section 20.2]).

We describe the orbit H_7 of the weight space of ϖ_7 under G_7 explicitly. The N_7 coordinates $\xi'(E)$ in $\mathbb{A}_7'$ are $\eta_i', \mu_{j,k}', \nu_{j,k}', \lambda_i'$ for $i, j, k \in \{1, \dots, 7\}$ and $j < k$. The equations for H_7 are described in [11] in terms of 56 coordinates x^{ij}, y_{ij} $(i < j \in \{1, \dots, 8\})$. They correspond to our variables as follows:

$$\eta_i' = x^{i8}, \qquad \mu_{k,l}' = y_{kl}, \qquad \nu_{k,l}' = x^{kl}, \qquad \lambda_i' = y_{i8}.$$

For the (1)-rulings D [7, Lemma 3.7], the relations p_D are u^{ijkl} and v_j^i as below. In the first column of Table 2, we list a symbol $D_I^{(n)}$ assigned to the (1)-ruling in the second column, and the third column gives the corresponding relation.

Let

$$u^{ijkl} = x^{ij}x^{kl} - x^{ik}x^{jl} + x^{il}x^{jk} + \sigma \cdot (y_{ab}y_{cd} - y_{ac}y_{bd} + y_{ad}y_{bc}),$$

where $i < j < k < l$ and $a < b < c < d$, with (i, j, k, l, a, b, c, d) a permutation of $(1, \dots, 8)$, and σ its sign. For $i \neq j$

$$v_j^i = \sum_{k \in (\{1,\dots,8\} \setminus \{i,j\})} x^{ik} y_{kj},$$

where $x^{ba} = -x^{ab}$ and $y_{ba} = -y_{ab}$ if $b > a$.

For the (2)-ruling $-K_7$, we have the following eight equations with 28 terms:

$$p_{-K_7}^{(i)} := v_i^i := -\frac{3}{4} \sum_{j \in (\{1,\dots,8\} \setminus \{i\})} x^{ij} y_{ij} + \frac{1}{4} \sum_{j<k \in (\{1,\dots,8\} \setminus \{i\})} x^{jk} y_{jk}.$$

Lemma 7. *For $\eta'_1, \dots, \eta'_7 \neq 0$, the 28 coordinates*

$$\eta'_i \quad \big(i \in \{1, \dots, 7\}\big), \qquad \mu'_{j,k} \quad \big(j < k \in \{2, \dots, 7\}\big), \qquad \nu'_{1,l} \quad \big(l \in \{2, \dots, 7\}\big)$$

in $\Xi'(E_1)_0$ and the 28 equations p_D for

$$D \in \big\{D^{(1)}_2, \dots, D^{(1)}_7, D^{(2)}_{1,2,3}, \dots, D^{(2)}_{1,6,7}, D^{(3)}_{1,2}, \dots, D^{(3)}_{1,7}, -K_7\big\}$$

define

$$\mu'_{1,i} \quad \big(i \in \{2, \dots, 7\}\big), \qquad \nu'_{j,k} \quad \big(j < k \in \{2, \dots, 7\}\big), \qquad \lambda'_l \quad \big(l \in \{1, \dots, 7\}\big),$$

resulting in a point on H_7.

Furthermore, we may replace p_{-K} by p_D for $D = D^{(3)}_{2,1}$.

Proof. As above, we may assume that $\eta'_1 = \dots = \eta'_7 = 1$ because of the action of $T_{\mathrm{NS}}(S_7)$. For the 27 (-1)-curves $E \in \mathcal{N}(E_1)_1$, the equation p_{E_1+E} defines $\xi'(E)$ directly in terms of the 28 variables in $\Xi'(E_1)_0$; we do not list the expressions here. By substituting these results, we use v^1_1 in order to express λ'_1 in terms of these variables:

$$\begin{aligned}
\lambda'_1 = {} & -\mu'_{2,3}\mu'_{4,5}\mu'_{6,7} + \mu'_{2,3}\mu'_{4,6}\mu'_{5,7} - \mu'_{2,3}\mu'_{4,7}\mu'_{5,6} + \mu'_{2,4}\mu'_{3,5}\mu'_{6,7} - \mu'_{2,4}\mu'_{3,6}\mu'_{5,7} \\
& + \mu'_{2,4}\mu'_{3,7}\mu'_{5,6} - \mu'_{2,5}\mu'_{3,4}\mu'_{6,7} + \mu'_{2,5}\mu'_{3,6}\mu'_{4,7} - \mu'_{2,5}\mu'_{3,7}\mu'_{4,6} + \mu'_{2,6}\mu'_{3,4}\mu'_{5,7} \\
& - \mu'_{2,6}\mu'_{3,5}\mu'_{4,7} + \mu'_{2,6}\mu'_{3,7}\mu'_{4,5} - \mu'_{2,7}\mu'_{3,4}\mu'_{5,6} + \mu'_{2,7}\mu'_{3,5}\mu'_{4,6} - \mu'_{2,7}\mu'_{3,6}\mu'_{4,5} \\
& - \mu'_{2,3}\lambda'_2 + \mu'_{2,3}\lambda'_3 - \mu'_{2,4}\lambda'_2 + \mu'_{2,4}\lambda'_4 - \mu'_{2,5}\lambda'_2 + \mu'_{2,5}\lambda'_5 \\
& - \mu'_{2,6}\lambda'_2 + \mu'_{2,6}\lambda'_6 - \mu'_{2,7}\lambda'_2 + \mu'_{2,7}\lambda'_7 - \mu'_{3,4}\lambda'_3 + \mu'_{3,4}\lambda'_4 \\
& - \mu'_{3,5}\lambda'_3 + \mu'_{3,5}\lambda'_5 - \mu'_{3,6}\lambda'_3 + \mu'_{3,6}\lambda'_6 - \mu'_{3,7}\lambda'_3 + \mu'_{3,7}\lambda'_7 \\
& - \mu'_{4,5}\lambda'_4 + \mu'_{4,5}\lambda'_5 - \mu'_{4,6}\lambda'_4 + \mu'_{4,6}\lambda'_6 - \mu'_{4,7}\lambda'_4 + \mu'_{4,7}\lambda'_7 \\
& - \mu'_{5,6}\lambda'_5 + \mu'_{5,6}\lambda'_6 - \mu'_{5,7}\lambda'_5 + \mu'_{5,7}\lambda'_7 - \mu'_{6,7}\lambda'_6 + \mu'_{6,7}\lambda'_7.
\end{aligned}$$

We check directly by substituting and expanding that the remaining equations defining H_7 are fulfilled.

As v^2_1 contains the term $\eta'_2\lambda'_1$, and $\eta'_2 \neq 0$, we may replace v^1_1 by v^2_1. □

4. Rescalings

Let $r \in \{6, 7\}$. We follow the strategy of the case $r = 5$ [17, Section 6.3] in order to describe conditions for good rescalings explicitly in terms of the rescaling factors. However, we use the results of the previous section to simplify this as follows:

Let

$$\mathcal{M}_6 := \big\{E_1 + E \bigm| E \in \mathcal{N}(E_1)_1\big\}$$

and let

$$\mathcal{M}_7 := \{E_1 + E \mid E \in \mathcal{N}(E_1)_1\} \cup \{D_{2,1}^{(3)}\}.$$

Let $\widetilde{H}_r \subset \mathbb{A}'_r$ be the variety defined by the equations g_D for $D \in \mathcal{M}_r$.

By Proposition 4, Lemmas 6 and 7, $H_r \cap U_r = \widetilde{H}_r \cap U_r$.

Remark 8. Because of $\mathcal{N}(E_1)_2 = \{C_1\}$, it could be considered more natural to use $-K_7 = E_1 + C_1$ instead of $D_{2,1}^{(3)} = E_2 + C_1$ in the definition of $\mathcal{M}_7$. However, we choose to avoid the (2)-ruling $-K_7$ for technical reasons.

Lemma 9. *A rescaling $\phi_r : \mathbb{A}_r \to \mathbb{A}'_r$ is good if and only if it embeds $\mathbb{A}(S_r)$ into $\widetilde{H}_r$.*

Proof. As $H_r \subset \widetilde{H}_r$, a good rescaling ϕ_r satisfies $\phi_r(\mathbb{A}(S_r)) \subset \widetilde{H}_r$. Conversely, we have

$$\phi_r(\mathbb{A}(S_r)) \cap U_r \subset \widetilde{H}_r \cap U_r = H_r \cap U_r$$

by Lemmas 6 and 7. Taking the closure and using that H_r is closed and that $\mathbb{A}(S_r)$ is irreducible by [2], we conclude that $\phi_r(\mathbb{A}(S_r)) \subset H_r$, so the rescaling is good. □

As in Section 2, let J_r be the ideal defining $\mathbb{A}(S_r)$ in $\mathbb{A}_r$.

In terms of the coordinate rings $\mathbb{K}[\mathbb{A}_r]$ and $\mathbb{K}[\mathbb{A}'_r]$ and in view of the previous lemma, a rescaling ϕ_r is good if, for all $D \in \mathcal{M}_r$, the ideal $J_r \subset \operatorname{rad}(J_r)$ contains $\phi_r^*(p_D)$, where p_D is the equation defining H_r corresponding to the (1)-ruling D.

As $\mathbb{K}[\mathbb{A}_r]$ and $\mathbb{K}[\mathbb{A}'_r]$ are both graded by $\operatorname{Pic}(S_r)$ and ϕ_r^* respects this grading, we need rescaling factors such that $\phi_r^*(p_D)$ of degree $D \in \mathcal{M}_r$ is a linear combination of the equations $F_{D,1}, \dots, F_{D,r-3} \in J_r$. For concrete calculations in the next sections, we describe this more explicitly:

Let $D \in \mathcal{M}_r$ be a (1)-ruling, which can be written in $r-1$ ways as the sum of two (-1)-curves E'_i, E''_i. For $i \in \{1, \dots, r-1\}$, let

$$\xi_i := \xi(E'_i)\xi(E''_i), \qquad \xi'_i := \xi'(E'_i)\xi'(E''_i), \qquad \xi''_i := \xi''(E'_i)\xi''(E''_i).$$

Then p_D has the form

$$p_D = \sum_{i=1}^{r-1} \epsilon_i \xi'_i \tag{1}$$

with $\epsilon_i \in \{\pm 1\}$.

As ξ_i vanishes exactly on $E'_i \cup E''_i$, the 2-dimensional space $H^0(S_r, \mathcal{O}(D))$ is generated by any two $\xi_i, \xi_{i'}$. Hence, all other $r-3$ elements ξ_j are linear combinations of $\xi_i, \xi_{i'}$, with non-vanishing coefficients. This gives $r-3$ relations of degree D in $\operatorname{Cox}(S_r)$. Rearranging $\xi_1, \dots, \xi_{r-1}$ such that the two elements $\xi_i, \xi_{i'}$ of our choice have the indices $r-2$ and $r-1$, we can write them as

$$F_{D,j} = \xi_j + \alpha_j \xi_{r-2} + \beta_j \xi_{r-1}, \tag{2}$$

for $j \in \{1, \dots, r-3\}$, where $\alpha_j, \beta_j \in \mathbb{K}^*$.

Suppose that $\phi_r^*(p_D)$ is a linear combination of the $F_{D,j}$ with factors λ_j:

$$\phi_r^*(p_D) - \sum_{j=1}^{r-3} \lambda_j F_{D,j} = 0.$$

Since $\phi_r^*(\xi'(E)) = \xi''(E) \cdot \xi(E)$, we have $\phi_r^*(\xi_i') = \xi_i'' \cdot \xi_i$ for the monomials of degree 2. Then the above equation is equivalent to the vanishing of

$$\sum_{i=1}^{r-3} (\epsilon_i \xi_i'' - \lambda_i)\xi_i + \left(\epsilon_{r-2}\xi_{r-2}'' - \sum_{j=1}^{r-3} \lambda_j \alpha_j\right)\xi_{r-2} + \left(\epsilon_{r-1}\xi_{r-1}'' - \sum_{j=1}^{r-3} \lambda_j \beta_j\right)\xi_{r-1}.$$

For $i \in \{1, \ldots, r-3\}$, we see by considering the coefficients of ξ_i that we must choose $\lambda_i = \epsilon_i \xi_i''$. With this, consideration of the coefficients of ξ_{r-2} and ξ_{r-1} results in the following conditions $g_{D,1}$, $g_{D,2}$ on the rescaling factors ξ_j'', which are homogeneous of degree $D \in \mathrm{Pic}(S_r)$:

$$g_{D,1} := \epsilon_{r-2}\xi_{r-2}'' - \sum_{j=1}^{r-3} \epsilon_j \alpha_j \xi_j'' = 0, \qquad g_{D,2} := \epsilon_{r-1}\xi_{r-1}'' - \sum_{j=1}^{r-3} \epsilon_j \beta_j \xi_j'' = 0.$$

Note that our choice of ξ_{r-2} and ξ_{r-1} in the definition of $F_{D,j}$ as discussed before (2) is reflected here in the sense that $g_{D,1}$ and $g_{D,2}$ express the corresponding ξ_{r-2}'' and ξ_{r-1}'' as linear combinations of $\xi_1'', \ldots, \xi_{r-3}''$ with non-zero coefficients.

This information can be summarized as follows:

Lemma 10. *For $r \in \{6, 7\}$, a rescaling is good if and only if the rescaling factors $\xi''(E)$ fulfill the equations $g_{D,1}$ and $g_{D,2}$ for each* (1)*-ruling $D \in \mathcal{M}_r$.*

As described above precisely, the non-zero coefficients ϵ_i are taken from the equations p_D (1) *defining H_r, and the non-zero α_j, β_j are taken from the equations $F_{D,j}$* (2) *defining $\mathbb{A}(S_r)$.*

Let $\Xi''(E)_k$ (respectively $\Xi''(E)_{>k}$) be the set of all $\xi''(E')$ for $E' \in \mathcal{N}(E)_k$ (respectively $E' \in \mathcal{N}(E)_{>k}$). Let

$$\Xi_{i,j}'' := \Xi''(E_1)_i \cap \Xi''(E_2)_j.$$

We claim that we may express the rescaling factors $\Xi''(E_1)_{>0} \cup \Xi''(E_2)_{>0}$ in terms of the other $N_{r-2} + 2$ rescaling factors $\{\eta_1'', \eta_2''\} \cup \Xi_{0,0}''$.

We will prove this for $r \in \{6, 7\}$ as follows: The $2 \cdot (N_r - N_{r-1} - 1)$ equations $g_{D,i}$ are homogeneous of degree D with respect to the $\mathrm{Pic}(S_r)$-grading of the variables $\xi''(E)$, and we are interested only in the solutions where all $\xi''(E)$ are non-zero. Because of the action of $T_{\mathrm{NS}}(S_r)$ on the rescaling factors and as $E_1, \ldots, E_r$ are part of a basis of $\mathrm{Pic}(S_r)$, we may assume $\eta_1'' = \cdots = \eta_r'' = 1$.

Consider a (1)-ruling $D = E_1 + E$ such that $(E_2, E) = 0$. Then

$$D = E_1' + E_1'' = \cdots = E_{r-3}' + E_{r-3}'' = E_1 + E = E_2 + E'$$

are the $r-1$ possibilities to write D as the sum of two intersecting (-1)-curves. Here, $E_i', E_i'' \in \mathcal{N}(E_1)_0 \cap \mathcal{N}(E_2)_0$. As in the discussion before Lemma 10, we may set up the equations $F_{D,j}$ such that $g_{D,1}$ and $g_{D,2}$ express $\xi''(E)$ and $\xi''(E')$ directly as a linear combination of $\xi''(E_i')\xi''(E_i'')$. This expresses all $\xi(E) \in \Xi_{1,0}''$ and all $\xi''(E') \in \Xi_{0,1}''$ in terms of variables in $\Xi_{0,0}''$.

For a (1)-ruling $D = E_1 + E$ such that $(E_2, E) = 1$, we have

$$D = E_1' + E_1'' = \cdots = E_{r-2}' + E_{r-2}'' = E_1 + E,$$

where we may assume $(E_2, E_i') = 0$ and $(E_2, E_i'') = 1$. Since $(E_1, E_i') = (E_1, E_i'') = 0$, we have $\xi''(E_i') \in \Xi_{0,0}''$ and $\xi''(E_i'') \in \Xi_{0,1}''$. Using the previous findings to express $\xi''(E_i'')$ in terms of variables $\Xi_{0,0}''$, the equation $g_{D,1}$ results in a condition on the variables $\Xi_{0,0}''$, while $g_{D,2}$ expresses $\xi''(E) \in \Xi_{1,1}''$ in terms of these variables.

In the case $r = 7$, the equation $g_{D,2}$ for $D = E_1 + C_2$ expresses $\lambda_2'' \in \Xi_{1,2}''$ in terms of variables in $\Xi_{0,1}''$, and $g_{D,1}$ gives a further condition on these variables. Furthermore, $g_{D,2}$ for the (1)-ruling $D = E_2 + C_1$ expresses $\lambda_1'' \in \Xi_{2,1}''$ in terms of $\Xi_{1,0}''$, while $g_{D,1}$ gives a further condition on them. Substituting the expressions for $\Xi_{0,1}''$ respectively $\Xi_{1,0}''$ in terms of $\Xi_{0,0}''$, we get expressions for λ_2'' and λ_1'', while the $g_{D,1}$ result in further condition on $\Xi_{0,0}''$.

We summarize this in the following lemma. For its proof, it remains to show in the following sections that the expressions for $\Xi''(E_1)_{>0} \cup \Xi''(E_2)_{>0}$ are non-zero, and that the further conditions vanish.

Lemma 11. *We can write the $N_r - N_{r-2} - 2$ rescaling factors in the set $\Xi''(E_1)_{>0} \cup \Xi''(E_2)_{>0}$ as non-zero expressions in terms of $N_{r-2} + 2$ rescaling factors $\Xi_{0,0}'' \cup \{\xi''(E_1), \xi''(E_2)\}$. With this, the $N_r - 2N_{r-1} + N_{r-2}$ further conditions on the rescaling factors are trivial.*

For an open subset of the $N_{r-2} + 2$ parameters $\{\eta_1'', \eta_2''\} \cup \Xi_{0,0}''$, all rescaling factors are non-zero, so we obtain good rescalings, which proves Theorem 1 once the proof of Lemma 11 is completed.

5. Degree 3

In this section, we prove Lemma 11 for $r = 6$ by solving the system of equations on the rescaling factors of Lemma 10: For each (1)-ruling $D \in \mathcal{M}_6$, we determine the coefficients of the equation p_D defining H_6 as in (1), and find the coefficients α_j, β_j of $F_{D,j}$ defining $\mathbb{A}(S_6)$ as in (2) in the list in [7, Section 3.3]. This allows us to write down the 20 equations $g_{D,i}$ on the rescaling factors $\xi''(E)$ explicitly. Let

$$\gamma_1 := ad - bc, \qquad \gamma_2 := (a-1)(d-1) - (b-1)(c-1)$$

for simplicity.

$$g_{E_1+m_{1,2},1} = -\eta_3''\mu_{2,3}'' - \eta_4''\mu_{2,4}'' - b\eta_5''\mu_{2,5}'' - d\eta_6''\mu_{2,6}'',$$
$$g_{E_1+m_{1,2},2} = \eta_1''\mu_{1,2}'' + \eta_4''\mu_{2,4}'' + \eta_5''\mu_{2,5}'' + \eta_6''\mu_{2,6}'',$$
$$g_{E_1+m_{1,3},1} = -\eta_1''\mu_{1,3}'' + \eta_4''\mu_{3,4}'' + \eta_5''\mu_{3,5}'' + \eta_6''\mu_{3,6}'',$$

$$\begin{aligned}
g_{E_1+m_{1,3},2} &= \eta_2''\mu_{2,3}'' + \eta_4''\mu_{3,4}'' + a\eta_5''\mu_{3,5}'' + c\eta_6''\mu_{3,6}'',\\
g_{E_1+m_{1,4},1} &= -\eta_1''\mu_{1,4}'' + \eta_3''\mu_{3,4}'' + (b-1)\eta_5''\mu_{4,5}'' + (1-d)\eta_6''\mu_{4,6}'',\\
g_{E_1+m_{1,4},2} &= -\eta_2''\mu_{2,4}'' + \eta_3''\mu_{3,4}'' + (b-a)\eta_5''\mu_{4,5}'' + (c-d)\eta_6''\mu_{4,6}'',\\
g_{E_1+m_{1,5},1} &= -\eta_2''\mu_{2,5}'' + a/b\eta_3''\mu_{3,5}'' + (a-b)/b\eta_4''\mu_{4,5}'' + \gamma_1/b\eta_6''\mu_{5,6}'',\\
g_{E_1+m_{1,5},2} &= -\eta_1''\mu_{1,5}'' + 1/b\eta_3''\mu_{3,5}'' + (1-b)/b\eta_4''\mu_{4,5}'' + (d-b)/b\eta_6''\mu_{5,6}'',\\
g_{E_1+m_{1,6},1} &= -\eta_1''\mu_{1,6}'' + 1/d\eta_3''\mu_{3,6}'' + (d-1)/d\eta_4''\mu_{4,6}'' + (b-d)/d\eta_5''\mu_{5,6}'',\\
g_{E_1+m_{1,6},2} &= -\eta_2''\mu_{2,6}'' + c/d\eta_3''\mu_{3,6}'' + (d-c)/d\eta_4''\mu_{4,6}'' - \gamma_1/d\eta_5''\mu_{5,6}'',\\
g_{E_1+Q_2,1} &= a(c-d)\eta_1''\lambda_2'' + (d-1)\eta_2''\lambda_1'' - \mu_{3,4}''\mu_{5,6}'' + \mu_{3,6}''\mu_{4,5}'',\\
g_{E_1+Q_2,2} &= \gamma_1\eta_1''\lambda_2'' + (b-d)\eta_2''\lambda_1'' - \mu_{3,5}''\mu_{4,6}'' - \mu_{3,6}''\mu_{4,5}'',\\
g_{E_1+Q_3,1} &= b(c-d)\eta_1''\lambda_3'' + (c-1)\eta_3''\lambda_1'' - \mu_{2,4}''\mu_{5,6}'' + \mu_{2,6}''\mu_{4,5}'',\\
g_{E_1+Q_3,2} &= \gamma_1\eta_1''\lambda_3'' + (a-c)\eta_3''\lambda_1'' - \mu_{2,5}''\mu_{4,6}'' - \mu_{2,6}''\mu_{4,5}'',\\
g_{E_1+Q_4,1} &= bc\eta_1''\lambda_4'' + (bc-b-c+1)\eta_4''\lambda_1'' - \mu_{2,3}''\mu_{5,6}'' + \mu_{2,6}''\mu_{3,5}'',\\
g_{E_1+Q_4,2} &= (ad-bc)\eta_1''\lambda_4'' + \gamma_2\eta_4''\lambda_1'' + \mu_{2,5}''\mu_{3,6}'' - \mu_{2,6}''\mu_{3,5}'',\\
g_{E_1+Q_5,1} &= (d-c)\eta_1''\lambda_5'' + \gamma_2\eta_5''\lambda_1'' - \mu_{2,4}''\mu_{3,6}'' + \mu_{2,6}''\mu_{3,4}'',\\
g_{E_1+Q_5,2} &= c\eta_1''\lambda_5'' + (a-c)(1-b)\eta_5''\lambda_1'' - \mu_{2,3}''\mu_{4,6}'' - \mu_{2,6}''\mu_{3,4}'',\\
g_{E_1+Q_6,1} &= (b-a)\eta_1''\lambda_6'' + \gamma_2\eta_6''\lambda_1'' + \mu_{2,4}''\mu_{3,5}'' - \mu_{2,5}''\mu_{3,4}'',\\
g_{E_1+Q_6,2} &= a\eta_1''\lambda_6'' + (c-a)(d-1)\eta_6''\lambda_1'' - \mu_{2,3}''\mu_{4,5}'' + \mu_{2,5}''\mu_{3,4}''.
\end{aligned}$$

As explained in the previous section, we may assume $\eta_1'' = \cdots = \eta_6'' = 1$.

Recall the discussion before the definition (2) of $F_{D,j}$ and before Lemma 10. If we had chosen $\xi_4 = \eta_i\lambda_1$ and $\xi_5 = \eta_1\lambda_i$ when writing down the equations $F_{E_1+Q_i,j}$ in [7, Section 3.3], then the resulting $g_{E_1+Q_i,2}$ would give λ_i'' directly as a quadratic expression in terms of $\mu_{j,k}''$. Furthermore, each of the five $g_{E_1+Q_i,1}$ would express λ_1'' as a quadratic equation in $\mu_{j,k}''$. Of course, we get the same result by solving the equivalent system of equations $g_{E_1+Q_i,j}$ as listed above.

The equations $g_{E_1+m_{1,i},j}$ for $i \in \{3,\dots,6\}$ and $g_{E_1+Q_2,j}$ allow us to express the variables $\mu_{1,i}''$, λ_2'' in $\Xi_{1,0}''$ and $\mu_{2,i}''$, λ_1'' in $\Xi_{0,1}''$ in terms of the six variables $\mu_{3,4}'',\dots,\mu_{5,6}'' \in \Xi_{0,0}''$. (As we have set the remaining elements $\eta_3'',\dots,\eta_6''$ of $\Xi_{0,0}''$ to the value 1, they do not occur in these expressions.)

With $\gamma_3 := d(a-c)(1-b) - c(b-d)(1-a)$, we obtain:

$$\begin{aligned}
\mu_{1,3}'' &= \mu_{3,4}'' + \mu_{3,5}'' + \mu_{3,6}'',\\
\mu_{2,3}'' &= -\mu_{3,4}'' - a\mu_{3,5}'' - c\mu_{3,6}'',\\
\mu_{1,4}'' &= \mu_{3,4}'' + (b-1)\mu_{4,5}'' + (1-d)\mu_{4,6}'',\\
\mu_{2,4}'' &= \mu_{3,4}'' + (b-a)\mu_{4,5}'' + (c-d)\mu_{4,6}'',\\
\mu_{1,5}'' &= 1/b\mu_{3,5}'' + (1-b)/b\mu_{4,5}'' + (d-b)/b\mu_{5,6}'',\\
\mu_{2,5}'' &= a/b\mu_{3,5}'' + (a-b)/b\mu_{4,5}'' + \gamma_1/b\mu_{5,6}'',
\end{aligned}$$

$$\mu''_{1,6} = 1/d\mu''_{3,6} + (d-1)/d\mu''_{4,6} + (b-d)/d\mu''_{5,6},$$
$$\mu''_{2,6} = c/d\mu''_{3,6} + (d-c)/d\mu''_{4,6} - \gamma_1/d\mu''_{5,6},$$
$$\lambda''_1 = -\gamma_1/\gamma_3\mu''_{3,4}\mu''_{5,6} - a(d-c)/\gamma_3\mu''_{3,5}\mu''_{4,6} - c(b-a)/\gamma_3\mu''_{3,6}\mu''_{4,5},$$
$$\lambda''_2 = (b-d)/\gamma_3\mu''_{3,4}\mu''_{5,6} + (1-d)/\gamma_3\mu''_{3,5}\mu''_{4,6} + (1-b)/\gamma_3\mu''_{3,6}\mu''_{4,5}.$$

For $E \in \{m_{1,2}, Q_3, \ldots, Q_6\}$, we consider the remaining equations $g_{E_1+E,i}$. We can use $g_{E_1+E,2}$ and substitution of our previous results in order to express $\xi''(E) \in \Xi''_{1,1}$ in terms of $\Xi''_{0,0}$:

$$\begin{aligned}
\mu''_{1,2} &= \mu''_{3,4} - a/b\mu''_{3,5} - c/d\mu''_{3,6} + (a-b)(b-1)/b\mu''_{4,5} \\
&\quad + (d-c)(d-1)/d\mu''_{4,6} + (b-d)\gamma_1/(bd)\mu''_{5,6}, \\
\lambda''_3 &= (a-c)/\gamma_3\mu''_{3,4}\mu''_{5,6} + a(1-c)/(b\gamma_3)\mu''_{3,5}\mu''_{4,6} + c(1-a)/(d\gamma_3)\mu''_{3,6}\mu''_{4,5} \\
&\quad + 1/(bd)\mu''_{4,5}\mu''_{4,6} - 1/d\mu''_{4,5}\mu''_{5,6} + 1/b\mu''_{4,6}\mu''_{5,6}, \\
\lambda''_4 &= \gamma_2/\gamma_3\mu''_{3,4}\mu''_{5,6} - 1/(bd)\mu''_{3,5}\mu''_{3,6} + (1-d)(c-d)(a-1)/(d\gamma_3)\mu''_{3,5}\mu''_{4,6} \\
&\quad - 1/d\mu''_{3,5}\mu''_{5,6} + (1-c)(b-1)(a-b)/(b\gamma_3)\mu''_{3,6}\mu''_{4,5} - 1/b\mu''_{3,6}\mu''_{5,6}, \\
\lambda''_5 &= 1/d\mu''_{3,4}\mu''_{3,6} - 1/d\mu''_{3,4}\mu''_{4,6} + (b-d)(1-a)\gamma_1/(d\gamma_3)\mu''_{3,4}\mu''_{5,6} \\
&\quad + a\gamma_2/\gamma_3\mu''_{3,5}\mu''_{4,6} + (b-1)(a-b)(a-c)/\gamma_3\mu''_{3,6}\mu''_{4,5} - \mu''_{3,6}\mu''_{4,6}, \\
\lambda''_6 &= -1/b\mu''_{3,4}\mu''_{3,5} - 1/b\mu''_{3,4}\mu''_{4,5} + (1-c)(b-d)\gamma_1/(b\gamma_3)\mu''_{3,4}\mu''_{5,6} \\
&\quad - \mu''_{3,5}\mu''_{4,5} + (d-1)(c-d)(a-c)/\gamma_3\mu''_{3,5}\mu''_{4,6} + c\gamma_2/\gamma_3\mu''_{3,6}\mu''_{4,5}.
\end{aligned}$$

Finally, we check by substituting and expanding that the five further conditions $g_{E_1+E,1}$ are trivial.

Using the restrictions on a, b, c, d imposed by the fact that $p_1, \ldots, p_6$ are in general position (e.g., a must be different from b and c, and all are neither 0 nor 1), we see that $\mu''_{1,2}, \ldots, \mu''_{2,6}, \lambda''_1, \ldots, \lambda''_6$ are non-zero polynomials in $\mu''_{3,4}, \ldots, \mu''_{5,6}$. Therefore, for an open subset of the $N_4 + 2 = 12$ parameters $\eta''_1, \ldots, \eta''_6, \mu''_{3,4}, \ldots, \mu''_{5,6}$, all rescaling factors are non-zero, resulting in good rescalings.

6. Degree 2

For the proof of Lemma 11 for $r = 7$, we proceed as in the case $r = 6$ and assume $\eta''_1 = \cdots = \eta''_7 = 1$.

Let $D := D_i^{(1)} \in \mathcal{M}_7$ for $i \in \{3, \ldots, 7\}$. We can arrange $F_{D,1}, \ldots, F_{D,4}$ in such a way that $g_{D,1}$ and $g_{D,2}$ express $\mu''_{1,i}$ and $\mu''_{2,i}$ in terms of $\mu''_{i,j}$ for $j \in \{3, \ldots, 7\} \setminus \{i\}$ (see the discussion before Lemma 10). Similarly, for $i \in \{3, \ldots, 7\}$ and $D := D_{1,2,i}^{(2)} \in \mathcal{M}_7$, we can arrange $g_{D,1}$ and $g_{D,2}$ such that they express $\nu''_{1,i}$ and $\nu''_{2,i}$ linearly in $\nu''_{1,2}$ and of degree 2 in $\mu''_{3,4}, \ldots, \mu''_{6,7}$. This expresses all variables in $\Xi''_{0,1} \cup \Xi''_{1,0}$ in terms of $\Xi''_{0,0}$.

Substituting this into an appropriately arranged $g_{D,2}$ for $D := D_2^{(1)} \in \mathcal{M}_7$ gives $\mu''_{1,2} \in \Xi''_{1,1}$ in terms of $\mu''_{3,4}, \ldots, \mu''_{6,7}$, and we check that $g_{D,1}$ becomes trivial.

Now, let $D := D^{(2)}_{1,i,j} \in \mathcal{M}_7$ for $i < j \in \{3,\dots,7\}$. We arrange $g_{D,1}$ and $g_{D,2}$ such that they express $\nu''_{1,i}$ respectively $\nu''_{i,j}$ in terms of $\nu''_{1,j}$ and expressions of degree 2 in $\mu''_{k,l}$. Using our previous findings, the first expression turns out trivial, and the second one gives $\nu''_{i,j} \in \Xi''_{1,1}$ in terms of $\nu''_{1,2}, \mu''_{3,4}, \dots, \mu''_{6,7} \in \Xi''_{0,0}$.

Finally, let $D \in \{D^{(3)}_{1,2}, \dots, D^{(3)}_{1,7}, D^{(3)}_{2,1}\} \subset \mathcal{M}_7$. We arrange $g_{D,1}$ and $g_{D,2}$ such that the first one is an expression in $\mu''_{j,k}$ and $\nu''_{j,k}$ which becomes trivial. The second one expresses λ''_i in terms of $\mu''_{j,k}$ and $\nu''_{j,k}$, and we substitute again our previous results.

This completes the proof of Lemma 11 and thus Theorem 1. In total, we obtain good rescalings for an open subset of a system $N_5 + 2 = 18$ parameters

$$\eta''_1, \dots, \eta''_7, \quad \mu''_{3,4}, \dots, \mu''_{6,7}, \quad \nu''_{1,2}.$$

Since it is straightforward to determine the exact expressions for the remaining 38 rescaling factors in terms of these parameters, and since the expressions are rather long, we choose not to list them here.

Remark 12. In principle, it would be possible to consider the conditions $g_{D,i}$ for all (1)-rulings without reducing to the subset $\mathcal{M}_r$ as we did in Section 3. While this is doable in degree 3 with some software help (`Magma`), especially the expressions corresponding to the (1)-ruling $D^{(5)}_i$ in degree 2 seem to be out of reach for direct computations. Furthermore, we would have to embed the relations v^i_i corresponding to the (2)-ruling $-K_7$, which causes further complications.

Remark 13. For $r \in \{5,6,7\}$, there is a $(N_{r-2} - r + 1)$-parameter family of images of $\mathbb{A}(S_r)$ under good embeddings in H_r by Remark 2. The dimension of $\mathbb{A}(S_r)$ is $r+3$, and there is a $(2 \cdot (r-4))$-parameter family of smooth Del Pezzo surfaces S_r of degree $9-r$. The dimension of H_r is $N_{r-1} + 1$.

In fact, [17, Section 6.3] shows that the closure of the union of all these images for all Del Pezzo surfaces of degree 4 equals H_5.

For $r = 6$, by comparing the dimensions and numbers of parameters, a similar result seems possible. However, for $r = 7$, we have

$$(N_{r-2} - r + 1) + (r+3) + 2 \cdot (r-4) = 26,$$

while H_7 has dimension $N_{r-1} + 1 = 28$. Consequently, the closure of the union of the corresponding images, over all Del Pezzo surfaces of degree 2, under all good embeddings cannot be H_7 for dimension reasons.

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